ON AN EXACT TRANSPORT EQUATION FOR TURBULENT DISSIPATION RATE FOR GENERALIZED NEWTONIAN FLUIDS BASED ON APPARENT VISCOSITY TRANSPORT EQUATION

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Abstract

The Flow of inelastic Non-Newtonian fluids is involved in many biological and industrial applications like nanofluids. Despite many years have passed since the beginning of the study of turbulent Non-Newtonian fluids, most of the studies carried out focus the attention on viscoelastic-fluids. In order to make accurate and low-cost prediction on turbulent inelastic Non-Newtonian fluids flow, a RANS Generalized Newtonian Fluid (GNF) turbulence model is required based on exact transport equation of turbulent variables. In a previous paper [52] we achieved the exact transport equations for turbulent kinetic energy and dissipation rate through the introduction of an apparent viscosity transport equation in 2D case for sake of simplicity. The object of this paper is to extend the results given in [52] in 3D case giving the full mathematical demonstration of the exact-equations. The modelling of the unknown terms it is left for a future work.

Keywords: Apparent viscosity, Turbulent Dissipation rate, Shear rate, Generalized Newtonian Fluid, Nanofluids.

1. Introduction

The flow of Non-Newtonian fluids is present in a wide range of industrial applications and biological problems. Slurries flow in pipes, wastewater treatment and aseptic food processing are just some examples of application involving Non-Newtonian fluids flow. Usually the flow rate is small in these processes and so the fluid flow is in laminar regime. Many theoretical solutions and numerical simulations are presented in the literature investigating the laminar flow of Non-Newtonian fluids such as those of Gori [1-2]. Nevertheless the turbulent regime may be encountered in some situation as sewage transport, drilling hydraulics and processing at high heat transfer rate and gives some advantages in comparison to the laminar regime. For example the turbulent regime increases the heat transfer coefficient and is found in the aseptic food processing due to the large temperature differences that reduce the viscosity and lead the transition from laminar to turbulent regime as shown in [3]. Another advantage of turbulent regime in pipe flow operations is that the specific energy consumption is lowest. In biology the most famous example of non-Newtonian fluid is blood. Despite in the most of the vascular network the flow rate is laminar there are some regions, such as bifurcations, in which turbulent conditions may occur promoting, if associated to biochemical factors, the formation of the atherosclerotic plaque [4].

The term "Non –Newtonian fluid" is very general and includes a wide range of fluids having very different constitutive relations. Among these the viscoelastic fluids are the most investigated because of the substantial drag reduction under turbulent flow regime called Tom's effect [5]. This property has been studied numerically in a wide number of papers employing different constitutive relations to describe the viscoelastic behaviour of the fluid.

Among these the FENE-P model is one of the most popular and many Direct Numerical Simulations (DNS) have been carried out to explain the phenomenon of drag reduction [6], to find relationships between flow and fluid rheological parameters at high medium and low regime (HDR, MDR, LDR) [7], to study the zero-pressure gradient turbulent boundary layers where the polymer is homogeneously distributed in the solvent [8], to investigate where and how polymers affect turbulence [9].

The DNS is a powerful instrument but its application is limited by computational resources. That's why many DNS at low and medium Reynolds number are used to develop different turbulence closure models. The models proposed by Poreh and Hassid in [10] and Durst and Rastogi in [11] were based on a similar approach consisting in modifying the damping function of the low Reynolds number k-ε model of Jones and Launder [12]. Others proposals can be found in [13] where a zero-equation model for the eddy viscosity has been proposed or in [14] where closures have been developed for turbulent correlations among flow and polymer conformation variables and incorporated into a single-point k–ε model.

The Giesekus model has been used as well as the FENE-P one in predicting drug reduction. Some example can be found in [15-16] where both models are used and in [17] where the numerical results are compared with Particle Image Velocimetry (PIV) experiments.

A mechanistic turbulence model for polymers was suggested in [18] in which it was argued that the dominant forces on a polymer fibber in the turbulent flow are elastic and centrifugal. The corrected velocity profiles resulting from dimensional analysis in turbulent boundary layer have been compared favourably with Virk's experiments [19].

Due to the complexity of the viscoelastic model, others Authors proposed a simpler constitutive equation inspired to the Generalized Newtonian Fluid Model (GNF). The elongation viscosity has been modelled as a function of magnitudes of strain and rotation rate tensors in Orlandi [20] and as a function of the third and second invariant of the rate-of-strain tensor in Den Tonder et Al. [21]. In Both works DNS results are provided in order to show the capability of the models to reproduce the drag reduction. In another paper Den Toonder et al. [22] studied two different constitutive equations, viscous anisotropic model and a viscoelastic anisotropic, by means of DNS and Laser Doppler Velocimetry (LDV). The viscous anisotropic model, characterized by a single scalar viscosity function related to the second and third invariants of the rate of shear tensor, showed a good qualitative agreement with the measurements.

Olivera and Pinho in [23] following an approach similar to the one adopted in [21] and choosing a *Bird-Carreau* constitutive equation for the viscosity depending on the second and third invariant of the rate of shear tensor qualitatively showed that the introduction of the third invariant of the rate of strain tensor in the viscosity contributed to an increase of viscous diffusion and turbulent dissipation rate.

The same constitutive equation has been employed by Pinho in [24] in order to derive a low Reynolds number *k-ε* model for drag reduction fluids. An algebraic equation has been proposed to correlate the instantaneous viscosity to the dissipation rate while the average viscosity and dissipation rate have been correlated through a normal logarithmic probability distribution. Appling the dimensional analysis it was possible to neglect many terms in all the transport equations while the remaining ones have been modelled. The final turbulent dissipation rate equation has been written in non conservative form because the explicit time derivative of the average viscosity is present.

Cruz and Pinho [25] completed the model presented in [24] basing it on the Nagano-Hishida one [26]. The values of parameters and forms of damping functions, derived taking into account viscometric and elastic near-wall effects, were given and comparison with experimental data were made by performing simulations of pipe flow viscoelastic polymer solutions. The model implemented has been improved successively in many other papers. In [27] Cruz et Al. added the new stress, i.e. the cross-correlation between the fluctuating viscosity and the fluctuating rate of strain. The new stresses modelled were proportional to the mean velocity gradient. In [28] the same model has been used modifying the damping functions and coefficients while in [29] the Launder-Sharma model [30] has been used instead of the Nagano-Hishida one [26].

Inspired by [24, 25, and 27] a full Reynolds stress model has been developed in [31]. The performances of the new model were compared with those in [27] and other experimental data. New developments were made to account separated flows, removing the dependence of the velocity gradient by the friction velocity in the recirculation zone. All the results showed that the Reynolds stress model perform better than the k- ε one.

Few numerical investigations dealt with turbulent flow of pseudo-plastic (shear-thinning fluid) and dilatant (shear-thickening fluid) fluids because of the lack of models with one or two point closure and, for this reason, some investigators performed DNS. Rudman and Blackburn used the Spectral Element-Fourier Method (SEM) in a duct flow, [4], and compared the DNS results of a power law fluid with small consistency index and a Herschel-Bulkley fluid with experimental data [32]. A turbulent model for a non-

Newtonian power law fluid has been developed in [33], in analogy to the turbulent viscosity, determining the temperature distribution for soybean milk flowing inside a tubular heat exchanger.

Turbulent flow of a non-Newtonian fluid is important also in the medical field. A model to predict the turbulent flow of a power-law fluid in a bio-reactor for anaerobic digestion has been developed in [34] with the classical k-ε model and the power-law viscosity. The *k-ε* equations have been derived in [35 - 36] for a power-law and Herschel-Bulkley fluid using the apparent viscosity of a non-Newtonian fluid in the RANS equations for a Newtonian fluid, but the agreement was not good enough.

Another very promising field of application is the nanofluid one. Nanofluids are dilute liquid suspensions of nanoparticles with at least one of their critical dimensions smaller than about 100 nm [37]. Many experimental studies confirmed that the viscosity of these fluids is temperature, amount of particle and shear rate dependent. In some of these works the viscosity seems to have a shear-thinning behaviour [38-45] while in others a shear-thickening one [46]. Few numerical simulations have been carried out in laminar regime [47-50]. An interesting model has been developed in [51] where the effect of the nanoparticle/base-fluid relative velocity is described more mechanistically than in the dispersion models however the fluid has been considered Newtonian.

Although there are many applications in which the turbulent motion of Generalized Newtonian fluids is involved there is no paper at the Authors knowledge, besides the Pinho one [24] which is addressed to viscoelastic fluids, in which a strict derivation of the turbulent dissipation rate equation has been carried out. The aim of our previous paper [52] was to derive such an equation in conservative form for a simple 2D domain.

The present work is aimed to extend the two-dimensional set of exact conservative equations developed in [52] in three dimensions. Viscosity is assumed dependent only on the second invariant of the rate of shear since as expressed in [23] the third invariant is related to extensional viscosity which is not interesting in the class of fluids, like the nanofluids, we are interested in moreover as stated in [53] there is some evidence that this may be reasonable for real fluids.

The transport equation of ε is deduced in this work by the use of the transport equation for the apparent viscosity, introduced in [51] and extended here to 3D case, which does not require a constitutive link between apparent viscosity and shear rate, does not need any hypotheses on the dependence of the turbulent dissipation rate on the fluctuating part of the rate of strain tensor, as required in [14].

The paper follows the same structure of [51] deriving first the transport equations for the average momentum and the turbulent kinetic energy which are same as in 2D case. Afterwards the transport equation for the rate of shear tensor and the shear-rate are derived which are different from the one in 2D case. The differential equation for the apparent viscosity is deduced using the same approach in [51]. From this equation it is possible to derive the equation of dissipation rate in conservative form and to give a physical interpretation to the new terms. The method used in this work allows explaining each term and classifying it as transport, production and dissipation one.

