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Highlights.

- We analyze the Parsimonious-Flooding Protocol on Mobile Opportunistic Networks yielded by the Geometric Random-Walk Model.
- For the first time, we provide analytical bounds on the completion time. Such bounds are optimal for a wide range of the network parameters.
- Departing significantly from previous analysis of the standard flooding protocol, our proof technique determines the geometric shape of the information spreading (i.e. the *infection wave*) over the time.

Parsimonious Flooding in Geometric Random-Walks*

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Abstract

We study the epidemic process yielded by the k -Flooding Protocol in geometric Mobile Ad-Hoc Networks. We consider n agents on a square performing independent random walks. At any time step, every active agent informs every non-informed agent which is within distance R from it. An informed agent is active only for k time steps. Initially, a source agent is informed and we look at the completion time of the protocol. We prove optimal bounds on the completion time of the process. Our method of analysis provides a clear picture of the geometric shape of the information spreading over the time.

Keywords: Information/Virus Spreading, Broadcasting Protocols, Geometric Random-Walks, Mobile Ad-Hoc Networks.

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1 Introduction

In the *Geometric Random-Walk Model* [7, 10, 12], n agents perform independent random walks over a square of side length L . We consider the following *epidemic* process performed by the agents. Every agent can be in three different states: non-informed (white), informed-active (red), informed-removed (black). During a time step, every agent performs one step of the random walk and every red agent informs (infects) all white agents lying within distance R . A white agent, informed (for the first time) at time step t , becomes red at time step $t + 1$. When an agent becomes red, he stays that for the next k time steps and, then, he becomes black and stays that forever.

At the initial time step, a source agent is in the red state. The completion time of the above infection process is the first time step in which every agent gets into the black state. If this time step does not exist then we say that the infection process does not complete.

Such a random process is inspired by two main scenarios: The *Susceptible-Infected-Removed (SIR)* process which is widely studied in Mathematical Epidemiology [1, 2, 4, 14, 20] and the (*Parsimonious*) *k-Flooding Protocol* [3] in opportunistic networks such as *geometric Mobile Ad-Hoc Networks (MANET)*.

While the standard flooding is inefficient in terms of agent’s energy consumption and message complexity, the k -flooding protocol (for small values of k) strongly reduces the agent’s energy consumption and the overall message complexity. However, as we will discuss later, the infection process yielded by the k -flooding (for small k) over a dynamic network is much more complex than that yielded by the standard flooding.

The k -flooding has been studied in [3] on *Edge-Markovian Evolving Graphs (Edge-MEG)*. An Edge-MEG [6, 7] is a Markovian random process that generates an infinite sequence of graphs over the same set of n agents. If an edge exists at time t then, at time $t + 1$, it dies with probability q . If instead the edge does not exist at time t , then it will come into existence at time $t + 1$ with probability p . The stationary distribution of an Edge-MEG with parameter p and q is the classic Erdős-Rényi random graph $\mathcal{G}(n, \tilde{p})$ where $\tilde{p} = \frac{p}{p+q}$. The work [3] gives tight bounds on the k -flooding on stationary Edge-MEG for arbitrary values of p , q and k . In particular, it derives the *reachability threshold* for the k -flooding, i.e., the smallest $k = k(n, p, q)$ over which the protocol completes.

Edge-MEG is an analytical model of dynamic networks capturing time-dependencies, an important feature observed in real scenarios such as (faulty) wireless networks and P2P networks. However, it does not model important features of MANET. Indeed, in Edge-MEG, edges are independent Markov chains while, in several real opportunistic networks, the correlation among edges cannot be neglected: agents use to act over a geometric space [7, 10, 12, 16].

Our Contribution. We study the k -flooding protocol in the geometric random-walk model (the resulting MANET will be called *geometric-MANET*). The move radius ρ determines the maximal distance an agent can travel in one time step. Even though network modeling issues are out of the aims of this work, both the transmission radius and the move radius play a crucial role in our analysis and, thus, a short discussion about them is needed. Both parameters depend on several factors. In mathematical epidemiology, they depend on the agent’s mobility, on the nature of infection and on the agent’s social behavior that all together determine the average rate of “positive” contacts. For instance, in some particular human viruses (such as HIV), the nature of contacts required for getting an effective infection yields a very low average infection-rate: this can be modeled by setting the move radius to be significantly larger than the transmission radius. In MANET, the move radius depends (besides on the

agent’s mobility) on the adopted protocol that tunes the transmission rate of the agents: the larger is the time between two consecutive transmissions the larger is ρ . A larger ρ could yield a better message complexity at the cost of a larger completion time (as for the standard flooding, the correct trade-offs is derived in [8]).

It turns out that the issue of setting the “most suitable” move radius concerns both the network modeling and the protocol design. For these reasons, we investigate the k -flooding for a wide range of parameters R and ρ .

If the transmission radius R is below the *connectivity threshold* (this threshold is here computed with respect to the uniform distribution of n (static) agents over the square), we prove that, *with high probability (w.h.p.)*¹, for any $\rho \geq 0$ and for any $k = O(1)$, the k -flooding does not complete.

We thus study the k -flooding when R is above the connectivity threshold, emphasizing that the “static” connectivity among (generic) agents says nothing about the dynamic *expansion*-properties yielded by the red agents over the time: this is the crucial issue in the k -flooding process.

For any $\rho \lesssim R$ and for any $k \geq 1$, we prove that the information spreads at “optimal” speed $\Theta(R)$, i.e., the k -flooding protocol w.h.p. completes within $O(L/R)$ time steps. Observe that, since $\rho \lesssim R$, this bound is asymptotically optimal.

Then, we consider a slightly-different version of the random-walk model with move radius ρ that can be up to any polynomial of R , i.e. $\rho \leq \text{POLY}(R)$. We prove that the information spreads at “optimal” speed $\Theta(\rho)$. So, for any $k \geq 1$, the k -flooding w.h.p. completes in time $O(L/\rho)$ which is optimal for any $\rho \geq R$. Notice that this optimal information speed makes the 1-flooding time smaller than the static diameter $O(L/R)$ of the stationary graph: our bound is thus the first analytical evidence that agent’s mobility actually speeds-up this infection process. Finally, we observe that, in both cases, the energy-efficient 1-flooding protocol is as fast as the standard flooding [7, 8].

Adopted Techniques and Further Results. The presence of black agents in the infection process makes the previous analysis techniques - those adopted for the flooding - almost useless. In particular, percolation theory [12, 10, 18], meeting and cover time of random walks on graphs [19], and the expansion/bootstrap arguments [7, 8] strongly rely on the fact that an informed agent will be active for all the flooding process. Furthermore, the analysis of k -flooding over the Edge-MEG model [3] strongly relies on the stochastic independence among the edges and the consequent high node-expansion property of $\mathcal{G}(n, p)$: properties that clearly do not hold in geometric-MANET.

Our method of analysis significantly departs from all those mentioned above. Besides the optimal bounds on the completion time, our analysis provides a clear characterization of the geometric evolution of the infection process. We make use of a grid partition of the square into cells of size $\Theta(R)$ and define a set of possible states a cell can assume over the time depending on the number of red, white and black agents inside it². We then derive the local state-evolution law of any cell. Thanks to the regularity of this law, we can characterize the evolution of the geometric *wave* formed by the *red cells* (i.e. cells containing some red agent). A crucial property we prove is that, at any time step, white cells (i.e. cells containing white agents only) will never be adjacent to black cells, so there is always a red wave “working” between

¹As usual, an event is said to hold *with high probability (w.h.p.)*, if its probability is at least $1 - 1/n^{\Theta(1)}$.

²When $\rho \gg R$, we also need a further grid partition into *supercells* of size $\Theta(\rho)$ and a more complex argument.

the black region and the white one. Furthermore, we show that the red wave eventually spans the entire region before all agents become black.

A similar epidemic phenomenon, also known as the *forest-fire spreading*, has been often observed in the context of human viruses over static social networks [14]: for the first time, we here provide its rigorous evidence for geometric-MANET.

