Curves and Jacobians

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Preface

Workshop on Curves and Jacobians took place at Sol Beach Resort Yangyang in October 18-21, 2010, bringing together more than forty participants. The workshop was a gathering of mathematicians as well as graduate students in certain inter-related fields in Algebraic geometry; classical Brill-Noether theory, Galois coverings, minimal free resolutions. The core of the workshop consisted of two series lectures, delivered by prominent experts; in addition to these, there were five specialized seminar talks.

The organizers, Jun-Muk Hwang and Young Rock Kim, and the audience were fortunate to have lectures of very high quality, and the present book represents all the lectures and talks delivered at the workshop. We hope that it reflects faithfully the present state of research in the fields covered, and that it may provide an access to these files for future investigations. We believe that the workshop becomes a cornerstone for mutual cooperation between the mathematics community and others.

The workshop was financially supported by National Researcher Program 2010-0020413 of NRF and MEST. In the name of all participants, the organizers thank NRF for the support which made the event possible and the speakers for contributing the manuscripts. Also we would like to extend our appreciation to the staff members of Korea Institute for Advanced Study for their devotion to the success of the workshop.

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The Organizers

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Lectures on Brill-Noether theory

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ABSTRACT. These notes are the summary of lectures given by the author, in the framework of *Joint Lectures of F. Flamini and E. Sernesi*, at the Workshop "Curves and Jacobians", held on October 18-21, 2010, Sol Beach Resort, Yangyang (Korea).

1. Preliminaries and notation

In this section, we will fix once and for all our assumptions and notation. We shall also recall some basic, well-known facts which will be frequently used in the next chapters. For non-reminded terminology, the reader is referred to standard references as **[3, 5]**

From now on, C will denote a smooth, irreducible, projective curve over $\mathbb{C}$, the field of complex numbers. The non-negative integer g denotes the geometric genus (or simply, the genus) of C.

$Div(C)$ will denote the group of Cartier divisors on C and, for any $D_1, D_2 \in Div(C)$, $D_1 \sim D_2$ will denote the linear equivalence of divisors. The symbol K_C will denote a *canonical divisor* on C and $\omega_C = \mathcal{O}_C(K_C)$ the *canonical line-bundle.*

Given $D \in Div(C)$, $\mathcal{O}_C(D) \in Pic(C)$ will denote the line bundle (equivalently invertible sheaf) on C determined by D, where $Pic(C)$ is the *Picard group* of C. Given $L = \mathcal{O}_C(D) \in Pic(C)$, for some $D \in Div(C)$, we will denote indifferently by

$$H^0(C, \mathcal{O}_C(D)) = H^0(C, L) = H^0(C, D)$$

the associated vector space of global sections.

1.1. Basics on linear systems. For any subspace $V \subseteq H^0(C, D)$, $|V|$ will denote the *linear system* (or *linear series*) determined by V; therefore $|V| = \mathbb{P}(V)$ canonically, where $\mathbb{P}(V)$ denote the projective space parametrizing one-dimensional subspaces of V. If $\deg(D) = n$ and $\dim(V) = r + 1 \geq 1$, then $|V|$ of degree n and of (projective) dimension r, or simpler, $|V|$ is a $\mathfrak{g}^r_n$. If $V = H^0(C, L) = H^0(C, D)$, then $|V| = |L| = |D|$ is the *complete linear system* associated to L.

Assume to have a $|V| = \mathfrak{g}^r_n$ with no base points (if $|V| = |L|$, this means L globally generated); thus, the $\mathfrak{g}^r_n$ defines a morphism

$$\varphi = \varphi_V : C \to \mathbb{P}(V^\vee) \cong \mathbb{P}^r, \tag{1}$$

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(where the isomorphism on the target space is NOT canonical), defined by

$$\varphi(p) := \{\sigma \in V \mid \sigma(p) = 0\}, \text{ for any } p \in C.$$

If otherwise $B :=$ base locus of$(|V|)$, then the morphism φ is defined as $\varphi := \varphi_{V(-B)}$. For complete linear system, we simply use either φ_L or φ_D, when $L = \mathcal{O}_C(D)$. Let

$$\Gamma := Im(\varphi) \subset \mathbb{P}(V^\vee).$$

By construction Γ is non-degenerate (i.e. not contained in a hyperplane of $\mathbb{P}(V^\vee)$).

If $|V| = \mathfrak{g}^r_n$ is base-point-free, then

$$n = \deg(\Gamma)\deg(\varphi).$$

(i) If $\deg(\varphi) = 1$, i.e. φ is birational onto Γ, then $|V|$ is said to be *simple* or *birational very-ample* and φ_V is the *normalization* of Γ.

(ii) If $\deg(\varphi) = k > 1$, then $|V|$ is said to be *composite* or *composed with an involution* f, i.e. one has a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & \Gamma \\ \searrow^{f} & & \nearrow_{\psi} \\ & \Gamma' & \end{array} \tag{2}$$

where
- Γ' is a smooth curve of genus g';
- f is an involution of degree k and genus g', i.e. a non constant, finite morphism of degree k between two smooth, projective curves of genera g and g', respectively;
- ψ is the normalization of Γ.

Recall that C is said to be *hyperelliptic* if it admits a $|\Delta| = \mathfrak{g}^1_2$ complete and base-point-free; in other words, φ_Δ is a rational involution of degree 2. Recall that the $\mathfrak{g}^1_2$ is uniquely determined on C (cf. [**1**, Ex. D-9, p. 41] or § 2).

DEFINITION 1.1. Let $|V_i| \subseteq |D_i|$ be non-empty linear series, where $D_i \in Div(C)$ effective, , $1 \leq i \leq 2$. The *minimal sum* $|V_1| + |V_2|$ is the smallest linear subseries of $|D_1 + D_2|$ containing every divisor of the form $\Delta_1 + \Delta_2$, where $\Delta_i \in |V_i|$, $1 \leq i \leq 2$.

Notation: if $|V| \subseteq |D|$, for some effective $D \in Div(C)$, then

$$h|V| := |V| + |V| + \cdots + |V|,$$

where the above sum contains h summands.

Remark 1.2. It is obvious that $\dim(|V_1| + |V_2|) \geq \dim(|V_1|) + \dim(|V_2|)$. In particular, $\dim(h|V|) \geq h\dim(|V|)$.

DEFINITION 1.3. Let $L \in Pic(C)$ be a line bundle. If L is very-ample, φ_L is an embedding and $\Gamma = \varphi_L(C)$ is said to be *linearly normal* in $\mathbb{P}(H^0(L)^\vee)$, i.e. Γ is not obtained as the birational projection of a curve from a higher dimensional projective space.

L is said to be *k-normal* if the natural multiplication map

$$m_k : Sym^k(H^0(L)) \longrightarrow H^0(L^{\otimes k}) \tag{3}$$

is surjective (observe that linearly normal means 1-normal). It is said to be *normally generated* if the multiplication maps m_k are surjective for all $k \geq 0$.

Remark 1.4. By definition of minimal sum, one has

$$k|L| = Im(m_k),$$

for any $k \geq 1$. Therefore, L normally generated means that the linear systems cut-out on $\Gamma = \varphi_L(C)$ by the hypersurfaces of every degree $k \geq 1$ in $\mathbb{P}(H^0(L)^\vee)$ are complete, i.e. $k|L| = |L^{\otimes k}|$. In such a case Γ is said to be *projectively normal* in $\mathbb{P}(H^0(L)^\vee)$.

Exercise Let $Q_2 \subset \mathbb{P}^3$ be a smooth quadric. Let ℓ_1, ℓ_2 be two skew lines in the same ruling of Q_2. Let Q_4 be a general quartic surface passing through ℓ_1 and ℓ_2. Show that

$$X := Q_2 \cap Q_4 = \ell_1 \cup \ell_2 \cup Y,$$

where Y is a smooth, irreducible curve of degree 6 which is linearly normal but not 2-normal in $\mathbb{P}^3$, where $L = \mathcal{O}_Y(1)$.

