

# QED Representation for the Net of Causal Loops

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*Dedicated to Roberto Longo on the occasion of his sixtieth birthday*

## Abstract

The present work tackles the existence of *local* gauge symmetries in the setting of Algebraic Quantum Field Theory (AQFT). The net of causal loops, previously introduced by the authors, is a model independent construction of a covariant net of local  $C^*$ -algebras on any 4-dimensional globally hyperbolic space-time, aimed to capture structural properties of any reasonable quantum gauge theory. Representations of this net can be described by causal and covariant connection systems, and local gauge transformations arise as maps between equivalent connection systems. The present paper completes these abstract results, realizing QED as a representation of the net of causal loops in Minkowski space-time. More precisely, we map the quantum electromagnetic field  $F_{\mu\nu}$ , *not free in general*, into a representation of the net of causal loops and show that the corresponding connection system and local gauge transformations find a counterpart in terms of  $F_{\mu\nu}$ .

**Keywords:** Quantum Field Theory, Operator Algebras, QED, Gauge Theory, non-Abelian cohomology.

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# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Preliminaries</b>	<b>5</b>
2.1	Nets of $C^*$ -algebras on the Minkowski space-time . . . . .	5
2.2	The quantum electromagnetic field . . . . .	6
<b>3</b>	<b>Surface integrals in QED</b>	<b>7</b>
3.1	Smearing simplices . . . . .	7
3.2	The electromagnetic field 2-form . . . . .	8
<b>4</b>	<b>A causal and covariant family of potentials</b>	<b>10</b>
4.1	The quantum potential 1-form . . . . .	10
4.2	The potential system . . . . .	14
<b>5</b>	<b>Representations induced by the electromagnetic field</b>	<b>16</b>
5.1	The simplicial set and the set of paths . . . . .	16
5.2	The net of causal loops . . . . .	18
5.3	Representations . . . . .	20
5.4	The electromagnetic field representation . . . . .	24
<b>6</b>	<b>Concluding remarks</b>	<b>27</b>
<b>A</b>	<b>Singular simplices and integration</b>	<b>27</b>
<b>B</b>	<b>Line integrals <i>via</i> the cone construction</b>	<b>29</b>

## 1 Introduction

The Standard Model of elementary particles is a successful physical theory tested by awesome experiments, although its constituents like QED or QCD are still out of a rigorous mathematical comprehension. Yet, similar situations hold for any quantum gauge theory and for the attempts to generalize the achievements of QFT to the realm of general relativity and gravity. Only some instances, like in quantum free fields models or in low dimensional space-time, have a rigorous description. On the other hand, more insights may derive from an axiomatic approach to the theory of quantum fields, see e.g. the recent book by F. Strocchi [46] for a general account on the axiomatic approaches to QFT focused on local gauge aspects.

In the present paper we interested in local gauge theory through AQFT, the axiomatic approach describing local observables in terms of nets of abstract  $C^*$ -algebras undergoing few basic physical requirements, see the reference book by R. Haag [29]. Gauge theories have been investigated in this framework mainly in terms of superselection sectors (equivalent classes of representations), the so called DHR analysis. In a 4-dimensional Minkowski space-time [20, 21], this is a real mathematical-physics outstanding result, grasping the rôle played by the DHR-sectors as the dual category of the group of *global* gauge symmetries [22]. This result disseminated over other cases of superselection theory, as in low-dimensional CFT, e.g. [31], in curved space-time [27, 7], also in combination with other non-commutative geometry features [14, 17]. Moreover, authors investigated other aspects of QFT using the language of AQFT, e.g. noncommutative space-time [19], the AdS/CFT correspondence [35], locally covariant QFT [5], perturbative QFT and the renormalization group [4, 24, 25].

It was clear from the beginning that DHR-sectors were not tailored for describing charges of electromagnetic type and also, as pointed out later, for the charges of purely massive theories. The right sectors for the massive theory case were introduced by Buchholz and Fredenhagen [BF11]. Instead, suitable requirements for charges of electromagnetic type have only recently been introduced by Buchholz and Roberts<sup>1</sup>: for representations in theories with long range forces, like QED, they used the notion of *charge class*, restricting the states of interest to observables localized on a light cone [BR13]. In spite of the limitations just outlined, the strength of the DHR-analysis is that both for BF-sectors and BR-sectors the mathematical machinery of DHR-analysis associating the group of *global* gauge transformations to sectors applies<sup>2</sup>.

What remains to be understood is the *local* gauge principle.

The abstract geometric formulation of the DHR analysis given by Roberts in terms of non-Abelian cohomology [Rob1, Rob11] yields a steering for this quest: it is motivated by the observation that the charge transporters of the global gauge theory satisfy a 1-cocycle equation. According to this, Roberts also proposed that a 2-cohomology underlies the local gauge principle in AQFT [Rob77]: starting from a unitary 2-cocycle  $w$  associated with the electromagnetic field, one should recover a potential  $u$ , i.e. a 1-cochain with coboundary  $w$ : local gauge transformations should arise as equivalence transformations of  $u$ . Roberts also suggested that the charge transporters of the DHR-analysis should be replaced in the QED-case by a “formal” gauge invariant expression as

$$\psi(\partial_0 p) e^{i \int_p A_\mu dp^\mu} \psi(\partial_1 p)^* , \quad (1.1)$$

uno

where  $p : \partial_1 p \rightarrow \partial_0 p$  is a path,  $A_\mu$  is the electromagnetic potential and  $\psi$  is the Dirac field, a sort of a finite Mandelstam string [M62].

Along the ideas of non-Abelian cohomology, a further step has been taken in [RR, RRV] where a geometric interpretation of the above picture has been furnished in terms of connections: the potential  $u$  can be interpreted as a connection having  $w$  as curvature, and the transformations of the principal bundle associated with  $u$  as local gauge transformations. But, from a physical point of view, the rôle played in gauge theories by observables associated to closed paths (Wilson loops) suggested a finer comprehension in terms of a net of local algebras<sup>3</sup>.

Actually, this is the motivation of our paper [CRV]. The *net of causal loops* turns out to be a combinatorial, model independent construction of a covariant net of local C\*-algebras over any 4-d globally hyperbolic space-time, i.e. covariant with respect to the global symmetries of the space-time and respecting the causality principle. The generators of these local algebras are closed paths (loops) associated with a suitable base  $K$  of the underlying space-time. Considering  $K$  as a partially ordered set with respect to the inclusion one can define a simplicial set: paths turn out to be compositions of 1-simplices of this simplicial set, and they can be figured out as finite coverings, by elements of  $K$ , of curves of the space-time.

The very relation to local gauge theories, along the ideas drawn by Roberts but also with significant differences, arose from the representations of this net. Covariant representations turn out to be causal and covariant 2-cochains  $w$ , i.e. unitary valued functions of 2-simplices localized on the boundary, not necessarily satisfying a cocycle equation. Furthermore, any such 2-cochain

<sup>1</sup>Actually, a long-standing project started by these authors and S. Doplicher.

<sup>2</sup>It is worth mentioning that more works in AQFT dealt with general on peculiar facets of QED: e.g. the analysis of the infrared problem and the localization of electromagnetic charge [8, 10], the interpretation of QED by local constraints [28], lattice approach [32], definition and superselection of free models [9, 15, 18], or proposal for interacting models by warped convolution [12].

<sup>3</sup>Among the others, we recall two interpretations of observables localized on lines and loops (holonomies), broadly related to the present work. Ashtekar and Corichi [1, 2] deepened the significance of the topological invariant of the Gauß linking number and its relation with the Fock-space inner product of the represented fields, [44] is a recent geometric survey. Moreover, the interpretation of electromagnetic charges in terms of the Wilson-'t Hooft operators in TQFT, e.g. the Kapustin-Witten lectures on electro-magnetic duality [30].

w induces, by a procedure akin to the reconstruction of the primitive of an exact 2-form, a causal and covariant connection system  $u$ : a family of connections, one for each element of  $K$ , in which causality and covariance arise as properties of the system and not of a single connection. Local gauge transformations naturally arise as maps between equivalent connection systems. A detailed description of these results is given in the Sections 5.1, 5.2, 5.3 where the net of causal loops is described for the specific case of double cones of the Minkowski spacetime.

The present paper completes the abstract arguments in [16], realizing QED as representation of the net of causal loops in Minkowski space-time. More precisely, we shall map the quantum electromagnetic field  $F_{\mu\nu}$ , *not free in general*, into a representation of the net of causal loops and show that all the abstract notions just described find a counterpart in terms of  $F_{\mu\nu}$ .

The map is constructed using a 2-form  $F_{\mu\nu}(y)$ , Definition 3.10, associated with the electromagnetic field  $F_{\mu\nu}$  through the convolution of the field with a test function, Section 3. Using a contracting homotopy of the Minkowski space-time, in Section 4, we reconstruct the primitive 1-form  $A_\mu^z(y)$ , called the electromagnetic potential form with respect to  $z$ , the point which the space-time is contracted to, Definition 4.4. The potential form  $A_\mu^z(y)$ , closely related to the quantum electromagnetic potential field, is neither local nor covariant. However, causality and covariance arise without recurring to the Gupta-Bleuler formalism, as properties of the system  $A_\mu := \{A_\mu^z, z \in \mathbb{R}^4\}$  of potential forms<sup>4</sup>, Definition 4.8. The notion of local gauge transformation is recast in terms of this system.

Then we consider, in Section 5, the net of causal loops defined over the set of double cones  $K$  of the Minkowski space-time. We slightly modify its definition with respect to [16] since we take into account two relevant facts: the flatness of the Minkowski space-time and the necessity of smearing the fields. The essential effect of this choice is to adjust the definition of the simplicial set, replacing the rôle played by the elements of  $K$  by the set of test functions supported within elements of  $K$ .

Practically the construction runs as follows: for any 2-simplex we associate the exponential of the integral of  $F_{\mu\nu}(y)$  over the triangular surface underling the 2-simplex. Thanks to the Stokes' theorem this defines a function  $w^{em}$  over the 2-simplices which has the right localization properties, turning to define a representation of the net of causal loops. Yet, we can define two connection systems:  $u^{em}$  which is obtained from  $w^{em}$  by using the combinatorial procedure outlined above; and  $u^{pot}$  obtained as the exponential of the line integral of  $A_\mu$ . The key result is that  $u^{pot}$  is equivalent to  $u^{em}$  in terms of a local gauge transformation, see Theorem 5.8. Moreover, we show that local gauge transformations of  $A_\mu$  agree with local gauge transformations of the connection systems.

Comparing our results to Roberts' approach sketched above, the unitary 2-cochain  $w^{em}$  associated with the electromagnetic field is *not*, in general, a 2-cocycle. The relevant physical information is carried by the localization property of  $w^{em}$ , as in the very philosophy of AQFT. This is enough to recover not a single connection but a *system* of connections in which causality and covariance however arise. Moreover, it suggests to replace the electromagnetic *potential field* by the electromagnetic *potential form* in the expression (I.1) for the charge transporters of QED, so we have

$$\psi(\partial_0 p) e^{i \int_p A_\mu^z dp^\mu} \psi(\partial_1 p)^* . \quad (1.2)$$

due

Hence we have a direct dependence on the point  $z$  of the space-time labelling the potential form. This parallels what happens in the analysis of sectors associated to charges of electromagnetic type [13], in which the theory is developed on the forward light-cone of a single point of the space-time, as outlined above.

<sup>4</sup>A covariant quantum electromagnetic potential was found in [34] for the case of infinite string-localized fields.

## 2 Preliminaries

We describe the setting and the notation used in the present paper. In the first subsection we recall some facts concerning the Minkowski space-time, the Poincaré group and the notion of causal nets of  $C^*$ -algebras. The properties of the quantum electromagnetic field, which are relevant for the aims of the present paper, are discussed in the second subsection.

### 2.1 Nets of $C^*$ -algebras on the Minkowski space-time

**The Minkowski space-time and the Poincaré group.** We recall some basic properties of the Minkowski space-time  $(\mathbb{R}^4, g)$  and establish some notations. We adopt the convention that the metric tensor  $g_{\mu\nu}$  has negative signature:  $g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$  and  $g_{\mu\nu} = 0$  if  $\mu \neq \nu$ . Furthermore, recall that  $g_{\mu\nu} = g^{\mu\nu}$  and  $g^\mu{}_\nu = g_\mu{}^\nu = \delta_{\mu\nu}$ . We use the contravariant notation to represent the components of element  $x$  of the Minkowski space-time:  $x = (x^0, x^1, x^2, x^3)$  or  $x = x^\mu$ , with  $\mu = 0, 1, 2, 3$ . The inner product induced by the metric tensor is

$$x \cdot y := x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 = x^\mu g_{\mu\nu} x^\nu = x^\mu x_\mu ,$$

where  $x_\mu = g_{\mu\nu} x^\nu$  is the covariant representation of  $x$ . Clearly  $x^\mu = g^{\mu\nu} x_\nu$ . Denoting the canonical scalar product of two element  $x, y$  of  $\mathbb{R}^4$  by  $(x, y)$  and by  $g$  the matrix associated with the metric tensor  $g_{\mu\nu}$ , the Minkowski inner product can be rewritten as  $x \cdot y = (x, gy)$ .

We shall say that two subsets  $X$  and  $Y$  of  $\mathbb{R}^4$  are *causally disjoint* if, and only if, the corresponding elements are spacelike separated. In symbol we write

$$X \perp Y \iff (x - y)^2 < 0 , \quad x \in X, y \in Y .$$

A Lorentz transformation is a linear transformation  $L$  leaving the inner product invariant,  $(Lx, gLy) = (x, gy)$ : in matrix notation  $L^T g L = g$ . In tensor notation, if  $(Lx)^\mu = L^\mu{}_\nu x^\nu$  and  $(Lx)_\mu = L_\mu{}^\nu x_\nu$ , where  $L_\mu{}^\nu = g_{\mu\alpha} L^\alpha{}_\beta g^{\beta\nu}$ , then  $L^\alpha{}_\mu g_{\alpha\beta} L^\beta{}_\nu = g_{\mu\nu}$ . So

$$(L^{-1})^\nu{}_\alpha = g_{\alpha\beta} L^\beta{}_\rho g^{\rho\nu} = L_\alpha{}^\nu .$$

The *restricted Lorentz group*  $\mathcal{L}_+^\uparrow$  is the subgroup of the Lorentz transformations whose matrices  $L^\mu{}_\nu$  have positive determinant and  $L^0{}_0 \geq 1$ . The *Poincaré group*  $\mathcal{P}_+^\uparrow$  is the semi-direct product  $\mathbb{R}^4 \rtimes \mathcal{L}_+^\uparrow$  with composition law defined as

$$(x, L) (x', L') := (x + Lx', LL') .$$

According to this relation,  $(-L^{-1}x, L^{-1})$  is the inverse of  $(x, L)$ . To economize notation, sometimes we shall denote an element of the Poincaré group by  $P := (x, L)$ , so when  $Y \subset \mathbb{R}^4$  or  $y \in \mathbb{R}^4$  we shall write the action as

$$PY = x + LY \quad , \quad Py = x + Ly .$$

**Nets of  $C^*$ -algebras.** The mathematical description of local observables in AQFT is given in terms of nets of  $C^*$ -algebras. We shall focus on the case of the Minkowski space-time and refer the reader to [43, 16] for more general situations. Let  $K$  denote the set of *double cones* of the Minkowski space-time [29]. Double cones form a base of the topology of  $\mathbb{R}^4$  which is stable under the action of the Poincaré group and upward directed under inclusion. By a *net* of  $C^*$ -algebras over the Minkowski space-time we shall mean an inclusion preserving (*isotonous*) correspondence  $\mathcal{A} : K \ni o \rightarrow \mathcal{A}_o \subseteq \mathcal{A}(\mathbb{R}^4)$ , i.e.

$$o_1 \subseteq o_2 \implies \mathcal{A}_{o_1} \subseteq \mathcal{A}_{o_2} ,$$

associating a C\*-subalgebra  $\mathcal{A}_o$  of a fixed target C\*-algebra  $\mathcal{A}(\mathbb{R}^4)$  to any double cone  $o$ . The net  $\mathcal{A}$  is said to be *causal* whenever

$$o_1 \perp o_2 \Rightarrow [\mathcal{A}_{o_1}, \mathcal{A}_{o_2}] = 0 ,$$

and it is said to be *covariant* if there is action of the Poincaré group  $\alpha : \mathcal{P}_+^\uparrow \rightarrow \mathbf{aut}\mathcal{A}(\mathbb{R}^4)$  such that

$$\alpha_P \circ \mathcal{A} = \mathcal{A} \circ \alpha_P , \quad P \in \mathcal{P}_+^\uparrow .$$

In the following we shall denote a causal and covariant net of C\*-algebras over the Minkowski space-time by  $(\mathcal{A}, \alpha)_K$ .