Nomenclature

Latin

S_{ij}	rate of strain tensor
k	mean turbulent kinetic energy
p	instantaneous static pressure
P	mean static pressure
S	shear rate
T_{ij}^{R}	mean Reynolds stress tensor
T_{ij}^{μ}	mean fluctuating-viscosity stress tensor
$T_{ij}^{R\prime}$	instantaneous Reynolds stress tensor
$T_{ij}^{\mu\prime}$	instantaneous fluctuating-viscosity stress tensor
t	time

Greek

$\dot{\gamma}$	shear rate
μ_{app}	apparent viscosity
$\overline{\mu_{app}}$	mean apparent viscosity
ρ	density
τ_c	yield stress
Ω_{ij}	rotation rate tensor
ε	mean dissipation rate
δ_{ij}	Kronecher delta

u_i	instantaneous i-velocity component
U_i	mean x i-velocity component
x_i	*i*- coordinate

2. Conservation equations of mass, momentum and turbulent kinetic energy

The present analysis is carried on for a Generalized Newtonian Fluid, GNF. The constitutive equation for the incompressible non-Newtonian fluid is written similarly to a Newtonian one with the apparent viscosity function of the shear-rate only for the reasons expressed in the previous section

$$T_{ij} = -p\delta_{ij} + 2\mu_{app} S_{ij} , \tag{2.1}$$

where T_{ij} is the stress tensor and p the static pressure.

The rate of strain tensor S_{ij} is

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} , \tag{2.2}$$

and the shear-rate $\dot{\gamma}$ is

$$\dot{\gamma} = \sqrt{2 S_{ij} S_{ij}} . \tag{2.3}$$

Defining S as

$$S = \frac{\dot{\gamma}}{\sqrt{2}} , \tag{2.4}$$

the shear-rate will be treated as S from now on.

The conservation equations for the instantaneous variables are the followings,

$$\frac{\partial u_k}{\partial x_k} = 0 , \tag{2.5}$$

for the mass, and

$$\rho\frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k}\left(2\mu_{app} S_{ik}\right) , \tag{2.6}$$

the momentum.

Each instantaneous variable is decomposed in mean and fluctuating components,

$$u_i = U_i + u_i' \,, \tag{2.7}$$

$$p = P + p' \,, \tag{2.8}$$

$$\mu_{app} = \overline{\mu_{app}} + \mu'_{app} \,. \tag{2.9}$$

The mean component of the stress tensor, Eq. (1), becomes

$$\overline{T_{ij}} = -P\delta_{ij} + 2\overline{\mu_{app}}\,\overline{S_{ij}} + 2\overline{\mu'_{app}S'_{ij}} \,, \tag{2.10}$$

while the fluctuating one

$$T'_{ij} = -p'\delta_{ij} + 2\mu'_{app}\overline{S_{ij}} + 2\overline{\mu_{app}}S'_{ij} + 2\mu'_{app}S'_{ij} - 2\overline{\mu'_{app}S'_{ij}} \,. \tag{2.11}$$

The third term of the mean component, Eq. (2.10), is null in Newtonian case and it is due to the viscosity fluctuations and we will refer to it as "fluctuating-viscosity stress tensor" or FV-stress tensor. It must be noted that this term is present in [23, 25-27, 28] as well and it has been called polymeric stress tensor because the fluid treated were polymers. The conservation equations of the mean variables are

$$\frac{\partial U_k}{\partial x_k} = 0 \,, \tag{2.12}$$

for the mass, and

$$\rho\frac{\partial U_i}{\partial t} + \rho U_k \frac{\partial U_i}{\partial x_k} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_k}\left(2\overline{\mu_{app}}\,\overline{S_{ik}} + T_{ik}^R + T_{ik}^{\mu}\right), \tag{2.13}$$

for the momentum.

In the mean flow, the Reynolds stress tensor is given by

$$T_{ij}^R = -\overline{\rho u_i' u_j'} \,, \tag{2.14}$$

while the FV-stress tensor is, according to [24]

$$T_{ij}^{\mu} = 2\overline{\mu'_{app}S'_{ij}} \,. \tag{2.15}$$

In a Newtonian fluid the Reynolds stresses, due to convection, are the only terms responsible for the energy transfer from mean to fluctuating scale, while, in a GNF, the FV-stresses are active in conjunction to the Reynolds ones in the energy transfer from mean to fluctuating scale. The FV-stress tensor is a new term that requires a modelling but is not the only one. Looking carefully to the equation we can see that we have the mean viscosity which is also an unknown variable that requires modelling.

In the conservation equation for the turbulent kinetic energy, derived in Appendix A, the material derivative of the variable is expressed as sum of five terms, as in a Newtonian fluid, but with some differences.

$$\frac{\partial}{\partial t}(\rho k)+U_j\frac{\partial}{\partial x_j}(\rho k)=P^k+\Pi^k+T^k+D^k-\rho\varepsilon\,. \tag{2.16}$$

The sum $\Pi^k+T^k+D^k$ gives the transport term in which we can see the pressure Π^k and the turbulent transport terms T^k which have the same definition in Newtonian case,

$$\Pi^k=-\frac{\partial}{\partial x_j}\left(\overline{p'u'_j}\right), \tag{2.17}$$

$$T^k=-\frac{\partial}{\partial x_j}\left(\overline{\rho k'u'_j}\right), \tag{2.18}$$

and the diffusive transport D^k which has a different definition due to the variable viscosity

$$D^k=\frac{\partial}{\partial x_j}\left(\overline{\mu_{app}}\frac{\partial k}{\partial x_j}+\frac{\overline{\mu_{app}}}{\rho}\frac{\partial T_{ij}^R}{\partial x_i}+\overline{\mu'_{app}\frac{\partial k'}{\partial x_j}}+\overline{\frac{\mu'_{app}}{\rho}\frac{\partial T_{ij}^{R'}}{\partial x_i}}+2\overline{\mu'_{app}u'_i}S_{ij}\right). \tag{2.19}$$

Since in the Newtonian case the viscosity is constant the mean of the viscosity is the viscosity itself and so the fluctuation of viscosity are zero, so the Newtonian expression can be found just replacing the mean viscosity with the viscosity and neglecting the terms in which the fluctuating viscosity appear. This means that in Newtonian case the diffusive transport is a term that doesn’t need to be modelled because it doesn’t introduce new correlation but for a GNF fluid it does and so the last three terms require a closure.

The term $P^k-\rho\varepsilon$ represents the balance between the production and dissipation of turbulent kinetic energy. Physically the dissipation is due to the work that the viscous stresses exert on the fluid and so should be defined as the mean of the product of the fluctuating viscous stress tensor and the fluctuating rate of strain tensor, but since in the momentum equation the FV-stess tensor appear, it is convenient give a different definition to the dissipation rate which will be

$$\varepsilon = 2\left(\overline{\mu_{app}}\,\overline{S'_{ij}S'_{ij}} + \overline{\mu'_{app}S'_{ij}S'_{ij}}\right)\Big/\rho \,. \tag{2.21}$$

And so the production term will be

$$P^k = \left(T^R_{ij} - T^\mu_{ij}\right)\overline{S_{ij}} \tag{2.22}$$

in accordance with [24]. In the mean momentum equation the Reynolds stresses component is added to the FV stresses while in the turbulent kinetic energy one they subtract, so if the Reynolds stresses transfer energy from the mean flow to feed the fluctuating components, the FV stresses reduce the effects of the Reynolds stresses. The conservation equations of mass, mean momentum and turbulent kinetic energy are the same reported in [51].

4. Transport equations for the mean and fluctuating shear rate and apparent viscosity

The transport equation for the rate of strain tensor, S_{ij}, is obtained considering the symmetrical part of the gradient of Eq. (2.6). Using the definition of Ω_{ij}

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right), \tag{4.2}$$

the transport equations for the S_{ij} components are

$$\begin{aligned}\rho\frac{\partial S_{ij}}{\partial t} + \rho u_k\frac{\partial S_{ij}}{\partial x_k} = &-\rho S_{ik}S_{jk} + \rho\Omega_{ik}\Omega_{jk} - \frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial x_j}\right) - \frac{1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial p}{\partial x_i}\right) + \frac{\partial}{\partial x_k}\left(\mu_{app}\frac{\partial S_{ij}}{\partial x_k}\right) + \\ &\frac{\partial}{\partial x_k}\left(\left(\frac{S_{ik}+\Omega_{ik}}{2}\right)\frac{\partial \mu_{app}}{\partial x_j} + \left(\frac{S_{jk}+\Omega_{jk}}{2}\right)\frac{\partial \mu_{app}}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\left(\frac{S_{ik}-\Omega_{ik}}{2}\right)\frac{\partial \mu_{app}}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left(\left(\frac{S_{jk}-\Omega_{jk}}{2}\right)\frac{\partial \mu_{app}}{\partial x_k}\right)\end{aligned} \,. \tag{4.3}$$

The details of the algebra are reported in Appendix B.

The transport equation for the square of the shear rate, S^2, is obtained multiplying Eq. (4.3) for each term of the rate of strain tensor by itself and summing all of them.

$$\rho\frac{\partial}{\partial t}\left(\frac{S^2}{2}\right)+\rho u_k\frac{\partial}{\partial x_k}\left(\frac{S^2}{2}\right)=-\rho S_{ij}S_{ik}S_{jk}+\rho S_{ij}\Omega_{ik}\Omega_{jk}+$$
$$\frac{\partial}{\partial x_k}\left(-S_{kj}\frac{\partial p}{\partial x_j}+\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{S^2}{2}\right)+\left(S_{ij}\left(S_{ik}+\Omega_{ik}\right)+S_{ik}\left(S_{ij}-\Omega_{ij}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}\right)+. \tag{4.4}$$
$$\frac{\partial S_{kj}}{\partial x_k}\frac{\partial p}{\partial x_j}-\mu_{app}\frac{\partial S_{ij}}{\partial x_k}\frac{\partial S_{ij}}{\partial x_k}-\left(\left(S_{ik}+\Omega_{ik}\right)\frac{\partial S_{ij}}{\partial x_k}+\left(S_{ij}-\Omega_{ij}\right)\frac{\partial S_{ik}}{\partial x_k}\right)\frac{\partial\mu_{app}}{\partial x_j}$$