Remark 1.5. [Geometric consequence of k-normality] Assume L to be very ample and k-normal. Let $\Gamma = \varphi_L(C) \subset \mathbb{P}(H^0(L)^\vee) \cong \mathbb{P}^r$. Consider the exact sequence defining Γ as a subscheme of $\mathbb{P}^r$:

$$0 \to \mathcal{I}_{\Gamma/\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_\Gamma \to 0$$

and tensor it by $\mathcal{O}_{\mathbb{P}^r}(k)$; one gets

$$0 \to \mathcal{I}_{\Gamma/\mathbb{P}^r}(k) \to \mathcal{O}_{\mathbb{P}^r}(k) \to \mathcal{O}_\Gamma(k) \to 0.$$

Passing to cohomology, one gets

$$0 \to H^0(\mathcal{I}_{\Gamma/\mathbb{P}^r}(k)) \to H^0(\mathcal{O}_{\mathbb{P}^r}(k)) \xrightarrow{\rho_k} H^0(\mathcal{O}_\Gamma(k)) \to H^1(\mathcal{I}_{\Gamma/\mathbb{P}^r}(k)) \to 0.$$

Since Γ is non-degenerate, then

$$Sym^k(H^0(L)) \cong H^0(\mathcal{O}_{\mathbb{P}^r}(k)),\ H^0(\mathcal{O}_\Gamma(k)) \cong H^0(L^{\otimes k}),\ \rho_k = m_k.$$

Therefore Γ is k-normal if and only if $h^1(\mathcal{I}_{\Gamma/\mathbb{P}^r}(k)) = 0$, i.e.

$$h^0(\mathcal{I}_{\Gamma/\mathbb{P}^r}(k)) = \binom{r+k}{k} - h^0(L^{\otimes k}), \tag{4}$$

where the last summand is easily computable via Riemann-Roch Theorem on C (cf. Thorem 1.6). Therefore, if Γ is k-normal one knows exactly the number of linearly independent hypersurfaces of degree k in $\mathbb{P}(H^0(L)^\vee)$ containing Γ.

Recall the following well-known results:

Theorem 1.6. *For any $D \in Div(C)$, one has*

(i) *[Riemann-Roch Theorem]* $h^0(D) - h^1(D) = \deg(D) - g + 1$;
(ii) *[Serre duality]* $h^1(D) = h^0(K_C - D)$.

The non-negative integer $h^1(D)$ is called the *index of speciality* of $|D|$, which is usually also denoted by $i(D)$. The divisor D is said to be *special* if $i(D) > 0$, *non-special* otherwise. From Serre duality, if D is special then, $\deg(D) \leq 2g-2$ and all the divisors in $|D|$ are of the same speciality; thus, $|D|$ is said to be a *special linear system.* Observe, in particular, that $|\omega_C|$ is a linear system of speciality 1.

If $|D| = \mathfrak{g}^r_n$ of speciality $i := i(D)$, the Riemann-Roch theorem states

$$r - i = n - g. \tag{5}$$

If $g = 0, 1$, there are no special divisors on C. On the other hand, for $g \geq 2$, any $D \in Div^n(C)$ with $n \leq g - 1$ is special (cf. also Remark 2.5-(1)).

The study of special linear systems on a given curve C plays a central role in the theory of algebraic curves and their moduli. Indeed, from the point of view of NON-SPECIAL linear systems, all curves of the same genus g look alike. It is at the level of SPECIAL linear systems that differences appear.

Example If C is of genus $g \geq 3$, then it admits a $\mathfrak{g}^1_2$ (of speciality $i = g - 1$) if and only if it is hyperelliptic. On the other hand, we shall recall in § 4, that the *general* curve of genus $g \geq 3$ is non-hyperelliptic.

As it follows from the previous example, the existence of "particular" special linear systems on a given curve gives strong constraints on its geometry. With this set-up, natural questions which arise are the following:

Problems Let C be a smooth, projective curve of genus $g \geq 2$.

(i) What are the possible values of r and n for which there exists a special $\mathfrak{g}^r_n$ on C?
(ii) If C admits a special $\mathfrak{g}^r_n$, how many other special $\mathfrak{g}^r_n$'s C actually admits?
(iii) How does the set (or the scheme) describing such $\mathfrak{g}^r_n$'s looks like?
(iv) Which kind of geometric properties are induced on projective embeddings of C by the existence on C of a (family of) special $\mathfrak{g}^r_n$?

Answers to the above questions are very intricate and will depend in general not only on g but also on the choice of the curve C of genus g. This is the core of the so called *Brill-Noether theory*, which we will discuss in § 4.

Anyhow, some first remarks can still be done. If $|V| \subseteq |D|$ is a $\mathfrak{g}^r_n$ on C of genus g, a necessary condition is $n \geq r$. More precisely, $\dim(|V|) \leq \dim(|D|) = n - g + i(D)$; therefore, $i(D) < g$ and $n > r$ unless $g = 0$; in this latter case, necessarily $i(D) = 0$, $|V| = |D|$ and $n = r$. In other words, C has a $\mathfrak{g}^r_r$ (necessarily complete and non-special) if and only if $g = 0$.

If $g = 0$, for every $P \in \mathbb{P}^1$, one has $|P| = \mathfrak{g}^1_1$. Therefore any two points on $\mathbb{P}^1$ are linearly equivalent; it follows that, for any $r \geq 1$, there is only one $\mathfrak{g}^r_r := |rP|$ on $\mathbb{P}^1$, which is necessarily very-ample.

Consequence *Every irreducible, non-degenerate curve $\Gamma \subset \mathbb{P}^r$ has degree at least r; equality holds if and only if Γ is a smooth, rational curve of degree r.*

1.2. Rational normal curves. Let $P \in \mathbb{P}^1$ be any point. For any $r \in \mathbb{Z}$, consider $\mathcal{O}_{\mathbb{P}^1}(rP)$; from what recalled above, if $r \geq 1$, $\mathcal{O}_{\mathbb{P}^1}(rP)$ is very ample on $\mathbb{P}^1$, so the morphism φ_{rP} associated to $|rP|$ is an embedding, which is called the *Veronese embedding of* $\mathbb{P}^1$ into $\mathbb{P}^r$. Its image $\Gamma := \varphi_{rP}(\mathbb{P}^1) \subset \mathbb{P}^r$ is called a *rational normal curve*; from the above discussion, it is a rational, non-degenerate curve of degree r, which is the minimal possible degree of an irreducible, non-degenerate curve. Up to projective transformations in $\mathbb{P}^r$, the map $\varphi_{rP} : \mathbb{P}^1 \longrightarrow \mathbb{P}^r$ is nothing but

$$[T_0, T_1] \xrightarrow{\varphi_{rP}} [T_0^r, T_0^{r-1}T_1, \ldots, T_0T_1^{r-1}, T_1^r]. \tag{6}$$

Proposition 1.7. *Any rational normal curve $\Gamma \subset \mathbb{P}^r$ is projectively normal, for any $r \geq 2$. In particular, $\mathcal{O}_\Gamma(1)$ is normally generated.*

Proof. Since by the r-tuple Veronese embedding, we have the identification $\mathcal{O}_\Gamma(1) \cong \mathcal{O}_{\mathbb{P}^1}(rP)$, for any $P \in \mathbb{P}^1$, then $\mathcal{O}_\Gamma(k) \cong \mathcal{O}_{\mathbb{P}^1}(krP)$, for any $k \geq 1$. For dimensional reasons, one has $k|rP| = |krP|$ on $\mathbb{P}^1$, for any $k \geq 1$. By Remark 1.4, one concludes. □

From (4) and Proposition 1.7, it follows that $h^0(\mathcal{I}_{\Gamma/\mathbb{P}^r}(2)) = \binom{r}{2}$, i.e. any rational normal curve is contained in $\binom{r}{2}$ linearly independent quadrics. A basis for $H^0(\mathcal{I}_{\Gamma/\mathbb{P}^r}(2))$ can be easily constructed by using (6); indeed, up to projective transformations, this basis is given by the maximal minors of the $(2 \times r)$-matrix of linear forms

$$\begin{pmatrix} X_0 & X_1 & \cdots & X_{r-1} \\ X_1 & X_2 & \cdots & X_r \end{pmatrix}. \tag{7}$$

It is easy to check that the intersections of these independent quadrics is exactly Γ. One could even be more precise; indeed,

Proposition 1.8. *Γ is scheme-theoretically the intersection of the $\binom{r}{2}$ quadrics generating $H^0(\mathcal{I}_{\Gamma/\mathbb{P}^r}(2))$.*

Proof. The reader is referred to [**2, 7**]. □

1.3. Surfaces of minimal degree. Let $r \geq 3$ and let $S \subset \mathbb{P}^r$ be a reduced, irreducible, non-degenerate surface. From § 1.2, $\deg(S) \geq r-1$; indeed $r-1$ is the minimal possible degree for a general hyperplane section of S, which must be irreducible and non-degenerate.

Definition 1.9. Let $S \subset \mathbb{P}^r$ be a reduced, irreducible, non-degenerate surface of $\deg(S) = r-1$. Then S is said to be a reduced, irreducible surface of *minimal degree*.