**Hilbert spaces.** In the present paper we write  $\mathcal{H}, \mathcal{H}', \dots$  to denote Hilbert spaces. We denote the C\*-algebra of bounded, linear operators on  $\mathcal{H}$  by  $\mathcal{BH}$ , and the group of unitary operators by  $\mathcal{UH}$ . In AQFT it is customary to realize a net  $\mathcal{A}$  in terms of operators on the Hilbert space  $\mathcal{H}$  (the vacuum representation). In this case the Poincaré symmetry is unitarily implemented, that is, there is a unitary representation  $U : \mathcal{P}_+^\uparrow \rightarrow \mathcal{UH}$  such that

$$\alpha_P(T) = U(P) T U(P)^* , \quad T \in \mathcal{A}(\mathbb{R}^4) , P \in \mathcal{P}_+^\uparrow .$$

## 2.2 The quantum electromagnetic field

OAa

In the present paper, with the term *quantum electromagnetic field* we shall mean a pair  $(U, F)$  where:

- $U$  is a unitary representation  $U$  of  $\mathcal{P}_+^\uparrow$  over a Hilbert space  $\mathcal{H}$ .
- $f \mapsto F_{\mu\nu}(f)$  is an operator valued distribution assigning to any real test function  $f \in \mathcal{S}(\mathbb{R}^4)$  an essentially self-adjoint operator  $F_{\mu\nu}(f)$  of  $\mathcal{H}$  defined on a dense *domain*  $\mathcal{D} \subset \mathcal{H}$ , such that  $F_{\mu\nu}(f)\mathcal{D} \subseteq \mathcal{D}$  for any test function  $f$ . We assume that the essential self-adjointness property is conserved under real linear combinations of  $F_{\mu\nu}(f)$ .
- The representation  $U$  leaves the domain  $\mathcal{D}$  invariant and the *covariance* property holds,

$$U(x, L) F_{\mu\nu}(f) U(x, L)^* = L^{-1}{}^\alpha{}_\mu L^{-1}{}^\beta{}_\nu F_{\alpha\beta}(f_{(x,L)}) , \quad f \in \mathcal{S}(\mathbb{R}^4) , \quad (2.1) \quad \text{OA:1}$$

for any  $(x, L) \in \mathcal{P}_+^\uparrow$ , where  $f_{(x,L)} := f \circ (x, L)^{-1}$ .

- *Causality* of the field holds,

$$[F_{\mu\nu}(f), F_{\mu\nu}(g)] = 0 , \quad \text{supp}(f) \perp \text{supp}(g) ,$$

where the commutation relation is intended to hold on  $\mathcal{D}$  and the symbol  $\perp$  stands for causal disjointness. However in the present paper we require the further property of *strong causality*,

$$[\exp(iF_{\mu\nu}(f)), \exp(iF_{\alpha\beta}(g))] = 0 , \quad \text{supp}(f) \perp \text{supp}(g) , \quad (2.2) \quad \text{OA:2}$$

where, to ease notation, we are denoting  $F_{\mu\nu}(f)$  and its closure by the same symbol.

- The field  $F$  is *antisymmetric*

$$F_{\mu\nu} = -F_{\nu\mu} , \quad (2.3) \quad \text{OA:3}$$

and satisfies the *first Maxwell equation*

$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 . \quad (2.4) \quad \text{OA:4}$$

Let us briefly comment the terminology used in the above definitions. As usual, the term *operator valued distribution* means that the mapping  $f \mapsto (\phi, F_{\mu\nu}(f)\psi)$  is a tempered distribution, i.e. a continuous linear functional on the Frechet space  $\mathcal{S}(\mathbb{R}^4)$  for any  $\phi, \psi \in \mathcal{D}$ .

The first Maxwell equation holds in the weak sense i.e.  $\partial_\alpha F_{\mu\nu}(f) := -F_{\mu\nu}(\partial_\alpha f)$ . We do not make assumptions on the second Maxwell equation, that is, we do not make hypothesis on the current

$$J_\nu := \partial^\mu F_{\mu\nu} , \quad (2.5) \quad \text{OA:5}$$

that may be, possibly, non-trivial. The explicit construction of  $(U, F)$  can be made in the free case ( $J_\nu \equiv 0$ ) by using Fock spaces [23] or the Wightman reconstruction theorem [45]. At the present time, no quantum electromagnetic field with non-trivial current is known in the 4-dimensional Minkowski space-time outside the perturbative setting [45].

Concerning the strong form of causality, this is necessary since using the exponential of the fields we will constructs representations of the net of *causal* loops. This strong form of causality is verified by the free electromagnetic field [45, Section 5.3] and by some interacting fields [26].

### 3 Surface integrals in QED

OB

In the present section we define the integral of the electromagnetic field over a 2-simplex, and prove that this operator is covariant in a suitable sense and localized on the boundary. To this end, we shall use the fact that the electromagnetic field defines a closed, exact, 2-form and the Stokes' theorem.

For what follows we shall use the language of algebraic topology, in which the integrations of smeared fields over surfaces and over curves are treated on the same footing and where the Stokes' theorem applies in a more general form. To this end, we shall use smearing functions to generalize the notion of  $n$ -simplex. To keep contact with the geometric intuition we shall mainly work with  $n$ -simplices rather than  $n$ -chains, but our results easily extend to these latter.

#### 3.1 Smearing simplices

OAB

Both for the purpose of smearing the fields and for treating abstract simplices given by subsets of the space-time, as in [16], we refine the notion of piecewise smooth  $n$ -simplex. In this regards, we shall denote the set of singular  $n$ -simplices of  $\mathbb{R}^4$ , with  $n \in \mathbb{N}$  the set of natural numbers including 0, by  $\Sigma_n(\mathbb{R}^4)$  and the associated face operators by  $\partial_i : \Sigma_n(\mathbb{R}^4) \rightarrow \Sigma_{n-1}(\mathbb{R}^4)$ ,  $i = 0, \dots, n$ . Accordingly,  $C_n(\mathbb{R}^4)$  shall denote the set of singular  $n$ -chains and  $\partial : C_n(\mathbb{R}^4) \rightarrow C_{n-1}(\mathbb{R}^4)$  the associated boundary operator. We refer the reader to the Appendix A for notations and standard results on singular, piecewise smooth homology (integration, Stokes' theorem, the cone construction).

Let  $\Sigma_n(\mathbb{R}^4)$  be the set of singular (piecewise smooth) simplices, i.e. piecewise smooth maps  $\chi : \Delta_n \rightarrow \mathbb{R}^4$ , where  $\Delta_n$  is the standard  $n$ -simplex, and let  $\partial_i : \Sigma_n(\mathbb{R}^4) \rightarrow \Sigma_{n-1}(\mathbb{R}^4)$  be the associated face operator, see Appendix A. We define the set of (singular, piecewise smooth) *smearing  $n$ -simplices* by

$$\Sigma_n(\mathbb{R}^4, \mathcal{S}) := \{(\chi, f) : \chi \in \Sigma_n(\mathbb{R}^4), f \in \mathcal{S}(\mathbb{R}^4), \text{supp}(f) \ni 0\} , \quad (3.1) \quad \text{OA:11}$$

and call  $f$  the *smearing function* of  $(\chi, f)$ . The Poincaré group acts on  $\Sigma_n(\mathbb{R}^4, \mathcal{S})$  by

$$P(\chi, f) := (P\chi, f_L) , \quad P = (x, L) \in \mathcal{P}_+^\uparrow , (\chi, f) \in \Sigma_n(\mathbb{R}^4, \mathcal{S}) , \quad (3.2) \quad \text{OA:12}$$

where  $(P\chi(t))^\mu := x^\mu + L^\mu{}_\nu \chi^\nu(t)$ , with  $t \in \Delta_n$ , and  $f_L \equiv f_{(0,L)}$ . Notice that translations do not act on the smearing function  $f$ , so  $\text{supp}(f_L) \ni 0$ .

Similarly to singular homology, we define the  $\mathbb{Z}$ -module of *smearing  $n$ -chain*  $C_n(\mathbb{R}^4, \mathcal{S})$  as the set of finite formal linear combinations of smearing  $n$ -simplices,  $\sum_i m_i(\chi_i, f_i)$  with  $m_i \in \mathbb{Z}$  and  $(\chi_i, f_i) \in \Sigma_n(\mathbb{R}^4, \mathcal{S})$ . The Poincaré action extends on  $C_n(\mathbb{R}^4, \mathcal{S})$  correspondingly.

The face maps lift, naturally, from  $\Sigma_n(\mathbb{R}^4)$  to  $\Sigma_n(\mathbb{R}^4, \mathcal{S})$  by defining  $\partial_i(\chi, f) := (\partial_i\chi, f)$  for any  $i$ , and the boundary operator  $\partial$  lifts to smearing  $n$ -chain accordingly. In particular, the *boundary* of a smearing  $n$ -simplex is the  $(n-1)$ -chain  $\partial(\chi, f) := (\partial\chi, f)$ . Clearly  $\partial^2 = 0$ , and this leads to the (*smearing*)  *$n$ -homology module*  $H_n(\mathbb{R}^4, \mathcal{S}) := Z_n(\mathbb{R}^4, \mathcal{S})/B_n(\mathbb{R}^4, \mathcal{S})$ , where  $Z_n(\mathbb{R}^4, \mathcal{S}) := \text{Ker}(\partial)$  the module of (*smearing*)  $n$ -cycles and  $B_n(\mathbb{R}^4, \mathcal{S}) = \text{Im}(\partial)$  the module of (*smearing*)  $n$ -boundaries. Notice that  $H_n(\mathbb{R}^4, \mathcal{S})$  is invariant under the Poincaré action and that it vanishes for  $n > 0$ .

The reason why we introduce smearing simplices relies on the following observation: to any smearing  $n$ -simplex  $(\chi, f)$  there corresponds an  $n$ -form  $\chi[f]$  defined by

$$\chi[f]^{\alpha_1 \dots \alpha_n}(x) := \int_{\Delta_n} f(x - \chi(t)) \chi^{\alpha_1 \dots \alpha_n}(t) d^n t \quad , \quad x \in \mathbb{R}^4 \quad , \quad (3.3) \quad \boxed{\text{OA:13}}$$

where  $\chi^{\alpha_1 \dots \alpha_n}$  is the  $n$ -volume form of  $\chi$ , see Appendix [A](#). This  $n$ -form plays the rôle of the  $n$ -volume element for the integration of a quantum field over an  $n$ -simplex. It is easily seen that  $\chi[f]^{\alpha_1 \dots \alpha_n} \in \mathcal{S}(\mathbb{R}^4)$  and that the partial Riemann sums defining the integral (3.3) converge in the sense of the topology of  $\mathcal{S}(\mathbb{R}^4)$  [\[42\]](#). It is natural to define the *support* of a smearing  $n$ -simplex as the support of the corresponding  $n$ -form, i.e.

$$|(\chi, f)| := \text{supp}(\chi[f]) \quad , \quad (\chi, f) \in \Sigma_n(\mathbb{R}^4, \mathcal{S}) \quad , \quad (3.4) \quad \boxed{\text{OA:14}}$$

that turns out to be Poincaré covariant,  $|P(\chi, f)| = P|(\chi, f)|$  for any  $P \in \mathcal{D}_+^\uparrow$ , and it holds

$$|(\chi, f)| \subseteq \text{supp}(f) + \chi(\Delta_n) \quad . \quad (3.5) \quad \boxed{\text{OA:15}}$$

The above reasoning extends to  $n$ -chains setting  $[\varphi]^{\alpha_1 \dots \alpha_n} := \sum_i m_i \chi_i [f_i]^{\alpha_1 \dots \alpha_n}$  for any an  $n$ -chain  $\varphi = \sum_i m_i(\chi_i, f_i)$ . This is still an  $n$ -form with coefficients in  $\mathcal{S}(\mathbb{R}^4)$ . The support of the  $n$ -chain is defined as the support of the associated  $n$ -form, and the property of covariance easily generalizes.

### 3.2 The electromagnetic field 2-form

As already said, we want to integrate the electromagnetic field on a 2-simplex. The idea is to integrate on the given 2-simplex the convolution of  $F_{\mu\nu}$  by a test function. To this end, for any  $f \in \mathcal{S}(\mathbb{R}^4)$  define

$$f_y(x) := f(x - y) \quad , \quad x, y \in \mathbb{R}^4 \quad ,$$

and observe that  $f_y$  is a test function as well for any  $y$ . Since  $y \mapsto f_y$  fits the topology of Schwartz functions, the mapping  $y \mapsto F_{\mu\nu}(f_y)$  is smooth in the sense of distributions, i.e. the functions  $y \mapsto (\phi, F_{\mu\nu}(f_y)\psi)$  are smooth for any  $\phi, \psi \in \mathcal{D}$  and for any  $\mu, \nu$  (see [\[42\]](#)). Furthermore, covariance implies

$$F_{\mu\nu}(f_y) = U(y)F_{\mu\nu}(f)U(y)^* \quad , \quad y \in \mathbb{R}^4 \quad , \quad (3.6) \quad \boxed{\text{OB:1}}$$

where, with an abuse of notation,  $U(y)$  stands for  $U(y, \mathbb{1})$ . Finally, we note that  $\partial_{y^\rho} f_y = -(\partial_\rho f)_y$  and this implies  $\partial_{y^\rho} F_{\mu\nu}(f_y) = F_{\mu\nu}((\partial_\rho f)_y)$ . So

$$\begin{aligned} \partial_{y^\rho} F_{\mu\nu}(f_y) + \partial_{y^\nu} F_{\rho\mu}(f_y) + \partial_{y^\mu} F_{\nu\rho}(f_y) &= \\ &= F_{\mu\nu}((\partial_\rho f)_y) + F_{\rho\mu}((\partial_\nu f)_y) + F_{\nu\rho}((\partial_\mu f)_y) \\ &= U(y)(F_{\mu\nu}(\partial_\rho f) + F_{\rho\mu}(\partial_\nu f) + F_{\nu\rho}(\partial_\mu f))U(y)^* = 0 \quad , \end{aligned} \quad (3.7) \quad \boxed{\text{OB:2}}$$

that is,  $y \mapsto F_{\mu\nu}(f_y)$  is a closed 2-form in the sense of distributions.



**The surface integral of  $F_{\mu\nu}$ .** The rough idea is to define our integral on  $\sigma \in \Sigma_2(\mathbb{R}^4)$  as

$$\int_{\sigma} F(f_{\sigma}) d\sigma := \frac{1}{2} \int_{\Delta_2} F_{\mu\nu}(f_{\sigma(t)}) \sigma^{\mu\nu}(t) d^2t, \quad (3.8) \quad \boxed{\text{OB:2o}}$$

where  $f \in \mathcal{S}$  and the factor  $1/2$  derives from the fact that  $F_{\mu\nu}$  is antisymmetric and  $\mu, \nu$  varies independently in  $\{0, 1, 2, 3\}$ . To make rigorous this formal expression involving unbounded operators, we consider the smearing surface  $(\sigma, f) \in \Sigma_2(\mathbb{R}^4, \mathcal{S})$  with the corresponding 2-form, see  $\boxed{\text{OA:13}}$  (3.3),

$$\sigma[f]^{\mu\nu}(x) := \int_{\Delta_2} f(x - \sigma(t)) \sigma^{\mu\nu}(t) d^2t, \quad x \in \mathbb{R}^4, \quad (3.9) \quad \boxed{\text{OB:2c}}$$

and define

$$F \langle \sigma, f \rangle := \frac{1}{2} F_{\mu\nu}(\sigma[f]^{\mu\nu}), \quad (\sigma, f) \in \Sigma_2(\mathbb{R}^4, \mathcal{S}). \quad (3.10) \quad \boxed{\text{OB:4}}$$

The next results show the properties of this operator. In particular, the following Lemma  $\boxed{\text{OB:5}}$  illustrates how  $\boxed{\text{OB:4}}$  (3.10) does fit the formal expression  $\boxed{\text{OB:2o}}$  (3.8).