A transport equation for the fluctuating components of the rate of strain tensor can be obtained by subtracting the mean equation from the instantaneous one. The following expression is obtained

$$\rho\frac{\partial S'_{ij}}{\partial t}+\rho U_k\frac{\partial S'_{ij}}{\partial x_k}=-\rho\left(\overline{S_{jk}}S'_{ik}+\overline{S_{ik}}S'_{jk}-\overline{\Omega_{jk}}\Omega'_{ik}-\overline{\Omega_{ik}}\Omega'_{jk}\right)-\rho\left(S'_{ik}S'_{jk}-\Omega'_{ik}\Omega'_{jk}\right)-\rho u'_k\frac{\partial\overline{S_{ij}}}{\partial x_k}-\rho u'_k\frac{\partial S'_{ij}}{\partial x_k}+$$
$$-\frac{\partial}{\partial x_i}\left(\frac{1}{2}\frac{\partial p'}{\partial x_j}\right)-\frac{\partial}{\partial x_j}\left(\frac{1}{2}\frac{\partial p'}{\partial x_i}\right)+\frac{\partial}{\partial x_k}\left(\mu_{app}\frac{\partial S'_{ij}}{\partial x_k}+\mu'_{app}\frac{\partial\overline{S_{ij}}}{\partial x_k}\right)-\frac{\partial}{\partial x_k}\left(\frac{1}{2}\left(\frac{\partial}{\partial x_j}\left(T^R_{ki}+T^\mu_{ik}\right)+\frac{\partial}{\partial x_i}\left(T^R_{kj}+T^\mu_{jk}\right)\right)\right)+$$
$$\frac{\partial}{\partial x_k}\left(\left(\frac{S'_{ik}+\Omega'_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_j}+\left(\frac{S'_{jk}+\Omega'_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_i}\right)+\frac{\partial}{\partial x_j}\left(\left(\frac{S'_{ik}-\Omega'_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left(\left(\frac{S'_{jk}-\Omega'_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}\right)+ \quad .(4.5)$$
$$\frac{\partial}{\partial x_k}\left(\left(\frac{\overline{S_{ik}}+\overline{\Omega_{ik}}}{2}\right)\frac{\partial\mu'_{app}}{\partial x_j}+\left(\frac{\overline{S_{jk}}+\overline{\Omega_{jk}}}{2}\right)\frac{\partial\mu'_{app}}{\partial x_i}\right)+\frac{\partial}{\partial x_j}\left(\left(\frac{\overline{S_{ik}}-\overline{\Omega_{ik}}}{2}\right)\frac{\partial\mu'_{app}}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left(\left(\frac{\overline{S_{jk}}-\overline{\Omega_{jk}}}{2}\right)\frac{\partial\mu'_{app}}{\partial x_k}\right)$$

The conservation equation for the dissipative terms of the turbulent kinetic energy is written using the conservation equation for the variable $S'_{mn}S'_{mn}$, obtained multiplying the transport equation of each term for the rate of strain tensor, Eq. (4.5), by itself and summing all of them.

$$\rho\frac{\partial}{\partial t}\left(\frac{S'_{mn}S'_{mn}}{2}\right)+\rho U_k\frac{\partial}{\partial x_k}\left(\frac{S'_{mn}S'_{mn}}{2}\right)=-2\rho\left(S'_{ij}S'_{ik}\overline{S_{jk}}-S'_{ij}\Omega'_{ik}\overline{\Omega_{jk}}\right)-\rho\left(S'_{ij}S'_{ik}S'_{jk}-S'_{ij}\Omega'_{ik}\Omega'_{jk}\right)+$$
$$-\rho u'_k S'_{ij}\frac{\partial\overline{S_{ij}}}{\partial x_k}+\frac{\partial}{\partial x_k}\left(-S'_{kj}\frac{\partial p'}{\partial x_j}-\rho u'_k\frac{S'_{mn}S'_{mn}}{2}+\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{S'_{mn}S'_{mn}}{2}\right)+\frac{1}{2}T^{\mu'}_{ij}\frac{\partial\overline{S_{ij}}}{\partial x_k}\right)+$$
$$\frac{\partial}{\partial x_k}\left(\left(S'_{ij}\left(S'_{ik}+\Omega'_{ik}\right)+S'_{ik}\left(S'_{ij}-\Omega'_{ij}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}\right)+\frac{\partial}{\partial x_k}\left(\left(S'_{ij}\left(\overline{S_{ik}}+\overline{\Omega_{ik}}\right)+S'_{ik}\left(\overline{S_{ij}}-\overline{\Omega_{ij}}\right)\right)\frac{\partial\mu'_{app}}{\partial x_j}\right)+ \tag{4.6}$$
$$-\frac{\partial}{\partial x_k}\left(S'_{ij}\frac{\partial}{\partial x_j}\left(T^R_{ik}+T^\mu_{ik}\right)\right)+\frac{\partial p'}{\partial x_j}\frac{\partial S'_{kj}}{\partial x_k}-\mu_{app}\frac{\partial S'_{ij}}{\partial x_k}\frac{\partial S'_{ij}}{\partial x_k}-\mu'_{app}\frac{\partial S'_{ij}}{\partial x_k}\frac{\partial\overline{S_{ij}}}{\partial x_k}+\frac{\partial}{\partial x_j}\left(T^R_{ik}+T^\mu_{ik}\right)\frac{\partial S'_{ij}}{\partial x_k}+$$
$$-\left(\left(S'_{ik}+\Omega'_{ik}\right)\frac{\partial S'_{ij}}{\partial x_k}+\left(S'_{ij}-\Omega'_{ij}\right)\frac{\partial S'_{ik}}{\partial x_k}\right)\frac{\partial\mu_{app}}{\partial x_j}-\left(\left(\overline{S_{ik}}+\overline{\Omega_{ik}}\right)\frac{\partial S'_{ij}}{\partial x_k}+\left(\overline{S_{ij}}-\overline{\Omega_{ij}}\right)\frac{\partial S'_{ik}}{\partial x_k}\right)\frac{\partial\mu'_{app}}{\partial x_j}+$$

The details of the algebra are reported in Appendix C. The conservation equation for the dissipation rate can be obtained by Eq. (2.21), using Eq. (4.6) and applying the principle of conservation.

For a Newtonian fluid the conservation equation of the product of two variables, i.e. the Reynolds stresses, can be obtained easily because each transport equation has the same structure and is sufficient to multiply each variable for the relative transport equation of the other and sum to derive the transport equation for the product of the two terms. This operation is not trivial for a GNF because viscosity is variable and a transport equation for the apparent viscosity that has the same structure of the term for which the viscosity is multiplied is necessary. We proposed a way to solve this problem in [51] where the apparent viscosity has been considered function of the square of the shear rate instead of function of the shear rate itself**.** To avoid confusion between the second and the first derivatives with respect the square of a variable let assume

$$S^2 = A. \tag{4.7}$$

The apparent viscosity is then

$$\mu_{app} = f\left(A\left(x_j, t\right)\right), \tag{4.8}$$

and, using the following relations

$$\frac{\partial \mu_{app}}{\partial t} = \frac{df}{dA}\frac{\partial S^2}{\partial t}, \quad \frac{\partial \mu_{app}}{\partial x_j} = \frac{df}{dA}\frac{\partial S^2}{\partial x_j}, \tag{4.9}$$

the transport equation for apparent viscosity can be finally written as

$$\frac{\partial \mu_{app}}{\partial t} + u_j \frac{\partial \mu_{app}}{\partial x_j} = \frac{df}{dA}\frac{\partial S^2}{\partial t} + \frac{df}{dA} u_j \frac{\partial S^2}{\partial x_j} = \frac{df}{dA}\left(\frac{\partial S^2}{\partial t} + u_j \frac{\partial S^2}{\partial x_j}\right). \tag{4.10}$$

Defining

$$C = \frac{df}{dA}, \tag{4.11}$$

and multiplying Eq. (4.4) by C, Eq. (4.10) can be written as

$$\rho\frac{\partial\mu_{app}}{\partial t}+\rho u_k\frac{\partial\mu_{app}}{\partial x_k}=-2\rho C\left(S_{ij}S_{ik}S_{jk}-S_{ij}\Omega_{ik}\Omega_{jk}\right)+$$
$$\frac{\partial}{\partial x_k}\left(-2CS_{kj}\frac{\partial p}{\partial x_j}+\mu_{app}\frac{\partial\mu_{app}}{\partial x_k}+\left(2CS_{ij}\left(S_{ik}+\Omega_{ik}\right)+2CS_{ik}\left(S_{ij}-\Omega_{ij}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}\right)+\frac{\partial}{\partial x_k}\left(2CS_{kj}\right)\frac{\partial p}{\partial x_j}. \qquad (4.12)$$
$$-\mu_{app}\frac{\partial}{\partial x_k}\left(2CS_{ij}\right)\frac{\partial S_{ij}}{\partial x_k}-\left(\left(S_{ik}+\Omega_{ik}\right)\frac{\partial}{\partial x_k}\left(2CS_{ij}\right)+\left(S_{ij}-\Omega_{ij}\right)\frac{\partial}{\partial x_k}\left(2CS_{ik}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}$$

In conclusion, Eq. (4.12) is the transport equation of the apparent viscosity which is a differential equation rather than an algebraic expression. The form of Eq. (4.12) is similar to the $\$^2$ transport equation. The transport equations derived here for the apparent viscosity, shear rate and variable $\$^2$ are more general than those presented in [51] because here, due to the tridimensionality is not possible to perform the simplifications adopted in 2D case.

