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Lipschitz continuity and local semiconcavity for exit time problems with state constraints

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Abstract

In the classical time optimal control problem, it is well known that the so-called Petrov condition is necessary and sufficient for the minimum time function to be locally Lipschitz continuous. In this paper, the same regularity result is obtained in the presence of nonsmooth state constraints. Moreover, for a special class of control systems we obtain a local semiconcavity result for the constrained minimum time function.

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1. Introduction

The minimum time problem is a classical problem in control theory. Given a nonempty closed set $\mathcal{K} \subset \mathbb{R}^n$ and a control system

$$\begin{cases} \dot{y}(t) = f\big(t, y(t), u(t)\big), & u(t) \in U, \\ y(0) = x, \end{cases} \tag{1.1}$$

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where $U \in \mathbb{R}^m$ is a compact set and u is a measurable function, the problem consists of finding a time optimal control u steering the solution $y^{x,u}(t)$ of (1.1) to the target $\mathcal{K}$. The minimum time needed to steer x to $\mathcal{K}$, regarded as a function of x, is called the minimum time function and is denoted by

$$\mathcal{T}_{\mathcal{K}}(x) = \inf_u \tau_{\mathcal{K}}(x, u),$$

where $\tau_{\mathcal{K}}(x, u) = \inf\{t \geqslant 0\colon\ y^{x,u}(t) \in \mathcal{K}\}$. Observe that $\mathcal{T}_{\mathcal{K}}(x) \in [0, \infty]$, in general. The controllable set $\mathcal{C}$ consists of all points $x \in \mathbb{R}^n$ such that $\mathcal{T}_{\mathcal{K}}(x)$ is finite. The regularity of the minimum time function, being related to the controllability properties of system (1.1), has been the object of an extensive literature. For example, one can show that system (1.1) is small time controllable on $\mathcal{K}$ if and only if $\mathcal{T}_{\mathcal{K}}$ is continuous on $\partial\mathcal{K}$, see, e.g., [3]. In a similar way, stronger regularity properties of $\mathcal{T}_{\mathcal{K}}$ can be proved to be equivalent to stronger controllability properties of system (1.1). More precisely, one can show that $\mathcal{T}_{\mathcal{K}}$ is locally Lipschitz in the controllable set—and dominated by the distance from $\mathcal{K}$ near the target—if and only if, for some constant $\mu > 0$,

$$\min_{u \in U} \bigl\langle f(t, x, u), \nu \bigr\rangle \leqslant -\mu|\nu|, \quad \forall t \geqslant 0,\ \forall x \in \partial\mathcal{K}$$

for all normal vectors ν to $\mathcal{K}$ at x. The above condition was introduced by Petrov [16] for a point target, and extended later to more general sets, see, e.g., [4,19]. Further regularity properties of the minimum time function are also known: under Petrov's condition, $\mathcal{T}_{\mathcal{K}}$ is semiconcave if $\mathcal{K}$ has the interior sphere property (see [9]), or if the set $f(t, x, U)$ of admissible velocities is sufficiently smooth (see [8,13,18]).

Note that the above discussion is restricted to unconstrained systems. It is an interesting question whether any of these properties remains true for problems with state constraints. In fact, as far as the semiconcavity of $\mathcal{T}_{\mathcal{K}}$ is concerned, it is easy to see that such a property brakes down if constraints are present, see Example 4.4. What about Lipschitz continuity, then? This paper aims at finding positive answers to such a question, as well as further addressing the issue of semiconcavity.

We consider, first, system (1.1) subject to nonsmooth state constraints, that is, given a domain $\Omega \subset \mathbb{R}^n$ and a point $x \in \Omega$, we take as admissible controls only the measurable functions u such that the corresponding trajectory $y^{x,u}(t)$ stays in $\overline{\Omega}$ for all $t \in [0, \tau_{\mathcal{K}}(x, u)]$. Adapting a technique due to Frankowska and Rampazzo [15], we will show that the minimum time function is Lipschitz continuous in $\mathcal{C}$ if, in addition to the assumptions that are usually imposed on control systems, the following conditions are satisfied:

(a) Ω has a suitable uniformly hypertangent conical field (see Section 3);
(b) $f(\cdot, x, u)$ is locally Lipschitz continuous.

