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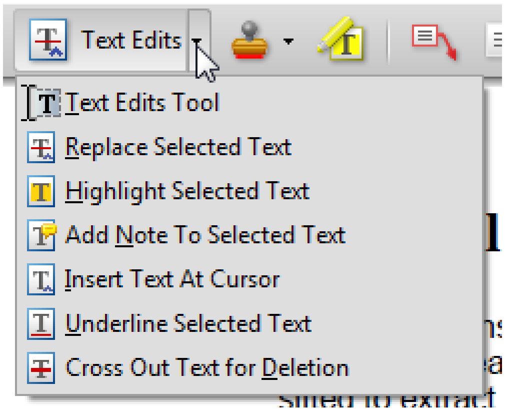

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Unique continuation and approximate controllability for a degenerate parabolic equation

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This article studies unique continuation for weakly degenerate parabolic equations in one-space dimension. A new Carleman estimate of local type is obtained to deduce that all solutions that vanish on the degeneracy set, together with their conormal derivative, are identically equal to zero. An approximate controllability result for weakly degenerate parabolic equations under Dirichlet boundary condition is deduced.

Keywords: degenerate parabolic equations; unique continuation; approximate controllability; local Carleman estimate

AMS Subject Classifications: 35K65; 93B05; 35A23; 93C20

1. Introduction

We consider a parabolic equation degenerating at the boundary of the space, which is related to a motivating example of a Crocco-type equation coming from the study of the velocity field of a laminar flow on a flat plate (see, e.g. [1]).

The null controllability of degenerate parabolic operators in one-space dimension has been well-studied for locally distributed controls. For instance, in [2,3], the problem

$$\begin{cases} u_t - (x^\alpha u_x)_x = \chi_\omega h, & (t,x) \in Q := (0,1) \times (0,T), \\ u(1,t) = 0, & t \in (0,T), \\ \text{and } \begin{cases} u(0,t) = 0, & \text{for } 0 \le \alpha < 1, \\ (x^\alpha u_x)(0,t) = 0, & \text{for } 1 \le \alpha < 2, \end{cases} & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,1), \end{cases}$$

where χ_ω denotes the characteristic function of $\omega = (a,b)$ with $0 < a < b < 1$, is shown to be null controllable in $L^2(0,1)$ in any time $T > 0$. Generalizations of the

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above result to semilinear problems and nondivergence form operators can be found in [4] and [5,6], respectively. The global Carleman estimate derived in [3] was also used in [7] to prove Lipschitz stability estimates for inverse problems relative to degenerate parabolic operators.

It is a commonly accepted viewpoint that, if a system is controllable via locally distributed controls, then it is also controllable via boundary controls and vice versa. This is indeed the case for uniformly parabolic operators. For degenerate operators, on the contrary, no null controllability result is available in the literature – to the best of our knowledge – when controls act on 'degenerate' parts of the boundary. Indeed, in this case, switching from locally distributed to boundary controls is by no means automatic for at least two reasons. In the first place, Dirichlet boundary data can only be imposed in weakly degenerate settings (i.e. when $0 \le \alpha < 1$), since otherwise solutions may not define a trace on the boundary, see [8, section 5]. Secondly, the standard technique which consists in enlarging the space domain and placing an 'artificial' locally distributed control in the enlarged region would lead to an unsolved problem in the degenerate case. Indeed, such a procedure requires being able to solve the null controllability problem for an operator which degenerates in the interior of the space domain, with controls acting only on one side of the domain with respect to the point of degeneracy.

In this article, we establish a simpler result, that is, the approximate controllability via controls at the 'degenerate' boundary point for the weakly degenerate parabolic operator

$$Pu := u_t - (x^\alpha u_x)_x, \quad \text{in } Q(0 \le \alpha < 1).$$

In order to achieve this, we follow the classical duality argument that reduces the problem to the unique continuation for the adjoint of P, that is, the operator

$$Lu := u_t + (x^\alpha u_x)_x, \quad \text{in } Q,$$

with boundary conditions

$$u(0,t) = (x^\alpha u_x)(0,t) = 0. \tag{1}$$

To solve such a problem, in Section 2, we derive new local Carleman estimates for L, in which the weight function exhibits a decreasing behaviour with respect to x (Theorem 2.3). Then, in Section 3, we obtain our unique continuation result proving that any solution u of $Lu = 0$ in Q, which satisfies (1), must vanish identically in Q (Theorem 3.1). Finally, in Section 4, we show how to deduce the approximate controllability with Dirichlet boundary control for the weakly degenerate problem $(0 \le \alpha < 1)$

$$\begin{cases} u_t - (x^\alpha u_x)_x = 0, & (t,x) \in Q, \\ u(t,1) = 0, & t \in (0,T), \\ u(t,0) = g(t), & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases}$$

The outline of this article is as follows. In Section 2, we derive our local Carleman estimate. Then, in Section 3, we apply such an estimate to deduce a unique continuation result for L. Finally, in Section 4, we obtain approximate controllability with Dirichlet boundary controls as a consequence of unique continuation.

2. A Carleman estimate with decreasing-in-space weight functions

We begin by recalling the definition of the function spaces that will be used throughout this article. The reader is referred to [3,4] for more details on these spaces.

For any $\alpha \in (0,1)$ we define $H^1_\alpha(0,1)$ to be the space of all absolutely continuous functions u: $[0,1] \to \mathbb{R}$ such that

$$\int_0^1 x^\alpha |u_x(x)|^2 \, \mathrm{d}x < \infty,$$

where u_x denotes the derivative of u. Like the analogous property of standard Sobolev spaces, one can prove that $H^1_\alpha(0,1) \subset C([0,1])$. So, one can also set

$$H^1_{\alpha,0}(0,1) = \left\{ u \in H^1_\alpha(0,1) \colon u(0) = u(1) = 0 \right\}.$$

Now, define the operator A: $D(A) \subset L^2(0,1) \to L^2(0,1)$ by

$$\begin{cases} D(A) := \left\{ u \in H^1_{\alpha,0}(0,1) \colon x^\alpha u_x \in H^1(0,1) \right\}, \\ Au = (x^\alpha u_x)_x \quad \forall u \in D(A). \end{cases}$$

We recall that A is the infinitesimal generator of an analytic semigroup of contractions on $L^2(0,1)$, and $D(A)$ is a Banach space with the graph norm

$$|u|_{D(A)} = \|u\|_{L^2(0,1)} + \|Au\|_{L^2(0,1)}.$$

Example 2.1 As one can easily check by a direct calculation, $x \mapsto 1 - x^{1-\alpha}$ belongs to $H^1_\alpha(0,1)$ and

$$(x^\alpha u_x)_x = 0 \quad \forall x \in [0,1].$$

However, $f \notin D(A)$ since $f(0) = 1$.