5. Transport equation for turbulent dissipation rate

The turbulent dissipation rate equation can be obtained by the sum of Eq. (4.4), multiplied by $4\frac{\mu_{app}}{\rho}$, and Eq. (4.12), multiplied by $\frac{2}{\rho}\$^2$. The instantaneous dissipation rate is finally averaged to give the detailed form of the transport equation

$$\rho\frac{\partial\varepsilon}{\partial t}+\rho U_k\frac{\partial\varepsilon}{\partial x_k}=P^\varepsilon+\Pi^\varepsilon+T^\varepsilon+D^\varepsilon-\rho\varepsilon^\varepsilon \qquad (5.1)$$

$$P^\varepsilon=-4\overline{CS'_{mn}S'_{mn}S_{ij}\left(S_{ik}S_{jk}-\Omega_{ik}\Omega_{jk}\right)}-4\overline{\mu_{app}u'_kS'_{ij}}\frac{\partial\overline{S_{ij}}}{\partial x_k}+\frac{2}{\rho}\frac{\partial}{\partial x_j}\left(T^R_{ik}+T^\mu_{ik}\right)\frac{\partial T^\mu_{ij}}{\partial x_k}+\frac{1}{\rho}\frac{\partial}{\partial x_k}\left(2\overline{\mu_{app}S_{ij}}\right)\frac{\partial T^\mu_{ij}}{\partial x_k}+$$
$$-4\overline{\mu_{app}S'_{ij}\left(\left(S'_{ik}S'_{jk}+S'_{ik}\overline{S_{jk}}+S'_{jk}\overline{S_{ik}}\right)-\left(\Omega'_{ik}\Omega'_{jk}+\Omega'_{ik}\overline{\Omega_{jk}}+\Omega'_{jk}\overline{\Omega_{ik}}\right)\right)}+ \qquad (5.2)$$

$$\Pi^\varepsilon=\frac{\partial}{\partial x_k}\left(-\frac{4}{\rho}\overline{\mu_{app}S'_{kj}\frac{\partial p'}{\partial x_j}}-\frac{4}{\rho}\overline{CS'_{mn}S'_{mn}S_{kj}\frac{\partial p}{\partial x_j}}\right)+\frac{4}{\rho}\overline{\frac{\partial}{\partial x_k}\left(\mu_{app}S'_{kj}\right)\frac{\partial p'}{\partial x_j}}+\frac{4}{\rho}\overline{\frac{\partial}{\partial x_k}\left(CS'_{mn}S'_{mn}S_{kj}\right)\frac{\partial p}{\partial x_j}} \qquad (5.3)$$

$$T^\varepsilon=\frac{\partial}{\partial x_k}\left(-\rho\overline{u'_k\varepsilon'}\right) \qquad (5.4)$$

$$D^{\varepsilon}=\frac{\partial}{\partial x_k}\left(\overline{\mu_{app}\frac{\partial\varepsilon'}{\partial x_k}}+\frac{2}{\rho}\overline{\mu_{app}T_{ij}^{\mu'}}\frac{\partial\overline{S_{ij}}}{\partial x_k}\right)+\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\overline{\mu_{app}\left(S'_{ij}\left(\overline{S_{ik}}+\overline{\Omega_{ik}}\right)+S'_{ik}\left(\overline{S_{ij}}-\overline{\Omega_{ij}}\right)\right)\frac{\partial\mu'_{app}}{\partial x_j}}\right)+$$

$$\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\overline{\left(CS'_{mn}S'_{mn}\left(S_{ij}\left(S_{ik}+\Omega_{ik}\right)+S_{ik}\left(S_{ij}-\Omega_{ij}\right)\right)+\mu_{app}\left(S'_{ij}\left(S'_{ik}+\Omega'_{ik}\right)+S'_{ik}\left(S'_{ij}-\Omega'_{ij}\right)\right)\right)\frac{\partial\mu_{app}}{\partial x_j}}\right)+ \quad (5.5)$$

$$-\frac{\partial}{\partial x_k}\left(\frac{2}{\rho}T_{ij}^{\mu}\frac{\partial}{\partial x_j}\left(T_{ik}^{R}+T_{ik}^{\mu}\right)\right)$$

$$\varepsilon^{\varepsilon}=\frac{4}{\rho^2}\overline{\mu_{app}S'_{ij}\frac{\partial S'_{ij}}{\partial x_k}\frac{\partial\mu_{app}}{\partial x_k}}+\frac{4}{\rho^2}\overline{\frac{\partial}{\partial x_k}\left(\overline{\mu_{app}}\,\overline{S_{ij}}\right)\frac{\partial}{\partial x_k}\left(\mu'_{app}S'_{ij}\right)}+$$

$$\frac{4}{\rho^2}\overline{\left(\left(S'_{ik}+\Omega'_{ik}\right)\frac{\partial}{\partial x_k}\left(\mu_{app}S'_{ij}\right)+\left(S'_{ij}-\Omega'_{ij}\right)\frac{\partial}{\partial x_k}\left(\mu_{app}S'_{ik}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}}+\frac{4}{\rho^2}\overline{\mu_{app}\frac{\partial}{\partial x_k}\left(\mu_{app}S'_{ij}\right)\frac{\partial S'_{ij}}{\partial x_k}}+$$

$$\frac{4}{\rho^2}\overline{\left(\left(\overline{S_{ik}}+\overline{\Omega_{ik}}\right)\frac{\partial}{\partial x_k}\left(\mu_{app}S'_{ij}\right)+\left(\overline{S_{ij}}-\overline{\Omega_{ij}}\right)\frac{\partial}{\partial x_k}\left(\mu_{app}S'_{ik}\right)\right)\frac{\partial\mu'_{app}}{\partial x_j}}+\frac{4}{\rho^2}\overline{\mu'_{app}\frac{\partial}{\partial x_k}\left(\mu_{app}S'_{ij}\right)\frac{\partial\overline{S_{ij}}}{\partial x_k}}+ \quad (5.6)$$

$$\frac{4}{\rho^2}\overline{\left(\left(S_{ik}+\Omega_{ik}\right)\frac{\partial}{\partial x_k}\left(CS'_{mn}S'_{mn}S_{ij}\right)+\left(S_{ij}-\Omega_{ij}\right)\frac{\partial}{\partial x_k}\left(CS'_{mn}S'_{mn}S_{ik}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}}+\frac{4}{\rho^2}\overline{\mu_{app}S'_{mn}S'_{mn}\frac{\partial}{\partial x_k}\left(CS_{ij}\right)\frac{\partial S_{ij}}{\partial x_k}}$$

The details of the algebra are reported in Appendix D. The Newtonian expression can be found just replacing the mean viscosity with the viscosity and neglecting the terms in which the fluctuating viscosity its derivatives and the variable *C* appear. It is clear from the equations above that introducing the dependence on the second invariant of shear rate tensor on the viscosity dramatically changes the structure of the transport equation of turbulent dissipation rate introducing many new terms, respect to the Newtonian case, which require modelling.

It is possible verify that from equations (5.1 - 5.6) we can get the simplified expressions achieved in [51] for the 2D case which are reported below

$$P^{\varepsilon}\rightarrow\frac{2}{\rho}\overline{\mu_{app}\frac{\partial T_{ij}^{R'}}{\partial x_k}}\frac{\partial\overline{S_{ij}}}{\partial x_k}-4\overline{\mu_{app}\frac{\partial k'}{\partial x_k}}\frac{\partial\overline{S_{kj}}}{\partial x_j}+\frac{1}{\rho}\frac{\partial}{\partial x_k}\left(T_{ij}^{R}+T_{ij}^{\mu}+2\overline{\mu_{app}}\,\overline{S_{ij}}\right)\frac{\partial T_{ij}^{\mu}}{\partial x_k} \quad (5.7)$$

$$D^{\varepsilon}\rightarrow\frac{\partial}{\partial x_k}\left(\overline{\mu_{app}\frac{\partial\varepsilon'}{\partial x_k}}+\overline{2\varepsilon'\left(1+\frac{CS_{ij}S_{ij}}{\mu_{app}}\right)\frac{\partial\mu_{app}}{\partial x_k}}+\frac{4}{\rho}\overline{\mu_{app}S'_{ij}\frac{\partial}{\partial x_k}\left(\mu_{app}\overline{S_{ij}}\right)}\right)$$

$$-\frac{\partial}{\partial x_k}\left(\frac{1}{\rho}T_{ij}^{\mu}\frac{\partial}{\partial x_k}\left(T_{ij}^{R}+T_{ij}^{\mu}+2\overline{\mu_{app}}\,\overline{S_{ij}}\right)\right) \quad (5.8)$$

$$\varepsilon^{\varepsilon} \rightarrow \frac{4}{\rho^2} \overline{\frac{\partial}{\partial x_k}\left(\mu_{app} S'_{ij} + C\$^2 S_{ij}\right) \frac{\partial}{\partial x_k}\left(\mu_{app} S_{ij}\right)} \qquad (5.9)$$

6. Conclusions

The aim of this paper is to extend the results obtained in [52] for a 2D inelastic Generalized Newtonian Fluid in 3D case. The conservation equations for the mean momentum components, turbulent kinetic energy and turbulent dissipation rate, achieved through the transport equations of the instantaneous components of the rate of shear tensor, are investigated as in [52]. It can be concluded that no difference is present in the mean momentum and turbulent kinetic energy equations, while the 3D nature of the flow affects the turbulent dissipation rate equation. This is found in Newtonian fluids as well where the non-linear terms generate a different expression for the production of turbulent dissipation rate. In a GNF fluid the diffusive terms are non-linear as well and consequently the 3D motion affects all the terms in which the viscosity is present i.e. production, molecular transport and dissipation. From what said until now it may appear that the 2D study carried out in [52] has been useless but this is not the case. Although the terms obtained in the two different analyses don't match, the 2D terms have much simpler expression and since they are expected to have the same order of magnitude of the 3D ones, they can be used to get the closure relationship once the modelling of the exact equations will be undertaken.

The transport equations obtained in this work are exact and expressed in conservative form but not usable because the closure relationships are missing nevertheless what we proposed to do was to justify the introduction of a new conservation equation for the turbulent dissipation rate giving a strict mathematical derivation. The terms modelling problem can be approached in different ways. In [24] , in which a visco-elastic fluid has been considered, an order of magnitude analysis was used to neglect some term under the hypothesis that the instantaneous viscosity is a function of the instantaneous turbulent dissipation rate which was considered to follow a log-normal distribution.

For what concern us we plan to face the closure problem in the near future performing Direct Numerical Simulations on different non-Newtonian fluids at different Reynolds number. The numerical results obtained will allow us understanding the behaviour of the correlation of viscosity fluctuation and viscosity derivatives with the other terms, neglecting the unessential terms and modelling the others.

Appendix

The transport equations for a GNF are derived similarly to a Newtonian fluid with a variable apparent viscosity.