$\boxed{\text{OB:5}}$  **Lemma 3.1.** *Let  $(\sigma, f) \in \Sigma_2(\mathbb{R}^4, \mathcal{S})$ . Then  $F \langle \sigma, f \rangle$  is an essentially self-adjoint operator defined on  $\mathcal{D}$  such that  $F \langle \sigma, f \rangle \mathcal{D} \subseteq \mathcal{D}$ , and:*

- (i)  $U(P)F \langle \sigma, f \rangle U(P)^* = F \langle P(\sigma, f) \rangle, \quad P \in \mathcal{P}_+^{\uparrow};$
- (ii)  $(\phi, F \langle \sigma, f \rangle \psi) = 1/2 \int_{\Delta_2} (\phi, F_{\mu\nu}(f_{\sigma(t)})\psi) \sigma^{\mu\nu}(t) d^2t, \quad \phi, \psi \in \mathcal{D};$
- (iii) if  $|(\sigma', f')| \perp |(\sigma, f)|$ , then  $[\exp(iF \langle \sigma, f \rangle), \exp(iF \langle \sigma', f' \rangle)] = 0$ , where  $|\cdot|$  denotes the support  $\boxed{\text{OA:14}}$  (3.4) of the smearing surface.

*Proof.* The first properties follow by definition. (i) Let  $P = (x, A) \in \mathcal{P}_+^{\uparrow}$ . According to  $\boxed{\text{OA:1}}$  (2.1) we have  $U(P)F \langle \sigma, f \rangle U(P)^* = (1/2) L^{-1}_{\mu}{}^{\alpha} L^{-1}_{\nu}{}^{\beta} F_{\alpha\beta}(\sigma[f]^{\mu\nu} \circ P^{-1})$ , where

$$\begin{aligned} \sigma[f]^{\mu\nu} \circ P^{-1}(y) &= \int_{\Delta_2} f(L^{-1}y - L^{-1}x - \sigma(t)) \sigma^{\mu\nu}(t) d^2t \\ &= \int_{\Delta_2} f_L(y - x - L\sigma(t)) \sigma^{\mu\nu}(t) d^2t = \int_{\Delta_2} (f_L)_{x+L\sigma(t)}(y) \sigma^{\mu\nu}(t) d^2t. \end{aligned}$$

Since  $L^{-1}_{\mu}{}^{\alpha} = L^{\alpha}_{\mu}$  (see Section  $\boxed{\text{OA}}$  (2)), we get

$$\begin{aligned} 2 \cdot U(P)F \langle \sigma, f \rangle U(P)^* &= \\ &= \int_{\Delta_2} F_{\alpha\beta}((f_L)_{x+L\sigma(t)}) \cdot L^{-1}_{\mu}{}^{\alpha} L^{-1}_{\nu}{}^{\beta} \sigma^{\mu\nu}(t) d^2t = \int_{\Delta_2} F_{\alpha\beta}((f_L)_{x+L\sigma(t)}) \cdot (L\sigma)^{\alpha\beta}(t) d^2t \\ &= \int_{\Delta_2} F_{\alpha\beta}((f_L)_{x+L\sigma(t)}) \cdot (x + L\sigma)^{\alpha\beta}(t) d^2t \\ &= F_{\alpha\beta}(P\sigma[f_L]^{\alpha\beta}) = 2 \cdot F \langle P(\sigma, f) \rangle, \end{aligned}$$

completing the proof. (ii) follows from the observation, see Section  $\boxed{\text{OAB}}$  (3.1), that the map  $y \mapsto \int_{\Delta_2} f_{\sigma(t)}(y) \sigma^{\mu\nu}(t) d^2t$  is limit in the topology of  $\mathcal{S}(\mathbb{R}^4)$  of partial Riemann sums, so applying the distribution  $(\phi, F_{\mu\nu}(\cdot)\psi)$  we can pass under the sign of integral. (iii) follows by the definition  $\boxed{\text{OB:4}}$  (3.10) and strong causality  $\boxed{\text{OA:2}}$  (2.2).  $\square$

## 4 A causal and covariant family of potentials

OBb

We have observed that the electromagnetic field defines an exact 2-form of  $\mathbb{R}^4$ . Using the classical formula for the reconstruction of the primitive of an exact 2-form we shall define the (electromagnetic) potential 1-form, depending on the origin used to perform the integration; varying the origin in  $\mathbb{R}^4$  gives a system of potential forms that turns out to be causal and covariant in a suitable sense. We analyze the properties of these operators and define their line integral.

### 4.1 The quantum potential 1-form

We want to study the primitives of the closed 2-form defined by the electromagnetic field and their line integrals. To this end we may use the cone construction to obtain directly the integral of a primitive on a smearing curve, however we prefer to follow a bottom-up approach first finding the primitive 1-forms. The equivalence between these two formulations shall be proved in Appendix [B](#).

If  $\omega$  is a closed 2-form on  $\mathbb{R}^4$ , then the formula

$$\omega_\mu^z(y) = \int_0^1 t(y-z)^\alpha \omega_{\mu\alpha}(z+t(y-z)) dt \quad , \quad y \in \mathbb{R}^4 \quad , \quad (4.1)$$

OBb:0

gives a primitive of  $\omega$  for any  $z \in \mathbb{R}^4$ , as can be verified by an elementary computation. Indeed this is the argument used to prove the Poincaré Lemma, that is usually presented with the choice  $z = 0$ . As we shall see soon, the additional degree of freedom given by  $z \in \mathbb{R}^4$  is necessary to make explicit the covariance and the causality of the potential. On this grounds, given a test function  $f$  the formal definition of our quantum potential form is

$$A_\mu^z(y, f) := \int_0^1 t(y-z)^\alpha F_{\mu\alpha}(f_{z+t(y-z)}) dt \quad , \quad y \in \mathbb{R}^4 \quad .$$

We show below that this formula holds in the sense of sesquilinear forms. To give the rigorous definition we observe that for any smearing point  $(y, f) \in \Sigma_0(\mathbb{R}^4, \mathcal{S})$  there corresponds a 1-tensor  $y_f^z$ , defined as

$$y_f^{z,\alpha}(x) := \int_0^1 (y-z)^\alpha t f_{z+t(y-z)}(x) dt \quad , \quad y \in \mathbb{R}^4 \quad , \quad (4.2)$$

OBb:1

where, as usual, the components  $y_f^{z,\alpha}$  take values in  $\mathcal{S}(\mathbb{R}^4)$  and are limit, in the topology of  $\mathcal{S}(\mathbb{R}^4)$ , of the partial Riemann sums defining the integral. Furthermore,

$$\text{supp}(y_f^z) \subseteq \text{supp}(f) + \{z + t(y-z) \in \mathbb{R}^4 \mid t \in [0, 1]\} \quad . \quad (4.3)$$

OBb:2

OBb:3

**Definition 4.1.** For any  $z \in \mathbb{R}^4$ , the **(electromagnetic) potential 1-form based on  $z$**  is the operator-valued map

$$A_\mu^z(y, f) := F_{\mu\alpha}(y_f^{z,\alpha}) \quad , \quad (y, f) \in \Sigma_0(\mathbb{R}^4, \mathcal{S}) \quad . \quad (4.4)$$

OBb:3a

The point  $z$  is called **pole**.

Some basic properties, among which the fact that  $A_\mu^z(y, f)$  is a weak primitive of the electromagnetic field, are shown in the next lemma.

OBb:4

**Lemma 4.2.** Let  $(y, f) \in \Sigma_0(\mathbb{R}^4, \mathcal{S})$  and  $z \in \mathbb{R}^4$ . Then  $A_\mu^z(y, f)$  is an essentially self-adjoint operator on  $\mathcal{D}$  such that  $A_\mu^z(y, f)\mathcal{D} \subseteq \mathcal{D}$ . The mapping  $y \mapsto (\phi, A_\mu^z(y, f)\psi)$  is smooth and the relations

- (i)  $(\phi, A_\mu^z(y, f)\psi) = \int_0^1 t(y-z)^\alpha (\phi, F_{\mu\alpha}(f_{z+t(y-z)})\psi) dt,$   
(ii)  $(\phi, F_{\mu\nu}(f_y)\psi) = \partial_{y^\mu}(\phi, A_\nu^z(y, f)\psi) - \partial_{y^\nu}(\phi, A_\mu^z(y, f)\psi),$

hold for any  $\phi, \psi \in \mathcal{D}$ .

*Proof.* By definition it follows that  $A_\mu^z(y, f)$  is an essentially self-adjoint operator on  $\mathcal{D}$ . The equation (i) follows by the same argument used to prove Lemma 3.1(ii), that is, convergence of the integral defining the form  $y_f^z$  in the sense of Schwartz topology. To prove equation (ii) we define  $\tau_\mu(y) := (\phi, A_\mu^z(y, f)\psi)$ ,  $y \in \mathbb{R}^4$ , and using (i) we find

$$\partial_\mu \tau_\nu(y) = \frac{\partial}{\partial y^\mu} \int_0^1 t(y-z)^\alpha (\phi, F_{\nu\alpha}(f_{z+t(y-z)})\psi) dt .$$

The argument used to prove that (4.1) is a primitive now applies, so  $\tau$  is a primitive of the 2-form  $y \mapsto (\phi, F_{\mu\nu}(f_y)\psi)$  as desired.  $\square$

The electromagnetic potential form  $A_\mu^z(y, f)$  *should not be confused* with the electromagnetic potential field commonly used in QED. In fact, even if  $A_\mu^z(y, f)$  is a weak primitive of the electromagnetic field with respect to the convolution variable  $y$ , Lemma 4.2(ii), its exterior derivative with respect to the test function  $f$  does not give the electromagnetic field, as one can easily verify. Indeed, our notion of exterior derivative is based on the translation of smearing points  $(y, f) \in \Sigma_0(\mathbb{R}^4, \mathcal{S})$ , and it has a more geometric nature with respect to the usual differentiation of Wightman fields that is based instead on the corresponding distributional notion <sup>5</sup>. In spite of this difference,  $A_\mu^z$  manifests the same problematic features as the potential field in QED: *it is neither localized nor covariant*. To be precise  $A_\mu^z(y, f)$  is not localized on  $\text{supp}(f)$  but, according to Definition 4.1, on the support of the tensor  $y_f^z$ , whose localization is estimated by (4.3). Concerning covariance we have the following result.

**OBb:5** **Lemma 4.3.** *For any  $(y, f) \in \Sigma_0(\mathbb{R}^4, \mathcal{S})$  and  $P = (a, L) \in \mathcal{P}_+^\uparrow$  we have*

$$U(P) A_\mu^z(y, f) U(P)^* = L^{-1}{}_\mu{}^\delta A_\delta^{Pz}(P(y, f)) .$$

*Proof.* According to (2.1) we have

$$U(P) A_\mu^z(y, f) U(P)^* = U(P) F_{\mu\alpha}(y_f^{z,\alpha}) U(P)^* = L^{-1}{}_\mu{}^\delta L^{-1}{}_\alpha{}^\beta F_{\delta\beta}(y_f^{z,\alpha} \circ P^{-1}) .$$

Using (4.2) we compute

$$\begin{aligned} y_f^{z,\alpha} \circ P^{-1}(x) &= y_f^{z,\alpha}(L^{-1}(x-a)) = \int_0^1 (y-z)^\alpha t \cdot f(L^{-1}(x-a) - z - t(y-z)) dt \\ &= \int_0^1 (y-z)^\alpha t \cdot f_L(x-a-Lz-t(Ly-Lz)) dt \\ &= (L^{-1})^\alpha{}_\beta \int_0^1 (Py-Pz)^\beta t \cdot f_L(x-Pz-t(Py-Pz)) dt \\ &= (L^{-1})^\alpha{}_\beta (Py)_{f_L}^{Pz,\beta}(x) . \end{aligned}$$

<sup>5</sup>For a discussion of the "classical" quantum potential, particularly indicated for the purposes of AQFT, we refer the reader to [6]: in these papers, the exterior derivative in the usual sense of Wightman fields is discussed from the point of view of Borchers algebras.

Thus, the definition [\(4.2\)](#) <sup>OBb:1</sup> and the relation  $L^{-1} \alpha^\beta = L^\beta \alpha$  (see Section [2](#)) <sup>OA</sup> imply that

$$\begin{aligned} U(P) A_\mu^z(y, f) U(P)^* &= L_\mu^{-1\delta} L_\alpha^{-1\beta} F_{\delta\beta}(y_f^{z,\alpha} \circ P^{-1}) \\ &= L_\mu^{-1\delta} L_\alpha^{-1\beta} (L^{-1})^\alpha{}_\nu F_{\delta\beta}((Py)_{f_L}^{Pz,\nu}) = L_\mu^{-1\delta} F_{\delta\beta}((Py)_{f_L}^{Pz,\beta}) \\ &= L_\mu^{-1\delta} A_\delta^{Pz}(P(y, f)) , \end{aligned}$$

completing the proof. □

Thus covariance is violated because the Poincaré transformation affects the pole  $z$  with respect to which the potential form is reconstructed. We shall return on this point in the following sections.

**Gauge transformations.** The classical notion of local gauge transformation involves a gauge function  $g \in C^2(\mathbb{R}^4)$  whose gradient is added to the potential,  $A_\mu \mapsto A_\mu + \partial_\mu g$ . Coherently with the previous subsections, we apply a smearing  $f \in \mathcal{S}(\mathbb{R}^4)$  and add the convolution variable  $y \in \mathbb{R}^4$ , then we define, for any  $g \in L^\infty(\mathbb{R}^4)$ ,

$$A_\mu^z(y, f) \mapsto A_\mu^{g,z}(y, f) := A_\mu^z(y, f) + \partial_{y^\mu} g(y, f) \quad , \quad (y, f) \in \Sigma_0(\mathbb{R}^4, \mathcal{S}) ,$$

where we used the notation

$$g(y, f) := \int g(y-x) f(x) d^4x \quad , \quad y \in \mathbb{R}^4 ,$$

for the convolution. This ensures that the gauge transformation leaves the observable  $F_{\mu\nu}$  unaffected: in fact, Lemma [4.2\(ii\)](#) <sup>OBb:4</sup> implies that

$$\partial_{y^\nu} A_\mu^{g,z}(y, f) - \partial_{y^\mu} A_\nu^{g,z}(y, f) = F_{\mu\nu}(f_y) + (\partial_{y^\mu} \partial_{y^\nu} - \partial_{y^\nu} \partial_{y^\mu}) g(y, f) = F_{\mu\nu}(f_y) .$$

**The line integral of the potential form.** The last step is to analyze the integral of the quantum potential form on a curve  $\gamma$ . The idea is, as usual, to find a suitable test function giving a rigorous sense to the formal expression

$$\int_\gamma A_\mu^z(\gamma, f) d\gamma^\mu := \int_0^1 A_\mu^z(\gamma(s), f) \dot{\gamma}^\mu(s) ds . \quad (4.5) \quad \text{OBb:00}$$

In analogy to the case of the potential form, we pick a point  $z \in \mathbb{R}^4$  and associate a 2-tensor  $\gamma_f^z$  to any smearing curve  $(\gamma, f) \in \Sigma_1(\mathbb{R}^4, \mathcal{S})$ , by defining

$$\gamma_f^{z,\mu\nu}(x) := \int_0^1 \left( \int_0^1 (\gamma(s) - z)^\mu t f_{z+t(\gamma(s)-z)}(x) dt \right) \dot{\gamma}^\nu(s) ds \quad , \quad x \in \mathbb{R}^4 . \quad (4.6) \quad \text{OBb:10}$$

Again, the coefficients  $\gamma_f^{z,\mu\nu}$  are in  $\mathcal{S}(\mathbb{R}^4)$  and are limit of the partial Riemann sums defining the integral. Concerning the support, as in [\(4.3\)](#) <sup>OBb:2</sup>, we have

$$\text{supp}(\gamma_f^z) \subseteq \text{supp}(f) + \{z + t(\gamma(s) - z) \in \mathbb{R}^4 \mid t, s \in [0, 1]\} . \quad (4.7) \quad \text{OBb:11}$$

In words,  $\gamma_f^z$  is supported in the envelope between  $\text{supp}(f)$  and the set spanned by  $z + t(\gamma(s) - z)$  as  $t, s$  vary in  $[0, 1]$ .