Lemma 2.2 *Let $u \in D(A)$ be such that $x^\alpha u_x \to 0$ as $x \to 0$. Then*

$$|x^\alpha u_x(x)| \le |u|_{D(A)} \sqrt{x} \quad \forall x \in [0,1] \tag{2}$$

and

$$|x^{\alpha-1} u(x)| \le \frac{2}{3-2\alpha} |u|_{D(A)} \sqrt{x} \quad \forall x \in [0,1] \tag{3}$$

Moreover, for any $\beta > 0$,

$$\int_0^1 x^{2\alpha+\beta-4} u^2 \, \mathrm{d}x + \int_0^1 x^{2\alpha+\beta-2} u_x^2 \, \mathrm{d}x \le c(\alpha,\beta) \, |u|^2_{D(A)}, \tag{4}$$

where $c(\alpha,\beta) = \beta^{-1}[1 + 2(3-2\alpha)^{-2}]$.

Proof Let $u \in D(A)$ be such that $x^\alpha u_x \to 0$ as $x \to 0$. Since

$$x^\alpha u_x(x) = \int_0^x \frac{\mathrm{d}}{\mathrm{d}s}\left(s^\alpha \frac{\mathrm{d}u}{\mathrm{d}s}(s)\right)\mathrm{d}s,$$

(2) follows by Hölder's inequality. Then, owing to (2),

$$|u(x)| \le \int_0^x \left|s^\alpha \frac{\mathrm{d}u}{\mathrm{d}s}(s)\right| s^{-\alpha}\,\mathrm{d}s \le |u|_{D(A)} \int_0^x s^{\frac{1}{2}-\alpha}\,\mathrm{d}s,$$

which in turn yields (3). Next, in view of (2),

$$\int_0^1 x^{2\alpha+\beta-2} u_x^2\,\mathrm{d}x \ \le |u|^2_{D(A)} \int_0^1 x^{\beta-1}\,\mathrm{d}x = \frac{1}{\beta}\ |u|^2_{D(A)}.$$

Finally, on account of (3),

$$\int_0^1 x^{2\alpha+\beta-4} u^2\,\mathrm{d}x \le \left(\frac{2}{3-2\alpha}\right)^2 |u|^2_{D(A)} \int_0^1 x^{\beta-1}\,\mathrm{d}x.$$

The proof of (4) is thus complete. ■

2.1. *Statement of the Carleman estimate*

Let $T > 0$. Hereafter, we set

$$Q = (0,1) \times (0,T).$$

Moreover, for any integrable function f on Q, we will use the abbreviated notation

$$\int_Q f = \int_Q f(x,t)\mathrm{d}x\,\mathrm{d}t.$$

Let $0 < \alpha < 1$ and fix $\beta \in (1-\alpha, 1-\frac{\alpha}{2})$. Define the weight functions l, p and ϕ as

$$\forall t \in (0,T), \quad l(t) := \frac{1}{t(T-t)}, \tag{5}$$

$$\forall x \in (0,1), \quad p(x) := -x^\beta \tag{6}$$

and

$$\forall (x,t) \in Q, \quad \phi(x,t) := p(x)l(t). \tag{7}$$

For any function $v \in L^2(0,T;\ D(A)) \cap H^1(0,T;\ L^2(0,1))$, we set

$$Lv := v_t + (x^\alpha v_x)_x.$$

We will prove the following Carleman estimate:

THEOREM 2.3 *Let* $v \in L^2(0,T;\ D(A)) \cap H^1(0,T;\ L^2(0,1))$, *and suppose that, for a.e.* $t \in (0,T)$,

$$v(0,t) = (x^\alpha v_x)(0,t) = v(1,t) = (x^\alpha v_x)(1,t) = 0.$$

Then the functions

$$\begin{cases} (x,t) \mapsto [l^3(t)x^{2\alpha+3\beta-4} + l(t)x^{2\alpha+\beta-4}]v^2(x,t)e^{2s\phi(x,t)} \\ \\ (x,t) \mapsto l(t)x^{2\alpha+\beta-2}v_x^2(x,t)e^{2s\phi(x,t)} \end{cases} \quad \textit{and} \tag{8}$$

are integrable over Q. *Moreover, there exist constants* $C = C(T, \alpha, \beta) > 0$ *and* $s_0 = s_0(T, \alpha, \beta) > 0$ *such that, for all* $s \geq s_0$,

$$\int_Q \left[s^3 l^3 x^{2\alpha+3\beta-4} + slx^{2\alpha+\beta-4}\right] v^2 e^{2s\phi} + \int_Q slx^{2\alpha+\beta-2} v_x^2 e^{2s\phi} \leq C \int_Q |Lv|^2 e^{2s\phi}. \tag{9}$$

The proof is inspired by [9,10], where global Carleman estimates for uniformly parabolic equations were first obtained, and by [3,4,7], where this technique was adapted to degenerate parabolic operators by the choice of appropriate weight functions.

We now proceed to derive another Carleman estimate which follows from (9) and yields unique continuation, deferring the proof of Theorem 2.3 to the following section.

COROLLARY 2.4 *Let* $v \in L^2(0, T; D(A)) \cap H^1(0, T; L^2(0, 1))$ *and suppose that, for a.e.* $t \in (0, T)$,

$$v(0,t) = (x^\alpha v_x)(0,t) = v(1,t) = (x^\alpha v_x)(1,t) = 0.$$

Then there exist constants $C = C(T, \alpha, \beta) > 0$ *and* $s_0 = s_0(T, \alpha, \beta) > 0$ *such that for all* $s \geq s_0$,

$$\int_Q s^3 l^3 v^2 e^{2s\phi} + \int_Q slx^{2\alpha+\beta-2} v_x^2 e^{2s\phi} \leq C \int_Q |Lv|^2 e^{2s\phi}. \tag{10}$$