A. Transport equation for turbulent kinetic energy

The following equations for the fluctuating components are derived by subtraction of the mean equations from the instantaneous ones.

The conservation equation for mass is

$$\frac{\partial u_k'}{\partial x_k} = 0 . \tag{A.1}$$

The momentum conservation equation is

$$\rho \frac{\partial u_i'}{\partial t} + \rho U_k \frac{\partial u_i'}{\partial x_k} = -\rho u_k' \frac{\partial U_i}{\partial x_k} - \rho u_k' \frac{\partial u_i'}{\partial x_k} - \frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_k}\left(2\overline{\mu_{app}} S_{ik}' + 2\mu_{app}' \overline{S_{ik}} + 2\mu_{app}' S_{ik}' \right) - \frac{\partial}{\partial x_k}\left(T_{ki}^R + T_{ik}^\mu \right) . \tag{A.2}$$

The turbulent kinetic energy is obtained multiplying Eq. (A.2) by u_i', summing up all components and averaging. The same process than in a Newtonian fluid is used for the viscous terms with the derivation by part

$$\begin{aligned} & u_i' \frac{\partial}{\partial x_k}\left(2\overline{\mu_{app}} S_{ik}' \right) = \frac{\partial}{\partial x_k}\left(2\overline{\mu_{app}} u_i' S_{ik}' \right) - 2\overline{\mu_{app}} S_{ik}' \frac{\partial u_i'}{\partial x_k} = \\ & \frac{\partial}{\partial x_k}\left(\overline{\mu_{app}} \frac{\partial}{\partial x_k}\left(\frac{u_i' u_i'}{2} \right) + \overline{\mu_{app}} \frac{\partial u_i' u_k'}{\partial x_i} - \overline{\mu_{app}} u_k' \frac{\partial u_i'}{\partial x_i} \right) - 2\overline{\mu_{app}} S_{ik}' \frac{\partial u_i'}{\partial x_k} , \end{aligned} \tag{A.3}$$

$$u_i' \frac{\partial}{\partial x_k}\left(2\mu_{app}' \overline{S_{ik}} \right) = \frac{\partial}{\partial x_k}\left(2\mu_{app}' u_i' \overline{S_{ik}} \right) - 2\overline{S_{ik}} \mu_{app}' \frac{\partial u_i'}{\partial x_k} . \tag{A.4}$$

Summing up it is obtained the equation for the turbulent kinetic energy (16) where the terms with similar physical meaning are close each to the other, while the Reynolds stresses are closer to those with the fluctuating viscosity components.

B. Transport equation for the shear rate

The exact form of the variable apparent viscosity is obtained by using the fundamental expressions of the rate of strain tensor and the terms of the rotation rate tensor, Ω_{ij}, based on mass conservation and the relative definitions.

The following basic equations are employed

$$S_{ii} = 0, \tag{B.1}$$

$$-\frac{\partial S_{ij}}{\partial x_k} + \frac{1}{2}\left(\frac{\partial S_{ik}}{\partial x_j} + \frac{\partial S_{jk}}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial \Omega_{ik}}{\partial x_j} + \frac{\partial \Omega_{jk}}{\partial x_i}\right) = 0, \tag{B.2}$$

which allow to derive the conservation equations for the instantaneous variables in a form similar to those for a Newtonian fluid.

The non-Newtonian terms are obtained by changing the order of derivation, summing and subtracting the different terms. Applying the $\frac{1}{2}\frac{\partial}{\partial x_j}$ operator to u_i component and the $\frac{1}{2}\frac{\partial}{\partial x_i}$ operator to u_j component and summing up the transport equation for the rate of strain tensor components can be obtained.

$$\begin{aligned}\rho\frac{\partial S_{ij}}{\partial t} + \rho u_k \frac{\partial S_{ij}}{\partial x_k} = -\rho\frac{1}{2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k} - \rho\frac{1}{2}\frac{\partial u_k}{\partial x_j}\frac{\partial u_i}{\partial x_k} - \frac{\partial}{\partial x_j}\left(\frac{1}{2}\frac{\partial p}{\partial x_i}\right) - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\frac{\partial p}{\partial x_j}\right) + \\ \frac{\partial}{\partial x_k}\left(\mu_{app}\left(\frac{\partial S_{jk}}{\partial x_i} + \frac{\partial S_{ik}}{\partial x_j}\right) + S_{jk}\frac{\partial \mu_{app}}{\partial x_i} + S_{ik}\frac{\partial \mu_{app}}{\partial x_j}\right)\end{aligned} \tag{B.3}$$

Using the definitions of rate of strain tensor and rotation rate tensor we can find

$$\begin{aligned}-\rho\frac{1}{2}\frac{\partial u_k}{\partial x_j}\frac{\partial u_i}{\partial x_k} - \rho\frac{1}{2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k} = \\ -\rho\frac{1}{2}\left(S_{kj} + \Omega_{kj}\right)\left(S_{ik} + \Omega_{ik}\right) - \rho\frac{1}{2}\left(S_{ki} + \Omega_{ki}\right)\left(S_{jk} + \Omega_{jk}\right) = -\rho S_{ik}S_{jk} + \rho\Omega_{ik}\Omega_{jk}\end{aligned} . \tag{B.4}$$

Using equation (B.2) we can find

$$\begin{aligned}\frac{\partial}{\partial x_k}\left(\mu_{app}\left(\frac{\partial S_{jk}}{\partial x_i} + \frac{\partial S_{ik}}{\partial x_j}\right) + S_{jk}\frac{\partial \mu_{app}}{\partial x_i} + S_{ik}\frac{\partial \mu_{app}}{\partial x_j}\right) = \frac{\partial}{\partial x_k}\left(\mu_{app}\frac{\partial S_{ij}}{\partial x_k}\right) + \\ \frac{\partial}{\partial x_k}\left(\mu_{app}\frac{\partial}{\partial x_j}\left(\frac{S_{ik} - \Omega_{ik}}{2}\right) + \mu_{app}\frac{\partial}{\partial x_i}\left(\frac{S_{jk} - \Omega_{jk}}{2}\right) + S_{ik}\frac{\partial \mu_{app}}{\partial x_j} + S_{jk}\frac{\partial \mu_{app}}{\partial x_i}\right)\end{aligned} \tag{B.5}$$

Now swapping the derivation order

$$
\begin{aligned}
&\frac{\partial}{\partial x_k}\left(\mu_{app}\frac{\partial}{\partial x_j}\left(\frac{S_{ik}-\Omega_{ik}}{2}\right)+\mu_{app}\frac{\partial}{\partial x_i}\left(\frac{S_{jk}-\Omega_{jk}}{2}\right)\right)=\\
&\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}\left(\mu_{app}\frac{S_{ik}-\Omega_{ik}}{2}\right)+\frac{\partial}{\partial x_i}\left(\mu_{app}\frac{S_{jk}-\Omega_{jk}}{2}\right)-\left(\frac{S_{ik}-\Omega_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_j}-\left(\frac{S_{jk}-\Omega_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_i}\right)=\\
&\frac{\partial}{\partial x_k}\left(-\left(\frac{S_{ik}-\Omega_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_j}-\left(\frac{S_{jk}-\Omega_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_i}\right)+\frac{\partial}{\partial x_j}\left(\left(\frac{S_{ik}-\Omega_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left(\left(\frac{S_{jk}-\Omega_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}\right)+\\
&\frac{\partial}{\partial x_j}\left(\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{S_{ik}-\Omega_{ik}}{2}\right)\right)+\frac{\partial}{\partial x_i}\left(\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{S_{jk}-\Omega_{jk}}{2}\right)\right)
\end{aligned}
\tag{B.6}
$$

And remembering eq. (B.5)

$$
\begin{aligned}
&\frac{\partial}{\partial x_j}\left(\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{S_{ik}-\Omega_{ik}}{2}\right)\right)+\frac{\partial}{\partial x_i}\left(\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{S_{jk}-\Omega_{jk}}{2}\right)\right)=\frac{\partial}{\partial x_j}\left(\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{\partial u_k}{\partial x_i}\right)\right)+\frac{\partial}{\partial x_i}\left(\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{\partial u_k}{\partial x_j}\right)\right)=\\
&\frac{\partial}{\partial x_j}\left(\mu_{app}\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)\right)+\frac{\partial}{\partial x_i}\left(\mu_{app}\frac{\partial}{\partial x_j}\left(\frac{\partial u_k}{\partial x_k}\right)\right)=0
\end{aligned}
\tag{B.7}
$$

So finally we can obtain Eq. (4.3)

The multiplication of each equation for the transported variable allows obtaining:

$$
\begin{aligned}
&\rho\frac{\partial}{\partial t}\left(\frac{S_{ij}S_{ij}}{2}\right)+\rho u_k\frac{\partial}{\partial x_k}\left(\frac{S_{ij}S_{ij}}{2}\right)=-\rho S_{ij}S_{ik}S_{jk}+\rho S_{ij}\Omega_{ik}\Omega_{jk}-\frac{1}{2}\frac{\partial}{\partial x_i}\left(S_{ij}\frac{\partial p}{\partial x_j}\right)-\frac{1}{2}\frac{\partial}{\partial x_j}\left(S_{ij}\frac{\partial p}{\partial x_i}\right)+\\
&\frac{\partial}{\partial x_k}\left(\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{S_{ij}S_{ij}}{2}\right)\right)+\frac{\partial}{\partial x_k}\left(S_{ij}\left(\frac{S_{ik}+\Omega_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_j}+S_{ij}\left(\frac{S_{jk}+\Omega_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_i}\right)-\mu_{app}\frac{\partial S_{ij}}{\partial x_k}\frac{\partial S_{ij}}{\partial x_k}+\\
&\frac{\partial}{\partial x_j}\left(S_{ij}\left(\frac{S_{ik}-\Omega_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left(S_{ij}\left(\frac{S_{jk}-\Omega_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}\right)+\frac{1}{2}\frac{\partial S_{ij}}{\partial x_i}\left(\frac{\partial p}{\partial x_j}\right)+\frac{1}{2}\frac{\partial S_{ij}}{\partial x_j}\left(\frac{\partial p}{\partial x_i}\right)+\\
&-\left(\left(\frac{S_{ik}+\Omega_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_j}+\left(\frac{S_{jk}+\Omega_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_i}\right)\frac{\partial S_{ij}}{\partial x_k}-\left(\frac{S_{ik}-\Omega_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}\frac{\partial S_{ij}}{\partial x_j}-\left(\frac{S_{jk}-\Omega_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}\frac{\partial S_{ij}}{\partial x_i}
\end{aligned}
\quad . \tag{B.8}
$$

Summing all components finally the conservation equation for the square of the shear rate in the instantaneous form is obtained as Eq. (4.4).