Our method of analysis has further consequences. Thanks to the regularity of the red-wave shape, we are able to bound the time by which a given subregion will be infected for the first time. This bound is a function of the distance between the subregion and the initial position of the source agent. Actually, it is a function of the distance from the *closest* red agent at the starting time. So, our technique also works in the *multi-source* case, i.e. in the presence of an arbitrary set of source agents that aim to spread the same infection (or message). Under the same assumptions made for the single-source case, we can thus prove the completion time is w.h.p. $\Theta(\text{ECC}(A)/R)$ (or $\Theta(\text{ECC}(A)/\rho)$) where A is the set of the positions of the source agents at starting time and $\text{ECC}(A)$ is the *geometric eccentricity* of A in the square.

Related Works. As mentioned above, there are no analytical studies of the parsimonious flooding process over any geometric mobility model. In what follows, we briefly discuss some analytical results concerning the flooding over some models of MANET. In [15], the flooding time is studied over a restricted geometric-MANET. Approximately, it corresponds to the case $\rho = \Theta(L)$. Notice that, under this restriction, the stochastic dependence between two consecutive agent positions is negligible. In [13], the speed of data communication between two agents is studied over a class of *Random-Direction* models yielding uniform stationary agent’s distributions (including the geometric-MANET model). They provide an upper bound on this speed that can be interpreted as a *lower* bound on flooding time when the mobile network is very sparse and disconnected (i.e. $R, \rho = o(1)$). Their technique, based on Laplacian transform of independent journeys, cannot be extended to provide any upper bound on the time of any version of the flooding. We observe that, differently from our model, both [15] and [13] assume that when an agent gets informed then all agents of his *current* connected component (no matter how large it is) will get informed in one time step. The same unrealistic assumption is adopted in [18], where bounds on flooding time (and some other tasks) are obtained in the Poisson approximation of the geometric-MANET model. In [7, 8], the first almost tight bounds for the flooding time over geometric-MANET have been given. As mentioned before, their proofs strongly rely on the fact that informed agents stay always active. Flooding and gossip time for random walks over the grid graph (so, the agent’s space is a graph) have been studied in [19]. Here, an agent informs all agents lying in the same node. Besides the differences between k -flooding and flooding discussed before, it is not clear whether their results could be extended to the random walk model over geometric spaces. Especially, in their model, there is no way to consider arbitrary values of the move radius.

Roadmap of the Paper. The rest of the paper is organized as follows. After giving some preliminary definitions in Section 2, we warm-up by first analyzing the cases $\rho \lesssim R$ and $\rho \leq R^2/\sqrt{\log n}$ in Section 3 and Section 4, respectively. By presenting such easier cases first, we also aim at giving a preview of the key-ingredients of our approach: thus some standard probabilistic arguments are only sketched there. In Section 5, we first introduce a slightly-different geometric random-walk model and then provide the details of a more powerful version of our approach that extends the bound $O(L/\rho)$ to any $\rho \leq \text{POLY}(R)$. In Section 6, we first briefly describe the extension to the multi-source case and, then, derive the completion threshold of R for the k -flooding for any constant k . Finally, some open questions are discussed

in Section 7.

2 Preliminaries

We analyze geometric-MANET over a square \mathcal{Q} of side length $L > 0$ by considering some suitable partition $\mathcal{C}(\mathcal{Q})$ of \mathcal{Q} into a grid of square *cells* of side length ℓ .

We say that two cells are adjacent if they touch each other by side or by corner. The *cell-diameter* $D_{\square}(\mathcal{Q})$ of \mathcal{Q} is defined as follows. Given two cells $c, c' \in \mathcal{C}(\mathcal{Q})$, define their *cell-distance* $d_{\square}(c, c')$ as the length of the shortest cell-path $p = \langle c = c_0, c_1, \dots, c_s = c' \rangle$ such that, for every i , c_i is adjacent to c_{i+1} . Then, we define

$$D_{\square}(\mathcal{Q}) = \max\{ d_{\square}(c, c') \mid c, c' \in \mathcal{C}(\mathcal{Q}) \}$$

Similarly, we can define the *cell-distance* between a cell and any cell subset \mathcal{C} , i.e.,

$$d_{\square}(c, \mathcal{C}) = \min\{d_{\square}(c, c') \mid c' \in \mathcal{C}\}$$

Observe that the size of the square and the *cell-diameter* $D_{\square}(\mathcal{Q})$ are tightly related: $D_{\square}(\mathcal{Q}) = \Theta(L/\ell)$.

According to the geometric random-walk model, there are n agents that perform independent random walks over \mathcal{Q} . At any time step, an agent, located at point $x \in \mathcal{Q}$, can move uniformly at random to any position in $\mathcal{B}(x, \rho) \cap \mathcal{Q}$, where $\mathcal{B}(x, \rho)$ is the disk of center x and radius ρ . This is a special case of the *Random Trip Model* introduced in [16] where it is proved that it admits a unique stationary agents distribution and, in this case, it is almost uniform: we will formally state this property by an agent-density condition (see Eq. 1). In the sequel, we always assume that, at time $t = 0$, the agent positions are random w.r.t. the stationary distribution.

Let us consider n mobile agents acting over the square \mathcal{Q} according to the geometric random-walk model. Most of our technical arguments rely on a suitable density property yielded by the agent's stationary distribution and by the cell partition. We say that the resulting geometric-MANET satisfies the *density property* w.r.t. the cell partition of side length ℓ if, with probability at least $1 - (1/n)^4$, for every time step $t = 0, 1, \dots, n$ and for every cell $c \in \mathcal{C}(\mathcal{Q})$, the number $\#_c$ of agents in c at time step t satisfies the following inequalities

$$[Density Property] \quad \eta_1 \ell^2 \frac{n}{L^2} \leq \#_c \leq \eta_2 \ell^2 \frac{n}{L^2}, \text{ where } \eta_1, \eta_2 \text{ are positive constants} \quad (1)$$

We observe that a sufficient condition for the density property in our model is

$$\ell \geq \beta \frac{L}{\sqrt{n}} \sqrt{\log n} \text{ for a suitable constant } \beta > 0$$

In the k -flooding protocol, at any time step, every agent can be in three different states: white (non-informed), red (informed-active), and black (informed-inactive). Then, the configuration $\text{CONF}(t)$ is defined as the set of the positions of the n agents together with their respective states at the end of time t .

The analysis of the information spreading will be performed in terms of the number of infected cells. Given a cell $c \in \mathcal{C}(\mathcal{Q})$, the neighborhood $N(c)$ is the set of cells formed by c

and all its adjacent cells. Since the stationary agent's distribution is almost uniform and the maximal speed of any message in the geometric-MANET is $R + \rho$, we easily get the following lower bound.

Fact 1 For any $k \geq 0$, the k -flooding time is w.h.p. $\Omega(L/(R + \rho))$.

For the sake of simplicity, we will prove our results by assuming $L = \sqrt{n}$ (i.e., average agent density equal to 1). It is straightforward to scale all the definitions and results to an arbitrary average density of agents. Furthermore, the proofs of the upper bounds in Sections 3-5 are given for the case $k = 1$ (1-flooding), however, all our arguments can be easily extended to the k -flooding protocol, for any $k \geq 1$.

In our proofs, we will often use the Chernoff's Bound and the Method of Bounded Differences [17].