Examples of reduced, irreducible surfaces of minimal degree are the following:

(i) a cone projecting a rational normal curve $\Gamma \subset \mathbb{P}^{r-1} \subset \mathbb{P}^r$ from a point $O \in \mathbb{P}^r \setminus \mathbb{P}^{r-1}$;
(ii) the Veronese surface V_4 of degree 4 in $\mathbb{P}^5$ (i.e. $V_4 = v_2(\mathbb{P}^2)$, where v_2 is the Veronese double embedding of $\mathbb{P}^2$ via the complete linear system of conics);
(iii) a rational ruled surface (or *rational scroll*) $S_{k,r-k-1}$, $1 \leq k \leq r-2$, which is the joint variety of a rational normal curve $\Gamma_k \subset \Lambda_k$ and a rational normal curve $\Gamma_{r-k-1} \subset \Lambda_{r-k-1}$, where Λ_k and Λ_{r-k-1} are two linear subspaces which are skew in $\mathbb{P}^r$.

Observe that only Example (i) is singular.

Surfaces of minimal degree are characterized by the following well-known result (for a proof, see e.g. [**3**, p. 525]).

Theorem 1.10. [*Del Pezzo*] *Let $r \geq 3$. Every reduced, irreducible surface of minimal degree in $\mathbb{P}^r$ belongs to the list of examples above.*

Observe, in particular, that the general hyperplane section of a surface of minimal degree is a rational normal curve.

2. The canonical curve

Given C of genus g, consider its canonical line-bundle ω_C, which is of speciality 1 and degree $2g-2$.

If $g=0$, then $C \cong \mathbb{P}^1$; thus $\omega_C \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ and $|K_C| = \emptyset$. Otherwise, one has:

Proposition 2.1. *Let $g > 0$. Then $|K_C|$ is base-point-free; moreover, it is the unique $\mathfrak{g}^{g-1}_{2g-2}$ on C.*

Proof. As in [**3**, pg. 247]. □

From Proposition 2.1, $|K_C|$ defines a morphism φ_{K_C} (which will be also denoted by κ, [**8**, §2]), called the *canonical morphism*,

$$\kappa : C \to \mathbb{P}(H^0(\omega_C)^\vee) \cong \mathbb{P}(H^1(\mathcal{O}_C)). \tag{8}$$

From now on, $\mathbb{P}(H^0(\omega_C)^\vee) \cong \mathbb{P}(H^1(\mathcal{O}_C))$ will be simply denoted by $\mathbb{P}$ and called the *canonical space*; observe that $\mathbb{P} \cong \mathbb{P}^{g-1}$ not canonically.

For $g=1$, κ is constant. For $g=2$, κ is the hyperelliptic involution of degree 2 and genus 0 on C (in other words, κ represents C as a 2-1 cover of $\mathbb{P}^1$). Otherwise

Proposition 2.2. *Let $g \geq 3$. Then $|K_C|$ is very ample if and only if C is non-hyperelliptic.*

Proof. As in [**3**, pg. 247]. □

Consequences: Let C be of genus $g \geq 3$.

• If C is non-hyperelliptic, then κ is an embedding and $\kappa(C) = \Gamma \subset \mathbb{P}$, where Γ is a smooth curve of genus g and of degree $2g-2$ in $\mathbb{P}^{g-1}$. Since $\omega_C \cong \omega_\Gamma = \mathcal{O}_\Gamma(1)$, then Γ is called a *canonical curve.*

• If otherwise C is hyperelliptic, then it admits a $\mathfrak{g}^1_2$, call it $|\Delta|$; therefore, for any $p \in C$ there exists a $D_p \in |\Delta|$ such that $p \in Supp(D_p)$. This happens if and only if $h^0(K_C - p) = h^0(K_C - D_p) = g-1$, i.e. if and only if $|K_C|$ is composed. Looking to diagram (2), since $\deg(\kappa) \geq 2$, then $\deg(\Gamma) \leq g-1$. On the other hand, since $\kappa(C) = \Gamma \subset \mathbb{P}$ is non degenerate, then $\deg(\Gamma) \geq g-1$. Therefore, Γ is a rational normal curve of degree $g-1$ in $\mathbb{P} \cong \mathbb{P}^{g-1}$ (cf. §1.2). In particular, in this situation, diagram (2) is nothing but

$$\begin{array}{ccc} C & \xrightarrow{\kappa} & \Gamma \\ & \searrow^{\varphi_\Delta} \qquad \nearrow_{\varphi_{(g-1)q}} & \\ & \mathbb{P}^1 & \end{array} \tag{9}$$

where $q \in \mathbb{P}^1$ is any point, $\varphi_{(g-1)q}$ is the $(g-1)$-tuple Veronese embedding (cf. § 1.2). In particular, from Proposition 2.1, we refind that the hyperelliptic involution on C is uniquely determined (cf. § 1.1 and [**1**, Ex. D-9, p. 41]).

2.1. Geometric version of Riemann-Roch Theorem.

In this paragraph, we want to underline the geometric consequences which are behind the definition of speciality of a linear system and the Riemann-Roch theorem.

DEFINITION 2.3. Let $\Gamma \subset \mathbb{P}^r$ be a smooth curve and let $D \in Div^n(\Gamma)$ be any effective divisor. The *linear span of D*, denoted by $\langle D \rangle$, is the intersection of all the hyperplanes in $\mathbb{P}^r$ such that $\mathcal{O}_\Gamma(H-D)$ is effective (i.e. $D \subseteq H \cdot \Gamma$)

Assume C to be a smooth, non-hyperelliptic, projective curve of genus $g \geq 3$. From Proposition 2.2, let $\Gamma \subset \mathbb{P}$ be the canonical model of C, as a non-degenerate curve of genus g and degree $2g-2$. If $D \in Div^n(C)$ is of speciality $i := i(D)$, let $\kappa(D) \in \Gamma$. Since $\omega_C \cong \mathcal{O}_\Gamma(1)$ then, by Serre duality,

$$(10) \quad i = h^0(K_C - D) = \left\{ \begin{array}{l} \text{number of linearly independent hyperplanes} \\ \text{in } \mathbb{P} \text{ passing through } \kappa(D) \end{array} \right\}.$$

In other words, $i = codim_{\mathbb{P}}(\langle \kappa(D) \rangle)$. Therefore, $D \in Div^n(C)$ is special if and only if $\langle \kappa(D) \rangle$ has positive codimension, i.e. it is contained in at least one hyperplane of $\mathbb{P}$. In particular, this implies that $\langle \kappa(D) \rangle$ does not span the whole canonical space $\mathbb{P}$ (even if Γ does).

By Grassmann formula, we have

$$(11) \qquad \dim(\langle \kappa(D) \rangle) = g - 1 - i.$$

Proposition 2.4. [*Geometric version of Riemann-Roch Theorem*] *Let C be a smooth, non-hyperelliptic, projective curve of genus $g \geq 3$. Let $\Gamma = \kappa(C) \subset \mathbb{P}$ be its canonical model. If $D \in Div^n(C)$ is effective such that $|D| = \mathfrak{g}^r_n$, then*

$$\dim(\langle \kappa(D) \rangle) = n - r - 1.$$

In other words, if Γ has one linear space Λ of dimension $n-r-1$ which is a n-secant linear space for Γ, then Γ admits a (linear) family of dimension r of such n-secant spaces.

Proof. $\dim(\langle \kappa(D) \rangle) = g - 1 - i = n - r - 1$, as it follows by the Riemann-Roch theorem. □

Remark 2.5. (1) The study of effective divisors and linear systems (in particular the special ones) on a non-hyperelliptic curve translates into the study of projective geometry and linear secant spaces of the canonical model of C.
(2) Since, from (11), D is special if and only if $\dim(\langle \kappa(D) \rangle) \leq g - 2$, we refind that any D of degree at most $g - 1$ must be special.
(3) On a non-hyperelliptic curve C a special divisor $D \in Div^n(C)$ moves in a linear system whose dimension is proportional to the failure of the points $\kappa(D)$ to be in general position on the canonical model of C, i.e. $r = r(D)$ is the number of linearly independent relations among the n points $\kappa(D)$.

Examples (1) Let C of genus $g \geq 3$, non-hyperelliptic, and assume there exists a $|D| = \mathfrak{g}^1_3$ (necessarily base-point-free). This happens for example if C is *trigonal* (see Proposition 2.10-(b)). Then Γ admits a 1-dimensional family of 3-secant lines in $\mathbb{P}$

(2) Let C be non-hyperelliptic of genus $g = 4$. Its canonical model $\Gamma \subset \mathbb{P}^3$ is a space curve of degree 6. Let $D \in Div^5(C)$.
• If D is non special, $\dim(\langle \kappa(D) \rangle) = 3$ and $r = r(D) = 1$, i.e. any 5-tuples of points of this 1-dimensional family spans the whole $\mathbb{P}^3$.
• If D is special, then $r = i + 1$. Since $\Gamma \subset \mathbb{P}^3$, by (11) and the speciality of D, $i(D)$ is either 2 or 1.
(i) $i(D) = 2$ if and only if $\dim(\langle \kappa(D) \rangle) = 1$ and $r = 3$, which means that Γ admits a 3 dimensional family of 5-secant lines. This is in contradiction with projective geometry standard results as the Trisecant lemma (cf. [**1**, Lemma, p. 109])).