After these preliminary observations, the *line integral of the potential form* is the operator

$$A^z \langle \gamma, f \rangle := F_{\mu\nu}(\gamma_f^{z,\mu\nu}) \quad , \quad (\gamma, f) \in \Sigma_1(\mathbb{R}^4, \mathcal{S}) . \quad (4.8) \quad \text{OBb:12}$$

We give some properties of the above line integral. We start by observing that  $A^z \langle \gamma, f \rangle$  is an essentially self-adjoint operator defined on  $\mathcal{D}$  such that  $A^z \langle \gamma, f \rangle \mathcal{D} \subseteq \mathcal{D}$ . The fact that (4.5) holds in the weak sense is proved by the following result:

lem.OBb5

**Lemma 4.4.** For any  $\phi, \psi \in \mathcal{D}$ ,  $z \in \mathbb{R}^4$  and  $(\gamma, f) \in \Sigma_1(\mathbb{R}^4, \mathcal{S})$ , we have:

$$(i) \quad (\phi, A^z \langle \gamma, f \rangle \psi) = \int_0^1 (\phi, A_\nu^z(\gamma(s), f) \psi) \dot{\gamma}^\nu(s) ds;$$

$$(ii) \quad (\phi, A^z \langle \gamma, f \rangle \psi) = \int_0^1 \int_0^1 (\gamma(s) - z)^\mu t (\phi, F_{\mu\nu}(f_{z+t(\gamma(s)-z)}) \psi) \dot{\gamma}^\nu(s) ds dt.$$

*Proof.* (i) We note that  $(\gamma(s), f) \in \Sigma_0(\mathbb{R}^4, \mathcal{S})$  for any  $s \in \Delta_1$  and find, according to (4.2), that  $\gamma_f^{z, \mu\nu}(x) = \int_0^1 \gamma(s)_f^{z, \mu}(x) \dot{\gamma}^\nu(s) ds$  for any  $x \in \mathbb{R}^4$ . Noticing, as usual, that in this relation  $\gamma_f^{z, \mu\nu}$  is limit in the topology of  $\mathcal{S}(\mathbb{R}^4)$  of the partial Riemann sums defining the integral, we have

$$\begin{aligned} (\phi, A^z \langle \gamma, f \rangle \psi) &= (\phi, F_{\mu\nu}(\gamma_f^{z, \mu\nu}) \psi) = \int_0^1 (\phi, F_{\mu\nu}(\gamma(s)_f^{z, \mu}) \psi) \dot{\gamma}^\nu(s) ds \\ &\stackrel{\text{OBb:3a}_1}{=} \int_0^1 (\phi, A_\nu^z(\gamma(s), f) \psi) \dot{\gamma}^\nu(s) ds, \end{aligned}$$

as desired. Finally, (ii) follows by Lemma 4.2(ii).  $\square$

The definition (4.8) makes manifest that the line integral of the potential form is not localized on the support of the test function, but on the subset defined in (4.7). The action of the Poincaré group is computed in the following result, which, in particular, shows how the pole is involved by the transformation:

**Proposition 4.5.** For any  $z \in \mathbb{R}^4$ ,  $(\gamma, f) \in \Sigma_1(\mathbb{R}^4, \mathcal{S})$  and  $P \in \mathcal{P}_+^\uparrow$  we have

$$U(P) A^z \langle \gamma, f \rangle U(P)^* = A^{Pz} \langle P(\gamma, f) \rangle. \quad (4.9)$$

OBb:14

*Proof.* For  $P = (x, L)$ , we apply Lemma 4.4(i) and Lemma 4.3 obtaining

$$\begin{aligned} (\phi, U(P) A^z \langle \gamma, f \rangle U(P)^* \psi) &= L^{-1}{}_\mu{}^\delta \int_0^1 (\phi, A_\delta^{Pz}(P(\gamma(s), f)) \psi) \dot{\gamma}^\mu(s) ds \\ &= \int_0^1 (\phi, A_\delta^{Pz}(P(\gamma(s), f)) \psi) (\dot{P}\gamma)^\mu(s) ds = (\phi, A^{Pz} \langle P(\gamma, f) \rangle \psi). \end{aligned}$$

$\square$

Finally, we analyze the behaviour of the line integral under a gauge transformation  $g \in L^\infty(\mathbb{R}^4)$ . Let  $(\gamma, f) \in \Sigma_1(\mathbb{R}^4, \mathcal{S})$ . We compute

$$\begin{aligned} (\phi, A^{g,z} \langle \gamma, f \rangle \psi) &= \int_0^1 (\phi, A_\nu^z(\gamma(s), f) \psi) \dot{\gamma}^\nu(s) ds + (\phi, \psi) \int_0^1 \{\partial_{y^\nu} g(y, f)\}_{y=\gamma(s)} \dot{\gamma}^\nu(s) ds \\ &= (\phi, A^z \langle \gamma, f \rangle \psi) + (\phi, \psi) \{g(\gamma(1), f) - g(\gamma(0), f)\}, \end{aligned}$$

and find the familiar expression

$$A^{g,z} \langle \gamma, f \rangle = A^z \langle \gamma, f \rangle + g(\gamma(1), f) - g(\gamma(0), f), \quad (4.10)$$

eq.g1

which shows the appearance of the scalar factors  $g(\gamma(t), f)$ ,  $t = 0, 1$ . Thus, when  $\gamma$  is closed we find  $A^{g,z} \langle \gamma, f \rangle = A^z \langle \gamma, f \rangle$ .

**Stokes' theorem and loop observables.** We are now ready to prove a quantum version of the Stokes' theorem. As a preliminary remark, we note that if  $(\sigma, f), (\sigma', f) \in \Sigma_2(\mathbb{R}^4, \mathcal{S})$  have the same boundary, then

$$F \langle \sigma, f \rangle = F \langle \sigma', f \rangle , \quad (4.11) \quad \boxed{\text{lem.OB:6}}$$

as can be easily verified by considering the 2-form

$$\omega_{\mu\nu}(y) := (\phi, F_{\mu\nu}(f_y)\psi) , \quad y \in \mathbb{R}^4 , \quad (4.12) \quad \boxed{\text{def.om}}$$

that is defined for any  $\phi, \psi \in \mathcal{D}$ . In fact,  $\omega$  is exact by Eq. 3.7, so the integral of  $\omega$  on  $\sigma$  equals the integral of a primitive of  $\omega$  on the boundary  $\partial\sigma$ . This fact can be completely understood by proving the Stokes' theorem for the quantum electromagnetic field:

prop:OBb6

**Proposition 4.6.** *For any smearing closed curve  $(\ell, f) \in \Sigma_1(\mathbb{R}^4, \mathcal{S})$  and for any  $\sigma \in \Sigma_2(\mathbb{R}^4)$  such that  $\ell = \partial\sigma$ , we have  $A^z \langle \ell, f \rangle = F \langle \sigma, f \rangle$ .*

*Proof.* For any  $\phi, \psi \in \mathcal{D}$ , we consider the form  $\omega$  defined in (4.12) and define

$$\varphi_\nu^z(y) := (\phi, A_\nu^z(y, f)\psi) , \quad y \in \mathbb{R}^4 .$$

By Lemma 4.2(ii) we have  $d\varphi^z = \omega$ . Applying the Stokes' theorem we find

$$\begin{aligned} (\phi, A^z \langle \ell, f \rangle \psi) &\stackrel{\text{Lemma 4.4(ii)}}{=} \int_0^1 \varphi_\nu^z(\ell(s)) \dot{\ell}^\nu(s) ds = \int_\ell \varphi^z \\ &= \int_\sigma \omega = \int_{\Delta_2} (\phi, F_{\mu\nu}(f_{\sigma(s)})\psi) \sigma^{\mu\nu}(s) d^2s \stackrel{\text{Lemma 3.1(ii)}}{=} (\phi, F \langle \sigma, f \rangle \psi) , \end{aligned}$$

and the proof follows by density of  $\mathcal{D}$  in  $\mathcal{H}$ . □

OB:7

**Remark 4.7.** Some observations are in order.

1. The previous result proves that  $A_\mu^z$  is – up to exact forms – the "right" electromagnetic potential if the point is the one to implement the Stokes' theorem.
2. The proof of the previous proposition is based on the property of  $F_{\mu\nu}$  being closed as a 2-form (2.4), so we do not expect limitations from considering quantum fields with values in a generic Lie algebra, or defined over generic space-times with trivial second de Rham cohomology.
3. Let us consider a closed smearing curve  $(\ell, f) \in \Sigma_1(\mathbb{R}^4, \mathcal{S})$  and define

$$w(\ell, f) := \exp(iA^z \langle \ell, f \rangle) . \quad (4.13) \quad \boxed{\text{eq.w}}$$

Recalling the meaning of the loop integral  $A^z \langle \ell, f \rangle$  we have that  $w(\ell, f)$  yields a rigorous version of the notion of Wilson loop used in quantum gauge theories. As we shall see in the sequel,  $w(\ell, f)$  is the starting point for the construction of the representation of the net of causal loops defined by the electromagnetic field.

## 4.2 The potential system

As observed in the previous section the quantum potential form  $A_\mu^z$  is a weak primitive of the electromagnetic field, but it is neither covariant nor causal. Nevertheless, a remarkable fact is that these two properties underlay our construction and arise, as we shall see in the following lines, if we look at the collection  $\{A_\mu^z, z \in \mathbb{R}^4\}$ .

pot.system

**Definition 4.8.** Given a quantum electromagnetic field  $(U, F)$ , we call **the quantum potential system associated with**  $(U, F)$  the collection  $A_\mu := \{A_\mu^z, z \in \mathbb{R}^4\}$  of maps

$$A_\mu^z : (y, f) \mapsto A_\mu^z(y, f),$$

where  $A_\mu^z(y, f)$  is the operator defined according to (4.8) for any  $(y, f) \in \Sigma_0(\mathbb{R}^4, \mathcal{S})$ .

We can now give a new interpretation to the results of the previous sections in terms of the potential system  $A_\mu$ , that satisfies the following properties:

1. It is a *weak primitive* of  $F_{\mu\nu}$ , Lemma 4.2(ii), in the sense that

$$(\phi, F_{\mu\nu}(f_y)\psi) = \partial_{y^\mu}(\phi, A_\nu^z(y, f)\psi) - \partial_{y^\nu}(\phi, A_\mu^z(y, f)\psi);$$

2. It is *covariant*, Lemma 4.3, that is,

$$\text{ad}_{U(P)} \circ A_\mu^z = L_\mu^{-1\nu} A_\nu^{Pz} \circ P, \quad P = (x, L) \in \mathcal{P}_+^\uparrow;$$

3. It is *causal on the smearing loops*: let us consider smearing closed curves  $(\ell_1, f_1), (\ell_2, f_2) \in \Sigma_1(\mathbb{R}^4, \mathcal{S})$  such that: (a) they are causally disjoint, in the sense that  $\text{supp}(\ell_1[f_1]) \perp \text{supp}(\ell_2[f_2])$ ; (b) they form a trivial link. Then we have

$$[\exp(iA^{z_1} \langle \ell_1, f_1 \rangle), \exp(iA^{z_2} \langle \ell_2, f_2 \rangle)] = 0, \quad \forall z_1, z_2 \in \mathbb{R}^4. \quad (4.14)$$

eq.cloop

In fact, since (a) and (b) hold, it is always possible to find causally disjoint smearing surfaces  $(\sigma_1, f_1), (\sigma_2, f_2)$  having  $(\ell_1, f_1), (\ell_2, f_2)$  as boundaries, via suitable cone constructions (see Theorem 5.6 and following remarks). Thus, applying Proposition 4.6,  $A^{z_k} \langle \ell_k, f_k \rangle = F \langle \sigma_k, f_k \rangle$ , for  $k = 1, 2$ , and (4.14) follows by Lemma 3.1(iii).

**Gauge transformations.** The previous considerations suggest that the meaningful object defined by the quantum electromagnetic field is the potential system rather than one of the potential form  $A_\mu^z$ , for some  $z \in \mathbb{R}^4$ . Thus we must modify the notion of gauge transformation accordingly, that is, we must require that the potential system obtained by applying to  $A_\mu$  a gauge transformation fulfils the above three properties. On these grounds, we consider a family of functions

$$g := \{g^z \in L^\infty(\mathbb{R}^4), z \in \mathbb{R}^4\},$$

fulfilling the covariance property

$$g^{Pz}(y, f) = g^z(P^{-1}(y, f)), \quad (y, f) \in \Sigma_0(\mathbb{R}^4, \mathcal{S}), P \in \mathcal{P}_+^\uparrow, \quad (4.15)$$

eq.gt1

where, as before,  $g^z(y, f)$  stands for the convolution  $g^z * f(y)$ . This set of functions form an additive Abelian group  $\mathcal{G}$  that we call the *group of gauge transformations*. Accordingly, the potential system transforms, under a gauge transformation, as

$$A_\mu^{g,z}(y, f) := A_\mu^z(y, f) + \partial_\mu g^z(y, f), \quad g \in \mathcal{G}. \quad (4.16)$$

OBb:7

Clearly,  $A_\mu^g$  satisfies the above Property 1. Concerning Property 2, we note that  $\partial_\mu g^z(y, f) = \partial_\mu g^{Pz}(P(y, f)) = L^{-1}{}_\mu{}^\nu \partial_{(Py)^\nu} g^{Pz}(P(y, f))$ ; this relation, (4.16) and Lemma 4.3 give the desired covariance

$$\begin{aligned} U(P)A_\mu^{g,z}(y, f)U(P)^* &= U(P)A_\mu^z(y, f)U(P)^* + \partial_\mu g^z(y, f) \\ &= L^{-1}{}_\mu{}^\delta A_\delta^{Pz}(P(y, f)) + L^{-1}{}_\mu{}^\delta \partial_{(Py)^\delta} g^{Pz}(P(y, f)) \\ &= L^{-1}{}_\mu{}^\delta A_\delta^{g,Pz}(P(y, f)). \end{aligned}$$

Finally, since the gradients  $\partial_\mu g^z(y, f)$  do not contribute to the loop integrals,  $A_\mu^g$  satisfies Property 3.

In order to provide examples of nontrivial gauge transformations, writing the above covariance property in explicit terms we find  $\int g^{Pz}(y-x)f(x)d^4x = \int g^z(P^{-1}(y-x))f(x)d^4x$ , where  $P = (a, L)$ . Since this equality holds for any  $f \in \mathcal{S}(\mathbb{R}^4)$ , we conclude that (H.15) is equivalent to

$$g^{Pz} = g^z \circ P^{-1} \quad , \quad z \in \mathbb{R}^4 \quad , \quad P \in \mathcal{P}_+^\uparrow . \quad (4.17)$$

eq.gt2

This equation yields an important restriction on the family of the gauge transformations  $g$ : the component  $g^0$ , i.e. the pole  $z$  is the origin 0, is any  $L^\infty(\mathbb{R}^4)$  function invariant on the orbits of the Lorentz group on the Minkowski spacetime; the other on the Minkowski spacetime components  $g^z$ , for  $z \neq 0$ , are easily obtained from  $g^0$  by the action of the Poincaré group. As an example we may take  $g^z(y) := g((y-z)^2)$ , where  $g \in L^\infty(\mathbb{R})$  and  $(y-z)^2 = (y-z) \cdot (y-z)$ .

The analysis of constraints on gauge transformations deriving from physically meaningful *gauge fixing* is out of the aims of the general approach of the present paper.

## 5 Representations induced by the electromagnetic field

OC

In the present section we prove that the electromagnetic field induces a representation of the net of causal loops over the Minkowski space-time. Since we shall use as generators of loops *smearing, affine* 1-simplices, our causal loop net is different from that defined for an arbitrary globally hyperbolic space-time [16], anyway the construction is the same.