C. Transport equation for $\$^2$

The transport equation for the fluctuations of the rate of strain tensor is derived similarly to the instantaneous case. Applying the $\frac{1}{2}\frac{\partial}{\partial x_j}$ operator to u_i' component and the $\frac{1}{2}\frac{\partial}{\partial x_i}$ operator to u_j' component and summing up the transport equation for the rate of strain tensor components can be obtained.

$$\rho\frac{\partial S'_{ij}}{\partial t}+\rho U_k\frac{\partial S'_{ij}}{\partial x_k}=-\rho\frac{1}{2}\left(\frac{\partial U_k}{\partial x_j}\frac{\partial u'_i}{\partial x_k}+\frac{\partial U_j}{\partial x_k}\frac{\partial u'_k}{\partial x_i}+\frac{\partial U_k}{\partial x_i}\frac{\partial u'_j}{\partial x_k}+\frac{\partial U_i}{\partial x_k}\frac{\partial u'_k}{\partial x_j}+\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_k}{\partial x_j}+\frac{\partial u'_j}{\partial x_k}\frac{\partial u'_k}{\partial x_i}\right)+$$

$$-\rho u'_k\frac{\partial \overline{S_{ij}}}{\partial x_k}-\rho u'_k\frac{\partial S'_{ij}}{\partial x_k}-\frac{\partial}{\partial x_i}\left(\frac{1}{2}\frac{\partial p'}{\partial x_j}\right)-\frac{\partial}{\partial x_j}\left(\frac{1}{2}\frac{\partial p'}{\partial x_i}\right)+\frac{\partial}{\partial x_k}\left(\mu_{app}\left(\frac{\partial S'_{ik}}{\partial x_j}+\frac{\partial S'_{jk}}{\partial x_i}\right)+S'_{ik}\frac{\partial \mu_{app}}{\partial x_j}+S'_{jk}\frac{\partial \mu_{app}}{\partial x_i}\right)+ \tag{C.1}$$

$$\frac{\partial}{\partial x_k}\left(\mu'_{app}\left(\frac{\partial \overline{S_{ik}}}{\partial x_j}+\frac{\partial \overline{S_{jk}}}{\partial x_i}\right)+\overline{S_{ik}}\frac{\partial \mu'_{app}}{\partial x_j}+\overline{S_{jk}}\frac{\partial \mu'_{app}}{\partial x_i}\right)-\frac{\partial}{\partial x_k}\left(\frac{1}{2}\left(\frac{\partial}{\partial x_j}\left(T^R_{ik}+T^\mu_{ik}\right)+\frac{\partial}{\partial x_i}\left(T^R_{jk}+T^\mu_{jk}\right)\right)\right)$$

Where we have used the following relationship

$$\frac{\partial}{\partial x_k}\left(\overline{\rho u'_k S'_{ij}}\right)=\frac{\partial}{\partial x_k}\left(\overline{\rho u'_k\frac{1}{2}\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)}\right)=\frac{\partial}{\partial x_k}\left(-\frac{1}{2}\left(\frac{\partial T^R_{ki}}{\partial x_j}+\frac{\partial T^R_{kj}}{\partial x_i}\right)\right)-\rho\left(\overline{S'_{ik}S'_{jk}}-\overline{\Omega'_{ik}\Omega'_{jk}}\right)$$

$$\frac{\partial}{\partial x_k}\left(-\overline{\mu'_{app}\frac{\partial S'_{ij}}{\partial x_k}}-\overline{\left(\frac{S'_{ik}+\Omega'_{ik}}{2}\right)\frac{\partial \mu'_{app}}{\partial x_j}}-\overline{\left(\frac{S'_{jk}+\Omega'_{jk}}{2}\right)\frac{\partial \mu'_{app}}{\partial x_i}}\right)+\frac{\partial}{\partial x_j}\left(-\overline{\left(\frac{S'_{ik}-\Omega'_{ik}}{2}\right)\frac{\partial \mu'_{app}}{\partial x_k}}\right)+$$

$$\frac{\partial}{\partial x_i}\left(-\overline{\left(\frac{S'_{jk}-\Omega'_{jk}}{2}\right)\frac{\partial \mu'_{app}}{\partial x_k}}\right)=\frac{\partial}{\partial x_k}\left(-\overline{\mu'_{app}\frac{\partial S'_{ij}}{\partial x_k}}-\overline{\left(\frac{S'_{ik}+\Omega'_{ik}}{2}\right)\frac{\partial \mu'_{app}}{\partial x_j}}-\overline{\left(\frac{S'_{jk}+\Omega'_{jk}}{2}\right)\frac{\partial \mu'_{app}}{\partial x_i}}\right)+$$

$$\frac{\partial}{\partial x_j}\left(-\frac{\partial}{\partial x_k}\overline{\mu'_{app}\left(\frac{S'_{ik}-\Omega'_{ik}}{2}\right)}\right)+\frac{\partial}{\partial x_i}\left(-\frac{\partial}{\partial x_k}\overline{\mu'_{app}\left(\frac{S'_{jk}-\Omega'_{jk}}{2}\right)}\right)=$$

$$\frac{\partial}{\partial x_k}\left(-\overline{\mu'_{app}\frac{\partial S'_{ij}}{\partial x_k}}-\overline{\left(\frac{S'_{ik}+\Omega'_{ik}}{2}\right)\frac{\partial \mu'_{app}}{\partial x_j}}-\overline{\left(\frac{S'_{jk}+\Omega'_{jk}}{2}\right)\frac{\partial \mu'_{app}}{\partial x_i}}-\frac{\partial}{\partial x_j}\overline{\mu'_{app}\left(\frac{S'_{ik}-\Omega'_{ik}}{2}\right)}-\frac{\partial}{\partial x_i}\overline{\mu'_{app}\left(\frac{S'_{jk}-\Omega'_{jk}}{2}\right)}\right)=$$

$$\frac{\partial}{\partial x_k}\left(-\frac{1}{2}\left(\frac{\partial T^\mu_{ik}}{\partial x_j}+\frac{\partial T^\mu_{jk}}{\partial x_i}\right)\right)+$$

Using the definitions of rate of strain tensor and rotation rate tensor we can find

$$\begin{cases}\dfrac{\partial U_k}{\partial x_j}\dfrac{\partial u'_i}{\partial x_k}+\dfrac{\partial U_j}{\partial x_k}\dfrac{\partial u'_k}{\partial x_i}+\dfrac{\partial U_k}{\partial x_i}\dfrac{\partial u'_j}{\partial x_k}+\dfrac{\partial U_i}{\partial x_k}\dfrac{\partial u'_k}{\partial x_j}=\left(\overline{S_{jk}}-\overline{\Omega_{jk}}\right)\left(S'_{ik}+\Omega'_{ik}\right)+\left(\overline{S_{jk}}+\overline{\Omega_{jk}}\right)\left(S'_{ik}-\Omega'_{ik}\right)+\\ \left(\overline{S_{ik}}-\overline{\Omega_{ik}}\right)\left(S'_{jk}+\Omega'_{jk}\right)+\left(\overline{S_{ik}}+\overline{\Omega_{ik}}\right)\left(S'_{jk}-\Omega'_{jk}\right)=2\left(\overline{S_{jk}}S'_{ik}+\overline{S_{ik}}S'_{jk}-\overline{\Omega_{jk}}\Omega'_{ik}-\overline{\Omega_{ik}}\Omega'_{jk}\right)\\ \dfrac{\partial u'_i}{\partial x_k}\dfrac{\partial u'_k}{\partial x_j}+\dfrac{\partial u'_j}{\partial x_k}\dfrac{\partial u'_k}{\partial x_i}=\left(\left(S'_{ik}+\Omega'_{ik}\right)\left(S'_{jk}-\Omega'_{jk}\right)+\left(S'_{jk}+\Omega'_{jk}\right)\left(S'_{ik}-\Omega'_{ik}\right)\right)=2\left(S'_{ik}S'_{jk}-\Omega'_{ik}\Omega'_{jk}\right)\end{cases} \tag{C.2}$$

Using similar reasons made in equations (B.5-B.7),

$$
\begin{cases}
\dfrac{\partial}{\partial x_k}\left(\mu_{app}\dfrac{\partial S'_{ik}}{\partial x_j}+\mu_{app}\dfrac{\partial S'_{jk}}{\partial x_i}+S'_{ik}\dfrac{\partial \mu_{app}}{\partial x_j}+S'_{jk}\dfrac{\partial \mu_{app}}{\partial x_i}\right)=\dfrac{\partial}{\partial x_k}\left(\mu_{app}\dfrac{\partial S'_{ij}}{\partial x_k}\right)+ \\
\dfrac{\partial}{\partial x_k}\left(\left(\dfrac{S'_{ik}+\Omega'_{ik}}{2}\right)\dfrac{\partial \mu_{app}}{\partial x_j}+\left(\dfrac{S'_{jk}+\Omega'_{jk}}{2}\right)\dfrac{\partial \mu_{app}}{\partial x_i}\right)+\dfrac{\partial}{\partial x_j}\left(\left(\dfrac{S'_{ik}-\Omega'_{ik}}{2}\right)\dfrac{\partial \mu_{app}}{\partial x_k}\right)+\dfrac{\partial}{\partial x_i}\left(\left(\dfrac{S'_{jk}-\Omega'_{jk}}{2}\right)\dfrac{\partial \mu_{app}}{\partial x_k}\right) \\
\dfrac{\partial}{\partial x_k}\left(\mu'_{app}\dfrac{\partial \overline{S_{ik}}}{\partial x_j}+\mu'_{app}\dfrac{\partial \overline{S_{jk}}}{\partial x_i}+\overline{S_{ik}}\dfrac{\partial \mu'_{app}}{\partial x_j}+\overline{S_{jk}}\dfrac{\partial \mu'_{app}}{\partial x_i}\right)=\dfrac{\partial}{\partial x_k}\left(\mu'_{app}\dfrac{\partial \overline{S_{ij}}}{\partial x_k}\right)+ \\
\dfrac{\partial}{\partial x_k}\left(\left(\dfrac{\overline{S_{ik}}+\overline{\Omega_{ik}}}{2}\right)\dfrac{\partial \mu'_{app}}{\partial x_j}+\left(\dfrac{\overline{S_{jk}}+\overline{\Omega_{jk}}}{2}\right)\dfrac{\partial \mu'_{app}}{\partial x_i}\right)+\dfrac{\partial}{\partial x_j}\left(\left(\dfrac{\overline{S_{ik}}-\overline{\Omega_{ik}}}{2}\right)\dfrac{\partial \mu'_{app}}{\partial x_k}\right)+\dfrac{\partial}{\partial x_i}\left(\left(\dfrac{\overline{S_{jk}}-\overline{\Omega_{jk}}}{2}\right)\dfrac{\partial \mu'_{app}}{\partial x_k}\right)
\end{cases}
\tag{C.3}
$$