It turns out, however, that assumption (b) above can be too restrictive for applications. For instance, in the study of dislocation dynamics one needs a Lipschitz regularity result for the constrained minimum time function of the simple control system

$$\begin{cases} \dot{y}(t) = c\bigl(t, y(t)\bigr)u(t), \quad u(t) \in U \text{ for a.e. } t, \\ y(0) = x, \end{cases} \tag{1.2}$$

where $c : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is just a *bounded measurable* function with respect to t. In Section 4 of this paper, we will analyze the regularity of the minimum time function for such a system in detail, giving a straightforward proof of the Lipschitz continuity of $\mathcal{T}_\mathcal{K}$ for $C^{1,1}$-smooth constraints, and obtaining a local semiconcavity result in the autonomous case. The Lipschitz regularity of $\mathcal{T}_\mathcal{K}$, for more general dynamics, could also be derived from a recent result by Bettiol and Frankowska [6]. As an application, we obtain a Lipschitz regularity result also for the value function of the exit time problem

$$\mathcal{V}_\mathcal{K}(x) = \inf_{u(\cdot)} \int_0^{\tau_\mathcal{K}(x,u)} L\big(y^{x,u}(t), u(t)\big)\, dt$$

assuming $L(\cdot, u)$ to be a bounded Lipschitz continuous function.

This paper is organized as follows. In Section 2 we give notations, definitions, and we describe the control system that we will study. In Section 3 we prove the Lipschitz continuity of the minimum time function for systems with nonsmooth state constraints. In Section 4 we focus our attention on the special system $f(t, x, u) = c(t, x)u$, deriving Lipschitz continuity and local semiconcavity for $\mathcal{T}_\mathcal{K}$. In Section 5 we extend some of the previous results to an exit time problem.

2. Preliminaries

2.1. Notation

Throughout this paper we denote by $|\cdot|$, $\langle \cdot, \cdot \rangle$, respectively, the Euclidean norm and scalar product in $\mathbb{R}^n$. For any subset $S \subseteq \mathbb{R}^n$, $\overline{S}$ stands for its closure, ∂S for its boundary and $S^c = \mathbb{R}^n \setminus S$ the complement.

The *distance function* from a set S is the function $d_S : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$d_S(x) := \inf_{y \in S} |x - y|.$$

We can also define the *signed distance* from S, $\bar{d}_S : \mathbb{R}^n \to \mathbb{R}$, that measures the distance from the boundary ∂S, with negative values for points in S, i.e.,

$$\bar{d}_S(x) := d_S(x) - d_{S^c}(x).$$

For any $x \in \mathbb{R}^n$ and $r > 0$ we denote the ball of radius r centered at x by $B(x, r) = \{y \in \mathbb{R}^n \colon |y - x| < r\}$, and we also use the simplified notations

$$B_r(x) := B(x, r), \qquad B_r := B_r(0), \qquad B := B_1.$$

We denote the r-neighborhood of a set S by $S + B_r = \{x \in \mathbb{R}^n \colon d_S(x) < r\}$.

The *tangent cone* to S at a point $x \in S$ is given by

$$T_S(x) := \left\{ v \in \mathbb{R}^n \colon \limsup_{y \to x,\, t \searrow 0} \frac{d_S(y + tv) - d_S(y)}{t} = 0 \right\}.$$

Then we can also define the *normal cone* to S at a point $x \in S$ as

$$N_S(x) := \{ p \in \mathbb{R}^n \colon \langle p, v \rangle \leqslant 0 \ \forall v \in T_S(x) \}.$$

If ∂S is smooth, for any $x \in \partial S$ the gradient $\nabla \overline{d}_S(x)$ gives the outward unit normal to S, and $N_S(x) = [0, +\infty)\nabla \overline{d}_S(x) = \{ t \nabla \overline{d}_S(x) \colon t \in [0, +\infty) \}$.

Definition 2.1. A closed set $S \in \mathbb{R}^n$ has the *Interior Sphere Property* (or *ISP*) of radius $r > 0$ if, for any $x \in \partial S$, there exists a point y_x such that $x \in \overline{B}_r(y_x) \subseteq S$.

The ISP is a one-sided regularity property for the boundary of S.

Definition 2.2. An open set $S \in \mathbb{R}^n$ has the *Exterior Sphere Property* (or *ESP*) of radius $r > 0$ if, for any $x \in \partial S$, there exists a point y_x such that $x \in \overline{B}_r(y_x) \subseteq S^c$.