(ii) $i(D) = 1$ if and only if $\dim(\langle\kappa(D)\rangle) = 2$ and $r = 2$. This means that Γ admits a 2-dimensional family of 5-secant planes. We shall see that it actually occurs (cf. § 5).

2.2. Clifford's theorem. As already mentioned at the end of § 1.1, given an effective divisor D of degree n on C of genus g, are there any bounds for r, equivalently for i, for $|D| = \mathfrak{g}^r_n$? For $n \geq 2g-2$, the answer is trivial since $i = 0$, unless $D \sim K_C$, in which case $n = 2g-2$ and $i = 1$.

For $n \leq 2g-1$, a first answer is given by the following well-known result:

Theorem 2.6. [*Clifford's theorem* (1882)] *Let C be a smooth, projective curve of genus $g \geq 2$, $0 \leq n \leq 2g-1$ and $|D| = \mathfrak{g}^r_n$ on C. Then $2r \leq n$.*

Moreover, equality holds if and only if either

- *$D \sim 0$, or*
- *$D \sim K_C$, or*
- *C is hyperelliptic and $|D| = |r\Delta| = r|\Delta|$, where $|\Delta| = \mathfrak{g}^1_2$ on C.*

Proof. The reader is referred to e.g. [**1**, p. 107]. □

Remark 2.7. Let us comment the second part of the Theorem 2.6. By Riemann-Roch theorem, note that $2r = n$ if and only if $r + i = g$, i.e.

$$2r = n \Leftrightarrow r(D) + r(K_C - D) = g - 1 \Leftrightarrow |D| + |K_C - D| = |K_C|.$$

Thus, if $D \not\sim 0, K_C$, by Clifford's theorem C is hyperelliptic and the condition $|D| + |K_C - D| = |K_C|$ is equivalent to the surjectivity of the multiplication map

$$\mu_0(D) : H^0(D) \otimes H^0(K_C - D) \to H^0(K_C).$$

Therefore C is hyperelliptic if and only if $\mu_0(D)$ is surjective.

As we shall see in § 4, the multiplication map $\mu_0(D)$ will play a fundamental role in the Brill-Noether theory (cf. formula (20)).

2.3. Nother's theorem. We want to recall some fundamental results concerning the *intrinsic* geometry of the canonical model of a non-hyperelliptic curve.

Let $g \geq 3$ and C non-hyperelliptic of genus g. From Propositions 2.1 and 2.2, ω_C is globally generated and very ample. The evaluation map

$$H^0(\omega_C) \otimes \mathcal{O}_C \xrightarrow{ev} \omega_C$$

is so a surjective morphism of vector bundles on C. Denote by $M_\omega := Ker(ev)$, which is a vector bundle on C of rank $g-1$.

Exercise By using the canonical embedding $\kappa : C \to \mathbb{P}$ and the Euler sequence in $\mathbb{P}$ restricted to Γ, show that $M_\omega = \kappa^*(\Omega^1_{\mathbb{P}}) \otimes \omega_C$.

With this set-up, one has the exact sequence

$$0 \to M_\omega \to H^0(\omega_C) \otimes \mathcal{O}_C \to \omega_C \to 0. \tag{12}$$

Tensoring (12) by $\omega_C^{\otimes k}$, for any $k \geq 1$, and passing to cohomology one gets the maps

$$\cdots \to H^0(\omega_C) \otimes H^0(\omega_C^{\otimes k}) \xrightarrow{\sigma_{k+1}} H^0\left(\omega_C^{\otimes(k+1)}\right) \to \cdots, \tag{13}$$

which are simply multiplication maps of global sections.

By recalling (3) and by using the same diagram as in [**4**, p. 132], with $L = \omega_C$, $\rho_{k+1} = m_{k+1}$ and $\nu_k = \sigma_{k+1}$, it is easy to show that

$$m_k \text{ surjective } \Leftrightarrow \sigma_k \text{ surjective.}$$

Thus, one can translate the normal generation of the canonical bundle ω_C of a non-hyperelliptic curve of genus $g \geq 3$ into a surjectivity of multiplication maps of global sections. Morover, since for $k \geq 1$, $h^1(\omega_C^{\otimes k}) = 0$, the surjectivity of σ_{k+1} is equivalent to showing $h^1(M_\omega \otimes \omega_C^{\otimes k}) = 0$, which simply is a vanishing condition.

Using this, one can give a proof of the following fundamental result:

Theorem 2.8. [*Noether's theorem*] *Let C be a smooth, non-hyperelliptic, projective curve of genus $g \geq 3$. Then ω_C is normally generated. In particular, its canonical model $\Gamma = \kappa(C) \subset \mathbb{P}$ is projectively normal.*

Proof. The reader is referred to [**4**, Application 2.5]. □

Remark 2.9. If C is hyperelliptic, ω_C is not normally generated, because κ is not even very ample, as it is composed with the hyperelliptic involution on C. On the other hand, $\Gamma = \kappa(C) \subset \mathbb{P}$ is a rational normal curve, therefore it is projectively normal in $\mathbb{P}$ (cf. Proposition 1.7).

Exercise Let C be hyperelliptic of genus $g \geq 3$. Show that

$$m_2 : Sym^2(H^0(\omega_C)) \to H^0(\omega_C^{\otimes 2})$$

has corank $g - 2$

2.4. Enriques-Babbage's theorem and Petri's theorem. There are some fundamental consequences of Noether's theorem. Let C be a smooth, non-hyperelliptic, projective curve of genus $g \geq 3$ and let $\Gamma = \kappa(C)$.

- If $g = 3$, then Γ is a smooth plane quartic.
- If $g = 4$, Γ is a space curve of degree 6. From the fact that ω_C is normally generated and by recalling Remark 1.5, we have $h^0(\mathcal{I}_{\Gamma/\mathbb{P}^3}(2)) = 1$ and $h^0(\mathcal{I}_{\Gamma/\mathbb{P}^3}(2)) = 5$. Since Γ is non degenerate, the quadric containing it is either a smooth quadric or a quadric cone. Moreover, among the cubics containing Γ, there are some irreducible ones. By Bezout's theorem, Γ is set-theoretically a complete intersection in $\mathbb{P}^3$ of a quadric and a cubic (a stronger result actually holds, cf. Theorem 2.13).
- If $g \geq 5$, $h^0(\mathcal{I}_{\Gamma/\mathbb{P}}(2)) = \binom{g-2}{2}$. Note that, since $g \geq 5$, we have $\binom{g-2}{2} \geq g - 2 = codim_{\mathbb{P}}(\Gamma)$.

From the previous analysis, it is natural to ask if Γ could be the intersection of the quadrics containing it, for $g \geq 5$ (for $g = 3, 4$ it is false). Consider

$$S := \bigcap_{Q \in H^0(\mathcal{I}_{\Gamma/\mathbb{P}}(2))} Q.$$

Proposition 2.10. *Let $g \geq 5$ and let C be a smooth, non-hyperelliptic, projective curve of genus g. If either*
(a) $g = 6$ and C is isomorphic to a smooth, plane quintic (i.e. C admits a very ample $|D| = \mathfrak{g}^2_5$), or
(b) $g \geq 5$ and C is trigonal *(i.e. it admits a complete $\mathfrak{g}^1_3$), then*

$$\Gamma = \kappa(C) \subset S.$$

Proof. The reader is referred to [**1**, (3.1) Proposition, p. 124]. □

The preceding proposition is the central step in the proof of the following fundamental result stating that, with only the exceptions listed above, a canonical curve is set-theoretically an intersection of quadrics.