We can finally find Eq.(4.5)

The multiplication of Eq. (4.5) for the corresponding transported variable allows writing the following equation:

$$
\begin{aligned}
&\rho\frac{\partial}{\partial t}\left(\frac{S'_{ij}S'_{ij}}{2}\right)+\rho U_k\frac{\partial}{\partial x_k}\left(\frac{S'_{ij}S'_{ij}}{2}\right)=-\rho\left(S'_{ij}S'_{ik}\overline{S_{jk}}+S'_{ij}S'_{jk}\overline{S_{ik}}-S'_{ij}\Omega'_{ik}\overline{\Omega_{jk}}-S'_{ij}\Omega'_{jk}\overline{\Omega_{ik}}\right)+\\
&-\rho\left(S'_{ij}S'_{ik}S'_{jk}-S'_{ij}\Omega'_{ik}\Omega'_{jk}\right)-\rho u'_k S'_{ij}\frac{\partial\overline{S_{ij}}}{\partial x_k}-\frac{\partial}{\partial x_i}\left(\frac{1}{2}S'_{ij}\frac{\partial p'}{\partial x_j}\right)-\frac{\partial}{\partial x_j}\left(\frac{1}{2}S'_{ij}\frac{\partial p'}{\partial x_i}\right)+\\
&\frac{\partial}{\partial x_k}\left(-\rho u'_k\frac{S'_{ij}S'_{ij}}{2}+\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{S'_{ij}S'_{ij}}{2}\right)+\mu'_{app}S'_{ij}\frac{\partial\overline{S_{ij}}}{\partial x_k}-S'_{ij}\frac{1}{2}\left(\frac{\partial}{\partial x_j}\left(T^R_{ik}+T^{\mu}_{ik}\right)+\frac{\partial}{\partial x_i}\left(T^R_{jk}+T^{\mu}_{jk}\right)\right)\right)+\\
&\frac{\partial}{\partial x_k}\left(S'_{ij}\left(\frac{S'_{ik}+\Omega'_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_j}+S'_{ij}\left(\frac{S'_{jk}+\Omega'_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_i}+\left(\frac{\overline{S_{ik}}+\overline{\Omega_{ik}}}{2}\right)S'_{ij}\frac{\partial\mu'_{app}}{\partial x_j}+\left(\frac{\overline{S_{jk}}+\overline{\Omega_{jk}}}{2}\right)S'_{ij}\frac{\partial\mu'_{app}}{\partial x_i}\right)+\\
&\frac{\partial}{\partial x_j}\left(S'_{ij}\left(\frac{S'_{ik}-\Omega'_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}+\left(\frac{\overline{S_{ik}}-\overline{\Omega_{ik}}}{2}\right)S'_{ij}\frac{\partial\mu'_{app}}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left(S'_{ij}\left(\frac{S'_{jk}-\Omega'_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}+\left(\frac{\overline{S_{jk}}-\overline{\Omega_{jk}}}{2}\right)S'_{ij}\frac{\partial\mu'_{app}}{\partial x_k}\right)+\\
&\frac{1}{2}\frac{\partial p'}{\partial x_j}\frac{\partial S'_{ij}}{\partial x_i}+\frac{1}{2}\frac{\partial p'}{\partial x_i}\frac{\partial S'_{ij}}{\partial x_j}-\left(\mu_{app}\frac{\partial S'_{ij}}{\partial x_k}+\mu'_{app}\frac{\partial\overline{S_{ij}}}{\partial x_k}+\frac{1}{2}\left(\frac{\partial}{\partial x_j}\left(T^R_{ik}+T^{\mu}_{ik}\right)+\frac{\partial}{\partial x_i}\left(T^R_{jk}+T^{\mu}_{jk}\right)\right)\right)\frac{\partial S'_{ij}}{\partial x_k}+\\
&-\left(\left(\frac{S'_{ik}+\Omega'_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_j}+\left(\frac{S'_{jk}+\Omega'_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_i}+\left(\frac{\overline{S_{ik}}+\overline{\Omega_{ik}}}{2}\right)\frac{\partial\mu'_{app}}{\partial x_j}+\left(\frac{\overline{S_{jk}}+\overline{\Omega_{jk}}}{2}\right)\frac{\partial\mu'_{app}}{\partial x_i}\right)\frac{\partial S'_{ij}}{\partial x_k}+\\
&-\left(\left(\frac{S'_{ik}-\Omega'_{ik}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}+\left(\frac{\overline{S_{ik}}-\overline{\Omega_{ik}}}{2}\right)\frac{\partial\mu'_{app}}{\partial x_k}\right)\frac{\partial S'_{ij}}{\partial x_j}-\left(\left(\frac{S'_{jk}-\Omega'_{jk}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}+\left(\frac{\overline{S_{jk}}-\overline{\Omega_{jk}}}{2}\right)\frac{\partial\mu'_{app}}{\partial x_k}\right)\frac{\partial S'_{ij}}{\partial x_i}
\end{aligned}
\tag{C.4}
$$

The summation of all transport equations with the square of the fluctuating rate of strain tensor components allows obtaining Eq. (4.6) in its final form.

D. Transport equation for the dissipation rate of turbulent kinetic energy

The step by step algebra to derive the equation for the dissipation rate is the following.

D.1- First step

Multiplication of Eq. (C.1) by $4\frac{\mu_{app}}{\rho}$

$$\begin{aligned}
&\frac{4}{\rho}\mu_{app}\left(\rho\frac{\partial}{\partial t}\left(\frac{S'_{mn}S'_{mn}}{2}\right)+\rho U_k\frac{\partial}{\partial x_k}\left(\frac{S'_{mn}S'_{mn}}{2}\right)\right)=-\rho\frac{4}{\rho}\mu_{app}\left(S'_{ij}S'_{ik}\overline{S_{jk}}+S'_{ij}S'_{jk}\overline{S_{ik}}-S'_{ij}\Omega'_{ik}\overline{\Omega_{jk}}-S'_{ij}\Omega'_{jk}\overline{\Omega_{ik}}\right)+\\
&-\rho\frac{4}{\rho}\mu_{app}\left(S'_{ij}S'_{ik}S'_{jk}-S'_{ij}\Omega'_{ik}\Omega'_{jk}\right)-\rho\frac{4}{\rho}\mu_{app}u'_kS'_{ij}\frac{\partial\overline{S_{ij}}}{\partial x_k}+\\
&\frac{\partial}{\partial x_k}\left(-\frac{4}{\rho}\mu_{app}S'_{kj}\frac{\partial p'}{\partial x_j}-\rho u'_k\frac{4}{\rho}\mu_{app}\frac{S'_{mn}S'_{mn}}{2}+\mu_{app}\frac{4}{\rho}\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{S'_{mn}S'_{mn}}{2}\right)+\frac{1}{2}\frac{4}{\rho}\mu_{app}T^{\mu'}_{ij}\frac{\partial\overline{S_{ij}}}{\partial x_k}\right)+\\
&\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}\left(S'_{ij}\left(S'_{ik}+\Omega'_{ik}\right)+S'_{ik}\left(S'_{ij}-\Omega'_{ij}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}\right)+\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}\left(S'_{ij}\left(\overline{S_{ik}}+\overline{\Omega_{ik}}\right)+S'_{ik}\left(\overline{S_{ij}}-\overline{\Omega_{ij}}\right)\right)\frac{\partial\mu'_{app}}{\partial x_j}\right)+\\
&-\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\frac{\partial}{\partial x_j}\left(T^R_{ik}+T^\mu_{ik}\right)\right)+\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{kj}\right)\frac{\partial p'}{\partial x_j}-\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)\frac{\partial S'_{ij}}{\partial x_k}+\\
&\frac{\partial}{\partial x_j}\left(T^R_{ik}+T^\mu_{ik}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)-\mu'_{app}\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)\frac{\partial\overline{S_{ij}}}{\partial x_k}+\rho u'_k\frac{S'_{mn}S'_{mn}}{2}\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}\right)+\\
&-\left(\left(S'_{ik}+\Omega'_{ik}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)+\left(S'_{ij}-\Omega'_{ij}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ik}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}+\\
&-\left(\left(\left(\overline{S_{ik}}+\overline{\Omega_{ik}}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)+\left(\overline{S_{ij}}-\overline{\Omega_{ij}}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ik}\right)\right)\frac{\partial\mu'_{app}}{\partial x_j}\right)+
\end{aligned}\tag{D.1}$$