Proof Since $\beta < 1 - \frac{\alpha}{2}$, we have that $4\beta < 4 - 2\alpha$ and $2\alpha + 4\beta - 4 < 0$. Moreover, $2\alpha + 3\beta - 4 < 2\alpha + 4\beta - 4 < 0$ since $\beta > 0$. Consequently, $x^{2\alpha+3\beta-4} \geq 1$ for all $x \in (0, 1)$. Then,

$$\int_Q s^3 l^3 v^2 e^{2s\phi} \leq \int_Q s^3 l^3 x^{2\alpha+3\beta-4} v^2 e^{2s\phi}$$

and the proof is complete. ■

2.2. *Proof of Theorem 2.3*

Let $v \in L^2(0, T; D(A)) \cap H^1(0, T; L^2(0, 1))$ and suppose that, for a.e. $t \in (0, T)$,

$$v(0,t) = (x^\alpha v_x)(0,t) = v(1,t) = (x^\alpha v_x)(1,t) = 0. \tag{11}$$

LEMMA 2.5 *Let* $w := ve^{s\phi}$. *Then* w *belongs to* $L^2(0, T; D(A)) \cap H^1(0, T; L^2(0, 1))$ *and satisfies, for a.e.* $t \in (0, T)$,

$$w(0,t) = w(1,t) = 0 \tag{12}$$

and

$$(x^\alpha w_x)(0,t) = (x^\alpha w_x)(1,t) = 0. \tag{13}$$

Moreover, w satisfies $L_s w = e^{s\phi} L v$, *where* $L_s w = L_s^+ w + L_s^- w$, *and*

$$\begin{aligned} L_s^+ w &= (x^\alpha w_x)_x - s\phi_t w + s^2 x^\alpha \phi_x^2 w, \\ L_s^- w &= w_t - 2sx^\alpha \phi_x w_x - s(x^\alpha \phi_x)_x w. \end{aligned} \tag{14}$$

Furthermore, $L_s^+ w, L_s^- w \in L^2(Q)$ *and*

$$\begin{aligned} \int_Q L_s^+ w\, L_s^- w &= \frac{s}{2}\int_Q \phi_{tt} w^2 + s\int_Q x^\alpha (x^\alpha \phi_x)_{xx} w w_x + 2s^2 \int_Q x^\alpha \phi_x \phi_{tx} w^2 \\ &\quad + s\int_Q \big(2x^{2\alpha}\phi_{xx} + \alpha x^{2\alpha-1}\phi_x\big) w_x^2 + s^3 \int_Q \big(2x^\alpha \phi_{xx} + \alpha x^{\alpha-1}\phi_x\big) x^\alpha \phi_x^2 w^2. \end{aligned} \tag{15}$$

Proof One easily checks that, for a.e. $t \in (0,T)$,

$$x^\alpha w_x = sx^\alpha \phi_x v e^{s\phi} + x^\alpha v_x e^{s\phi}.$$

Note that, because of our choice (6), $\phi_x = -\beta l x^{\beta-1}$, so that $x^\alpha \phi_x = -\beta l x^{\alpha+\beta-1}$. Then, the fact that $w \in L^2(0,T;\, D(A)) \cap H^1(0,T;\, L^2(0,1))$, as well as (12) and (13), follows from Lemma 2.2 and (11). Similarly, one can show $L_s^+ w, L_s^- w \in L^2(Q)$. As for (15), one obtains

$$\begin{aligned} \int_Q L_s^+ w\, L_s^- w &= \frac{s}{2}\int_Q \phi_{tt} w^2 + s\int_Q x^\alpha (x^\alpha \phi_x)_{xx} w w_x + 2s^2 \int_Q x^\alpha \phi_x \phi_{tx} w^2 \\ &\quad + s\int_Q (2x^{2\alpha}\phi_{xx} + \alpha x^{2\alpha-1}\phi_x) w_x^2 + s^3 \int_Q (2x^\alpha \phi_{xx} + \alpha x^{\alpha-1}\phi_x) x^\alpha \phi_x^2 w^2 \\ &\quad + \int_0^T \Big[x^\alpha w_x w_t - s\phi_x (x^\alpha w_x)^2 + s^2 x^\alpha \phi_t \phi_x w^2 \\ &\qquad\quad - s^3 x^{2\alpha}\phi_x^3 w^2 - s x^\alpha (x^\alpha \phi_x)_x w w_x \Big]_{x=0}^{x=1} \mathrm{d}t. \end{aligned}$$

This formula can be derived integrating by parts as in [4, Lemma 3.4], once one has checked that every integral on the right-hand side does exist. For the sake of brevity, we shall defer the proof of the last assertion to the reasoning below, where we provide bounds for each integrand. One can thus complete the proof recalling Lemma 2.2, together with (12) and (13), to deduce that the boundary terms are all equal to zero, so that the above identity reduces to (15). ■

We can now proceed with the proof of Theorem 2.3. Let us first note that the integrability of the functions in (8) is a consequence of Lemma 2.2. Next, since $e^{s\phi} L v = L_s^+ w + L_s^- w$, identity (15) yields

$$\begin{aligned} \|e^{s\phi} L v\|_{L^2(Q)}^2 &\ge s\int_Q \phi_{tt} w^2 + 2s\int_Q x^\alpha (x^\alpha \phi_x)_{xx} w w_x + 4s^2 \int_Q x^\alpha \phi_x \phi_{tx} w^2 \\ &\quad + 2s\int_Q (2x^{2\alpha}\phi_{xx} + \alpha x^{2\alpha-1}\phi_x) w_x^2 + 2s^3 \int_Q (2x^\alpha \phi_{xx} + \alpha x^{\alpha-1}\phi_x) x^\alpha \phi_x^2 w^2. \end{aligned}$$

Let us denote by $\sum_{k=1}^{5} J_k$ the right-hand side of the above estimate. We will now use the properties of the weight functions in (5), (6) and (7) to bound each J_k.

First of all, we have

$$|J_1| = \left| s \int_Q \phi_{tt} w^2 \right| \le s \int_Q |l''| w^2.$$

Yet, one can easily check that there exists a constant $C = C(T) > 0$ such that, for all $t \in (0, T)$, $|l''(t)| \le C l^3(t)$. Then, there exists $C = C(T) > 0$ such that

$$|J_1| \le Cs \int_Q l^3 w^2. \tag{16}$$

Observe that, although $l^3(t)$ is unbounded in Q, the above integral is finite by construction. Indeed, the exponential weight $e^{s\phi}$ in the definition of w suffices for the convergence of the integral. Similar considerations that apply to all other terms J_k will not be repeated.