Proceeding as in [16], we show that representations of the net of causal loops have a geometric interpretation in terms of connection systems of  $\Sigma_*(\mathbb{R}^4, \mathcal{S})$ , and this leads to a natural notion of gauge transformation. The feature introduced in the present paper is that representations can be also equivalently described in terms of 2-cochains of  $\Sigma_*(\mathbb{R}^4, \mathcal{S})$ , as a consequence of the fact that  $\mathbb{R}^4$  is contractible.

This is a key observation: in fact we shall show that  $F_{\mu\nu}$  defines, *via* the associated 2-form, a 2-cochain  $w^{em}$ , hence a representation of the net of causal loops. Using the abstract procedure outlined above we consider the connection system  $u^{em}$  defined by  $w^{em}$  and show that  $u^{em}$  is equivalent to the potential system  $A_\mu$  introduced in the previous section. Finally, gauge transformations of  $A_\mu$  define gauge transformations of  $u^{em}$ .

### 5.1 The simplicial set and the set of paths

OCa

As already mentioned we modify the definition of the simplicial set made in [16], considering the affine subcomplex of  $\Sigma_*(\mathbb{R}^4, \mathcal{S})$ . This simplicial set is not pathwise connected, however all the key constructions made in [16] apply, that is, covariance under the action of the Poincaré group and the notion of covariant path-frame.

**The simplicial set  $\Sigma_*$ .** The corner stone of the net of causal loops defined in [16] is the simplicial set denoted here by  $\Sigma_*(K)$ , defined in terms of double cones of the Minkowski space-time and their inclusions. This choice encodes the essential properties of localization and covariance that reflect into the net of causal loops. Namely, we defined a non-Abelian free group generated by 1-simplices of  $\Sigma_*(K)$ , then we considered the subgroups generated by loops (suitable compositions of 1-simplices, see next paragraph), and then we used these groups to define the net.

Now, we want a simplicial set encoding localization and covariance, and able to deal with the smearing of quantum fields. To this end, the natural choice is the simplest simplicial subset of  $\Sigma_*(\mathbb{R}^4, \mathcal{S})$ , that of affine singular smearing simplices. This results to be the natural choice



because of the vanishing of the integral over *non-injective* affine simplices, namely degenerated simplices defined below.

An *affine singular smearing*  $n$ -simplex  $s$  is a pair  $(\varphi_s, f)$  where  $\varphi_s : \Delta_n \rightarrow \mathbb{R}^4$  is an affine function and  $f \in \mathcal{S}$ . We denote the set of affine singular  $n$ -simplices by  $\Sigma_n$ , and the corresponding simplicial set by  $\Sigma_*$ .

An affine smearing  $n$ -simplex  $s$  is determined by the knowledge of its vertices. In fact defining  $s_i := \varphi_s(e_i)$ , where each  $e_i$ ,  $i = 0, \dots, n$ , is a vertex of  $\Delta_n$ , we have

$$\varphi_s(t) = s_0 + \sum_{i=1}^n t_i (s_i - s_0) , \quad t \in \Delta_n . \quad (5.1) \quad \boxed{\text{OCa:0}}$$

So we write

$$s = (s_0, \dots, s_n; f) , \quad s \in \Sigma_n ; \quad (5.2) \quad \boxed{\text{OCa:1}}$$

a smearing affine  $n$ -simplex  $s$  is said to be *degenerated* if two of its vertices coincide.

We shall denote 0-, 1-, 2-simplices by the letters  $a, b$  and  $c$  respectively, and observe that 0-simplices are points, 1-simplices are segments, and 2-simplices are triangles. In particular, we denote the parametric function (5.1) of  $b \in \Sigma_1$  by  $r_b : \Delta_1 \rightarrow \mathbb{R}^4$  and that of  $c \in \Sigma_2$  by  $\sigma_c : \Delta_2 \rightarrow \mathbb{R}^4$ . Concerning 1-simplices, if  $b = (b_0, b_1; f)$  then  $\partial_0 b = (b_1; f)$  and  $\partial_1 b = (b_0; f)$ . So the 1-face corresponds to the 0-vertex and conversely. Instead, the faces of a 2-simplex  $c = (c_0, c_1, c_2; f)$  are given by  $\partial_0 c = (c_1, c_2; f)$ ,  $\partial_1 c = (c_0, c_2; f)$  and  $\partial_2 c = (c_0, c_1; f)$ . According to the orientation described in Appendix A, the *opposite*  $\bar{b}$  of  $b$  is the 1-simplex  $\bar{b} = (b_1, b_0; f)$  or, equivalently in terms of face relations,  $\partial_0 \bar{b} = \partial_1 b$  and  $\partial_1 \bar{b} = \partial_0 b$ . Instead, the *opposite* of  $c$  is the 2-simplex  $\bar{c} = (c_0, c_2, c_1; f)$  or, in terms of face relations,  $\partial_0 \bar{c} = \overline{\partial_0 c}$ ,  $\partial_1 \bar{c} = \partial_2 c$  and  $\partial_2 \bar{c} = \partial_1 c$ .

**Words, paths, loops, and path-frames.** We use  $\Sigma_1$  as an alphabet for generating words and, in particular, paths. A finite ordered sequence  $w = b_n b_{n-1} \dots b_1$  of 1-simplices is called a *word*. The *opposite* of a word  $w = b_n \dots b_1$  is the word  $\bar{w} := \bar{b}_1, \dots, \bar{b}_n$ . We shall denote the *empty* word by  $\mathbb{1}$ . We have to care of not confusing these words with the elements of  $C_1(\mathbb{R}^4, \mathcal{S})$ : actually, we shall see that these words define a *non-Abelian* free group.

The action of the Poincaré group extends from 1-simplices, see (3.2), to words, by

$$Pw := P b_n \dots P b_1 , \quad P \in \mathcal{P}_+^\uparrow , \quad (5.3) \quad \boxed{\text{OCa:4a}}$$

and  $P\mathbb{1} := \mathbb{1}$  on the empty word. A word  $w = b_n \dots b_1$  is said to be a *path* whenever its generators satisfy the relation

$$\partial_0 b_{i+1} = \partial_1 b_i , \quad i = 1, \dots, n-1 . \quad (5.4)$$

Note that this implies that the generators of a path have all the same smearing function. We set  $\partial_1 w := \partial_1 b_1$  and  $\partial_0 w := \partial_0 b_n$  and call these 0-simplices, respectively, the *starting* and the *ending point* of the path  $w$ . We shall also use the notation

$$w : a \rightarrow o$$

to denote a path from  $a$  to  $o$ . Given two paths  $p : o \rightarrow o'$  and  $q : o' \rightarrow o''$ , since  $\partial_0 p = o' = \partial_1 q$  the juxtaposition  $pq$  of the generators of  $q$  on  $p$  gives a new path  $qp : o \rightarrow o''$ , called the *composition*. Finally, a path  $w : o \rightarrow o$  is said to be a *loop over*  $o$ .

The boundary of  $c \in \Sigma_2$  in the sense of homology is the 1-cycle  $\partial_0 c - \partial_1 c + \partial_2 c$ . On the other side, we have a geometric notion of boundary, defined as the loop

$$\partial c : \partial_1 \partial_2 c \rightarrow \partial_1 \partial_2 c , \quad \partial c := \overline{\partial_1 c} \partial_0 c \partial_2 c , \quad c \in \Sigma_2 . \quad (5.5) \quad \boxed{\text{OCa:4b}}$$

We call  $\partial c$  the *path-boundary* of  $c$ . We stress the difference between the path-boundary  $\partial c$  and the boundary  $\partial c$  of a 2-simplex  $c$ : the former is a word, i.e. an *ordered sequence* of 1-simplices. Note that by the convention adopted in the definition of the opposite  $\bar{c}$  of a 2-simplex  $c$ , we have  $\partial \bar{c} = \bar{\partial}_1 \bar{c} \partial_0 \bar{c} \partial_2 \bar{c} = \bar{\partial}_2 \bar{c} \bar{\partial}_0 \bar{c} \bar{\partial}_1 \bar{c} = \bar{\partial} c$ .

The simplicial set  $\Sigma_*$  is not pathwise connected since no path joins two 0-simplices having different smearing functions; actually, the connected components are indexed by the smearing function itself. Thus, given  $a = (a_0; f) \in \Sigma_0$  we define

$$\Sigma_n^a := \{s \in \Sigma_n : s = (s_0, \dots, s_n; f)\}, \quad n \in \mathbb{N},$$

i.e. the subset of  $n$ -simplices having the same smearing function as the 0-simplex  $a$ . If  $a' \in \Sigma_0^a$ , then the 1-simplex  $e_{(a',a)}$  defined by

$$\partial_1 e_{(a',a)} := a, \quad \partial_0 e_{(a',a)} = a', \quad (5.6)$$

OCa:5

connects  $a$  to  $a'$  and is written explicitly as the segment  $e_{(a',a)} = (\varphi_{(a',a)}, f)$ ,  $\varphi_{(a',a)}(t) = a_0 + ta'_0$ ,  $t \in \Delta_1$ . In particular, we use the notation

$$e_a := e_{(a,a)}, \quad a \in \Sigma_0, \quad (5.7)$$

OCa:5a

which is the degenerated 1-simplex (see Subsection <sup>OCa</sup>5.1) whose vertices equals  $a_0$ , and having smearing function  $f$ . We call  $e_a$  the *trivial loop* over  $a$ .

A *path-frame* over a pole  $a \in \Sigma_0$  is a set of paths

$$\mathcal{P}_a = \{p_{(a,a')} : a' \rightarrow a, a' \in \Sigma_0^a\},$$

satisfying the condition that  $p_{(a,a)} = e_a$ . A *covariant path-frame system* is a collection of path-frames  $\mathcal{P} = \{\mathcal{P}_a, a \in \Sigma_0\}$  satisfying the relation  $P\mathcal{P}_a = \mathcal{P}_{Pa}$ , i.e.  $Pp_{(a,a')} = p_{Pa,Pa'}$  for any  $P \in \mathcal{P}_+^\uparrow$ . Notice that, if we set  $\mathcal{E}_a := \{e_{(a,a')} \mid a' \in \Sigma_0^a\}$  then the collection  $\mathcal{E} := \{\mathcal{E}_a \mid a \in \Sigma_0\}$  is a covariant path-frame system. We shall refer to  $\mathcal{E}$  as the *Euclidean* path-frame system.

## 5.2 The net of causal loops

OCb

We now follow the route of <sup>CRV</sup>[16] and define the net of causal loops using the simplicial set  $\Sigma_*$ . Since in <sup>CRV</sup>[16] we started from a different simplicial set the resulting net is different, nevertheless the procedure is the same up to the following (not substantial) points.

First, here we prefer to use also degenerate simplices, because this will simplify the construction of the representation induced by the electromagnetic field. The definition of net of causal loops remains unchanged, because degenerate 1-simplices turn out to be equivalent to the identity of the group.

Second, we omit the functorial picture of the net with respect to space-times, since we are interested in the case of the Minkowski space-time only.

**Groups of loops.** Our construction starts defining the group  $F$  generated by  $\Sigma_1$  with relations

$$b\bar{b} = \mathbb{1}, \quad e_a = \mathbb{1}, \quad b \in \Sigma_1, \quad a \in \Sigma_0, \quad (5.8)$$

OCb:1

where  $\mathbb{1}$  is the identity. These two relations are equivalent to

$$b^{-1} = \bar{b}, \quad e_a e_a = e_a, \quad b \in \Sigma_1, \quad a \in \Sigma_0.$$

The group  $F$  is (non-canonically) isomorphic to a free group whose elements are *reduced words*, that is, words  $w$  in which either pairs of the form  $b\bar{b}$  or degenerated 1-simplices  $e_a$  do not appear.

Hence any word  $w$  is associated to a unique *reduced* word denoted by  $w^r$ . Finally, we observe that if a word is a path  $w : a \rightarrow a'$ , then its reduced word is still a path  $w^r : a \rightarrow a'$ .

The notion of support of a 1-simplex extends to elements of  $F$  as follows: the *support*  $|w|$  of  $w \in F$  is the subset of  $\mathbb{R}^4$  obtained as the union of the supports of the generators of the reduced word  $w^r$ . For instance, if  $w = b_2 b \bar{b} b_1 e_a$  with  $b_1 \neq \bar{b}_2$ , then  $|w| = |b_2| \cup |b_1|$  since  $w^r = b_2 b_1$ .

Observing that  $P\bar{b} = \overline{Pb}$  for any  $b \in \Sigma_1$ ,  $P \in \mathcal{P}_+^\uparrow$ , we have that (5.3) defines an action of the Poincaré group on  $F$ . This action sends reduced words into reduced words, and this implies that

$$P|w| = x + L|w| = |Pw|, \quad P = (x, L) \in \mathcal{P}_+^\uparrow, \quad w \in F. \quad (5.9) \quad \boxed{\text{OCb:2}}$$

Note that if  $p : a \rightarrow a'$ , then  $Pp : Pa \rightarrow Pa'$ .

$\boxed{\text{OCb:3}}$

**Definition 5.1.** We call **group of loops** the subgroup  $L(\mathbb{R}^4)$  of  $F$  generated by loops.

An element of  $L(\mathbb{R}^4)$  is, by definition, a (reduced) word of the form  $w = p_n p_{n-1} \cdots p_1$ , where any  $p_i$  is a loop over  $a_i \in \Sigma_0$ . It is easily seen that  $L(\mathbb{R}^4)$  is stable under reduction of words. Furthermore,  $L(\mathbb{R}^4)$  inherits from  $F$  the Poincaré action.

**Causal loops.** Using the group of loops we construct a causal net of groups over the set of double cones of the Minkowski space-time. To begin with, we define

$$L_o := \{w \in L(\mathbb{R}^4) \mid |w| \subseteq o\}, \quad o \in K, \quad (5.10) \quad \boxed{\text{OCb:5}}$$

and observe that, according to the definition of the support of an element of  $L(\mathbb{R}^4)$ , this definition refers not to a generic word  $w$  but to its reduced word  $w^r$ . The set  $L_o$  is a subgroup of  $L(\mathbb{R}^4)$ : in fact, if  $|w|, |w'| \subseteq o$  then  $|\bar{w}| \subseteq o$  and  $|ww'| \subseteq o$ , moreover  $\mathbb{1} \in L_o$  for any double cone  $o$  by (5.8). Since  $L_o \subseteq L_{o'}$ ,  $o \subseteq o'$ , the mapping  $o \mapsto L_o$  forms a net of subgroups of  $L(\mathbb{R}^4)$  over the set of double cones. By (5.9) this net is covariant, i.e.  $PL_o = L_{Po}$ ,  $P \in \mathcal{P}_+^\uparrow$ .

We now impose on  $L(\mathbb{R}^4)$  the relations

$$w_1 w_2 = w_2 w_1, \quad w_i \in L_{o_i}, \quad i = 1, 2, \quad o_1 \perp o_2, \quad (5.11) \quad \boxed{\text{eq. c}}$$

defining the *group of causal loops*, that we denote by  $\widehat{L}(\mathbb{R}^4)$ . We then have an induced net

$$\widehat{L}_o \subseteq \widehat{L}(\mathbb{R}^4), \quad o \in K, \quad (5.12) \quad \boxed{\text{OCb:7}}$$

where each  $\widehat{L}_o$  is the subgroup generated by the image of  $L_o$  under the quotient defined by (5.11). Since Poincaré transformations preserve the causal disjointness relation of subsets of  $\mathbb{R}^4$ , we have that (5.3) induces the action

$$w \mapsto \beta_P(w) \in \widehat{L}_{Po}, \quad o \in K, \quad w \in \widehat{L}_o, \quad P \in \mathcal{P}_+^\uparrow. \quad (5.13) \quad \boxed{\text{OCb:8}}$$

Clearly, the net  $\widehat{L} = \{\widehat{L}_o\}_{o \in K}$  is causal by (5.11).

**The net of causal loops.** Let  $C^*$  denote the functor assigning the group  $C^*$ -algebra  $C^*G$  to the locally compact group  $G$ . If  $G$  is discrete then  $C^*G$  is unital. Furthermore, if  $G_1$ , and  $G_2$  are discrete and  $\rho : G_1 \rightarrow G_2$  is an injective group morphism, then  $C^*\rho : C^*G_1 \rightarrow C^*G_2$  is a unital, injective  $*$ -morphism.