Theorem 2.11. [*Enriques* (1919) - *Babbage* (1939)] *Let* $\Gamma \subset \mathbb{P}$ *be a canonical curve of genus* g*, with* $g \geq 5$*. Then*

$$S = \Gamma$$

unless Γ *is either*
- *trigonal of genus* $g \geq 5$*, or*
- $g = 6$ *and* Γ *is isomorphic to a smooth plane quintic.*

In all these cases, S *is a smooth surface of minimal degree (cf. § 1.3). More precisely,* S *is either a rational scroll or the Veronese surface in* $\mathbb{P}^5$*, respectively.*

Proof. The reader is referred to [**1**, Enriques-Babbage Theorem, p. 124]. □

Remark 2.12. (1) A curve of genus $g = 6$ cannot be simultaneously trigonal and isomorphic to a smooth plane quintic.
(2) For $g = 3$, $S = \emptyset$. As we said before, $\Gamma \subset \mathbb{P}^2$ is a smooth plane quartic, therefore it is trigonal (take the pencil of 3-secant lines through any point of Γ). For $g = 4$, $\Gamma \subset \mathbb{P}^3$ is a sextic curve. Since, as we saw before, it lies on either a smooth quadric or a quadric cone, in any case Γ is trigonal and $\Gamma \subset S$.

Enriques-Babbage theorem gives only a set-theoretical information and states nothing about the generators of the homogeneous ideal I_Γ of the canonical curve. By using the same strategy, via vector bundle methods as in the proof of Theorem 2.8, one can prove the following:

Theorem 2.13. [*Petri* (1922), *Saint-Donat* (1973)] *Let* $g \geq 4$ *and let* C *be a smooth, non-hyperelliptic, projective curve of genus* g*. Let* $\Gamma \subset \mathbb{P}$ *be its canonical model,* I_Γ *its homogeneous ideal.*

Then I_Γ *is generated by polynomials of degree* 2 *and* 3*. Moreover* I_Γ *is generated by only quadrics if and only if* C *is neither trigonal nor isomorphic to a smooth plane quintic.*

Proof. The reader is referred to [4, § 3. Proof of the Theorem, p.138]. □

3. Symmetric products and Jacobians

For any $n \geq 1$, denote by C_n the n-th symmetric product of C: it can be identified with the set of all effective divisors of degree n on C. It has a natural structure of smooth, projective variety of dimension n (cf. [**3**, 236]). Let

$$\mathcal{D}_n := \{(D, p) \mid p \in D\} \subset C_n \times C;$$

one has

$$\begin{array}{ccccc} \mathcal{D}_n & \stackrel{j}{\hookrightarrow} & C_n \times C & \stackrel{p}{\to} & C \\ & & \downarrow^{q_n} & & \\ & & C_n & & \end{array} \quad . \tag{14}$$

The fibre over $D \in C_n$ of the composition $q_n \circ j$ is the divisor $D \subset C_n$. For this reason, $\mathcal{D}_n$ is called the *universal (or tautological) divisor* of $C_n \times C$.

One has the following identification:

Proposition 3.1. *There is a natural identification*

$$T_{[D]}(C_n) \cong H^0(C, \mathcal{O}_D(D)),$$

where $T_{[D]}(C_n)$ *is the Zariski tangent space of* C_n *at the point* $[D] \in C_n$.

Proof. The reader is referred to [**1**, pp. 160-165]. □

The Picard group $Pic(C)$ decomposes according to the degree, as

$$Pic(C) = \coprod_{n \in \mathbb{Z}} Pic^n(C),$$

and $Pic^0(C)$ is a subgroup which has the structure of an *abelian variety*, i.e. a projective algebraic group, non singular and of dimension g. With this structure, $Pic^0(C)$ is often denoted by $J(C)$ and called the *jacobian variety* of C. By translation, the structure of algebraic variety is extended to any $Pic^n(C)$, for all $n \in \mathbb{Z}$, in such a way that $Pic^n(C) \cong J(C)$ as projective varieties, for all $n \in \mathbb{Z}$. On the other hand, these isomorphisms are not canonically defined if $n \neq 0$, so that in particular $Pic^n(C)$ does not inehrit the structure of algebraic group, for $n \neq 0$. By taking into account these isomorphisms at projective variety level, one has the following identification:

Proposition 3.2. *For any line bundle* L *of degree* n, *let* $[L] \in Pic^n(C)$ *be the corresponding point. There is a canonical identification*

$$T_{[L]}(Pic^n(C)) \cong H^1(C, \mathcal{O}_C),$$

where $T_{[L]}(Pic^n(C))$ *is the Zariski tangent space of* $Pic^n(C)$ *at* $[L]$.

Proof. The reader is referred to [**1**, pp. 166-170]. □

Due to the functorial definitions of $\mathcal{D}_n$ and $J(C)$ and the existence of a *Poincaré line-bundle*, one has a natural morphism of projective varieties

$$u_n : C_n \to Pic^n(C), \;\; D \to \mathcal{O}_C(D), \tag{15}$$

which is called the *Abel-Jacobi morphism* (at level n). By the very definition, the fibres of u_n are complete linear systems, in particular they are connected.

As we shall see in § 4, the Abel-Jacobi morphism is the key tool to introduce the Brill-Noether loci in $Pic^n(C)$.

Notation as in Diagram (14). Consider the exact sequence on $C_n \times C$

$$0 \to \mathcal{O}_{C_n \times C} \to \mathcal{O}_{C_n \times C}(\mathcal{D}_n) \to \mathcal{O}_{\mathcal{D}_n}(\mathcal{D}_n) \to 0;$$

since C_n is smooth, applying $q_{n,*}$ gives

(16)

$$0 \to \mathcal{O}_{C_n} \to q_{n,*}(\mathcal{O}_{C_n \times C}(\mathcal{D}_n)) \to q_{n,*}(\mathcal{O}_{\mathcal{D}_n}(\mathcal{D}_n)) \xrightarrow{\partial} R^1 q_{n,*}(\mathcal{O}_{C_n \times C}) \to R^1 q_{n,*}(\mathcal{O}_{C_n \times C}(\mathcal{D}_n)) \to 0.$$

From the functorial definitions of $\mathcal{D}_n$, of $J(C)$ and the existence of the Poincaré line-bundle, one has

Proposition 3.3. *There are natural identifications of vector bundles on* C_n:

$$T_{C_n} \cong q_{n,*}(\mathcal{O}_{\mathcal{D}_n}(\mathcal{D}_n)), \;\; u_n^*(T_{Pic^n(C)}) \cong R^1 q_{n,*}(\mathcal{O}_{C_n \times C});$$

moreover ∂ *coincides with* $u_{n,*}$, *the differential of the Abel-Jacobi morphism.*

Proof. The reader is referred e.g. to [**1**, (2.3) Lemma]. □

Remark 3.4. From Proposition 3.3 it follows that, when we restrict (16) to a point $[D] \in C_n$, we get the exact sequence of vector spaces

$$0 \to H^0(\mathcal{O}_C) \to H^0(\mathcal{O}_C(D)) \to H^0(\mathcal{O}_D(D)) \xrightarrow{\partial_D} H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_C(D)) \to 0. \tag{17}$$

In other words, given the natural exact sequence of sheaves on C

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \mathcal{O}_D(D) \to 0,$$

the coboundary map ∂_D of the associated exact sequence in cohomology identifies with the differential

$$du_{n,D} : T_D(C_n) \to T_{u_n(D)}(Pic^n(C)),$$

where $T_x(X)$ denotes the Zariski tangent space of the projective variety X at the point $x \in X$ (cf. also [**8**, Lemma 2.1]).

Corollary 3.5. *Let $D \in C_n$. Then*

$$\mathbb{P}(Im(\partial_D)) = \mathbb{P}(Im(du_{n,D})) = \langle \kappa(D) \rangle \subset \mathbb{P}(H^1(\mathcal{O}_C)) = \mathbb{P}.$$

In other words, the projiectivized image of the differential of the Abel-Jacobi map at $D \in C_n$ coincides with the linear span $\langle \kappa(D) \rangle$ in the canonical space $\mathbb{P}$.

Proof. The reader is referred to [**1**, pp. 189-190]. □

4. Basics on Brill-Nother theory

With notation as in the previous section, let $g \geq 3$. Consider

$$T_{C_n} \cong q_{n,*}(\mathcal{O}_{\mathcal{D}_n}(\mathcal{D}_n)) \xrightarrow{\partial} R^1 q_{n,*}(\mathcal{O}_{C_n \times C}) \cong u_n^*(T_{Pic^n(C)}),$$

which is a map of vector-bundles on C_n, the first of rank n, the second of rank g.

Definition 4.1. For any integer $0 \leq r \leq n$, define the closed (possibly empty) subscheme of C_n

$$C_n^r := \{[D] \in C_n \mid rk(du_{n,D}) \leq n - r\} \tag{18}$$

and

$$W_n^r := u_n(C_n^r) \subset Pic^n(C) \tag{19}$$

as its scheme-theoretic image.