Now, to estimate $J_2 = 2s \int_Q x^\alpha (x^\alpha \phi_x)_{xx} w w_x$ observe that $\phi_x(x, t) = -\beta l(t) x^{\beta-1}$. Then, $x^\alpha \phi_x(x, t) = -\beta l(t) x^{\alpha+\beta-1}$ and

$$x^\alpha (x^\alpha \phi_x)_{xx} = -\beta(\alpha + \beta - 1)(\alpha + \beta - 2) l(t) x^{2\alpha+\beta-3}. \tag{17}$$

Since

$$|x^\alpha (x^\alpha \phi_x)_{xx} w w_x| \le C(\alpha, \beta) l(t) [x^{2\alpha+\beta-4} w^2 + x^{2\alpha+\beta-2} w_x^2],$$

the integral in the definition of J_2 converges by (4). Moreover, in view of (12) and Lemma 2.2, we have

$$J_2 = s \int_Q x^\alpha (x^\alpha \phi_x)_{xx} \partial_x (w^2) = -s \int_Q (x^\alpha (x^\alpha \phi_x)_{xx})_x w^2,$$

where

$$(x^\alpha (x^\alpha \phi_x)_{xx})_x = -\beta(\alpha + \beta - 1)(\alpha + \beta - 2)(2\alpha + \beta - 3) l(t) x^{2\alpha+\beta-4}.$$

Let us show that the product $\beta(\alpha + \beta - 1)(\alpha + \beta - 2)(2\alpha + \beta - 3)$ is positive. First of all, since $1 - \alpha < \beta$, we have $\alpha + \beta - 1 > 0$. Since $\alpha < 1$ and $\beta < 1$, $\alpha + \beta - 2 < 0$. Also,

$$2\alpha + \beta - 3 < 2\alpha + 1 - \frac{\alpha}{2} - 3 = \frac{3}{2}\alpha - 2 < 0,$$

since $\alpha < 1$. Therefore, $\beta(\alpha + \beta - 1)(\alpha + \beta - 2)(2\alpha + \beta - 3) > 0$. Then, there exists $C = C(\alpha, \beta) > 0$ such that

$$J_2 \ge C(\alpha, \beta) s \int_Q l x^{2\alpha+\beta-4} w^2. \tag{18}$$

Next, we have

$$J_3 = 4s^2 \int_Q x^\alpha (-\beta x^{\beta-1} l(t))(-\beta x^{\beta-1} l'(t)) w^2 = 4s^2 \int_Q l(t) l'(t) \beta^2 x^{\alpha+2\beta-2} w^2,$$

where we observe that the integral is finite since, by (3),

$$x^{\alpha+2\beta-2}w^2(x,t) \le \frac{4}{(3-2\alpha)^2}\,|w(\cdot,t)|^2_{D(A)}x^{1+2\beta-\alpha}\,.$$

Also, $|l(t)l'(t)| \le Cl^3(t)$ for all $t \in (0,T)$ and some constant $C = C(T) > 0$. Then,

$$|J_3| \le Cs^2\int_Q l^3(t)x^{\alpha+2\beta-2}w^2. \tag{19}$$

Computing the derivatives in J_4, one has

$$\begin{aligned} J_4 &= 2s\int_Q(-2\beta(\beta-1)l(t)x^{2\alpha+\beta-2} - \alpha\beta l(t)x^{\alpha+\alpha-1+\beta-1})w_x^2 \\ &= 2s\int_Q l(t)\beta x^{2\alpha+\beta-2}(-2\beta+2-\alpha)w_x^2, \end{aligned}$$

where the integral is finite by (4). Moreover, $\beta < 1-\frac{\alpha}{2}$, so that $-2\beta-\alpha+2>0$. Then, for some $C = C(\alpha,\beta) > 0$

$$J_4 = C(\alpha,\beta)s\int_Q l(t)x^{2\alpha+\beta-2}w_x^2. \tag{20}$$

Finally, arguing in the same way for J_5 we have

$$\begin{aligned} J_5 &= 2s^3\int_Q(-2\beta(\beta-1)l(t)x^{\alpha+\beta-2} - \alpha\beta l(t)x^{\alpha-1+\beta-1})l^2(t)\beta^2x^{\alpha+2\beta-2}w^2 \\ &= 2s^3\int_Q \beta^3 l^3(t)(-2\beta+2-\alpha)x^{2\alpha+3\beta-4}w^2, \end{aligned}$$

where the convergence of the integral is again ensured by (4). Since $-2\beta+2-\alpha>0$, there exists $C = C(\alpha,\beta) > 0$ such that

$$J_5 = C(\alpha,\beta)s^3\int_Q l^3(t)x^{2\alpha+3\beta-4}w^2. \tag{21}$$

Coming back to (15), and using (16), (18), (19), (20) and (21), one has

$$\begin{aligned} \|e^{s\phi}Lv\|^2_{L^2(Q)} \ge &-Cs\int_Q l^3w^2 + C(\alpha,\beta)s\int_Q lx^{2\alpha+\beta-4}w^2 - Cs^2\int_Q l^3(t)x^{\alpha+2\beta-2}w^2 \\ &+ C(\alpha,\beta)s\int_Q l(t)x^{2\alpha+\beta-2}w_x^2 + C(\alpha,\beta)s^3\int_Q l^3(t)x^{2\alpha+3\beta-4}w^2. \end{aligned}$$

So, we can immediately deduce that, for some constant $C = C(T,\alpha,\beta) > 0$,

$$\begin{aligned} &\int_Q\big(s^3l^3(t)x^{2\alpha+3\beta-4}w^2 + sl(t)x^{2\alpha+\beta-4}\big)w^2 + \int_Q sl(t)x^{2\alpha+\beta-2}w_x^2 \\ &\quad\le C\bigg(\|e^{s\phi}Lv\|^2_{L^2(Q)} + s\int_Q l^3(t)w^2 + s^2\int_Q l^3(t)x^{\alpha+2\beta-2}w^2\bigg). \end{aligned} \tag{22}$$

Now, we are going to absorb the two rightmost terms of (22) on the left-hand side. First of all, we note that

$$2\alpha + 3\beta - 4 - (\alpha + 2\beta - 2) = \alpha + \beta - 2 < 0.$$

As a consequence, since $0 < x < 1$,

$$\int_Q l^3(t) x^{\alpha+2\beta-2} w^2 \le \int_Q l^3(t) x^{2\alpha+3\beta-4} w^2.$$

Moreover, we have already mentioned that $2\alpha + 3\beta - 4 < 0$, so that for all $x \in (0, 1)$, $1 \le x^{2\alpha+3\beta-4}$ and

$$\int_Q l^3(t) w^2 \le \int_Q l^3(t) x^{2\alpha+3\beta-4} w^2.$$

Then, (22) becomes

$$\begin{aligned} &\int_Q (s^3 l^3(t) x^{2\alpha+3\beta-4} + s l(t) x^{2\alpha+\beta-4}) w^2 + \int_Q s l(t) x^{2\alpha+\beta-2} w_x^2 \\ &\quad \le C\left(\|e^{s\phi} L v\|^2_{L^2(Q)} + (s + s^2) \int_Q l^3(t) x^{2\alpha+3\beta-4} w^2 \right), \end{aligned} \tag{23}$$

with $C = C(T, \alpha, \beta) > 0$. Now, there exists $s_0 = s_0(T, \alpha, \beta) > 0$ such that, for all $s \ge s_0$, $C(s + s^2) \le s^3/2$. Therefore, for all $s \ge s_0$ and some $C = C(T, \alpha, \beta) > 0$,

$$\int_Q \left(s^3 l^3(t) x^{2\alpha+3\beta-4} + s l(t) x^{2\alpha+\beta-4}\right) w^2 + \int_Q s l(t) x^{2\alpha+\beta-2} w_x^2 \le C \|e^{s\phi} L v\|^2_{L^2(Q)}. \tag{24}$$