Lemma 2 (Chernoff's Bound.) Let $X = \sum_{i=1}^n X_i$ where X_1, \dots, X_n are independent Bernoulli random variables (r.v.) and let ϵ be such that $0 < \epsilon < 1$. If $0 < \mu \leq \mathbf{E}[X]$, then it holds

$$\mathbf{P}(X \leq (1 - \epsilon)\mu) \leq e^{-\frac{\epsilon^2}{2}\mu} \quad \text{and} \quad \mathbf{P}(X \geq (1 + \epsilon)\mu) \leq e^{-\frac{\epsilon^2}{3}\mu}$$

Lemma 3 (The Method of Bounded Differences.) Let X_1, X_2, \dots, X_n be independent r.v. with X_k taking values in a set A_k . Assume that the real-valued function F defined on $\prod_{k=1}^n A_k$ satisfies

$$|F(\mathbf{x}) - F(\mathbf{x}')| \leq c \quad \text{for some } c > 0$$

whenever the vectors \mathbf{x}, \mathbf{x}' differ only in one co-ordinate. Let $\mu = \mathbf{E}[F]$. Then, for any real $t > 0$, it holds

$$\mathbf{P}(|F(\mathbf{x}) - \mu| \geq t) \leq e^{-\frac{2t^2}{nc^2}}$$

3 Warm-Up: High Transmission-Rate or Low-Mobility

We warm-up with the case where the move radius is smaller than the transmission radius. More precisely, we assume $\rho \leq R/(2\sqrt{2})$. We consider a grid partition of \mathcal{Q} into cells of side length $\ell = R/(2\sqrt{2})$. Hence, an agent lying in a cell C cannot escape from $N(C)$ in one step. This makes the analysis simpler than the cases with larger ρ . However, some basic ideas and arguments adopted here will be exploited for the harder cases too.

Theorem 4 Let $R \geq c_0 L \sqrt{\log n/n}$ for a sufficiently large constant c_0 and $\rho \leq R/(2\sqrt{2})$. Then, for any $k \geq 1$, the k -flooding time is w.h.p. $\Theta(L/R)$.

3.1 Proof of Theorem 4

The lower bound is an immediate consequence of Fact 1. Let us now consider the upper bound. For the sake of simplicity, we assume that the time step is divided into 2 consecutive phases: the *transmission phase* where every red agent transmits the information and the *move phase* where every agent performs one step of the random walk.

We now introduce the feasible states of a cell during the infection process.

Definition 5 At (the end of) any time step a cell C can be in 4 different states. It is white if it contains only white agents. It is red if it contains at least one red agent. It is black if it contains black agents only. Finally, it is grey if it is not in any of the previous 3 cases.

We will show that the infection process, at any time step, has (w.h.p.) a well defined shape in which no white cell is adjacent to a black one and there is no grey cell. This shape is formalized in the following definitions. A subset of white cells is a *white component* if it is a connected component w.r.t. the subset of all the white cells.

Definition 6 A configuration $\text{CONF}(t)$ is regular if the following properties hold:

- a) No cell is grey,
- b) Every white component is adjacent to a red cell,
- c) No white cell is adjacent to a black cell.

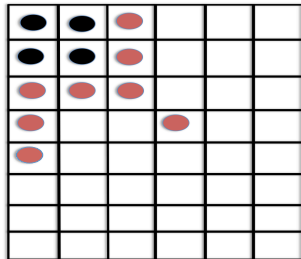


Figure 1. A Regular Configuration: black cells are surrounded by red ones and no grey cell exists.

Let us remark that the starting configuration $\text{CONF}(0)$ is regular.

Definition 7 A white cell is red-close if it is adjacent to a red cell.

The next lemma determines the local state-evolution of any cell in a regular configuration

Lemma 8 Consider any cell C at time step t and assume $\text{CONF}(t)$ be regular. Then the following properties hold:

- 1) If C is red-close then it becomes red in $\text{CONF}(t + 1)$ w.h.p.,
- 2) If C is (no red-close) white then it becomes white or red in $\text{CONF}(t + 1)$,
- 3) If C is red then it becomes red or black in $\text{CONF}(t + 1)$,
- 4) If C is black then it becomes red or black in $\text{CONF}(t + 1)$.

Proof.

1.) The proof of this claim relies on the density property (Eq. 1). After the transmission phase of time step $t + 1$, every agent in the red-close cell C becomes red. Then, thanks to the density property, we can apply Chernoff’s Bound and get that w.h.p. at least a constant fraction of these red agents remains in C after the move phase as well. Thus, at time $t + 1$, w.h.p. C becomes red.

2.) Observe that if a white cell w is not red-close then, since $\text{CONF}(t)$ is regular, w is surrounded by white cells only, then at time $t + 1$ no black agent can reach w . Notice that

w may become red at time $t + 1$. Indeed, if there is a red-close cell in $N(w)$, after the move phase of time step $t + 1$, some red agent can reach w.

3.) After the transmission phase of step $t + 1$, since $R = 2\sqrt{2}\ell$, all agents in $N(c)$ will be either red or black. So, after the move phase, no white agent can reach c since $\rho < \ell$.

4.) Since $\text{CONF}(t)$ is regular, a black cell B is initially surrounded by black or red cells only. After the transmission phase of step $t + 1$, all agents in $N(B)$ will be either red or black. So, after the move phase, no white agent can reach B since $\rho < \ell$.

□

As an easy consequence of the above lemma, we get the following fact.

Lemma 9 *For any $t \leq n$, if $\text{CONF}(t)$ is regular, then w.h.p. $\text{CONF}(t + 1)$ is regular as well.*

The above result provides a first characterization of the shape of the infection process: the process evolves along a sequence of regular configurations. We now analyze the speed of convergence to the absorbing black configuration. From the above lemma, we can now assume that all configurations are regular (w.h.p.).

Lemma 10 *For any $t < n$, let w be any white cell in $\text{CONF}(t + 1)$ and let $\text{Red}(t)$ be the set of red cells in $\text{CONF}(t)$. It holds w.h.p. that*

$$d_{\square}(w, \text{Red}(t + 1)) \leq d_{\square}(w, \text{Red}(t)) - 1$$

Proof. From Lemma 8, a white cell w in $\text{CONF}(t + 1)$ was white in $\text{CONF}(t)$ as well. Now, consider one shortest cell path p from w to a red cell r in $\text{Red}(t)$. Then, thanks to Lemma 8, the (red-close) white cell w' in p and adjacent to r will w.h.p. become red in $\text{CONF}(t + 1)$. Notice that some other white cell in p may become red in $\text{CONF}(t + 1)$. In any case, the lemma holds.

□

Starting from the initial configuration $\text{CONF}(0)$ (that has one red cell), Lemma 10 implies that *every* white cell in \mathcal{Q} w.h.p will become red within $O(D_{\square}(\mathcal{Q})) = O(L/R)$ time steps. Moreover, thanks to Lemmas 8 and 9, every red cell will become either red or black. Finally, when all cells are black or red, after the next time step there are no more white agents and the theorem follows.

4 Low Transmission-Rate or High-Mobility I

We consider the network formed by n mobile agents with transmission radius $R \geq c_0\sqrt{\log n}$ and move radius ρ such that $R/2 \leq \rho \leq \alpha R^2/\sqrt{\log n}$ for a sufficiently small constant $\alpha > 0$. We consider a grid partition of \mathcal{Q} into cells of side length $\ell = R/(4\sqrt{2})$. The constant c_0 is chosen in order to guarantee the density condition (Eq. 1).

Theorem 11 *Under the above assumptions, for any $k \geq 1$, the k -flooding time over \mathcal{Q} is w.h.p. bounded by $\Theta(L/\rho)$.*

4.1 Proof of Theorem 11

We adopt Definitions 5, 6, and 7 given in the previous section and, moreover, we add the following one.

Definition 12 *Two cells are ρ -close if the (Euclidean) distance between their geometric centers is at most ρ .*

As in the previous section, we provide the local state-evolution law of any cell in a regular configuration.

Lemma 13 *Consider any cell C at time step t and assume $\text{CONF}(t)$ be regular. Then, w.h.p., the following properties hold:*

- 1) *If C is red-close then it becomes red in $\text{CONF}(t+1)$,*
- 2) *If C is ρ -close to a red-close cell, then C becomes red in $\text{CONF}(t+1)$,*
- 3) *If C is white but it is not ρ -close to any red-close cell then it becomes white or red in $\text{CONF}(t+1)$,*
- 4) *If C is red or black but it is not ρ -close to any red-close cell then it becomes red or black in $\text{CONF}(t+1)$.*

Proof.