Since $rk(\partial_D) \leq n-r$ if and only if $\dim(ker(\partial_D)) \geq r$, set-theoretically we have:

$$Supp(C_n^r) = \{[D] \in C_n \mid \dim(|D|) \geq r\}$$

and

$$Supp(W_n^r) = \left\{[L] \in Pic^n(C) \mid h^0(C, L) \geq r + 1\right\}.$$

The scheme W_n^r is said to be the *Brill-Noether locus* parametrizing complete linear series of degree n and dimension at least r on C.

Note that C_n^r and W_n^r can be empty, singular, reducible, even non-reduced. We shall see several examples (cf. § 5). It is clear from the definitions that $C_n^0 = C_n$, whereas $W_n^0 := W_n = u_n(C_n) \subseteq Pic^n(C)$ is the subscheme parametrizing effective line bundles of degree n.

• If $n \geq g$, from Riemann-Roch theorem

$$C_n^0 = C_n^1 = \ldots = C_n^{n-g} = C_n \text{ and } W_n^0 = W_n^1 = \ldots = W_n^{n-g} = Pic^n(C).$$

In particular, u_n is surjective. If moreover $n \geq 2g-1$, since $i(D) = 0$ for any $[D] \in C_n$, C_n is a projective space bundle over $Pic^n(C)$ with fibres $\mathbb{P}^{n-g}$ and $C_n^{n-g+k} = W_n^{n-g+k} = \emptyset$, for any $k \geq 1$.

• If $n \leq g$, given $[D] \in C_n$ general, D imposes independent conditions to $|K_C|$, i.e. $h^0(K_C - D) = g - n$. Therefore, by Riemann-Roch theorem $r(D) = 0$, i.e. for $[D] \in C_n$ general $\dim(u_n^{-1}(u_n(D)) = \dim(|D|) = 0$. Since the fibres of u_n are always connected, in this case we have:

(i) W_n is birational to C_n; in particular $\dim(W_n) = n$;
(ii) $C_n \neq C_n^1$;
(iii) $W_n \neq W_n^1$;
(iv) u_n is generically injective.

Particular cases (1) For $n = 1$, $u_1 : C \to Pic^1(C)$ is injective, as it follows from what discussed above and from $g \geq 3$. Moreover since $r(P) = 0$, for any $P \in C$, $du_{n,P}$ is injective for any $P \in C$. This means that u_1 is an embedding. If we fix a point $P_0 \in C$, we can compose the embedding u_1 with the translation in $Pic(C)$, $t_{\mathcal{O}_C(-P_0)} : Pic^1(C) \xrightarrow{\cong} J(C)$. By composition, one determines an embedding of $C \hookrightarrow J(C)$, defined up to translation. From Corollary 3.5, du_1 induces the *Gauss map*

$$\gamma : C \to \mathbb{P}, \quad P \to \langle \kappa(P) \rangle,$$

which actually coincides with the canonical morphism κ.

(2) For $n = g$, u_g is birational, since it is surjective and generically injective. In particular, $Pic^g(C)$ (equivalently $J(C)$) is birational to C_g, $W_g^0 = Pic^g(C)$ and u_g contracts special linear systems.

(3) For $n = g-1$, $W_{g-1}^0 \subset Pic^{g-1}(C)$ is a divisor, parametrizing effective line bundles of degree $g-1$. This is called the *theta divisor*, denoted also by Θ, which is intrinsically defined in $Pic^{g-1}(C)$. The theta divisor and the rich projective geometry associated to it will be studied in more details in [**8**, § 3].

Let us comment some general facts.

Proposition 4.2. *If $n \leq g$, the map u_n is birational onto its image and induces an isomorphism $C_n \setminus C_n^1 \xrightarrow{\cong} W_n \setminus W_n^1$. Moreover W_n is reduced, irreducible and normal.*

Proof. The first part is a direct consequence of what we discussed above. The second part is a consequence of the Stein factorization theorem and the Zariski Main Theorem. □

One also has:

Lemma 4.3. *If $r \geq 1$ and $n \leq g$, W_n^r does not coincide with the whole $Pic^n(C)$. Moreover $C_n^{r+1} \subset \overline{C_n^r}$ and $W_n^{r+1} \subset \overline{W_n^r}$.*

Proof. For a proof, see e.g. [**1**, (3.5) Lemma] (cf. also [**8**, Proposition 2.2]). □

From the previuos lemma, one has:

Lemma 4.4. *Let $r \geq 1$ and $n \leq g$. Take $D \in C^r_n \setminus C^{r+1}_n$ and let*

$$\mu_0(D) : H^0(D) \otimes H^0(K_C - D) \to H^0(K_C) \tag{20}$$

the natural multiplication map, called the Petri map *of $\mathcal{O}_C(D)$.*

Then

$$T_{u_n(D)}(W^r_n) \cong (Im(\mu_0(D)))^{\perp_B} \subset H^1(\mathcal{O}_C) = T_{u_n(D)}(Pic^n(C)),$$

where the orthogonality is with respect to the natural, non-degenerate, bilinear form $B : H^0(K_C) \otimes H^1(\mathcal{O}_C) \to H^1(K_C)$. In particular, $\dim_{u_n(D)}(W^r_n) \leq g - \dim(Im(\mu_0(D))$.

Proof. For a proof, see e.g. [**1**, (4.2) Proposition]. □

Remark 4.5. Denote $L = \mathcal{O}_C(D) \in Pic^n(C)$. Then one can speak equivalently of the Petri map for the line bundle L as

$$\mu_0(L) : H^0(L) \otimes H^0(\omega_C \otimes L^\vee) \to H^0(\omega_C). \tag{21}$$

Since

$$(Im(\mu_0(L)))^{\perp_B} \cong Coker(\mu_0(L)),$$

from Lemma 4.4 one has that $|L| = \mathfrak{g}^r_n$ and that $\dim(T_{[L]}(W^r_n)) = \dim(Coker(\mu_0(L)) = g - h^0(L)h^0(\omega_C \otimes L^\vee) + \dim(ker(\mu_0(L))$. One defines

$$\rho(L) = \rho(g,r,n) := g - h^0(L)h^0(\omega_C \otimes L^\vee) = g - (r+1)(g-n+r) \tag{22}$$

the *Brill-Noether number*. Thus, one can restate what observed before in this way:

$$\dim(T_{[L]}(W^r_n)) = \rho(g,r,n) + \dim(ker(\mu_0(L)). \tag{23}$$

Given r and n, the structure of the morphism

$$u^r_n : C^r_n \to W^r_n,$$

induced by the Abel-Jacobi map, can be very complicated. Its fibres have dimension at least r, therefore it is clear that for every irreducible component $Z \subseteq C^r_n$ one has

$$\dim(u^r_n(Z)) \leq \dim(Z) - r$$

and equality holds if and only if $C^{r+1}_n \cap Z \neq Z$. In the special case $n \leq g-1$, $r = 0$, the morphism $u_n : C_n \to W_n$ is a resolution of singularities.

Natural questions (1) Can we say something more precise about the dimension of any irreducible component of W^r_n (if not empty)?
(2) Can we say something more precise about the smoothness of $[L] \in W^r_n$?

4.1. Martens theorem and the Brill-Noether inequality. It is not possible to give a formula for the dimension W^r_n on a curve of genus g only as a function of g, r, n because in general such dimension depends on the curve C. It may happen that $W^r_n = \emptyset$ for some curves of genus g and that $W^r_n \neq \emptyset$ for some other curves of the same genus: for examples, given C a curve of genus $g \geq 3$, $W^1_2(C) \neq \emptyset$ if and only C is hyperelliptic.

On the other hand, there are some results which give upper and lower bounds for $\dim(W^r_n)$ which hold for every curve of genus g and characterize the curves on which W^r_n has maximal dimension. It is evident that the only interesting cases occur for $r \geq 1$ and $2 \leq n \leq 2g-2$. Moreover, because of the natural isomorphism

$$W^r_n \cong W^{g-n+r-1}_{2g-2-n}$$

induced by the correspondence $L \to \omega_C \otimes L^\vee$ and the Riemann-Roch Theorem, we may limit ourselves to considering the cases $2 \leq n \leq g-1$ and $r \geq 1$.

Theorem 4.6. [*Brill-Noether* (1873); *Martens* (1967)] *Let g, r, n be integers such that $r \geq 1$, $2 \leq n \leq g-1$. Let C be a smooth, projective curve of genus g. For any irreducible component $Z \neq \emptyset$ of W^r_n, one has*

$$\rho(g,r,n) \leq \dim(Z) \leq n - 2r. \tag{24}$$

Moreover, the equality to the upper-bound holds, for some Z, if and only if C is hyperelliptic.