Eventually, recalling that $w = v e^{s\phi}$, we have

$$\int_Q \left(s^3 l^3(t) x^{2\alpha+3\beta-4} + s l(t) x^{2\alpha+\beta-4}\right) v^2 e^{2s\phi} + \int_Q s l(t) x^{2\alpha+\beta-2} w_x^2 \le C \|e^{s\phi} L v\|^2_{L^2(Q)}. \tag{25}$$

Moreover, $v_x e^{s\phi} = w_x - s\phi_x v e^{s\phi}$. Therefore,

$$\int_Q s l(t) x^{2\alpha+\beta-2} v_x^2 e^{2s\phi} \le 2 \int_Q s l(t) x^{2\alpha+\beta-2} w_x^2 + 2 s^3 \beta^2 \int_Q l^3 x^{2\beta-2+2\alpha+\beta-2} v^2 e^{2s\phi}.$$

Thus,

$$\int_Q s l(t) x^{2\alpha+\beta-2} v_x^2 e^{2s\phi} \le 2 \int_Q s l(t) x^{2\alpha+\beta-2} w_x^2 + 2 s^3 \beta^2 \int_Q l^3 x^{2\alpha+3\beta-4} v^2 e^{2s\phi}.$$

The proof of Theorem 2.3 is then completed thanks to (25).

3. A unique continuation result

In this section, our goal is to show the following unique continuation property for the 'adjoint operator'

$$L v = v_t + (x^\alpha v_x)_x \quad \text{in } Q.$$

THEOREM 3.1 *Let $v \in L^2(0,T;\ D(A)) \cap H^1(0,T,L^2(0,1))$ and suppose that, for a.e. $t \in (0,T)$,*

$$v(0,t) = (x^\alpha v_x)(0,t) = 0. \tag{26}$$

If $Lv \equiv 0$ in Q, then $v \equiv 0$ in Q.

Proof Let $0 < \delta < 1$ and $\Omega_\delta := \{x \in (0,1) \colon p(x) > -\delta\}$. The first step of the proof consists in proving that $v \equiv 0$ in $\Omega_\delta \times (T/4, 3T/4)$. First of all, let us note that

$$x \in \Omega_\delta \quad \text{if and only if } x < \delta^{1/\beta}. \tag{27}$$

Now, let us take $\eta \in (\delta, 1)$ and $\chi \in C^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$ and

$$\chi(x) = \begin{cases} 1, & x \in \Omega_\delta, \\ 0, & x \notin \Omega_\eta. \end{cases}$$

From the definition of χ above and (27), we deduce that

$$\forall x \in [0, \delta^{1/\beta}], \quad \chi(x) = 1, \tag{28}$$

and

$$\forall x \in [\eta^{1/\beta}, 1], \quad \chi(x) = 0. \tag{29}$$

Define $u \in L^2(0,T;\ D(A)) \cap H^1(0,T;\ L^2(0,1))$ by $u := \chi v$, and observe that

$$Lu = \partial_t u + (x^\alpha u_x)_x = \chi v_t + (x^\alpha (\chi v)_x)_x.$$

Hence, after some standard computations, we get

$$Lu = \chi'' x^\alpha v + \chi' \alpha x^{\alpha-1} v + 2\chi' x^\alpha v_x. \tag{30}$$

In order to appeal to Corollary 2.4, we have to check that u satisfies the required boundary conditions. First of all, for a.e. $t \in (0,T)$, $u(0,t) = \chi(0)v(0,t) = 0$ by (26), and $u(1,t) = \chi(1)v(1,t) = 0$ by(29). Moreover, $u_x = \chi_x v + \chi v_x$, so that $x^\alpha u_x = x^\alpha \chi_x v + \chi x^\alpha v_x$. Using assumption (26) and property (28) for χ, one gets that $(x^\alpha u_x)(0,t) = 0$ for a.e. $t \in (0,T)$. Also, using property (29) for χ, one has $(x^\alpha u_x)(1,t) = 0$ for a.e. $t \in (0,T)$. Thus, we are in a position to apply Corollary 2.4 to u. We obtain

$$\int_Q s^3 l^3 u^2 e^{2s\phi} + \int_Q s l x^{2\alpha+\beta-2} u_x^2 e^{2s\phi} \le C \int_Q |Lu|^2 e^{2s\phi}.$$

Replacing Lu by the expression in (30), we immediately deduce that there exists $C = C(T, \alpha, \beta) > 0$ such that

$$\begin{aligned} &\int_Q s^3 l^3 u^2 e^{2s\phi} + \int_Q s l x^{2\alpha+\beta-2} u_x^2 e^{2s\phi} \\ &\le C\left(\int_Q \left(|\chi''|^2 x^{2\alpha} + |\chi'|^2 \alpha^2 x^{2\alpha-2}\right) v^2 e^{2s\phi} + \int_Q |\chi'|^2 x^{2\alpha} v_x^2 e^{2s\phi} \right). \end{aligned} \tag{31}$$

First of all, using (28) and (29),

$$\int_Q |\chi''|^2 x^{2\alpha} v^2 e^{2s\phi} \le \int_{\delta^{1/\beta}}^{\eta^{1/\beta}} \int_0^T |\chi''|^2 v^2 e^{2s\phi}. \tag{32}$$

As for the second term, we have

$$\int_Q |\chi'|^2 \alpha^2 x^{2\alpha-2} v^2 e^{2s\phi} = \int_{\delta^{1/\beta}}^{\eta^{1/\beta}} \int_0^T |\chi'|^2 \alpha^2 x^{2\alpha-2} v^2 e^{2s\phi}$$

because of (29). Then,

$$\int_Q |\chi'|^2 \alpha^2 x^{2\alpha-2} v^2 e^{2s\phi} \le \int_{\delta^{1/\beta}}^{\eta^{1/\beta}} \int_0^T \delta^{(2\alpha-2)/\beta} \alpha^2 |\chi'|^2 v^2 e^{2s\phi}. \tag{33}$$