1). Immediately after the transmission phase of time step $t+1$, all agents in C become red. Thanks to the density condition and the bound $\rho \leq \alpha(R^2/\sqrt{\log n})$, we can apply Chernoff's Bound and get that, w.h.p., at least one of such red agents remains in C after the move phase.

2). We first observe that, after the transmission phase of time step $t+1$, all agents in the red-close cell C' become red. Define X to be the indicator r.v. that equals 1 iff C is hit by some red agent immediately after the move phase. From the density condition and the mutual independence among the random moves of the agents, it holds that

$$\mathbf{P}(X = 0) \leq \left(1 - \frac{\ell^2}{\pi\rho^2}\right)^{n\ell^2} \leq \exp\left(-\frac{\eta_1\ell^4}{\pi\rho^2}\right)$$

The upper bound $\rho \leq R^2/\sqrt{\log n}$ implies that

$$\mathbf{P}(X = 0) \leq \exp(-\Theta(\log n))$$

So, w.h.p. cell C becomes red.

3). We assume, by way of contradiction, that C becomes black or grey at time $t+1$. Since C is white, this implies that, immediately after the transmission phase of $t+1$, a black agent exists that is within Euclidean distance ρ from C . Let b be the black agent closest to C and let B be the cell containing it. Let Z be the cell which is the closest one to C among those in $N(B)$. We observe that the Euclidean distance between the centers of C and Z is equal to the Euclidean distance between B and C . This implies that C is ρ -close to Z . The cell Z cannot be black since $\text{CONF}(t)$ is regular and, thus, a red cell ρ -close to C would exist "between" C and Z : this would contradict the hypothesis of Claim 3. The cell Z cannot be white as well: indeed, since $\text{CONF}(t)$ is regular, B would be red and, thus, C would be ρ -close to a red-close cell. It follows that, at time t , Z must be red: this would contradict the hypothesis that C is not ρ -close to a red-close cell at time step t since, being $\text{CONF}(t)$ regular, there must exist a red-close cell "between" Z and C .

4). Immediately after the transmission phase of time step $t + 1$, let us consider the white agent w closest to cell c . If $d(w, c) > \rho$, we easily get the claim. Assume thus $d(w, c) \leq \rho$ and let w be the cell containing w . Let z be the cell which is the closest one to c among those in $N(w)$. Since $\text{CONF}(t)$ is regular, the cell z can be neither red nor black at time step t . So, cell z must be white at time t . After the transmission of time step $t + 1$, all agents in z must go into the red state since otherwise w would not be the closest white agent to c . Now, since z is ρ -close to c , by using the same argument of Claim 2, we have that, after the move phase, at least a red agent hits cell c : hence the latter becomes red as well. \square

As an easy consequence of the above lemma, we get the following fact.

Lemma 14 *For any $t < n$, if $\text{CONF}(t)$ is regular, then w.h.p. $\text{CONF}(t + 1)$ is regular as well.*

In what follows, we assume that all configurations are regular (w.h.p.) and analyze the speed of the infection process. Differently from the previous section, we will show this “speed” is $\Theta(\rho)$.

Lemma 15 *For any $t < n$, let w be any white cell in $\text{CONF}(t + 1)$ and let $\text{Red-close}(t)$ be the set of red-close cells in $\text{CONF}(t)$. It holds, w.h.p., that*

$$d_{\square}(w, \text{Red-close}(t + 1)) \leq \max\{d_{\square}(w, \text{Red-close}(t)) - \Theta(\rho/\ell), 0\}$$

Proof. From Lemma 13, a white cell w in $\text{CONF}(t + 1)$ was white in $\text{CONF}(t)$ as well. Consider one shortest (white) cell path $p(w, r)$ from w to $\text{Red-close}(t)$ in $\text{CONF}(t)$. Then, thanks to Claim 2 of Lemma 13, every cell w' in $p(w, r)$ which is ρ -close to r will w.h.p. become red in $\text{CONF}(t + 1)$. Notice that the number of such new red cells is at least $\lfloor \rho/(\sqrt{2}\ell) \rfloor$ and some other white cell in $p(w, r)$ may become red in $\text{CONF}(t + 1)$. In any case, the lemma holds. \square

The starting configuration $\text{CONF}(0)$ is regular and contains one red cell. Hence, from Lemma 15, we get that the cell-distance of any white cell from its closest red-close cell decreases at any step by an amount of $\Theta(\rho/\ell)$. Since at time $t = 0$, the maximal cell-distance is $D_{\square}(\mathcal{Q})$, we easily get that, w.h.p., every white cell in \mathcal{Q} will become red within time

$$O\left(\frac{D_{\square}(\mathcal{Q})\ell}{\rho}\right) = O\left(\frac{L}{\rho}\right)$$

Moreover, thanks to Lemma 13, every red cell will become either red or black. Finally, when all cells are black or red, in the next time step, there will be no more white agents and the theorem follows.

5 Low Transmission-Rate or High-Mobility II

In this section, we study the case when the move radius ρ can be significantly larger than $\alpha R^2/\sqrt{\log n}$. More precisely, the move radius can be an arbitrary polynomial of the transmission radius, i.e. $\rho = O(\text{POLY}(R))$. We also assume $R \geq c_0\sqrt{\log n}$ (for a sufficiently large c_0) and $\rho \geq 5R$.

A slightly different version of the geometric random-walk model is here adopted, the *cellular random walk*: the square \mathcal{Q} is partitioned in squared *supercells* of side length ρ . Then an agent

lying in any position of a supercell C selects his next position independently and uniformly at random over the supercell neighborhood $N(C)$. Moreover, an agent can be informed by another (red) agent only if they both belong to the same supercell. So, in the cellular random walk model, the *influence* of an agent lying in a supercell C , in the next time step, is always restricted to the subregion $N(C)$. Observe that the cellular random walk model preserves all significant features of the standard one while, on the other hand, it avoids several geometric technicalities that would yield a much more elaborate analysis without increasing the relevance of the obtained results.

We consider a cell partition of \mathcal{Q} with $\ell = R/\sqrt{2}$ such that the grid of supercells is a sub-grid of the grid of cells of side length ℓ . In what follows, the cells of side length ρ are called *supercells* and those of side length ℓ are called cells, simply. As usual, we assume $R \geq c_0 L \sqrt{\log n/n}$ for a sufficiently large constant c_0 that guarantees the density property w.r.t. the cells (and, consequently, w.r.t. the supercells).

Theorem 16 *Under the above assumptions and for ρ such that $5R \leq \rho \leq \text{POLY}(R)$, for any $k \geq 1$, the k -flooding time over \mathcal{Q} is w.h.p. $\Theta(L/\rho)$.*

5.1 Proof of Theorem 16

The higher agent mobility forces us to analyze the infection process over the supercells (besides over the cells). During the information spreading, the state a supercell can assume is defined by some bounds on the numbers of, respectively, red and white agents inside it. Informally speaking, in order to guarantee a fast infection process, the number of white agents must never be too small w.r.t. the number of red agents. Since the infection process is rather complex, we need to consider a quite large number of possible supercell states. Our analysis will first show that every supercell eventually evolves from the initial white state to the final black state according to a monotone process over a set of intermediate states. Then, we will show that the speed of this process is asymptotically optimal, i.e., proportional to the move radius.

For a supercell C and a time step t , let $\#_r^t(C)$, $\#_w^t(C)$, and $\#_b^t(C)$ be, respectively, the number of red, white, and black agents in C at time t . We introduce the parameter \hat{h} that determines the (constant) number of possible states a supercell can assume. Informally speaking, this parameter somewhat represents the maximal number of intermediate states (depending on the numbers of, respectively, white and red agents inside the supercell) that a supercell takes starting from the first “infection” until it becomes black. We define

$$\hat{h} = \left\lceil \log_{R^2} \left(c_0 \frac{\rho^2}{R^2} \log n \right) \right\rceil \quad (2)$$

We observe that the hypothesis $\rho = O(\text{POLY}(R))$ implies that $\hat{h} = \Theta(1)$.