We now come to the definition of the net of causal loops. We first consider the  $C^*$ -algebras  $C^*\widehat{L}(\mathbb{R}^4)$  and  $C^*\widehat{L}_o$ ,  $o \in K$ . Since  $\widehat{L}_o \subseteq \widehat{L}_{o'}$  for any  $o \subseteq o'$ , by functoriality there is a unital, injective  $*$ -morphism  $J_{o'o} : C^*\widehat{L}_o \rightarrow C^*\widehat{L}_{o'}$  satisfying  $J_{o''o'} \circ J_{o'o} = J_{o''o}$  for any  $o \subseteq o' \subseteq o''$ . For the same reason we have unital, injective  $*$ -morphisms  $\vec{J}_o : C^*\widehat{L}_o \rightarrow C^*\widehat{L}(\mathbb{R}^4)$  for any  $o \in K$ , such that  $\vec{J}_{o'} \circ J_{o'o} = \vec{J}_o$  for any  $o \subseteq o'$ . On these grounds:

OCb:9

**Definition 5.2.** Let  $\mathcal{A}$  denote the mapping  $\mathcal{A} : K \ni o \rightarrow \mathcal{A}_o \subseteq \mathcal{A}(\mathbb{R}^4)$ , where

$$\mathcal{A}(\mathbb{R}^4) := C^*\widehat{L}(\mathbb{R}^4) \quad , \quad \mathcal{A}_o := \vec{j}_o(C^*\widehat{L}_o) \quad , \quad o \in K \quad ,$$

and let  $\alpha : \mathcal{P}_+^\uparrow \rightarrow \mathbf{aut}\mathcal{A}(\mathbb{R}^4)$  be the action of the Poincaré group defined by applying the functor  $C^*$  to (5.13). We call the pair  $(\mathcal{A}, \alpha)$  **the net of causal loops** over  $K$ .

The map  $\mathcal{A}$  is clearly a net, which is causal by (5.11) and covariant by (5.13). We conclude by noting that  $\mathcal{A}$  is not trivial, and that the  $C^*$ -algebras  $\mathcal{A}_o$  are non-Abelian (see [16]).

### 5.3 Representations

Cc

We now discuss representations of  $(\mathcal{A}, \alpha)_K$  and point out their geometric meaning.

We start by recalling that these representations are in 1-1 correspondence with a particular class of representations of the group of loops  $L(\mathbb{R}^4)$ . This equivalence yields a geometric interpretation of representations in terms of causal and covariant connection systems. Using the fact that  $\mathbb{R}^4$  is contractible, we also show that representations of  $(\mathcal{A}, \alpha)_K$  can be equivalently described by a class of 2-cochains of  $\Sigma_*$ . This is a key result: in fact, we shall use the latter equivalence to prove that the electromagnetic field induces, *via* integration on 2-simplices, a representation of  $(\mathcal{A}, \alpha)_K$  (see Subsection 5.4).

**Representations and connection systems.** A covariant representation of the net of causal loops  $(\mathcal{A}, \alpha)_K$  is a pair  $(\pi, U)$ , where  $\pi : \mathcal{A}(\mathbb{R}^4) \rightarrow \mathcal{BH}$  is a (non degenerated) representation and  $U : \mathcal{P}_+^\uparrow \rightarrow \mathcal{UH}$  is such that

$$\mathrm{ad}_{U(P)} \circ \pi = \pi \circ \alpha_P \quad , \quad P \in \mathcal{P}_+^\uparrow \quad .$$

These representations are in 1-1 correspondence with *causal and covariant representations*  $(\lambda, U)$  of the group of loops  $L(\mathbb{R}^4)$ : that is, we have unitary representations

$$\lambda : L(\mathbb{R}^4) \rightarrow \mathcal{UH} \quad , \quad U : \mathcal{P}_+^\uparrow \rightarrow \mathcal{UH} \quad ,$$

satisfying the following properties:

- (a)  $[\lambda(p), \lambda(q)] = 0$  for loops  $p \in L_{o_1}$ ,  $q \in L_{o_2}$  with  $o_1 \perp o_2$ ;
- (b)  $\mathrm{ad}_{U(P)} \circ \lambda = \lambda \circ P$  for any  $P \in \mathcal{P}_+^\uparrow$ .

We refer the reader to [16] for the (easy) proof of the above 1-1 correspondence. Instead we focus on the relation between representations and connection systems.

A *connection system*<sup>6</sup> is a family  $u$  of maps

$$u_a : \Sigma_1^a \rightarrow \mathcal{UH} \quad , \quad a \in \Sigma_0 \quad ,$$

satisfying

$$u_a(\bar{b}) = u_a(b)^* \quad , \quad u_a(e_a) = \mathbb{1} \quad , \quad a \in \Sigma_0 \quad , \quad b \in \Sigma_1^a \quad . \quad (5.14) \quad \text{OCc:1}$$

We extend  $u$  to paths as follows: for any 0-simplex  $a$  and any path  $p = b_n \cdots b_1$  with  $b_i \in \Sigma_1^a$  for any  $i$ , we define

$$u_a(p) := u_a(b_n) \cdots u_a(b_2) u_a(b_1) \quad . \quad (5.15) \quad \text{OCc:2}$$

<sup>6</sup>The definition of connection system given here generalizes that introduced in [16] to a non-pathwise connected simplicial set. One can easily see that this coincides with the previous one when restricted to any connected component.

We say that  $u$  is *causal* whenever for any pair of loops  $p : a \rightarrow a$  and  $p' : a' \rightarrow a'$  such that  $p \in L_o$  and  $p' \in L_{o'}$  with  $o \perp o'$ , we have

$$[u_a(p), u_{a'}(p')] = 0 . \quad (5.16) \quad \boxed{\text{OCc:3}}$$

Notice that, in general,  $p$  and  $p'$  are loops in different connected components, i.e. they may have different smearing functions. We say that  $u$  is *covariant* whenever there is a unitary representation  $U : \mathcal{P}_+^\uparrow \rightarrow \mathcal{UH}$  such that

$$\text{ad}_{U(P)} \circ u_a = u_{Pa} \circ P , \quad P \in \mathcal{P}_+^\uparrow . \quad (5.17) \quad \boxed{\text{OCc:4}}$$

Covariant connection systems  $(u, U)$  and  $(u', U')$  are *equivalent* if there is a family of unitary mappings  $t = \{t_a : \Sigma_0^a \rightarrow \mathcal{U}(\mathcal{H}, \mathcal{H}') , a \in \Sigma_0\}$  satisfying

$$t_a(\partial_0 b) u_a(b) = u'_a(b) t_a(\partial_0 b) , \quad t_a(a') U(P) = U'(P) t_{Pa}(Pa') , \quad (5.18) \quad \boxed{\text{OCc:5}}$$

for any  $a \in \Sigma_0$ ,  $b \in \Sigma_1^a$ ,  $a' \in \Sigma_0^a$  and  $P \in \mathcal{P}_+^\uparrow$ .

$\boxed{\text{OCc:6}}$

**Lemma 5.3.** *There exists, up to equivalence, a 1-1 correspondence between causal and covariant representations of the group of loops  $L(\mathbb{R}^4)$  and causal and covariant connection systems.*

*Proof.* Let  $(\lambda, U)$  be a causal and covariant representation of  $L(\mathbb{R}^4)$  and  $\mathcal{P}$  a path-frame system. We define, for any path-frame  $\mathcal{P}_a$ ,

$$u_{\mathcal{P}_a}^\lambda(b) := \lambda\left(\bar{p}_{(a, \partial_0 b)} b p_{(a, \partial_1 b)}\right) , \quad b \in \Sigma_1^a , \quad (5.19) \quad \boxed{\text{OCc:7}}$$

where  $\bar{p}_{(a, \partial_0 b)}$  denotes the opposite of  $p_{(a, \partial_0 b)}$ . By definition we have  $u_{\mathcal{P}_a}^\lambda(\bar{b}) = u_{\mathcal{P}_a}^\lambda(b)^*$  and  $u_{\mathcal{P}_a}^\lambda(e_a) = \mathbb{1}$ . If  $p$  is a loop over  $a$ ,  $p = b_n \cdots b_1$ , then

$$\begin{aligned} u_{\mathcal{P}_a}^\lambda(p) &= u_{\mathcal{P}_a}^\lambda(b_n) \cdots u_{\mathcal{P}_a}^\lambda(b_2) u_{\mathcal{P}_a}^\lambda(b_1) \\ &= \lambda\left(\bar{p}_{(a, \partial_0 b_n)} b_n p_{(a, \partial_1 b_n)}\right) \cdots \lambda\left(\bar{p}_{(a, \partial_0 b_2)} b_2 p_{(a, \partial_1 b_2)}\right) \lambda\left(\bar{p}_{(a, \partial_0 b_1)} b_1 p_{(a, \partial_1 b_1)}\right) \\ &= \lambda\left(\bar{p}_{(a, \partial_0 b_n)} b_n p_{(a, \partial_1 b_n)} \cdots \bar{p}_{(a, \partial_0 b_2)} b_2 p_{(a, \partial_1 b_2)} \bar{p}_{(a, \partial_0 b_1)} b_1 p_{(a, \partial_1 b_1)}\right) \\ &= \lambda(p) , \end{aligned}$$

because the reduced loop of  $\bar{p}_{(a, \partial_0 b_n)} b_n p_{(a, \partial_1 b_n)} \cdots \bar{p}_{(a, \partial_0 b_2)} b_2 p_{(a, \partial_1 b_2)} \bar{p}_{(a, \partial_0 b_1)} b_1 p_{(a, \partial_1 b_1)}$  is  $p$  since  $\partial_0 b_i = \partial_1 b_{i+1}$ . From this and the property (b) of  $\lambda$ , causality for  $u_{\mathcal{P}}^\lambda$  follows. The covariance of  $(u_{\mathcal{P}}^\lambda, U)$  follows by the one of  $\lambda$  and of  $\mathcal{P}$  through a direct computation.

Conversely, let  $(u, U)$  be a causal and covariant connection system. Define

$$\lambda^u(p_n \cdots p_2 p_1) := u_{a_n}(p_n) \cdots u_{a_2}(p_2) u_{a_1}(p_1) , \quad p_i : a_i \rightarrow a_i , i = 1, \dots, n . \quad (5.20) \quad \boxed{\text{OCc:8}}$$

By the properties of a connection system, it is easily seen that  $(\lambda^u, U)$  is a causal and covariant representation of  $L(\mathbb{R}^4)$ . Notice also that

$$u_{\mathcal{P}_a}^{\lambda^u}(b) = \lambda^u(\bar{p}_{(a, \partial_0 b)} b p_{(a, \partial_1 b)}) = u_a(\bar{p}_{(a, \partial_0 b)}) u_a(b) u_a(p_{(a, \partial_1 b)}) .$$

So, if we set  $t_a(a') := u_a(p_{(a, a')})$  for any  $a' \in \Sigma_0^a$ , we have that  $t_a(\partial_0 b) u_a^{\lambda^u}(b) = u_a(b) t_a(\partial_1 b)$  and

$$\text{ad}_{U(Q)}(t_a(a')) = \text{ad}_{U(Q)}(u_a(p_{(a, a')})) = u_{Qa}(p_{Qa, Qa'}) = t_{Qa}(Qa') ,$$

for any  $Q \in \mathcal{P}_+^\uparrow$ . Hence  $u$  and  $u_{\mathcal{P}}^\lambda$  are equivalent. Conversely, given a representation of the group of loops  $\lambda$  and a loop  $p : a \rightarrow a$ ,  $p = b_n \dots b_1$ , we have

$$\lambda^{u_{\mathcal{P}}^\lambda}(p) = u_{\mathcal{P}_a}^\lambda(p) = \lambda \left( \overline{p}_{(\partial_0 b_n, a)} b_n p_{(\partial_1 b_n, a)} \cdots \overline{p}_{(\partial_0 b_2, a)} b_2 p_{(\partial_1 b_2, a)} \overline{p}_{(\partial_0 b_1, a)} b_1 p_{(\partial_1 b_1, a)} \right) = \lambda(p) .$$

So we have, up to equivalence, a 1-1 correspondence.  $\square$

It is worth pointing out that (5.19) can be seen as the combinatorial counterpart of the analytical procedure outlined in Subsection 5.4, which associates the primitive 1-form to a closed 2-form. We shall see in Subsection 5.4 that these two procedures agree.

**Gauge transformations.** Following [16], gauge transformations are now introduced as transformations sending a connection system to an equivalent connection system. More precisely, a *gauge transformation* of a causal and covariant connection system  $(u, U)$  is a family

$$g := \{g_a : \Sigma_0^a \rightarrow \mathcal{U}\mathcal{H} \ , \ a \in \Sigma_0\} , \quad (5.21) \quad \text{eq. gauge}$$

such that

- (a)  $\text{ad}_{U(P)}(g_a(a')) = g_{Pa}(Pa')$  for any  $a' \in \Sigma_0^a$  and  $P \in \mathcal{P}_+^\uparrow$ ;
- (b)  $\text{ad}_{g_a(a)}(u(\mathcal{A}_{\tilde{a}})) = u(\mathcal{A}_{\tilde{a}})$  for any  $\tilde{a} \supseteq a$ ,

where  $u(\mathcal{A}_{\tilde{a}})$  is the image of  $\mathcal{A}_{\tilde{a}}$  under the representation defined by  $u$ . Gauge transformations form a group  $\mathcal{G}^u$  under the multiplication  $(g \cdot \tilde{g})_a(a') := g_a(a') \tilde{g}_a(a')$ . We call  $\mathcal{G}^u$  the group of *gauge transformations* of  $(u, U)$ .

Let now  $g \in \mathcal{G}^u$ . Then the pair  $(u^g, U)$ , where

$$u_a^g(b) := g_a(\partial_0 b) u_a(b) g_a(\partial_1 b)^* , \quad b \in \Sigma_1^a , \quad (5.22) \quad \text{OCc:9}$$

is a causal and covariant connection system equivalent to  $(u, U)$ . In fact covariance is clear whilst, concerning causality, if  $p : a \rightarrow a$  is a loop then

$$u_a^g(p) = u_a^g(b_n) \cdots u_a^g(b_2) u_a^g(b_1) = g_a(a) u_a(p) g_a(a)^* .$$

So by property (b) of a gauge transformation we have that  $u_a^g$  is also causal.

It is worth pointing out that the degree of freedom in choosing different path-frame systems results to be a gauge transformation. In fact, let  $(\lambda, U)$  be a causal and covariant representation of  $L(\mathbb{R}^4)$  and let  $\mathcal{P}$  and  $\mathcal{Q}$  be covariant path-frame systems. Moreover, let

$$u_{\mathcal{P}_a}^\lambda(b) = \lambda \left( \overline{p}_{(a, \partial_0 b)} b p_{(a, \partial_1 b)} \right) , \quad u_{\mathcal{Q}_a}^\lambda(b) = \lambda \left( \overline{q}_{(a, \partial_0 b)} b q_{(a, \partial_1 b)} \right)$$

be the connection systems associated, *via*  $\lambda$ , to the path-frame systems  $\mathcal{P}$  and  $\mathcal{Q}$ . Then, defining

$$g_a(a') := \lambda(q_{(a, a')} \overline{p}_{(a, a')}) , \quad a' \in \Sigma_0^a , \quad (5.23) \quad \text{OCc:10}$$

it is easily seen that  $g$  is a gauge transformation of  $(u_{\mathcal{P}}^\lambda, U)$  such that  $u_{\mathcal{P}}^{\lambda, g} = u_{\mathcal{Q}}^\lambda$ . Notice, in particular, that  $g_a(a) = 1$  for any  $a \in \Sigma_0$ , so the action (b) on  $u(\mathcal{A}_{\tilde{a}})$  is trivial.

**Causal and covariant 2-cochains.** We now give an equivalent description of representations of the net of causal loops on  $\mathbb{R}^4$  in terms of 2-cochains. The idea is to observe that any connected component  $\Sigma_*^a \subset \Sigma_*$ ,  $a \in \Sigma_0$ , is contractible. So we shall see, *via* the cone construction, that any loop admits a natural “triangulation” in terms of 2-simplices. As anticipated this is a key result since, as we shall see in the next section, it allows the electromagnetic field to induce representations of the net of causal loops.