Eventually, the last term satisfies the bound

$$\int_Q |\chi'|^2 x^{2\alpha} v_x^2 e^{2s\phi} \le \int_{\delta^{1/\beta}}^{\eta^{1/\beta}} \int_0^T |\chi'|^2 x^{\alpha} v_x^2 e^{2s\phi} \tag{34}$$

since $0 \le x \le 1$. Coming back to (31) and using (32), (33) and (34), we conclude that there exists a constant $C = C(T, \alpha, \beta, \delta, \eta) > 0$ such that

$$\int_Q s^3 l^3 u^2 e^{2s\phi} + \int_Q s l x^{2\alpha+\beta-2} u_x^2 e^{2s\phi} \le C \int_{\delta^{1/\beta}}^{\eta^{1/\beta}} \int_0^T (|\chi''|^2 + |\chi'|^2)(v^2 + x^{\alpha} v_x^2) e^{2s\phi}.$$

Therefore, for some constant $C = C(T, \alpha, \beta, \delta, \eta) > 0$,

$$\int_Q s^3 l^3 u^2 e^{2s\phi} + \int_Q s l x^{2\alpha+\beta-2} u_x^2 e^{2s\phi} \le C \int_{\delta^{1/\beta}}^{\eta^{1/\beta}} \int_0^T (v^2 + x^{\alpha} v_x^2) e^{2s\phi}.$$

Hence,

$$\int_Q s^3 l^3 u^2 e^{2s\phi} \le C \int_{\delta^{1/\beta}}^{\eta^{1/\beta}} \int_0^T (v^2 + x^{\alpha} v_x^2) e^{2s\phi}. \tag{35}$$

Our goal is to estimate the weight $e^{2s\phi}$ from above in order to simplify the right-hand side of (35). First note that, for all $t \in (0, T)$, $l(t) \ge l(T/2) = 4T^{-2}$. Also, since p is negative and decreasing, for all $(x, t) \in (\delta^{1/\beta}, \eta^{1/\beta}) \times (0, T)$,

$$2sp(x)l(t) \le \frac{8sp(x)}{T^2} \le \frac{8sp(\delta^{1/\beta})}{T^2}.$$

Then,

$$\int_{\delta^{1/\beta}}^{\eta^{1/\beta}} \int_0^T (v^2 + x^{\alpha} v_x^2) e^{2s\phi} \le \exp\left(\frac{8sp(\delta^{1/\beta})}{T^2}\right) \|v\|^2_{L^2(0,T;H^1_a(0,1))}. \tag{36}$$

Now, we want to estimate $e^{2s\phi}$ from below, so that we may simplify the left-hand side of (35). We set

$$Q_0 := \left\{(x,t)\in Q : p(x) > -\frac{\delta}{3}, \quad \frac{T}{4} < t < \frac{3T}{4}\right\}.$$

First, since $l(t) \geq 4T^{-2}$ for all $t \in (0,T)$, we have

$$\int_Q s^3 l^3 u^2 e^{2s\phi} \geq \int_Q s^3 \left(\frac{4}{T^2}\right)^3 u^2 e^{2s\phi} \geq \int_{Q_0} s^3 \left(\frac{4}{T^2}\right)^3 u^2 e^{2s\phi}.$$

Moreover, $l(t) \leq 16/3T^2$ for all $T/4 < t < 3T/4$. So, for all $(x,t)\in Q_0$ one has

$$2sp(x)l(t) \geq s\frac{32}{3T^2}p(x) \geq \frac{4}{3}\frac{8sp((\delta/3)^{1/\beta})}{T^2}.$$

Consequently,

$$\begin{aligned}\int_{Q_0} s^3\left(\frac{4}{T^2}\right)^3 u^2 e^{2s\phi} &\geq s^3 \exp\left(\frac{4}{3}\frac{8sp((\delta/3)^{1/\beta})}{T^2}\right)\int_{Q_0}\left(\frac{4}{T^2}\right)^3 u^2,\\ &= s^3 \exp\left(\frac{4}{3}\frac{8sp(\delta/3)}{T^2}\right)\int_{Q_0}\left(\frac{4}{T^2}\right)^3 \chi^2 v^2.\end{aligned}$$

Note that $p(x) > -\delta/3 \Leftrightarrow x \in (0, (\delta/3)^{1/\beta})$. So, on account of (28),

$$s^3 \exp\left(\frac{4}{3}\frac{8sp((\delta/3)^{1/\beta})}{T^2}\right)\int_{Q_0}\left(\frac{4}{T^2}\right)^3 \chi^2 v^2 = s^3 \exp\left(\frac{4}{3}\frac{8sp((\delta/3)^{1/\beta})}{T^2}\right)\int_{Q_0}\left(\frac{4}{T^2}\right)^3 v^2.$$

Finally,

$$\int_Q s^3 l^3 u^2 e^{2s\phi} \geq s^3 \exp\left(\frac{4}{3}\frac{8sp((\delta/3)^{1/\beta})}{T^2}\right)\int_{Q_0}\left(\frac{4}{T^2}\right)^3 v^2. \tag{37}$$

Coming back to (35), and using (36) and (37) we have

$$\begin{aligned}&s^3\left(\frac{4}{T^2}\right)^3 \|v\|^2_{L^2(Q_0)} \exp\left(\frac{4}{3}\frac{8sp((\delta/3)^{1/\beta})}{T^2}\right)\\ &\quad\leq C(T,\alpha,\beta,\delta)\exp\left(\frac{8sp((\delta/3)^{1/\beta})}{T^2}\right)T^2\|v\|^2_{L^2(0,T;H^1_\alpha(0,1))},\end{aligned}$$

from which we immediately deduce that

$$\|v\|^2_{L^2(Q_0)} \leq C(T,\alpha,\beta,\delta)\|v\|^2_{L^2(0,T;H^1_\alpha(0,1))}\frac{1}{s^3}\exp\left(\frac{8s}{T^2}\left[p(\delta^{1/\beta}) - \frac{4}{3}p((\delta/3)^{1/\beta})\right]\right).$$

Now, $p(\delta^{1/\beta}) - 4p((\delta/3)^{1/\beta})/3 = -\delta + 4\delta/9 = -5\delta/9$. Passing to the limit when $s \to \infty$, we have that $\|v\|^2_{L^2(Q_0)} = 0$. In conclusion,

$$v \equiv 0 \quad \text{in } \left(0, \left(\frac{\delta}{3}\right)^{1/\beta}\right) \times \left(\frac{T}{4}, \frac{3T}{4}\right).$$

To complete the proof, observe that the classical unique continuation for parabolic equations implies that $v \equiv 0$ in $(0,1)\times(T/4,3T/4)$. Equivalently, $e^{(T-t)A}v(T)=0$ for all $t\in(T/4,3T/4)$, where e^{tA} is the semigroup generated by A. Since e^{tA} is analytic for $t>0$, this implies that $v\equiv 0$ in $(0,1)\times(0,T)$. ∎

4. From unique continuation to approximate controllability

Let $0<\alpha<1$ and fix $T>0$. We are interested in the following initial-boundary-value problem

$$\begin{cases} u_t-(x^\alpha u_x)_x=0, & (x,t)\in Q=(0,1)\times(0,T),\\ u(0,t)=g(t), & t\in(0,T),\\ u(1,t)=0, & t\in(0,T),\\ u(x,0)=u_0(x), & x\in(0,1). \end{cases} \tag{38}$$

We aim at proving approximate controllability at time T for the above equation, which amounts to showing that for any final state u_T and any arbitrarily small neighbourhood $\mathcal{V}$ of u_T, there exists a control g driving the solution of (38) to $\mathcal{V}$ at time T.