Proof. The inequality on the left easily follows from the definition of W^r_n as a degeneracy-locus of a vector bundle map on C_n. For a proof of the inequality on the right and the last part of the statement, the reader is referred to e.g. [**1**, (5.1) Theorem, p. 191]. □

The inequality on the lef side of (24) is called the *Brill-Noether inequality*; the other inequality together with the last part of the statement are the content of *Martens' theorem*.

When C is hyperelliptic, one can stress the above statements. Indeed, one has

Proposition 4.7. *Let C be a smooth, hyperelliptic curve of genus $g \geq 2$ and let $|\Delta|$ be the (unique) $\mathfrak{g}^1_2$ on C. Then for any special $|A| = \mathfrak{g}^r_n$, with $A \in Pic^n(C)$, there exist points $P_1, \ldots, P_{n-2r} \in C$ such that $|A| = r|\Delta| + P_1 + \cdots + P_{n-2r}$.*

In particular, $W^r_n(C) \cong C_{n-2r}$.

Proof. The reader is referred to [**1**, Exercise D-9, p.41]. □

From Theorem 4.6, if $W^r_n \neq \emptyset$, it follows that

$$expdim(W^r_n) = \rho(g,r,n) \quad \text{and} \quad expdim(C^r_n) = \rho(g,r,n) + r. \tag{25}$$

Thus, from (23), we have the following:

Theorem 4.8. [*Smoothness via the Petri map*] *Assume $W^r_n \neq \emptyset$ and let $[L] \in W^r_n \setminus W^{r+1}_n$. Then*
(i) W^r_n is smooth and of the expected dimension $\rho(g,r,n)$ at $[L]$ if and only if the Petri map $\mu_0(L)$ is injective, and
(ii) C^r_n is smooth and of the expected dimension $\rho(g,r,n)+r$ at any $[D] \in u_n^{-1}([L])$ if and only if the Petri map $\mu_0(L)$ is injective.

Proof. Statement (i) trivially follows from what discussed up to now.

For what concerns (ii), observe that $ker(\mu_0(L)) = 0$ if and only if $rank(du_{n,D}) = n - r$, for any $D \in |L| = u_n^{-1}([L])$. Recall that (17) reads as

$$0 \to T_{[D]}(|L|) \to T_{[D]}(C^n) \overset{du_{n,D}}{\longrightarrow} T_{[L]}(W^r_n);$$

therefore, $rank(du_{n,D}) = n - r$ means that $du_{n,D}$ is surjective. Since from the first part $[L]$ is a smooth point of W^r_n, the same holds for any $[D] \in u_n^{-1}([L])$, as a point of C^r_n. □

Exercise Let $g \geq 4$, C be a smooth, non-hyperelliptic curve of genus g and let $L \in Pic^{g-1}(C)$ such that $|L| = \mathfrak{g}^1_{g-1}$ is base-point-free. Show that the map $\mu_0(L)$ fails to be injective if and only if L is a *theta characteristic*, i.e. $L^{\otimes 2} \cong \omega_C$.

What about the singularities of W^r_n? A preliminary answer is given by the following

Theorem 4.9. [*Singularity*] *Let* $[L] \in W^{r+1}_n$. *Then*

$$T_{[L]}(W^r_n) = T_{[L]}(Pic^n(C)).$$

In particular, if W^r_n *has the expected dimension* $\rho(g,r,n)$ *and* $r > n-g$ *(i.e.* $g > \rho(g,r,n)$*) then* $[L] \in \mathrm{Sing}(W^r_n)$. *In other words,* $W^{r+1}_n \subseteq Sing(W^r_n)$.

Proof. The reader is referred to [**1**, (4.2) Proposition, p. 189]. □

We conclude this section by recalling a refinement of Martens theorem.

DEFINITION 4.10. Let C be a smooth, non-hyperelliptic curve of genus g. Then C is said to be *bielliptic* if C admits an elliptic involution of degree 2, i.e. there exists a surjective morphism $\epsilon : C \to E$ such that $\deg(\epsilon) = 2$ and E is elliptic.

It is not difficult to show that C is bielliptic if and only if $\Gamma := \kappa(C)$ lies on a cone $\Sigma \subset \mathbb{P}^{g-1}$ projecting from a point $P \notin \Gamma$ a non-degenerate curve of genus g, degree $g-1$ in $\mathbb{P}^{g-2}$ and Γ is a 2-section of Σ (left as an exercise).

Then, one has

Theorem 4.11. [*Mumford's theorem*] *Let* C *be a smooth, non-hyperelliptic curve of genus* $g \geq 4$ *and let* $n \leq g-2$. *Assume there exists a component* $Z \subset W^r_n$ *s.t.* $\dim(Z) = n - 2r - 1$. *Then* C *is either trigonal, or isomorphic to a smooth plane quintic or bielliptic.*

Observe that the first two cases in Mumford's theorem have been already encountered as exceptional cases of the Enriques-Babbage theorem (cf. Theorem 2.11) and of the Petri's theorem (cf. Theorem 2.13). This kind of curves determines also exceptional cases of the behaviour of Θ_{Sing}, i.e. of the singular subscheme of the theta divisor (cf. [**8**, § 5]).

Exercise Let $g \geq 4$ and $n \leq g-2$. Show that, if C is bielliptic then $\dim(W^2_n(C)) = n-5$.

5. Some examples

In this section we want to discuss some examples.

(1) Let C be of genus 2; $J(C)$ is an abelian surface.
• u_1 is an embedding and $u_1(C) = \Theta \subset Pic^1(C)$ (cf. Particular cases (1) and (3), before Proposition 4.2). In other words, Θ is smooth and isomorphic to C.
• u_2 is bijective outside the points in $u_2^{-1}([\omega_C]) = |K_C| = \mathfrak{g}^1_2$, i.e. u_2 contracts a rational curve $E \subset C_2$. From the Castelnuovo criterion of contraction, E is a (-1)-curve in C_2. In other words, C_2 is birational to $J(C)$, obtained by blowing-up a point on this abelian surface. W^1_2 is smooth and irreducible (just one reduced point, cf. Proposition 4.7).

(2) Let C be of genus 3. u_1 is always an embedding.
• $u_2(C_2) = \Theta \subset Pic^2(C)$. Two cases have to be considered.
C non-hyperelliptic: for any $D \in C_2$, by Riemann-Roch Theorem and Clifford's theorem, $r(D) = 0$ and $i(D) = 1$, which implies that u_2 is an isomorphism onto its image, i.e. Θ is smooth.

C hyperelliptic: the unique $|\Delta| = \mathfrak{g}^1_2$ on C determines a rational curve $E \subset C_2$ which is contracted at a point $u_2(E) \in Pic^2(C)$. Since $\Theta = W^0_2 \subset Pic^2(C)$ has the expected dimension, from Theorem 4.9, its only singularity is the point $u_2(E) \in W^1_2$ (by using the Riemann singularity theorem [**1**, p. 226], it is a double point of Θ; cf. also [**8**, § 3]).

• u_3 is an isomorphism on the locus of non-special divisor. As above, we have two cases:

C non-hyperelliptic: recall that when C is non-hyperelliptic, its canonical image is a smooth, plane quartic Γ, so C is trigonal. Indeed, from Proposition 2.4, $D \in C_3$ special on C implies that $\langle \kappa(D) \rangle$ is a line. The triples $\langle \kappa(D) \rangle$ are cut out by pencils of lines through a point. The fourth point of intersection is general in Γ, i.e. $Im(u_3) \cong C = \{\text{set of fourth points}\}$. In other words, u_3 is not an isomorphism over a curve $\gamma \subset Pic^3(C)$ isomorphic to C and $Pic^3(C)$ is obtained from C_3 by collapsing each ruling of a ruled surface birationally isomorphic to $C \times \mathbb{P}^1$.

C hyperelliptic: if $|\Delta| = \mathfrak{g}^1_2$, any special $D \in C_3$ is $|D| = |\Delta| + P$, for $P \in C$. Therefore u_3 as above contracts each ruling of $C \times \mathbb{P}^1$ to a point.

(3) Let C be a smooth curve of genus 4.

C non-hyperelliptic: its canonical image is a smooth sextic $\Gamma \subset \mathbb{P}^3$ and, from Noether's theorem, it lies on a unique irreducible quadric $Q \subset \mathbb{P}^3$. We have two possibilities.