Definition 17 *For any time t and for any supercell C , we define some possible states of C at time t .*

State $h = 0$ (White State): $\#_r^t(C) = 0$ AND $\#_b^t(C) = 0$.

State $h = 1, \dots, \hat{h} - 1$ (Intermediate States): *The values of $\#_r^t(C)$ and $\#_w^t(C)$ satisfy*

$$a_h R^{2h} \leq \#_r^t(C) \leq b_h R^{2h} \quad \text{AND} \quad \#_w^t(C) \geq c_h \rho^2$$

where a_h, b_h, c_h are constants that satisfy

$$a_h < b_h, \quad a_1 \geq \dots \geq a_{\hat{h}-1} > 0, \quad 0 < b_1 \leq \dots \leq b_{\hat{h}-1}, \quad \text{and} \quad c_1 \geq \dots \geq c_{\hat{h}-1} > 0$$

State \hat{h} (Red State): $\#_r^t(C) \geq 90 \frac{\rho^2}{R^2} \log n$.

State $\hat{h} + 1$ (Black State): $\#_w^t(C) = 0$.

All the above states are mutually disjoint but the last three ones, i.e., $\hat{h} - 1, \hat{h}, \hat{h} + 1$. For every supercell C and for any time step t , in order to indicate that C satisfies the condition of state h , we will write $h^t(C) = h$.

Definition 18 *The configuration $\text{CONF}(t)$ is regular if, for every supercell C , it holds:*

- a) an $h \in \{0, 1, \dots, \hat{h}, \hat{h} + 1\}$ exists such that $h^t(C) = h$,
- b) if $h^t(C) = \hat{h} + 1$, then, $\forall C' \in N(C)$, $h^t(C') = \hat{h} \vee h^t(C') = \hat{h} + 1$.

In the sequel, we exchange the order of the phases in a time step: the move phase now comes before the transmission one. Clearly, this does not change our asymptotical results.

The next technical lemmas allow to control the 1-step evolution of the state of any supercell in terms of the number of red and white agents and how such agents spread over its cells. Let us recall that constants η_1, η_2 are defined in the density property (Eq. 1).

Lemma 19 *Let C be a supercell such that $\#_w^t(N(C)) \geq \lambda \rho^2$, for some constant $\lambda \geq 720/c_0^2$. Then, immediately after the move phase of time $t + 1$ (and before the transmission phase), w.h.p., for every cell C in C , it holds that the number of white agents in C is at least $(\lambda/36)R^2$.*

Proof. Since the area of C is $\Theta(\rho^2)$, the proof relies on the fact that the overall number of white agents in $N(C)$ suffices to fill a large fraction of cells in C . Let X be the number of white agents in cell C immediately after the move phase of time step $t + 1$. At time t , for every white agent j in $N(C)$, let Y_j be the indicator r.v. that is 1 iff agent j moves to cell C . Notice that $\mathbf{P}(Y_j = 1) \geq R^2/(18\rho^2)$. Clearly, $X = \sum_j Y_j$ and

$$\mathbf{E}[X] \geq \#_w^t(N(C)) \frac{R^2}{18\rho^2} \geq \frac{\lambda}{18} R^2$$

Since the r.v. Y_j are independent, we can apply Chernoff's Bound and get that, w.h.p., $X \geq \frac{\lambda}{36} R^2$. The lemma follows by applying the union bound with respect to all cells in C . \square

Lemma 20 *Let C be a supercell such that $\#_r^t(C) \geq \lambda R^k$, for some constants $\lambda \geq 1800/c_0^2$ and $k \geq 2$. Then, immediately after the move phase of time $t + 1$, w.h.p., in every supercell $C' \in N(C)$, the number of cells in C' , hit by some red agent, is not smaller than $\min\{(\lambda/30)R^k, \rho^2/(2R^2)\}$.*

Proof. We distinguish two cases.

- We first consider the case

$$\rho \geq c_0 R \sqrt{\frac{\lambda}{18} \log n} \tag{3}$$

Let E be a subset of the red agents in C at time t of size $M = \min\{\lambda R^k, (18\rho^2)/R^2\}$. Let Z be the number of cells in C' hit by some red agent in E immediately after the move phase of time $t + 1$. We first bound the expected value of Z . For any cell c in C' , let Y_C be the indicator r.v. that is 1 iff c is hit. It holds that

$$\mathbf{P}(Y_C = 0) \leq \left(1 - \frac{R^2}{18\rho^2}\right)^M \leq \exp\left(-\frac{R^2}{18\rho^2}M\right)$$

Hence,

$$\mathbf{P}(Y_C = 1) \geq 1 - \exp\left(-\frac{R^2}{18\rho^2}M\right) \geq (1 - 1/e)\frac{R^2}{18\rho^2}M$$

where the second inequality follows from the inequality $\exp(-x) \leq 1 - (1 - 1/e)x$, for any $0 \leq x \leq 1$. We then get

$$\mathbf{E}[Z] = \sum_C \mathbf{E}[Y_C] \geq \frac{2\rho^2}{R^2}(1 - 1/e)\frac{R^2}{18\rho^2}M \geq \frac{M}{15}$$

Notice that the r.v. Y_C are not independent. So, in order to prove a concentration result for Z , we use the *method of bounded differences* [17] (Lemma 3). For every red agent j in E , let X_j be the r.v. that is equal to the cell c in C' hit by agent j immediately after the move phase of time $t + 1$ and $X_j = 0$ if j does not hit any cell in C' . Observe that the r.v. X_j are independent. Now, consider the function $F(X_1, \dots, X_M)$ that returns the number of cells in C' hit by some red agent. Now, we need to find the constant c for F in Lemma 3. Since every single X_j can affect at most one cell, then we can fix $c = 1$. Moreover, it holds that $F(X_1, \dots, X_M) = Z$. By applying Lemma 3, we get

$$\begin{aligned} \mathbf{P}\left(Z \leq \min\left\{\frac{\lambda R^k}{30}, \frac{\rho^2}{2R^2}\right\}\right) &\leq \mathbf{P}\left(Z \leq \frac{M}{30}\right) \leq \mathbf{P}\left(Z \leq \frac{1}{2}\mathbf{E}[Z]\right) \\ &\leq 2\exp\left(-\frac{M}{450}\right) \end{aligned}$$

From Eq. 3, the assumption $R \geq c_0\sqrt{\log n}$, and the lemma's hypothesis on λ , we get that $M \geq \beta \log n$, for a sufficiently large constant $\beta > 0$. The thesis then follows by applying the union bound with respect to all the supercells in $N(C)$.

- Now, we consider the case

$$\rho < c_0 R \sqrt{\frac{\lambda}{18} \log n} \tag{4}$$

Define

$$q = \left\lfloor \sqrt{\frac{\lambda}{18} \frac{R^2}{\rho}} \right\rfloor$$

Notice that $q \geq 1$ since Eq. 4 holds. We partition every cell into subcells of side length $s = \ell/q$. Let $m = (9\rho^2)/s^2$. Observe that m is an upper bound on the number of subcells in $N(C)$. It holds that

$$m = \frac{9\rho^2}{s^2} = \frac{9\rho^2}{\ell^2} q^2 \leq \frac{18\rho^2}{R^2} \left(\sqrt{\frac{\lambda}{18} \frac{R^2}{\rho}}\right)^2 = \lambda R^2$$

Thus, $m \leq \lambda R^k$. Let E be a subset of the red agents in C at time t of size m . Let Z be the number of subcells in C' hit by some red agent in E immediately after the move phase of time

$t + 1$. We first bound the expected value of Z . For any subcell s in C' , let Y_s be the indicator r.v. that is 1 iff s is hit. It holds that

$$\mathbf{P}(Y_s = 0) \leq \left(1 - \frac{s^2}{9\rho^2}\right)^m \leq \exp\left(-\frac{s^2}{9\rho^2}m\right) = \frac{1}{e}$$