Boundary control problems can be recast in abstract form in a standard way, see, e.g. [11]. Here, we follow a simpler method working directly on the parabolic problem, where the boundary control is reduced to a suitable forcing term. We begin by discussing the existence and uniqueness of solutions for (38).

4.1. *Well-posedness of (38)*

THEOREM 4.1 *For all $u_0\in H^1_{\alpha,0}(0,1)$ and all $g\in H^1_0(0,T)$, problem* (38) *has a unique mild solution $u\in L^2(0,T;H^1_\alpha(0,1)\cap C([0,1];L^2(0,1))$. Moreover,*

$$\sup_{t\in[0,T]}\|u(t)\|^2_{L^2(0,1)}+\|x^{\alpha/2}u_x\|^2_{L^2(0,T;L^2(0,1))}\le C(T)\Big(\|g\|^2_{H^1_0(0,T)}+\|u_0\|^2_{L^2(0,1)}\Big). \tag{39}$$

Furthermore, $(x^\alpha u_x)_x\in L^2(0,T;\ L^2(0,1))$ and (38) *is satisfied almost everywhere.*

Proof Let $u_0\in H^1_{\alpha,0}(0,1)$ and $g\in H^1_0(0,T)$. Let us introduce the initial-boundary-value problem with homogeneous boundary conditions

$$\begin{cases} y_t-(x^\alpha y_x)_x=-(1-x^{1-\alpha})\,g_t, & (x,t)\in Q,\\ y(0,t)=0, & t\in(0,T),\\ y(1,t)=0, & t\in(0,T),\\ y(x,0)=u_0(x), & x\in(0,1). \end{cases} \tag{40}$$

Let us first prove the existence of a solution of (38). Using the fact that A is the infinitesimal generator of an analytic semigroup, we know that problem (40) has a unique solution $y\in L^2(0,T;\ D(A))\cap H^1(0,T;\ L^2(0,1))$ (see, e.g. [2,7]). Moreover, multiplying the first equation of (40) by y and integrating over Q,

$$\sup_{t\in[0,T]}\|y(t)\|^2_{L^2(0,1)}+\|x^{\alpha/2}y_x\|^2_{L^2(0,T;L^2(0,1))}\le C(T,\alpha)\Big(\|g\|^2_{H^1_0(0,T)}+\|u_0\|^2_{L^2(0,1)}\Big). \tag{41}$$

Set, for a.e. $(x, t) \in Q$,

$$u(x,t) := y(x,t) + (1 - x^{1-\alpha})\, g(t). \tag{42}$$

Then, $u \in H^1(0, T; L^2(0,1)) \cap L^2(0, T; H^1_\alpha(0,1))$ and, as we observed in Example 2.1, $(x^\alpha u_x)_x = (x^\alpha y_x)_x \in L^2(0, T; L^2(0,1))$. Moreover,

$$\begin{aligned} u_t(x,t) &= y_t(x,t) + (1 - x^{1-\alpha})\, g_t(t) \\ &= (x^\alpha y_x)_x(x,t) - (1 - x^{1-\alpha})\, g_t(t) + (1 - x^{1-\alpha})\, g_t(t) \\ &= (x^\alpha y_x)_x(x,t) = (x^\alpha u_x)_x(x,t), \end{aligned}$$

for a.e. $(x, t) \in Q$. Since $u \in L^2(0, T; H^1_\alpha(0,1))$, for a.e. $t \in (0, T)$, $u(0,t)$ and $u(1,t)$ exist. Therefore, using (42), $u(0,t) = g(t)$ and $u(1,t) = 0$. Also, for a.e. $x \in (0,1)$, $u(x,0) = y(x,0) = u_0(x)$ since $g \in H^1_0(0,T)$. Consequently, u is a mild solution of (38) satisfying $(x^\alpha u_x)_x \in L^2(0,T; L^2(0,1))$ and $u \in H^1(0, T; L^2(0,1))$. Finally, estimate (39) follows from (41) and (42).

Next, let us prove uniqueness. Let u_1 and u_2 be two solutions of (38). Then, the difference $w := u_1 - u_2$ is a solution of (40), with $g \equiv 0$ and $u_0 \equiv 0$. Because of the uniqueness property of problem (40), $w \equiv 0$. ■

4.2. *Approximate controllability*

Our goal now is to show the following theorem.

THEOREM 4.2 *Let $u_0 \in H^1_{\alpha,0}(0,1)$. For all $u_T \in L^2(0,1)$ and all $\epsilon > 0$ there exists $g \in H^1_0(0,T)$ such that the solution u_g of problem* (38) *satisfies*

$$\|u_g(T) - u_T\|_{L^2(0,1)} \le \epsilon.$$

We start the proof with a lemma.

LEMMA 4.3 *If the conclusion of Theorem* 4.2 *is true for $u_0 \equiv 0$, then it is true for any $u_0 \in H^1_{\alpha,0}(0,1)$.*

Proof Let $u_0 \in H^1_{\alpha,0}(0,1)$ and $u_T \in L^2(0,1)$. Let $\epsilon > 0$. Let us introduce $\hat{u}$ the (mild) solution of

$$\begin{cases} \hat{u}_t - (x^\alpha \hat{u}_x)_x = 0, & (x,t) \in Q, \\ \hat{u}(0,t) = 0, & t \in (0,T), \\ \hat{u}(1,t) = 0, & t \in (0,T), \\ \hat{u}(x,0) = u_0(x), & x \in (0,1). \end{cases}$$

Then, $\hat{u}(T) \in L^2(0,1)$. Therefore, using the assumption of Lemma 4.3, there exists $g \in H^1_0(0,T)$ such that the solution υ_g of

$$\begin{cases} \upsilon_t - (x^\alpha \upsilon_x)_x = 0, & (x,t) \in Q, \\ \upsilon(0,t) = g(t), & t \in (0,T), \\ \upsilon(1,t) = 0, & t \in (0,T), \\ \upsilon(x,0) = 0, & x \in (0,1). \end{cases}$$

satisfies

$$\left\| \upsilon_g(T) - (u_T - \hat{u}(T)) \right\|_{L^2(0,1)} \le \epsilon.$$

Yet, one can easily see that $u_g(T) = \upsilon_g(T) + \hat{u}(T)$, so that the proof of Lemma 4.3 is achieved. ■

We now assume that $u_0 \equiv 0$.