D.2- Second step

Collection of the different terms

$$\begin{aligned}
&\frac{4}{\rho}\frac{S'_{mn}S'_{mn}}{2}\left(\rho\frac{\partial\mu_{app}}{\partial t}+\rho U_k\frac{\partial\mu_{app}}{\partial x_k}\right)=-4CS'_{mn}S'_{mn}\left(S_{ij}S_{ik}S_{jk}-S_{ij}\Omega_{ik}\Omega_{jk}\right)-\rho\frac{S'_{mn}S'_{mn}}{2}u'_k\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}\right)+\\
&\frac{\partial}{\partial x_k}\left(-\frac{4}{\rho}CS'_{mn}S'_{mn}S_{kj}\frac{\partial p}{\partial x_j}+\mu_{app}\frac{4}{\rho}\frac{S'_{mn}S'_{mn}}{2}\frac{\partial\mu_{app}}{\partial x_k}+\frac{4}{\rho}CS'_{mn}S'_{mn}\left(S_{ij}\left(S_{ik}+\Omega_{ik}\right)+S_{ik}\left(S_{ij}-\Omega_{ij}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}\right)+\\
&\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}CS'_{mn}S'_{mn}S_{kj}\right)\frac{\partial p}{\partial x_j}-\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\frac{S'_{mn}S'_{mn}}{2}\right)\frac{\partial\mu_{app}}{\partial x_k}-\mu_{app}\frac{4}{\rho}S'_{mn}S'_{mn}\frac{\partial}{\partial x_k}\left(CS_{ij}\right)\frac{\partial S_{ij}}{\partial x_k}+\\
&-\left(\left(S_{ik}+\Omega_{ik}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}CS'_{mn}S'_{mn}S_{ij}\right)+\left(S_{ij}-\Omega_{ij}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}CS'_{mn}S'_{mn}S_{ik}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}+
\end{aligned}\tag{D.2}$$

D.3- Third step

Summation of Eq. (D.1) and Eq. (D.2) gives

$$
\begin{aligned}
&\frac{4}{\rho}\mu_{app}\left(\rho\frac{\partial}{\partial t}\left(\frac{S'_{mn}S'_{mn}}{2}\right)+\rho U_k\frac{\partial}{\partial x_k}\left(\frac{S'_{mn}S'_{mn}}{2}\right)\right)+\frac{4}{\rho}\frac{S'_{mn}S'_{mn}}{2}\left(\rho\frac{\partial\mu_{app}}{\partial t}+\rho U_k\frac{\partial\mu_{app}}{\partial x_k}\right)=-4\mu_{app}\left(S'_{ij}S'_{ik}S'_{jk}-S'_{ij}\Omega'_{ik}\Omega'_{jk}\right)+\\
&-4CS'_{mn}S'_{mn}\left(S_{ij}S_{ik}S_{jk}-S_{ij}\Omega_{ik}\Omega_{jk}\right)-4\mu_{app}\left(S'_{ij}S'_{ik}\overline{S_{jk}}+S'_{ij}S'_{jk}\overline{S_{ik}}-S'_{ij}\Omega'_{ik}\overline{\Omega_{jk}}-S'_{ij}\Omega'_{jk}\overline{\Omega_{ik}}\right)-4\mu_{app}u'_kS'_{ij}\frac{\partial\overline{S_{ij}}}{\partial x_k}+\\
&\frac{\partial}{\partial x_k}\left(-\frac{4}{\rho}\mu_{app}S'_{kj}\frac{\partial p'}{\partial x_j}-\frac{4}{\rho}CS'_{mn}S'_{mn}S_{kj}\frac{\partial p}{\partial x_j}-\rho u'_k\frac{4}{\rho}\mu_{app}\frac{S'_{mn}S'_{mn}}{2}+\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}\frac{S'_{mn}S'_{mn}}{2}\right)+\frac{1}{2}\frac{4}{\rho}\mu_{app}T_{ij}^{\mu'}\frac{\partial\overline{S_{ij}}}{\partial x_k}\right)+\\
&\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}CS'_{mn}S'_{mn}\left(S_{ij}\left(S_{ik}+\Omega_{ik}\right)+S_{ik}\left(S_{ij}-\Omega_{ij}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}+\frac{4}{\rho}\mu_{app}\left(S'_{ij}\left(S'_{ik}+\Omega'_{ik}\right)+S'_{ik}\left(S'_{ij}-\Omega'_{ij}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}\right)+\\
&\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}\left(S'_{ij}\left(\overline{S_{ik}}+\overline{\Omega_{ik}}\right)+S'_{ik}\left(\overline{S_{ij}}-\overline{\Omega_{ij}}\right)\right)\frac{\partial\mu'_{app}}{\partial x_j}\right)-\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\frac{\partial}{\partial x_j}\left(T_{ik}^{R}+T_{ik}^{\mu}\right)\right)+\\
&\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{kj}\right)\frac{\partial p'}{\partial x_j}+\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}CS'_{mn}S'_{mn}S_{kj}\right)\frac{\partial p}{\partial x_j}-\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)\frac{\partial S'_{ij}}{\partial x_k}-\mu_{app}S'_{ij}\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}S'_{ij}\right)\frac{\partial\mu_{app}}{\partial x_k}+\\
&\frac{\partial}{\partial x_j}\left(T_{ik}^{R}+T_{ik}^{\mu}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)-\mu_{app}\frac{4}{\rho}S'_{mn}S'_{mn}\frac{\partial}{\partial x_k}\left(CS_{ij}\right)\frac{\partial S_{ij}}{\partial x_k}-\mu'_{app}\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)\frac{\partial\overline{S_{ij}}}{\partial x_k}+\\
&-\left(\left(S'_{ik}+\Omega'_{ik}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)+\left(S'_{ij}-\Omega'_{ij}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ik}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}+\\
&-\left(\left(\overline{S_{ik}}+\overline{\Omega_{ik}}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)+\left(\overline{S_{ij}}-\overline{\Omega_{ij}}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ik}\right)\right)\frac{\partial\mu'_{app}}{\partial x_j}+\\
&-\left(\left(S_{ik}+\Omega_{ik}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}CS'_{mn}S'_{mn}S_{ij}\right)+\left(S_{ij}-\Omega_{ij}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}CS'_{mn}S'_{mn}S_{ik}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}+
\end{aligned}
\tag{D.3}
$$

D.6- Fifth step

Introduction of the instantaneous dissipation rate $\frac{2}{\rho}\mu_{app}S'_{mn}S'_{mn}=\varepsilon'$, multiplication and division for C of the apparent viscosity gives

$$
\begin{aligned}
&\rho\frac{\partial\varepsilon'}{\partial t}+\rho U_k\frac{\partial\varepsilon'}{\partial x_k}=-4CS'_{mn}S'_{mn}S_{ij}\left(S_{ik}S_{jk}-\Omega_{ik}\Omega_{jk}\right)-4\mu_{app}u'_kS'_{ij}\frac{\partial\overline{S_{ij}}}{\partial x_k}+\\
&-4\mu_{app}S'_{ij}\left(\left(S'_{ik}S'_{jk}+S'_{ik}\overline{S_{jk}}+S'_{jk}\overline{S_{ik}}\right)-\left(\Omega'_{ik}\Omega'_{jk}+\Omega'_{ik}\overline{\Omega_{jk}}+\Omega'_{jk}\overline{\Omega_{ik}}\right)\right)+\\
&\frac{\partial}{\partial x_k}\left(-\frac{4}{\rho}\mu_{app}S'_{kj}\frac{\partial p'}{\partial x_j}-\frac{4}{\rho}CS'_{mn}S'_{mn}S_{kj}\frac{\partial p}{\partial x_j}-\rho u'_k\varepsilon'+\mu_{app}\frac{\partial\varepsilon'}{\partial x_k}+\frac{1}{2}\frac{4}{\rho}\mu_{app}T^{\mu'}_{ij}\frac{\partial\overline{S_{ij}}}{\partial x_k}\right)+\\
&\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\left(CS'_{mn}S'_{mn}\left(S_{ij}\left(S_{ik}+\Omega_{ik}\right)+S_{ik}\left(S_{ij}-\Omega_{ij}\right)\right)+\mu_{app}\left(S'_{ij}\left(S'_{ik}+\Omega'_{ik}\right)+S'_{ik}\left(S'_{ij}-\Omega'_{ij}\right)\right)\right)\frac{\partial\mu_{app}}{\partial x_j}\right)+\\
&\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}\left(S'_{ij}\left(\overline{S_{ik}}+\overline{\Omega_{ik}}\right)+S'_{ik}\left(\overline{S_{ij}}-\overline{\Omega_{ij}}\right)\right)\frac{\partial\mu'_{app}}{\partial x_j}\right)-\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\frac{\partial}{\partial x_j}\left(T^R_{ik}+T^{\mu}_{ik}\right)\right)+\\
&\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{kj}\right)\frac{\partial p'}{\partial x_j}+\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}CS'_{mn}S'_{mn}S_{kj}\right)\frac{\partial p}{\partial x_j}-\mu_{app}\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)\frac{\partial S'_{ij}}{\partial x_k}-\mu_{app}S'_{ij}\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}S'_{ij}\right)\frac{\partial\mu_{app}}{\partial x_k}+\\
&\frac{\partial}{\partial x_j}\left(T^R_{ik}+T^{\mu}_{ik}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)-\mu_{app}\frac{4}{\rho}S'_{mn}S'_{mn}\frac{\partial}{\partial x_k}\left(CS_{ij}\right)\frac{\partial S_{ij}}{\partial x_k}-\mu'_{app}\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)\frac{\partial\overline{S_{ij}}}{\partial x_k}+\\
&-\left(\left(S'_{ik}+\Omega'_{ik}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)+\left(S'_{ij}-\Omega'_{ij}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ik}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}+\\
&-\left(\left(\overline{S_{ik}}+\overline{\Omega_{ik}}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ij}\right)+\left(\overline{S_{ij}}-\overline{\Omega_{ij}}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}\mu_{app}S'_{ik}\right)\right)\frac{\partial\mu'_{app}}{\partial x_j}+\\
&-\left(\left(S_{ik}+\Omega_{ik}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}CS'_{mn}S'_{mn}S_{ij}\right)+\left(S_{ij}-\Omega_{ij}\right)\frac{\partial}{\partial x_k}\left(\frac{4}{\rho}CS'_{mn}S'_{mn}S_{ik}\right)\right)\frac{\partial\mu_{app}}{\partial x_j}+
\end{aligned}
\tag{D.4}
$$

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