From now on it will be useful to switch from the description of simplices in terms of vertices and faces to the parametric description, and conversely. To begin with, let us introduce the following notion:

OCc:14

**Definition 5.4.** A *causal and covariant 2-cochain* is a pair  $(w, U)$ , where  $U : \mathcal{P}_+^\uparrow \rightarrow \mathcal{UH}$  is a unitary representation and  $w : \Sigma_2 \rightarrow \mathcal{UH}$  is a 2-cochain satisfying the properties

(a)  $w(c)^* = w(\bar{c})$  and  $w(c) = 1$  if  $c$  is degenerated;

(b)  $[w(c_1), w(c_2)] = 0$  if  $|c_1| \perp |c_2|$ ;

(c)  $\text{ad}_{U(P)} \circ w = w \circ P$ , for any  $P \in \mathcal{P}_+^\uparrow$ ,

where in (b) the symbol  $|\cdot|$  refers to the support of a 2-simplex, equation [\(3.4\)](#) <sup>OA:14</sup>.

We need a preliminary observation in order to prove that such 2-cochains provide an equivalent description of the representations of the net of causal loops.

Let  $a = (a_0, f) \in \Sigma_0$ . For any  $b = (r_b, f) \in \Sigma_1^a$ , written in parametric form, we consider the smearing 2-simplex  $h^a(b) \in \Sigma_2^a$  defined by the cone construction [\(A.3\)](#) <sup>OA:8a</sup>. Namely

$$h^a(b) := (h^{a_0}(r_b), f) \quad , \quad b = (r_b, f) \in \Sigma_1^a . \quad (5.24) \quad \text{OCc:14a}$$

Note that  $h^a(b)$  is characterized as the unique smearing 2-simplex having faces

$$\partial_0 h^a(b) = b \quad , \quad \partial_2 h^a(b) = e_{(\partial_1 b, a)} \quad , \quad \partial_1 h^a(b) = e_{(\partial_0 b, a)} . \quad (5.25) \quad \text{OCc:14b}$$

Equivalently, this is the unique 2-simplex whose path-boundary is the loop  $\partial h^a(b) = \bar{e}_{(\partial_0 b, a)} b e_{(\partial_1 b, a)}$ . This implies that  $h^a(\bar{b}) = \overline{h^a(b)}$  for any  $b \in \Sigma_1^a$ , since  $\partial h^a(\bar{b}) = \bar{e}_{(\partial_1 b, a)} \bar{b} e_{(\partial_0 b, a)} = \overline{\partial h^a(b)}$ .

On these grounds, given a causal and covariant 2-cochain  $(w, U)$ , define

$$\lambda^w(p) := w(h^a(b_n)) \cdots w(h^a(b_1)) \quad , \quad p = b_n \cdots b_1 : a \rightarrow a . \quad (5.26) \quad \text{OCc:15}$$

This definition is well posed because, as observed before,  $h^a(\bar{b}) = \overline{h^a(b)}$ , so  $\lambda^w(p)$  is independent of the reduction of the loop  $p$ . For the same reason we have

$$\lambda^w(\bar{p}) = \lambda^w(p)^* \quad , \quad \lambda^w(pq) = \lambda^w(p)\lambda^w(q) \quad , \quad p, q : a \rightarrow a \quad ,$$

and this implies that  $\lambda^w$  is a representation of the group of loops.

Given a loop  $p : a \rightarrow a$ ,  $p = b_n \cdots b_1$ , we can assume without loss of generality that it is reduced. If  $|b_i| \subseteq o$  for any  $i$ , then  $|h^a(b_i)| \subseteq o$  for any  $i$  since double cones are convex, and this implies that  $\lambda^w$  is causal. Finally, covariance follows from the transformation properties of  $\Sigma_*$ , and we conclude that  $(\lambda^w, U)$  is causal and covariant.

Conversely, it is easily seen that if  $(\lambda, U)$  is a causal and covariant representation of  $L(\mathbb{R}^4)$ , then the pair  $(w^\lambda, U)$ , where

$$w^\lambda(c) := \lambda(\partial c) \quad , \quad c \in \Sigma_2, \quad (5.27) \quad \text{OCc:16}$$

is a causal and covariant 2-cochain. This correspondence is 1-1: in fact, if  $c$  is a 2-smearing simplex with path-boundary  $\partial c : a \rightarrow a$ , then

$$w^{\lambda^w}(c) = \lambda^w(\partial c) = w(h^a(\overline{\partial_1 c})) w(h^a(\partial_0 c)) w(h^a(\partial_2 c)) = w(h^a(\partial_0 c)) = w(c) ,$$

because, as can be easily seen by (b.25),  $h^a(\overline{\partial_1 c})$  and  $h^a(\partial_2 c)$  are degenerated 2-simplices ( $a$  being a vertex of both  $\partial_1 c$  and  $\partial_2 c$ ) and  $h^a(\partial_0 c) = c$ . On the other hand, for any loop  $p : a \rightarrow a$ ,  $p = b_n \cdots b_1$ , we have, using a calculation similar to that used in Lemma 5.3,

$$\lambda^{w^\lambda}(p) = w^\lambda(h^a(b_n)) \cdots w^\lambda(h^a(b_1)) = \lambda(\partial h^a(b_n)) \cdots \lambda(\partial h^a(b_1)) = \lambda(p) .$$

The results of the present section are summed up in the following theorem.

OCc:17

**Theorem 5.5.** *There exists, up to equivalence, a 1-1 correspondence between:*

- (i) *Covariant representations  $(\pi, U)$  of the net of causal loops  $(\mathcal{A}, \alpha)_K$ ;*
- (ii) *Causal and covariant representations  $(\lambda, U)$  of the group of loops  $L(\mathbb{R}^4)$ ;*
- (iii) *Causal and covariant connection systems  $(u, U)$ ;*
- (iv) *Causal and covariant 2-cochains  $(w, U)$ .*

## 5.4 The electromagnetic field representation

OCe

We now show that the electromagnetic field induces a representation of the net of causal loops  $(\mathcal{A}, \alpha)_K$  in terms of a causal and covariant 2-cochain  $(w^{em}, U)$ . This yields the connection system  $u^{em}$  with the relative group of gauge transformations, and we study the relation between them and the analytical procedure associating to  $F_{\mu\nu}$  the potential system  $A_\mu$  with the corresponding gauge transformations. The result is that we find a complete coherence: the exponential of the line integral of  $A_\mu$  defines a connection system  $u^{pot}$  which turns out to be gauge-equivalent to  $u^{em}$ , and gauge transformations of  $A_\mu$  define gauge transformations of  $u^{em}$ .

**The electromagnetic 2-cochain.** We now show that the electromagnetic field defines a causal and covariant 2-cochain, hence a covariant representation of the net of causal loops. To this end we introduce the following ‘‘homological’’ deformation of a 2-simplex.

Let  $c = (\sigma_c, f) \in \Sigma_2^a$  be a 2-simplex written in the parametric form, where  $\sigma_c$  is the triangular surface associated with the vertices of  $c$ . Given  $a = (a_0, f)$ , a 0-simplex having the same smearing function as  $c$ , let  $h^a(c) = (h^{a_0}(\sigma_c), f)$  be the 3-simplex obtained via the cone construction. According to the definition (A.3) we have

$$\partial_0 h^a(c) = c , \quad \partial_1 h^a(c) = h^a(\partial_0 c) , \quad \partial_2 h^a(c) = h^a(\partial_1 c) , \quad \partial_3 h^a(c) = h^a(\partial_2 c) ,$$

and the smearing 2-chain

$$h^a(\partial c) = h^a(\partial_0 c) - h^a(\partial_1 c) + h^a(\partial_2 c) , \quad c \in \Sigma_2^a , \quad (5.28)$$

OCc:19

has boundary  $\partial c$  by (A.2).

Now, let us consider the electromagnetic field  $F_{\mu\nu}$  and the corresponding unitary representation  $U : \mathcal{P}_+^\uparrow \rightarrow \mathcal{UH}$  defined in Section 2. Using (B.10), we define

$$w^{em}(c) := \exp(iF \langle \sigma_c, f \rangle) \in \mathcal{UH} , \quad c \in \Sigma_2 . \quad (5.29)$$

OCc:21



OCc:22

**Theorem 5.6.** *The pair  $(w^{em}, U)$  is a causal and covariant 2-cochain, defining a covariant representation of the net of causal loops  $(\mathcal{A}, \alpha)_K$  and fulfilling the following causality property. Given  $c \in \Sigma_2^a$ ,  $c' \in \Sigma_2^{a'}$  with  $a = (a_0, f)$ ,  $a' = (a'_0, f') \in \Sigma_0$ , we have*

$$|h^a(\partial c)| \perp |h^{a'}(\partial c')| \Rightarrow [w^{em}(c), w^{em}(c')] = 0, \quad (5.30)$$

eq.link

where  $|h^a(\partial c)|, |h^{a'}(\partial c')|$  denote the supports of the smearing 2-chains  $h^a(\partial c)$ ,  $h^{a'}(\partial c')$  respectively, see (3.5).

*Proof.* According to the Definition 5.4, the first part of the statement follows from Lemma 3.1 and Theorem 5.5. Concerning the second statement, we observe that, given  $c = (\sigma_c, f) \in \Sigma_2^a$ , the surface  $\sigma_c$  and the 2-chain  $h^{a_0}(\partial c)$  have the same boundary  $\partial c$ . So applying (4.11) we have  $w^{em}(c) = \exp(iF(h^{a_0}(\partial c), f))$ , and the proof follows by Lemma 3.1(iii).  $\square$

The latter property of  $w^{em}$  stated in the previous theorem is a refinement of the causality used in [16], in fact it does not involve double cones containing  $|c|, |c'|$ . It is easily seen that (5.30) applies to 2-simplices  $c, c'$  whose boundaries are causally disjoint and form a trivial link, so the previous theorem says that  $w^{em}(c)$  and  $w^{em}(c')$  commute in accord with the considerations in [37, Section 1].

**Connection systems and gauge transformations.** We have already pointed out that the combinatorial procedure used to extract the connection system from a 2-cochain is analogous to the analytical one, used to define the primitive 1-form starting from a closed 2-form. Now our aim is to prove that these two procedures are, up to equivalence, the same. To be precise, we shall compare the connection system associated with  $w^{em}$  with the one defined by the exponential of the line integral of  $A_\mu$ : we shall prove that these are the same, up to a gauge transformation in the sense of (5.21).

Consider the causal and covariant representation  $(\lambda^{em}, U)$  of the group of loops associated with  $(U, F)$ . This is defined, according to (5.26), by

$$\lambda^{em}(p) = w^{em}(h^a(b_n)) \cdots w^{em}(h^a(b_1)), \quad p = b_n \cdots b_1 : a \rightarrow a. \quad (5.31)$$

OCc:23

**Remark 5.7.** It is worth observing that Theorem 5.6 says that  $\lambda^{em}$  is *localized on loops which are path-boundaries of 2-simplices*: given two loops  $\partial c$  and  $\partial c'$ ,  $c, c' \in \Sigma_2$ , having causally disjoint supports and forming a trivial link, from Theorem 5.6 we obtain

$$[\lambda^{em}(\partial c), \lambda^{em}(\partial c')] = 0. \quad (5.32)$$

OCc:24

In fact, since the loops are not linked together, it is possible to find  $a$  and  $a'$  as in the recalled theorem and  $\lambda^{em}(\partial c) = w^{em}(c)$  by definition. Notice that in the case of free fields the relation (5.32) should also hold for arbitrary loops, because the composition of two Weyl operators gives the sum of the exponents up to a phase factor. So, if  $p$  is the boundary of a 2-surface, then  $\lambda^{em}(p)$  corresponds to the exponential of the integral of the fields on this surface up to some phase factor. Since phase factors do not affect commutativity, (5.32) should hold.

Let  $\mathcal{P}$  denote a covariant path-frame system. We consider the causal and covariant connection system  $(u_{\mathcal{P}}^{em}, U)$  associated with  $(\lambda^{em}, U)$  which is defined, according to (5.19), by

$$u_{\mathcal{P}_a}^{em}(b) = \lambda^{em}(\bar{p}_{(a, \partial_0 b)} p_{(a, \partial_1 b)}), \quad a \in \Sigma_0, b \in \Sigma_1^a. \quad (5.33)$$

OCc:25

As observed in Subsection <sup>Cc</sup>5.3, a changing of the path-frame system leads to an equivalent connection system by means of a gauge transformation. In particular, considering the Euclidean path-frame system  $\mathcal{E}$  and a generic one  $\mathcal{P}$  we have

$$g_a(\partial_0 b) u_{\mathcal{E}_a}^{em}(b) = u_{\mathcal{P}_a}^{em}(b) g_a(\partial_1 b), \quad b \in \Sigma_1^a, \quad (5.34) \quad \text{OCc: 25a}$$

where

$$g_a(a') := \lambda^{em}(p_{(a,a')} \bar{e}_{(a,a')}) , \quad a' \in \Sigma_0^a. \quad (5.35) \quad \text{OCc: 25b}$$

Concerning the connection system  $u_{\mathcal{E}}^{em}$ , since  $h^a(e_{(a,\partial_i b)})$ ,  $i = 0, 1$ , is a degenerated 2-simplex, applying (5.31) and (5.33) we find  $\lambda^{em}(\bar{e}_{(a,\partial_0 b)} b e_{(a,\partial_1 b)}) = w^{em}(h^a(b))$  and this implies

$$u_{\mathcal{E}_a}^{em}(b) = \exp(iF \langle h^{a_0}(r_b), f \rangle), \quad b \in \Sigma_1^a. \quad (5.36) \quad \text{eq. uF}$$

We now construct a connection system starting from the potential system  $A_\mu$ . We begin by defining a family  $u^{pot} := \{u_a^{pot} : \Sigma_1^a \rightarrow \mathcal{UH}\}$  of unitary 1-cochains, for  $a = (a_0; f) \in \Sigma_0$ , as

$$u_a^{pot}(b) := \exp(iA^{a_0} \langle r_b, f \rangle), \quad b = (r_b, f) \in \Sigma_1^a. \quad (5.37)$$

It is possible to check directly that  $(u^{pot}, U)$  is a causal and covariant connection system, but we do not need to do that because  $u^{pot} = u_{\mathcal{E}}^{em}$ . In fact, Proposition <sup>OBb: 16</sup>B.1 implies  $A^{a_0} \langle r_b, f \rangle = F \langle h^{a_0}(r_b), f \rangle$ , so by (5.36) we have  $u_{\mathcal{E}}^{em} = u^{pot}$  as desired. In conclusion, the equation <sup>OCc: 25a</sup>5.34 gives:

equivalence

**Theorem 5.8.** *Let  $(U, F)$  denote a quantum electromagnetic field. Under the above notations, for any covariant path-frame system  $\mathcal{P}$  we have*

$$(u_{\mathcal{P}}^{em}, U) \cong (u_{\mathcal{E}}^{em}, U) = (u^{pot}, U), \quad (5.38)$$

where  $\mathcal{E}$  is the Euclidean path-frame system and the equivalence  $\cong$  is realized by the gauge transformation <sup>OCc: 25b</sup>(5.35).

In words, the abstract connection system  $u_{\mathcal{P}}^{em}$  coincides with the one obtained as the exponential of the line integral of  $A_\mu$ , up to gauge transformations.

**Remark 5.9.** Two observation are in order.

1. If  $g = \{g^z\}_{z \in \mathbb{R}^4} \in \mathcal{G}$  is a gauge transformation of  $A_\mu$  in the sense of Subsection <sup>OBb</sup>4, then setting

$$g_a^u(a') := \exp(ig^{a_0}(a'_0, f)), \quad a' = (a'_0, f) \in \Sigma_0^a,$$

we easily find that  $g^u$  is a gauge transformation of  $u^{pot}$  leaving any local algebra  $\mathcal{A}_o$ ,  $o \in K$ , pointwise fixed.