• if Q is smooth, it is doubly ruled. Each of the two rulings on Q defines a $\mathfrak{g}^1_3$ on C. Let $L_1, L_2 \in Pic^3(C)$ be the two line bundles corresponding to these two (inequivalent) $\mathfrak{g}^1_3$'s. From the fact that $|L_i|$ is a pencil, it follows that $Ker(\mu_0(L_i)) \cong H^0(\omega_C \otimes L_i^{\otimes 2})$ (use the *Base-point-free pencil trick*). Since $\mathcal{O}_\Gamma(1) \cong \omega_C$, then $h^0(\omega_C \otimes L_i^{\otimes 2}) = 0$ because its global sections correspond to hyperplanes in $\mathbb{P}^3$ containing two lines of the same ruling of Q and lines of the same ruling are skew. Therefore $\mu_0(L_i)$ is injective, $1 \leq i \leq 2$. Since $\rho(4,1,3) = 0$, from Theorem 4.8, $W^1_3 = \{L_1, L_2\}$ is smooth, reducible and of the expected dimension.

• if Q is a quadric cone, it is ruled. In other words, the two distinct $\mathfrak{g}^1_3$'s of the previous case become equivalent $L_1 = L_2 = L$. This means that the support of W^1_3 is just one point. On the other hand $\dim(Ker(\mu_0(L))) = h^0(\omega_C \otimes L^{\otimes 2}) = 1$ (i.e. L is a *theta characteristic*), which means that W^1_3 is of the expected dimension but not reduced. Note also that $W^2_3 = \emptyset$ by Clifford's theorem; therefore this example also shows that in general the inclusion in Theorem 4.9 $W^{r+1}_n \subseteq Sing(W^r_n)$ is strict.

C hyperelliptic: any $W^r_n \cong C_{n-2r}$. $\Theta = W^0_3$ is birational to C_3; since any element of W^1_3 is of the form $|\Delta| + P$, where $|\Delta| = \mathfrak{g}^1_2$ and $P \in C$, as above $W^1_3 = \gamma \cong C$ is a singular curve for Θ and C_3 obtained by blowing-up this curve. In particular, $\dim(W^1_3) = 1$ as Martens' theorem predicts (note $\rho(4,1,3) = 0$).

(4) Let $\Gamma \subset \mathbb{P}^2$ be a smooth quintic. Its abstract model is a curve C of genus 6 admitting a very-ample $\mathfrak{g}^2_5 = |L|$, where $L \cong \mathcal{O}_\Gamma(1)$.

First of all, this $\mathfrak{g}^2_5$ is uniquely determined on C (Exercise), so $Supp(W^2_5(C)) = \{L\}$; in particular $W^2_5 \neq \emptyset$ even if $\rho(6,2,5) = -3 < 0$.

Now, take $P \in C$ be a general point. Thus $L(-P)$ determines a $\mathfrak{g}^1_4$ on C. Therefore $W^1_4(C) \cong C$ (as Mumford's theorem 4.11 predicts) even if $\rho(6,1,4) = 0$.

6. Final remarks and curves with general moduli

All the results and the examples discussed up to now show that, in general, the behaviour of the W^r_n's - even in low genera - seems to be unpredictable and actually depends on C. On the other hand, one can be more precise, at least for some questions related to W^r_n's.

Theorem 6.1. [*Existence theorem - Kempf and Kleiman-Laksov* (1972)] *Let C be any smooth, projective curve of genus g. Assume $\rho(g,n,r) \geq 0$, for integers $n \geq 1$, $r \geq n-g$. Then $W^r_n(C) \neq \emptyset$ and every irreducible component has at least the expected dimension $\rho(g,r,n)$.*

Proof. The proof involves ampleness of suitable vector bundles on $Pic^n(C)$ and non-emptyness of related degeneracy loci of vector bundle maps. The reader is referred to [**1**, Chapter VII]. □

Theorem 6.2. [*Connectedness theorem - Fulton and Lazarsfeld* (1981)] *Assume $\rho(g,n,r) \geq 1$. Then $W^r_n(C)$ is connected.*

Proof. The reader is referred to [**1**, Chapter VII]. □

If C is a *general curve* of genus g, i.e. if it corresponds to a general point of the moduli space $\mathcal{M}_g$, the situation is clearer.

Theorem 6.3. [*Dimension theorem or Brill-Noether theorem - Griffiths and Harris* (1980)] *Let C be a general curve of genus g. Assume $\rho(g,n,r) < 0$, for integers $n \geq 1$, $r \geq n-g$. Then $W^r_n(C) = \emptyset$*

Proof. The reader is referred to [**1**, p. 214] and the original article. □

The nice aspect of C to be with general moduli is that also the infinitesimal structure of $W^r_n(C)$, as in Lemma 4.4 and (23), is much more precise.

Theorem 6.4. [*Smoothness theorem or Petri's theorem - Gieseker* (1982)] *Let C be a general curve of genus g. Let $n \geq 1$, $r \geq n-g$ be integers. Then $W^r_n(C)$ is smooth and of the expected dimension $\rho(g,r,n)$ at each point $[L] \in W^r_n(C) \setminus W^{r+1}_n(C)$*

Proof. The reader is referred to [**1**, p. 214] and the original article. □

The proofs of both the previous results are based on degeneration techniques. In particualar, one has:

Corollary 6.5. *Let C be a general curve of genus g. Let $n \geq 1$, $r \geq n-g$ be integers.*
(i) If $\rho(g,r,n) \geq 1$, then $W^r_n(C)$ si irreducible.
(ii) $Sing(W^r_n(C)) = W^{r+1}_n(C)$.

Proof. (i) follows at once from Theorems 6.2 and 6.4. (ii) follows from Thorems 4.9 and 6.4. □

It is noteworthy that, for special curves C, $W^r_n(C)$ does not respect the previous statements: it may well be reducible even if $\rho > 0$ and if it has the expected dimension ρ (see e.g. C trigonal of genus 5 in [**1**]) and $W^{r+1}_n(C) \subset Sing(W^r_n(C))$ (cf. Example (3), when the canonical model of C lies on a quadric cone).

By recalling Theorem 4.8, an alternative version of Theorem 6.4 is the following:

Theorem 6.6. *[Smoothness theorem or Petri's theorem - VERSION II] Let C be a general curve of genus g. Let $n \geq 1$, $r \geq n - g$ be integers. For any $[L] \in W^r_n(C) \setminus W^{r+1}_n(C)$, the Petri map (21) is injective.*

This is the version that Gieseker proved (1982) and this is the one that K. Petri (1925) stated.

We conclude this section by showing some nice applications of the previous results on curves with general moduli. Recall first the following:

DEFINITION 6.7. Let C be a smooth, projective curve. If C admits a complete and base-point-free $\mathfrak{g}^1_k$ but no $\mathfrak{g}^1_{k-1}$'s, then C is said to be *k-gonal* and k is denoted by $gon(C)$.

Proposition 6.8. *Let C be a curve of genus g with general moduli. Then $gon(C) = \lfloor \frac{g+3}{2} \rfloor$.*

Proof. By Theorem 6.3, a $\mathfrak{g}^1_k$ exists if and only if $\rho(g,1,k) \geq 0$. On the other hand, $\rho(g,1,k) \leq 1$. Indeed, if $\rho(g,1,k) \geq 2$, this would give $0 \leq 2k-g-4 = \rho(g,1,k-1)$ so, by Theorem 6.1, $W^1_{k-1} \neq \emptyset$, against assumptions. Therefore $0 \leq \rho(g,1,k) \leq 1$, which gives $k = \frac{g+2}{2}$, if g is even, and $k = \frac{g+3}{2}$, if g is odd, as we had to show. □

Proposition 6.9. *Let C be a curve of genus g with general moduli. Let $[L] \in W^r_n(C) \setminus W^{r+1}_n(C)$, with $r \geq 3$. Then φ_L si an embedding.*

Proof. Let $L = \mathcal{O}_C(D)$, for some $D \in C_n$. Then $r(D) = r$ and $i(D) = i$. Let $\Delta \in C_2$ be any effective divisor. Thus $0 < r-2 \leq r(D-\Delta) \leq r$.

If $r(D-\Delta) = r$, then we would have $|D| = |D-\Delta|$. On the other hand, $r(D-\Delta) = r$ implies $i(D-\Delta) = i+2$ so, by Theorem 6.3, we have $g-(r+1)i = \dim(W^r_d) > \dim(W^r_{d-2}) = g-(r+1)i-2(r+1)$, since $2(r+1) \geq 6$: contradiction.

If $r(D-\Delta) = r-1$, then any $|D-\Delta| = |D-P|$, for some $P \in C$. In this case $r(D-P) = r-1$ and $i(D-P) = i$. But, from Theorem 6.3, we have $g - ri = \dim(W^{r-1}_{d-1}) > \dim(W^{r-1}_{d-2}) = g-ri-r$, since $r \geq 3$: contradiction.

Therefore $r(D-\Delta) = r-2$, for any $\Delta \in C_2$, which means that $|D|$ separates points and tangent vectors. □

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