Hence, $\mathbf{P}(Y_s = 1) \geq 1 - 1/e$. We then get

$$\mathbf{E}[Z] = \sum_s \mathbf{E}[Y_s] \geq \frac{\rho^2}{s^2}(1 - 1/e) \geq \frac{m}{15}$$

For every red agent j in E , let X_j be the r.v. that is equal to the subcell s in C' hit by agent j immediately after the move phase of time $t + 1$ and $X_j = 0$ if j does not hit any subcell in C' . Observe that the r.v. X_j are independent. We again apply the method of bounded differences. Consider the function $F(X_1, \dots, X_m)$ that returns the number of subcells in C' hit by some red agent. From the definitions of X_j s and F , we easily have that $F(X_1, \dots, X_m) = Z$, and, moreover, the constant c in Lemma 3 can be set to 1: hence, by that lemma, we get

$$\mathbf{P}\left(Z \leq \frac{m}{30}\right) \leq \mathbf{P}\left(Z \leq \frac{1}{2}\mathbf{E}[Z]\right) \leq 2 \exp\left(-\frac{m}{450}\right)$$

Observe that $m \geq (\lambda R^2)/4$. Then, from the latter bound and the hypothesis on λ , we get, as in the previous case, a strong concentration result on Z . Observe that the number of cells in C' hit by red agents is at least Z/q^2 . This implies that, w.h.p., the number of cells in C' hit by red agents is at least

$$\frac{m}{30} \frac{1}{q^2} \geq \frac{9\rho^2}{30s^2q^2} = \frac{9\rho^2}{30\ell^2} = \frac{18\rho^2}{30R^2} \geq \frac{\rho^2}{2R^2} \geq \min\left\{\frac{\lambda R^k}{30}, \frac{\rho^2}{2R^2}\right\}$$

The thesis follows by applying the union bound with respect to all the supercells in $N(C)$. \square

Lemma 21 *Let C be any supercell such that $h^t(C) = \hat{h}$. Then, immediately after the move phase of time $t + 1$, w.h.p., for every supercell $C' \in N(C)$, all the cells in C' are hit by some red agent.*

Proof. By hypothesis, we have that $\#_r^t(C) \geq K$ where

$$K = 90 \frac{\rho^2}{R^2} \log n$$

Let c be any cell in $N(C)$ and let Y be the indicator r.v. that is 1 iff cell c is hit by some red agent immediately after the move phase of time $t + 1$. It holds that

$$\mathbf{P}(Y = 0) \leq \left(1 - \frac{\ell^2}{9\rho^2}\right)^K \leq \exp\left(-\frac{\ell^2}{9\rho^2}K\right) = \exp\left(-\frac{R^2}{18\rho^2}90\frac{\rho^2}{R^2}\log n\right) = e^{-5\log n}$$

Then, the thesis follows by applying the union bound with respect to all the cells in $N(C)$. \square

For any time step t and any supercell C , let us define

$$\widehat{\#_r^t(C)} = \max\{\#_r^t(C') \mid C' \in N(C)\}$$

Lemma 22 *Let C be any supercell. For any $t \geq 1$, if $\widehat{\#_r^t(C)} \leq M$ for some $M > 0$, then, w.h.p., it holds that $\#_r^{t+1}(C) \leq 68\eta_2 MR^2$.*

Proof. Assume that $\widehat{\#_r^t(C)} \leq M$, then $\#_r^t(N(C)) \leq 9M$. So, immediately after the move phase of time $t + 1$, the red agents in C are at most $9M$. Notice that any red agent can transmit over at most 15 cells³. From the density property, every cell w.h.p. contains at most $\eta_2 \ell^2$ agents. It thus holds w.h.p. that

$$\#_r^{t+1}(C) \leq 9M \cdot 15\eta_2 \ell^2 \leq 68\eta_2 MR^2$$

□

Setting the constants⁴ in Definition 17. We can now set constants a_h , b_h , and c_h for $h = 1, 2, \dots, \hat{h} - 1$:

$$a_h = \frac{\eta_1^h}{2 \cdot 2160^{h-1} 20^{\frac{(h-1)(h-2)}{2}}} \quad (5)$$

$$b_h = 15 \cdot 68^{h-1} \eta_2^h \quad (6)$$

$$c_h = \frac{\eta_1}{2 \cdot 20^{h-1}} \quad (7)$$

We are now able to provide the local state-evolution law of any supercell in regular configurations. Let us define

$$m^t(C) = \max\{h \mid \exists C' \in N(C) : h^t(C') = h\}$$

Lemma 23 *If $\text{CONF}(t)$ is regular then, for every supercell C , the following implications hold w.h.p.:*

- | | |
|---|--|
| 1) $m^t(C) = 0$ | $\Rightarrow h^{t+1}(C) = 0$ |
| 2) $1 \leq m^t(C) \leq \hat{h} - 1$ | $\Rightarrow h^{t+1}(C) = m^t(C) + 1$ |
| 3) $m^t(C) = \hat{h} \wedge (h^t(C) = h \text{ with } h < \hat{h})$ | $\Rightarrow h^{t+1}(C) = \hat{h}$ |
| 4) $m^t(C) = \hat{h} \wedge h^t(C) = \hat{h}$ | $\Rightarrow h^{t+1}(C) = \hat{h} + 1$ |
| 5) $m^t(C) = \hat{h} + 1$ | $\Rightarrow h^{t+1}(C) = \hat{h} + 1$ |

Proof.

1). The statement directly follows from the definitions of State 0 and of the cellular random-walk model.

2). Let $m = m^t(C)$ and let $\bar{C} \in N(C)$ be a supercell such that $h^t(\bar{C}) = m$. Since $1 \leq m \leq \hat{h} - 1$, from the definition of intermediate states we get that $\#_w^t(N(C)) \geq c_m \rho^2$. From Lemma 19 (it holds that $c_m \geq 720/c_0^2$, for sufficiently large c_0) it follows that, immediately after the move phase of time $t + 1$, every cell in C contains at least $(c_m/36)R^2$ white agents, w.h.p.

Since $h^t(\bar{C}) = m$, it holds that $\#_r^t(\bar{C}) \geq a_m R^{2m}$. In virtue of Lemma 20 applied to \bar{C} (it holds that $a_m \geq 1800/c_0^2$, for sufficiently large c_0) we obtain that, immediately after the move phase of time $t + 1$, w.h.p. the cells in C hit by red agents are at least $\min\{(a_m/30)R^{2m}, \rho^2/(2R^2)\}$.

³The number 15 is a “rough” upper bound that considers the worst-case configuration where the agent is located to some border of the cell.

⁴The specific values of such constants are suitably chosen in order to let the technical calculations in the proof of Lemma 23 work correctly.

To summarize, immediately before the transmission phase of time $t+1$, every cell in C contains at least $(c_m/36)R^2$ white agents and at least $\min\{(a_m/30)R^{2m}, \rho^2/(2R^2)\}$ of those cells are hit by red agents, w.h.p. After the transmission, all the white agents in any hit cell get red, thus, w.h.p., it holds that

$$\#_r^{t+1}(C) \geq \frac{c_m}{72} \min \left\{ \frac{a_m}{15} R^{2m+2}, \rho^2 \right\} \quad (8)$$

We distinguish two cases.