2. Let  $a = (a_0; f) \in \Sigma_1$  and  $b, b' \in \Sigma_1^a$  with  $\partial_1 b_1 = \partial_0 b'_1$ ; then the composition of the corresponding lines  $r_b$  and  $r_{b'}$  yields the curve  $r_b * r_{b'}$ . It is worth to stress that, also in the simpler case of *free* electromagnetic field, the operator  $\exp(iA^{a_0} \langle r_b * r_{b'}, f \rangle)$  is different from the one obtained by the combinatorial product of the two single paths,

$$u^{pot}(bb') = u^{pot}(b)u^{pot}(b') = \exp(iA^{a_0} \langle r_b, f \rangle) \cdot \exp(iA^{a_0} \langle r_{b'}, f \rangle).$$

Hence, path composition is not preserved by applying the exponential.

## 6 Concluding remarks

In a previous paper <sup>CRV</sup> [16] we used loops in a globally hyperbolic space-time to generate a causal and covariant net of  $C^*$ -algebras, called the net of causal loops. We presented some of its representations in terms of connection systems, i.e. families of “abstract” connections fulfilling causality and covariance as properties of the family and not of a single connection. Local gauge transformations were defined as maps between equivalent connection systems, leaving element-wise invariant the loop algebras. The connections are recovered by the representations using a path-frame: a choice of paths joining any point of the space-time with a fixed point, the pole. Letting vary the pole yields the connection system.

In the present paper, using only the (possibly charged) quantum electromagnetic field  $F_{\mu\nu}$  we reconstructed a potential 1-form  $A_\mu^z$  with reference to the pole  $z \in \mathbb{R}^4$ , i.e. the centre of a contracting homotopy of the Minkowski space-time. Actually this homotopy corresponds to the abstract path-frame used to define a connection. According to different choices of  $z$  a potential system  $A_\mu = \{A_\mu^z, z \in \mathbb{R}^4\}$  is obtained, and a series of outcomes that agree with the abstract formulation given by the authors in <sup>CRV</sup> [16] follows. In particular,  $A_\mu$  gives a connection system and, in turns, a covariant representation of the net of causal loops. Furthermore, *local* gauge transformations defined in terms of the potential system  $A_\mu$  coincide with the ones defined in terms of the corresponding connection system.

For these results, an important outlook is the comparison with the global conditions defining the charge classes for theories with long-range interaction developed in <sup>DR</sup> [13]. In that paper a result of DR-duality type is obtained for the case of simple charge classes, giving a *global* Abelian gauge group. In particular, motivated by sound physical reasons, the charge classes are defined over a time-like cone in the Minkowski space-time, using a family of its subsets called hypercones.

To better understand these dual aspects of long-range interactions and local gauge theory on loops, it seems to be useful to study the geometric facets of the two cases, e.g. the choice of a time-like cone with its apex and hypercones on the one side, and of a path-frame with its pole on the other side.

In this regard, a hopeful hang is the realization <sup>due</sup> for QED of an abstract connection in terms of the charge transporter proposed in equation (I.2), in fact the presence of the charged massive field  $\psi$  and of the massless potential form  $A_\mu^z$  should make possible to shift charges to infinity. Compatibly with the two frameworks, the pole  $z$  may reveal the choice of a future time-like cone.

Under a different perspective, it may be of interest to explore the relation between the approach of this paper and others, e.g. TQFT, referring to the interpretation of observables localized on loops.

## A Singular simplices and integration

OAb

The suited language for dealing with the Stokes’ theorem is that of the algebraic topology. We start by introducing the simplicial set of singular piecewise smooth simplices in  $\mathbb{R}^4$ . Given the *standard  $n$ -simplex*  $\Delta_n := \{(t_1, t_2, \dots, t_n) \in \mathbb{R}^n \mid t_i \geq 0, \sum_{i=1}^n t_i \leq 1\}$ , a *singular piecewise smooth  $n$ -simplex*  $\chi$  is a piecewise smooth map  $\chi : \Delta_n \rightarrow \mathbb{R}^4$ . The vertices of  $\chi$  are the images of the vertices of  $\Delta_n$ : so, given the canonical base  $\{e_k\}$  of  $\mathbb{R}^n$  and  $i \in \{0, \dots, n\}$ , the  *$i$ -vertex* of  $\chi$  is defined by  $\chi(e_i)$  when  $i \in \{1, \dots, n\}$ , and by  $\chi(0)$  when  $i = 0$ . The order of the vertices endows  $\chi$  with a natural orientation. We say that an  $n$ -simplex  $\chi'$  has *opposite orientation* with respect to  $\chi$  whenever  $\chi' = \chi \circ T$ , where  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an affine transformation making an odd permutation of the vertices of  $\Delta_n$ . Denoting the set of singular piecewise smooth  $n$ -simplices by

$\Sigma_n(\mathbb{R}^4)$ , the *face maps*  $\partial_i : \Sigma_n(\mathbb{R}^4) \rightarrow \Sigma_{n-1}(\mathbb{R}^4)$  are defined by

$$\partial_i \chi(t_1, \dots, t_{n-1}) := \begin{cases} \chi(1 - \sum_{i=1}^{n-1} t_i, t_1, t_2, \dots, t_{n-1}), & i = 0, \\ \chi(t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}), & i > 0. \end{cases} \quad (\text{A.1}) \quad \boxed{\text{0A:5a}}$$

One can easily see that the collection  $\Sigma_*(\mathbb{R}^4) = \{\Sigma_n(\mathbb{R}^4), n \in \mathbb{N}\}$ , is a simplicial set.

**Singular Homology.** We denote the  $\mathbb{Z}$ -module of *singular  $n^{\text{th}}$ -chains* by  $C_n(\mathbb{R}^4)$ : this is the set of finite formal linear combinations  $\sum_i m_i \chi_i$  with  $m_i \in \mathbb{Z}$  and  $\chi_i \in \Sigma_n(\mathbb{R}^4)$ . The *boundary* of  $\chi \in \Sigma_{n+1}(\mathbb{R}^4)$  is the  $n$ -chain  $\partial\chi = \sum_{i=0}^{n+1} (-1)^i \partial_i \chi$ , and an  *$n$ -cycle* is an  $n$ -simplex  $\chi$  such that  $\partial\chi = 0$ . Now, as well known, since  $\partial \circ \partial = 0$  it turns out that *the set  $B_n(\mathbb{R}^4)$  of boundaries* of  $C_{n+1}(\mathbb{R}^4)$  is a submodule of *the set  $Z_n(\mathbb{R}^4)$  of  $n$ -cycles*, and  $H_n(\mathbb{R}^4) := Z_n(\mathbb{R}^4)/B_n(\mathbb{R}^4)$  is the  $n$ -th module of singular homology of  $\mathbb{R}^4$ , which vanishes for  $n > 0$ . The proof relies on the existence of contracting homotopies for  $\mathbb{R}^4$ . The easiest example is the *cone construction* which associates to any  $z \in \mathbb{R}^4$  a map  $h^z : C_n(\mathbb{R}^4) \rightarrow C_{n+1}(\mathbb{R}^4)$  such that

$$\partial \circ h^z + h^z \circ \partial = id_{C_n(\mathbb{R}^4)}. \quad (\text{A.2}) \quad \boxed{\text{eq. h}}$$

Explicitly,  $h^z$  is defined on  $\chi \in \Sigma_n(\mathbb{R}^4) \subset C_n(\mathbb{R}^4)$  by

$$h^z \chi(t_1, \dots, t_n, t_{n+1}) := \left( \sum_{i=1}^{n+1} t_i \right) \chi \left( \frac{t_2}{\sum_{i=1}^{n+1} t_i}, \dots, \frac{t_{n+1}}{\sum_{i=1}^{n+1} t_i} \right) + \left( 1 - \sum_{i=1}^{n+1} t_i \right) z, \quad (\text{A.3}) \quad \boxed{\text{0A:8a}}$$

and extended by linearity on  $n$ -chains. In particular  $\partial_0 h^z \chi = \chi$  and  $\partial_i h^z \chi = h^z \partial_{i-1} \chi$  for  $i > 0$ . This implies that any  $\chi \in C_1(\mathbb{R}^4)$  which is a closed curve ( $\chi(0) = \chi(1)$ ) is the boundary of the 2-simplex  $h^z \chi$ .

**Integration.** Given a smooth  $n$ -form  $\omega = \sum_{\alpha_1, \dots, \alpha_n} \omega_{\alpha_1 \dots \alpha_n} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_n}$ , where the indices  $\alpha_1, \dots, \alpha_n$  vary independently in  $0, 1, 2, 3$ , the integral over an  $n$ -simplex is defined by

$$\int_{\chi} \omega := \int_{\Delta_n} \omega_{\alpha_1 \dots \alpha_n}(\chi(t)) \chi^{\alpha_1 \dots \alpha_n}(t) d^n t, \quad \chi \in \Sigma_n(\mathbb{R}^4), \quad (\text{A.4}) \quad \boxed{\text{0A:7}}$$

where  $\chi^{\alpha_1 \dots \alpha_n}(t) := \det \left( \frac{\partial \chi_{\alpha_k}}{\partial t_k} \right)$  with  $h, k = 1, \dots, n$ . The Stokes' theorem hence reads as follows: given an  $(n-1)$ -form  $\omega$  and  $\chi \in \Sigma_n(\mathbb{R}^4)$ , then

$$\int_{\chi} d\omega = \int_{\partial\chi} \omega = \sum_{i=0}^n (-1)^i \int_{\partial_i \chi} \omega, \quad \chi \in \Sigma_n(\mathbb{R}^4). \quad (\text{A.5}) \quad \boxed{\text{0A:8}}$$

If  $\varphi = \sum_j m_j \chi_j \in C_n(\mathbb{R}^4)$  and  $\omega$  is an  $n$ -form, then  $\int_{\varphi} \omega := \sum_j m_j \int_{\chi_j} \omega$  and the Stokes' theorem extends to chains by linearity.

In what follows we shall deal mainly with singular 0-, 1-, 2- simplices in  $\mathbb{R}^4$ : a 0-simplex is a point  $x \in \mathbb{R}^4$ ; a 1-simplex is a piecewise smooth curve  $\gamma$ ; a 2-simplex is a piecewise smooth surface  $\sigma$ . Note that the boundary of a curve  $\gamma$  is the 0-chain  $\partial\gamma = \partial_0\gamma - \partial_1\gamma$  that in terms of vertices is  $\gamma(1) - \gamma(0)$ . The boundary of a surface  $\sigma$  is the 1-chain  $\partial\sigma = \partial_0\sigma - \partial_1\sigma + \partial_2\sigma$ . According to the above definition of orientation, we note that there is only one curve having opposite orientation of a given curve  $\gamma$ ; this is the curve  $\bar{\gamma}(t) = \gamma(1-t)$ ,  $t \in \Delta_1$ , that we call the *opposite* of  $\gamma$ . Note that  $\partial\bar{\gamma} = -\partial\gamma$ . Instead in the case of surfaces the opposite is not unique:

as a convention, we call the *opposite* of  $\sigma$  the surface  $\bar{\sigma} := \sigma \circ T$ , where  $T := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  is written in the canonical base of  $\mathbb{R}^2$ . The boundary of  $\bar{\sigma}$  results to be the 1-chain

$$\partial \bar{\sigma} = \partial_0 \bar{\sigma} - \partial_1 \bar{\sigma} + \partial_2 \bar{\sigma} = -\partial_0 \sigma - \partial_2 \sigma + \partial_1 \sigma = -\partial \sigma . \quad (\text{A.6}) \quad \boxed{\text{OA:9}}$$

Concerning the integration over singular simplices, we observe that the infinitesimal line element of  $\gamma \in \Sigma_1(\mathbb{R}^4)$  is  $\dot{\gamma}^\mu(t)dt$ ; the infinitesimal surface element of  $\sigma \in \Sigma_2(\mathbb{R}^4)$  is  $\sigma^{\mu\nu}(t)d^2t$ , where

$$\sigma^{\mu\nu}(t) = \left( \frac{\partial \sigma^\mu}{\partial t_1} \frac{\partial \sigma^\nu}{\partial t_2} - \frac{\partial \sigma^\mu}{\partial t_2} \frac{\partial \sigma^\nu}{\partial t_1} \right) (t) , \quad t \in \Delta_2 . \quad (\text{A.7}) \quad \boxed{\text{OA:10}}$$

Note that  $\dot{\gamma}^\mu$  and  $\partial \sigma^\mu / \partial t_k$  may have discontinuity points, nevertheless any of them is bounded.

## B Line integrals *via* the cone construction

**APP.cone**

We show that the line integral  $\int_{\gamma} f$  can be equivalently defined by using the cone construction. Given a smearing curve  $(\gamma, f) \in \Sigma_1(\mathbb{R}^4, \mathcal{S})$ , we consider the smearing surface  $h^z(\gamma, f) := (h^z \gamma, f) \in \Sigma_2(\mathbb{R}^4, \mathcal{S})$ , where  $h^z$  is defined by  $\int_{\gamma} f$ . Explicitly

$$h^z \gamma(t, s) := (t+s)\gamma\left(\frac{s}{t+s}\right) + (1-t-s)z , \quad (t, s) \in \Delta_2 , \quad (\text{B.1}) \quad \boxed{\text{OBb:15}}$$

and, as can be verified by a direct calculation, the surface element  $(h^z \gamma)^{\mu\nu}(t, s) ds dt$  is given by

$$(h^z \gamma)^{\mu\nu}(t, s) = \left( \gamma\left(\frac{s}{t+s}\right) - z \right)^\mu \dot{\gamma}^\nu\left(\frac{s}{t+s}\right) - \left( \gamma\left(\frac{s}{t+s}\right) - z \right)^\nu \dot{\gamma}^\mu\left(\frac{s}{t+s}\right) . \quad (\text{B.2}) \quad \boxed{\text{OBb:15a}}$$

**OBb:16**

**Proposition B.1.** For any  $z \in \mathbb{R}^4$  and any smearing curve  $(\gamma, f) \in \Sigma_1(\mathbb{R}^4, \mathcal{S})$  we have

$$A^z \langle \gamma, f \rangle = F \langle h^z \gamma, f \rangle .$$

*Proof.* Let  $\gamma_f^{z, \mu\nu}$  be the 2-tensor defined by  $\int_{\gamma} f$ . We change the variable as  $s_1 := ts$  and  $t_1 := t(1-s)$ . Then  $t = t_1 + s_1$ ,  $s = s_1/(t_1 + s_1)$  and the Jacobian of this transformation equals  $1/(t_1 + s_1)$ . Hence

$$\begin{aligned} \gamma_f^{z, \mu\nu}(x) &= \int_{\Delta_2} \left( \gamma^\mu\left(\frac{s_1}{t_1 + s_1}\right) - z^\mu \right) \dot{\gamma}^\nu\left(\frac{s_1}{t_1 + s_1}\right) f_{z+(t_1+s_1)\left(\gamma\left(\frac{s_1}{s_1+t_1}\right)-z\right)}(x) ds_1 dt_1 \\ &= \int_{\Delta_2} \left( \gamma\left(\frac{s_1}{t_1 + s_1}\right) - z \right)^\mu \dot{\gamma}^\nu\left(\frac{s_1}{t_1 + s_1}\right) f_{h^z \gamma(t_1, s_1)}(x) ds_1 dt_1 . \end{aligned}$$

Recalling  $\int_{\gamma} f$  we get

$$\gamma_f^{z, \mu\nu}(x) - \gamma_f^{z, \nu\mu}(x) = \int_{\Delta_2} f_{h^z \gamma(t_1, s_1)}(x) (h^z \gamma)^{\mu\nu}(t_1, s_1) ds_1 dt_1 = h^z \gamma[f]^{\mu\nu}(x) ,$$

and, by antisymmetry of  $F_{\mu\nu}$ , we obtain

$$A^z \langle \gamma, f \rangle = F_{\mu\nu}(\gamma_f^{z, \mu\nu}) = \frac{1}{2} F_{\mu\nu}(\gamma_f^{z, \mu\nu} - \gamma_f^{z, \nu\mu}) = \frac{1}{2} F_{\mu\nu}(h^z \gamma[f]^{\mu\nu}) = F \langle h^z \gamma, f \rangle ,$$

completing the proof.  $\square$

Note that when  $\gamma$  is closed, Proposition B.1 is a particular case of Proposition 4.6.

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