LEMMA 4.4 *For all* $g \in H_0^1(0,T)$, *for all* $v \in L^2(0,1)$,

$$\left(u_g(T), v\right)_{L^2(0,1)} = \int_0^T (x^\alpha \hat{v}_x)(0,t)\, g(t)\mathrm{d}t, \tag{43}$$

where $\hat{v} \in C([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_{\alpha,0})$ *is the solution of*

$$\begin{cases} \hat{v}_t + (x^\alpha \hat{v}_x)_x = 0, & (x,t) \in Q, \\ \hat{v}(t,0) = 0, & t \in (0,T), \\ \hat{v}(t,1) = 0, & t \in (0,T), \\ \hat{v}(T,x) = v(x), & x \in (0,1). \end{cases} \tag{44}$$

Proof Let us multiply by $\hat{v}$ the equation satisfied by u_g. Then, integrating by parts with respect to the space variable, one has, for almost all $t \in (0,T)$,

$$\left(u_{g,t}(t), \hat{v}(t)\right)_{L^2(0,1)} + \int_0^1 x^{\alpha/2} u_{g,x}(t) x^{\alpha/2} \hat{v}_x(t)\mathrm{d}x = 0. \tag{45}$$

Moreover, for all $\eta > 0$, $\hat{v} \in L^2(0, T-\eta; D(A)) \cap H^1(0, T-\eta; L^2(0,1))$. We multiply by u_g the equation satisfied by $\hat{v}$ on $(0, T-\eta)$. After a standard integration by parts with respect to the space variable, one has, for a.e. $t \in (0, T-\eta)$,

$$\left(u_g(t), \hat{v}_t(t)\right)_{L^2(0,1)} - \int_0^1 x^{\alpha/2} u_{g,x}(t) x^{\alpha/2} \hat{v}_x(t)\mathrm{d}x = (x^\alpha \hat{v})_x(0,t)\, g(t). \tag{46}$$

Adding (45) and (46), one gets, for a.e. $t \in (0, T-\eta)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u_g(t), \hat{v}(t)\right)_{L^2(0,1)} = (x^\alpha \hat{v})_x(0,t)\, g(t).$$

Now, integrating over $(0, T-\eta)$ and recalling that $u_g(0) = u_0 = 0$, one obtains

$$\left(u_g(T-\eta), \hat{v}(T-\eta)\right)_{L^2(0,1)} = \int_0^{T-\eta} (x^\alpha \hat{v})_x(0,t)\, g(t)\mathrm{d}t. \tag{47}$$

Since $u_g \in C([0,T]; L^2(0,1))$, $\hat{v} \in C([0,T]; L^2(0,1))$ and $\hat{v}(T) = v$, one gets

$$\left(u_g(T), v\right)_{L^2(0,1)} = \int_0^T (x^\alpha \hat{v}_x)(0,t)\, g(t)\mathrm{d}t,$$

passing to the limit as $\eta \downarrow 0$. ■

Finally, define the control operator B by

$$B\colon H_0^1(0,T) \longrightarrow L^2(0,1), \quad B\colon g \longmapsto u_g(T).$$

According to (39), $B \in \mathcal{L}(H_0^1(0,T), L^2(0,1))$. Then, problem (38) is approximately controllable if and only if the range of B is dense in $L^2(0,1)$. This is equivalent to the fact that the orthogonal of $\mathcal{R}(B)$ is reduced to $\{0\}$.

LEMMA 4.5 *If* $v \in \mathcal{R}(B)^{\perp}$, *then* $(x^\alpha \hat{v}_x)(\cdot, 0) \equiv 0$.

Proof Take $v \in \mathcal{R}(B)^{\perp}$. According to (43), for all $g \in H_0^1(0,T)$,

$$\int_0^T (x^\alpha \hat{v}_x)(0,t)\, g(t) \mathrm{d}t = 0.$$

Even if $t \longmapsto (x^\alpha \hat{v}_x)(0,t)$ is not *a priori* in $L^2(0,T)$, we can conclude that $(x^\alpha \hat{v}_x)(\cdot,0) \equiv 0$. Indeed, take $\eta > 0$. Take $g \in \mathcal{D}(0, T-\eta)$ and set $g \equiv 0$ on $(T-\eta, T)$. Then $g \in H_0^1(0,T)$ and

$$0 = \int_0^T (x^\alpha \hat{v}_x)(0,t)\, g(t) \mathrm{d}t = \int_0^{T-\eta} (x^\alpha \hat{v}_x)(0,t)\, g(t) \mathrm{d}t.$$

Yet, $t \longmapsto (x^\alpha \hat{v}_x)(0,t) \in L^2(0, T-\eta)$, so that, by density, for all $g \in L^2(0, T-\eta)$,

$$\int_0^{T-\eta} (x^\alpha \hat{v}_x)(0,t)\, g(t) \mathrm{d}t = 0.$$

Therefore, $(x^\alpha \hat{v}_x)(\cdot, 0) \equiv 0$ on $(0, T-\eta)$ for all $\eta > 0$. ■

In order to complete the proof of Theorem 4.2, we just need to apply our unique continuation result: since the solution $\hat{v}$ of (44) satisfies $(x^\alpha \hat{v}_x)(\cdot, 0) \equiv 0$ on $(0,T)$, we have that $\hat{v}(T) = v = 0$.

Remark 1 Theorem 4.2 yields the approximate controllability in $L^2(0,1)$ of problem (38), as is easily seen arguing as follows. Let $T > 0$, $\epsilon > 0$ and u_0, $u_T \in L^2(0,1)$. Set $u_1 = e^{TA/2} u_0$ and observe that, since the semigroup is analytic, $u_1 \in H^1_{\alpha,0}(0,1)$. Therefore, owing to Theorem 4.2, there exists $g_1 \in H_0^1(T/2, T)$ such that the solution of the problem

$$\begin{cases} u_t - (x^\alpha u_x)_x = 0, & (x,t) \in (0,1) \times (T/2, T), \\ u(0,t) = g_1(t), & t \in (T/2, T), \\ u(1,t) = 0, & t \in (T/2, T), \\ u(x, T/2) = u_1(x), & x \in (0,1) \end{cases}$$

satisfies $\|u(T) - u_T\|_{L^2(0,1)} \leq \epsilon$. Thus, a boundary control g for (38) which steers the system into an ϵ-neighbourhood of u_T is given by

$$g(t) = \begin{cases} 0, & t \in [0, T/2), \\ g_1(t), & t \in [T/2, T]. \end{cases}$$

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