- We first consider the case $m \leq \hat{h} - 2$. From the definition of \hat{h} , it holds that $(a_m/15)R^{2m+2} \leq (a_m/15)R^{2\hat{h}-2} \leq \rho^2$. Thus, Eq. 8 becomes

$$\#_r^{t+1}(C) \geq \frac{c_m a_m}{1080} R^{2m+2}$$

From Definitions 5 and 7, it holds that $a_{h+1} \leq a_h c_h / 1080$ (for $h = 1, \dots, \hat{h} - 2$). Thus, the lower bound on red agents of state $m + 1$ is satisfied:

$$\#_r^{t+1}(C) \geq a_{m+1} R^{2(m+1)} \quad (9)$$

Now we prove the upper bound. Since $\text{CONF}(t)$ is regular and for any $C' \in N(C)$ it holds $h^t(C') \leq m$, we obtain that

$$\#_r^t(C') \leq b_m R^{2m}$$

From Lemma 22, since $\widehat{\#_r^t(C)} \leq b_m R^{2m}$, w.h.p., it holds that

$$\#_r^{t+1}(C) \leq 68\eta_2 b_m R^{2m+2}$$

From Definition 6, it holds that $b_{h+1} \geq 68\eta_2 b_h$ (for $h = 1, \dots, \hat{h} - 2$). Thus, the upper bound on red agents of state $m + 1$ is satisfied:

$$\#_r^{t+1}(C) \leq b_{m+1} R^{2(m+1)} \quad (10)$$

It remains to show the lower bound on white agents. Let W be the number of white agents in the supercell C immediately after the move phase of time $t + 1$. The white agents in C at time $t + 1$ are those that were in C immediately after the move phase and that have not gotten red (after the transmission phase). It follows that

$$\#_w^{t+1}(C) \geq W - \#_r^{t+1}(C)$$

Recall that, immediately after the move phase of time $t + 1$, every cell in C contains at least $(c_m/36)R^2$ white agents, w.h.p. Thus, w.h.p., it holds that

$$W \geq \frac{2\rho^2}{R^2} \frac{c_m}{36} R^2 = \frac{c_m}{18} \rho^2$$

By combining this inequality with Eq. 10, w.h.p. we have that

$$\#_w^{t+1}(C) \geq \frac{c_m}{18} \rho^2 - b_{m+1} R^{2(m+1)}$$

From the definition of \hat{h} and the hypothesis $m \leq \hat{h} - 2$, it holds that

$$R^{2(m+1)} \leq R^{2\hat{h}-2} \leq c_0 \frac{\rho^2}{R^2} \log n \leq \frac{\rho^2}{c_0}$$

Hence,

$$\#_w^{t+1}(C) \geq \frac{c_m}{18} \rho^2 - b_{m+1} \frac{\rho^2}{c_0} = \left(\frac{c_m}{18} - \frac{b_{m+1}}{c_0} \right) \rho^2$$

From Definitions 6 and 7, it holds that $c_{h+1} \leq c_h/18 - b_{h+1}/c_0$ (for $h = 1, \dots, \hat{h} - 2$), for sufficiently large c_0 . Thus, the lower bound on white agents of state $m + 1$ is satisfied:

$$\#_w^{t+1}(C) \geq c_{m+1} \rho^2 \quad (11)$$

Finally, from Eq.s 9, 10, and 11, w.h.p., we get that $h^{t+1}(C) = m + 1$.

- We now consider the case $m = \hat{h} - 1$. From Eq. 8 it holds that w.h.p.

$$\begin{aligned} \#_r^{t+1}(C) &\geq \frac{c_{\hat{h}-1}}{72} \min \left\{ \frac{a_{\hat{h}-1}}{15} R^{2\hat{h}}, \rho^2 \right\} \\ &\geq \frac{c_{\hat{h}-1}}{72} \min \left\{ \frac{a_{\hat{h}-1}}{15} c_0 \frac{\rho^2}{R^2} \log n, \rho^2 \right\} \\ &= c_0 \frac{c_{\hat{h}-1} a_{\hat{h}-1}}{15 \cdot 72} \frac{\rho^2}{R^2} \log n \quad \left(\text{since } \frac{a_{\hat{h}-1} c_0 \log n}{15 R^2} \leq \frac{a_{\hat{h}-1}}{15 c_0} \leq 1 \right) \\ &\geq 90 \frac{\rho^2}{R^2} \log n \quad \left(\text{for sufficiently large } c_0 \right) \end{aligned}$$

and thus $h^{t+1}(C) = \hat{h} = m + 1$.

3). Since $m^t(C) = \hat{h}$, there is a supercell \bar{C} adjacent to C such that $h^t(\bar{C}) = \hat{h}$. From Lemma 21, immediately after the move phase of time $t + 1$, w.h.p. all the cells in C are hit by some red agent. Since $h^t(C) < \hat{h}$ and $\text{CONF}(t)$ is regular, it holds that $\#_w^t(C) \geq c_{\hat{h}-1} \rho^2$. It follows that, immediately after the transmission phase, w.h.p., the number of red agents in C is at least

$$c_{\hat{h}-1} \rho^2 \geq 90 \frac{\rho^2}{R^2} \log n$$

for sufficiently large c_0 . This implies that $h^{t+1}(C) = \hat{h}$.

4). Since $h^t(C) = \hat{h}$, from Lemma 21, immediately after the move phase of time $t + 1$, w.h.p. all the cells in C are hit by some red agent. It follows that, immediately after the transmission phase, there are no white agents in C . Thus, w.h.p. $\#_w^{t+1}(C) = 0$ and $h^{t+1}(C) = \hat{h} + 1$.

5). Let $\bar{C} \in N(C)$ be a supercell such that $h^t(\bar{C}) = \hat{h} + 1$. Since $\text{CONF}(t)$ is regular, it must be the case that $h^t(C) \geq \hat{h}$ (given that $C \in N(\bar{C})$). This implies that either $h^t(C) = \hat{h}$ or $h^t(C) = \hat{h} + 1$. We distinguish two cases.

- If $h^t(C) = \hat{h}$, from Lemma 21, immediately after the move phase of time $t + 1$, w.h.p. all the cells in C are hit by some red agent. It follows that, immediately after the transmission phase, there are no white agents in C . Thus $h^{t+1}(C) = \hat{h} + 1$.

- Consider now the case $h^t(C) = \hat{h} + 1$. Since $\text{CONF}(t)$ is regular, $\forall C' \in N(C)$ we have that $h^t(C') = \hat{h} \vee h^t(C') = \hat{h} + 1$. We distinguish two sub-cases.

i) $\exists C' \in N(C) : h^t(C') = \hat{h}$. From Lemma 21, it follows that $h^{t+1}(C) = \hat{h} + 1$.

ii) $\forall C' \in N(C) h^t(C') = \hat{h} + 1$. It holds that $\#_w^t(N(C)) = 0$ and thus no white agent can move into supercell C during time step $t + 1$. It follows that $h^{t+1}(C) = \hat{h} + 1$.

□

As a consequence of the above lemma, we get

Lemma 24 For any $t < n$, if $\text{CONF}(t)$ is regular, then, w.h.p., $\text{CONF}(t+1)$ is regular as well.

Proof. From Lemma 23, it is easy to verify that, if $\text{CONF}(t)$ is regular then, w.h.p., it holds that $h^{t+1}(C) \in \{0, 1, \dots, \hat{h}, \hat{h} + 1\}$, for every supercell C .

Let C be a supercell such that $h^{t+1}(C) = \hat{h} + 1$. From Lemma 23, it must be the case that either $h^t(C) = \hat{h}$ or $h^t(C) = \hat{h} + 1$. We distinguish two cases.

- i) $h^t(C) = \hat{h}$. It holds that, $\forall C' \in N(C)$, $m^t(C') \geq \hat{h}$. From Lemma 23 (Claims (3), (4), and (5)), it follows that, w.h.p., $h^{t+1}(C') = \hat{h} \vee h^{t+1}(C') = \hat{h} + 1$
- ii) $h^t(C) = \hat{h} + 1$. W.h.p., it holds that, $\forall C' \in N(C)$, $m^t(C') \geq \hat{h} + 1$. From Lemma 23 (Claim (5)), it follows that, w.h.p., $h^{t+1}(C') = \hat{h} + 1$.

The lemma is proved by applying the union bound over every supercell. □

Lemma 25 The initial configuration $\text{CONF}(0)$ is regular and, w.h.p., a supercell C exists such that $h^0(C) = 1$.

Proof. At the very beginning of time step 0, all the agents are white except the source one which is red. Immediately after the move phase, the source agent moves to a cell c . Immediately after the transmission phase, w.h.p. all the agents (except one) in the cell c get red. Let C be the supercell containing cell c . From the density condition, w.h.p., it holds that

$$\eta_1 \ell^2 \leq \#_r^0(C) \leq 15\eta_2 \ell^2$$

since at most 15 cells are affected by the transmission of the source agent. Thus, the bounds on red agents of state 1 are satisfied: $a_1 R^2 \leq \#_r^0(C) \leq b_1 R^2$, since $a_1 = \eta_1/2$ and $b_1 = 15\eta_2$. Let us now consider the number of white agents. In the supercell C , there are at least $\rho^2/\ell^2 - 15$ cells that are not affected by the transmission. From the density condition, those cells contain at least $\eta_1 \ell^2 (\rho^2/\ell^2 - 15)$ white agents. W.h.p., it follows that

$$\#_w^0(C) \geq \eta_1 \ell^2 \left(\frac{\rho^2}{\ell^2} - 15 \right) = \eta_1 \rho^2 - 15\eta_1 \ell^2 \geq \frac{\eta_1}{2} \rho^2$$

where the last inequality holds since $\rho \geq 5R$. Hence, $\#_w^0(C) \geq c_1 \rho^2$ and $h^0(C) = 1$. Clearly all the other supercells are in the state 0. □

For any time step t , let $\text{Red}(t)$ be the set of supercells whose state at time t is at least 1. For any supercell C , denote by $d_{\square}^t(C)$ the distance w.r.t. supercells between C and $\text{Red}(t)$. Clearly, if $C \in \text{Red}(t)$ then $d_{\square}^t(C) = 0$.

Lemma 26 For any $t \leq n$, if $\text{CONF}(t)$ is regular, then it holds w.h.p. that, for any supercell W such that $h^t(W) = 0$,

$$d_{\square}^{t+1}(W) \leq d_{\square}^t(W) - 1$$

Proof. Let P be a shortest path of supercells between W and $\text{Red}(t)$. Path P contains a supercell W' such that $h^t(W') = 0$ and W' is adjacent to a supercell in $\text{Red}(t)$. From Lemma 23, it follows that the state of W' at time $t+1$ is at least 1 and thus $W' \in \text{Red}(t+1)$. Hence, $d_{\square}^{t+1}(W) \leq d_{\square}^t(W) - 1$. □

Theorem 16 is an easy consequence of Lemmas 24, 25 and 26.

6 The Multi-Source Case and the Completion Threshold

The multi-source k -flooding. Consider the k -flooding process with n agents over the square \mathcal{Q} whose starting configuration contains an arbitrary subset of source agents. Every source agent has the same message (infection) and, again, the goal is to bound the completion time of the k -flooding process. Let A be the set of positions of the source agents at the starting time.

For any point $x \in \mathcal{Q}$, define

$$d(x, A) = \min\{d(x, a) \mid a \in A\}$$

$$\text{ECC}(A) = \max\{d(x, A) \mid x \in \mathcal{Q}\}$$

The parameter $\text{ECC}(A)$ is the geometric eccentricity of A in \mathcal{Q} .

The crucial fact is that, in all parameter ranges studied in the previous sections (i.e. those of Sections 3, 4, and 5), the obtained local state-evolution laws of the cells work for *any* starting configuration which is a regular one. Then, it is easy to verify that, for any choice of the source subset, the starting configuration is w.h.p. regular. Moreover, our previous analysis of the speed of the information spreading does not change at all: the initial distance between every white cell (supercell) and its closest red cell (red-close supercell) decreases w.h.p. by 1 at any time step. In the case of more source agents, such initial distance is bounded by $\text{ECC}(A)$. We thus obtain the following result.

Theorem 27 *Under the same assumptions of the single-source case, for any choice of the source positions A , w.h.p., the k -flooding time is:*

- i) $\Theta(\text{ECC}(A)/R)$, for any $\rho \leq R/(2\sqrt{2})$;*
- ii) $\Theta(\text{ECC}(A)/\rho)$, for any $R/2 \leq \rho \leq \text{POLY}(R)$.*

The completion threshold for R . Let us consider the k -flooding protocol on the geometric-MANET of n agents over the square \mathcal{Q} of side length \sqrt{n} . Then the following negative result holds.

Theorem 28 *Let k be any positive constant, let R be such that $R < \gamma\sqrt{\log n}$ for a sufficiently small positive constant γ , and let ρ be such that $0 \leq \rho \leq \sqrt{n}$. Then, w.h.p., the k -flooding protocol does not complete.*

Proof. We will prove that, w.h.p., there is a subset of agents that will be isolated (so they cannot communicate any information) for some constant number of time steps. Then, by choosing one of them as the source, we easily get the thesis.

Let us consider the starting random disk graph $G(n, R)$ whose nodes are the positions of the n agents over \mathcal{Q} at time $t = 0$ and there is an edge between nodes i and j iff their relative Euclidean distance is at most R . We first prove that in this graph there are w.h.p. at least $\Theta(\sqrt{n})$ isolated nodes (i.e. agents with no neighbor).

For any node i , define the indicator r.v. X_i that equals 1 iff node i is isolated in $G(n, R)$. We get that

$$\mathbf{P}(X_i = 1) \geq \left(1 - \frac{\pi R^2}{n}\right)^{n-1} \geq e^{-2\pi\gamma \log n}$$

This implies that

$$\mathbf{E} \left[X = \sum_i X_i \right] \geq n^{1-2\pi\gamma} \quad (12)$$

Unfortunately, the r.v. X_i 's are not independent, so we need to apply the method of bounded differences. For any $i \in [n]$, consider the r.v. Z_i that equals the position (i.e. coordinates) of agent i in \mathcal{Q} . Observe that such r.v. are independent. Then we define the function $Z = F(Z_1, \dots, Z_n)$ that is the number of isolated nodes and observe that $Z = X$.

In order to apply Lemma 3 to function F , we need to find out the correct bound for the constant c . We claim that, in our geometric case, $c \leq 7$. Indeed, any change of the position of any node (i.e. agent) in the square can affect the R -disks of at most 6 isolated nodes. So, by setting $c = 7$ and $t = (1/2) \cdot \mathbf{E} [F(Z_1, \dots, Z_n)]$ in Lemma 3, we can choose $\gamma > 0$ sufficiently small (for instance $\gamma = 1/5\pi$) so that, Lemma 3 implies

$$\mathbf{P} \left(X \leq n^{1/2} \right) = \mathbf{P} \left(F(Z_1, \dots, Z_n) \leq n^{1/2} \right) \leq \exp \left(-n^{\Theta(1)} \right)$$

We now show how to extend the above argument in order to prove that, w.h.p., for $t = 2, \dots, k$ (for $k = \theta(1)$), there will be $\Omega(n^{\Theta(1)})$ isolated nodes (agents) in the random disk graph of time step t .

Let us define $I \subseteq [n]$ as the subset of isolated agents after the first step and consider them after the next move step. Thanks to the density property, the probability that one of the I -agents is still isolated is $\Omega(e^{-R^2})$. Since $|I| = \Omega(\sqrt{n})$, after the second step, the expected number of I -agents that remain isolated is $\Omega(n^\alpha)$, for some positive constant $\alpha < 1/2$. Again, we can get a strong concentration result on this random variable by applying Lemma 3, as shown for the starting configuration. Notice that, after every time step, the number of isolated agents decreases by some root factor. This implies that our argument works for a constant number of steps (only). □

7 Conclusions

The probabilistic analysis of information/virus spreading in dynamic geometric graphs is a challenging issue that is the subject of several current research projects and some relevant advances have been obtained in the last five years by using approaches based on time/space discrete approximation of the evolving systems and discrete probability [3, 5, 6, 10, 13, 18, 19]. We believe this *discrete* approach is rather promising to address several important related issues which are still far to be solved. Among them, we emphasize the study of other infection processes such as the *Susceptible-Infective-Susceptible* model [11]. A further open issue is to extend the analysis of the parsimonious flooding to other explicit models of MANET such as the random way-point model [16, 9].

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