Remark 2.3. An open set $S \subseteq \mathbb{R}^n$ has the ESP if and only if S^c has the ISP.

Remark 2.4. If both S and $\overline{S^c}$ have the ISP, then S has a $C^{1,1}$ boundary (and vice versa). We recall that ∂S is of class $C^{1,1}$ if the signed distance $\overline{d}_S(\cdot)$ is of class $C^{1,1}$ in a neighborhood of ∂S.

Proposition 2.5. *Let $S \subseteq \mathbb{R}^n$ be a closed set with the ISP of radius r. Then, for any $x \in \partial S$ and $y \in \overline{S^c}$*

$$\langle p_x, x - y \rangle \leqslant \frac{1}{2r} |x - y|^2,$$

where $p_x := \frac{x - y_x}{|x - y_x|}$, and y_x is given by Definition 2.1.

Proof. If $x \in \partial S$ we know that the ball $\overline{B}_r(y_x) \subseteq S$, and $x \in \partial B_r(y_x)$.

Now, let $y \in \overline{S^c}$ be such that

$$0 \leqslant \langle p_x, x - y \rangle \leqslant 2r. \tag{2.1}$$

Call $\bar{y}$ the projection of y on $x - p_x\mathbb{R}$, and call $\hat{y}$ the intersection of the segment $[\bar{y}, y]$ with $\partial B_r(y_x)$ (see Fig. 1). Note that the intersection $[\bar{y}, y] \cap \partial B_r(y_x)$ is a singleton because of (2.1). It is an obvious fact that

$$|x - y|^2 \geqslant |x - \bar{y}|^2 + |\bar{y} - \hat{y}|^2.$$

Moreover $|\bar{y} - \hat{y}|^2 = 2r|x - \bar{y}| - |x - \bar{y}|^2$, and then

$$|x - y|^2 \geqslant 2r|x - \bar{y}| = 2r\langle p_x, x - y \rangle.$$

On the other hand, if $\langle p_x, x - y \rangle < 0$, then the inequality is trivial, since $\frac{1}{2r}|x - y|^2 \geqslant 0$. Finally, for $\langle p_x, x - y \rangle > 2r$, we have that $|x - y| > 2r$. But this ensures that

$$2r\langle p_x, x - y \rangle \leqslant 2r|x - y| \leqslant |x - y|^2. \qquad \square$$

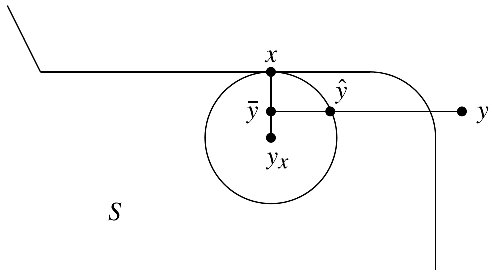

Fig. 1. A geometric proof.

Remark 2.6. For a set S with ESP, in view of Remark 2.3, we have a similar estimate as in Proposition 2.5, with $x \in \partial S$ and $y \in \overline{S}$.

The interior sphere property, like many other geometric properties, has important analytical applications. For example, it is deeply connected with semiconcavity, a regularity property whose definition we recall next.

Definition 2.7. A continuous function $u : \mathcal{O} \to \mathbb{R}$, with $\mathcal{O} \subseteq \mathbb{R}^n$, is called (*linearly*) *semiconcave* if there exists $C > 0$ such that

$$\lambda u(x) + (1-\lambda)u(y) - u\big(\lambda x + (1-\lambda)y\big) \leqslant C\lambda(1-\lambda)|x-y|^2,$$

for all $x \in \mathcal{O}$, and for all $x, y \in \mathbb{R}^n$ such that $[x, y] \subseteq \mathcal{O}$. In this case, we say that C is a *semiconcavity constant* for u in $\mathcal{O}$. We call $\mathrm{SC}(\mathcal{O})$ the class of the semiconcave functions on $\mathcal{O}$.

The following proposition establishes a useful connection between a certain nondegeneracy property of a semiconcave function u, and the interior sphere property of the level sets of u. Such a connection will be crucial for the discussion of Example 4.4. We recall that the superdifferential of a function $u : \mathcal{O} \to \mathbb{R}$ at a point $x \in \mathcal{O}$ is defined by

$$D^+u(x) = \left\{ p \in \mathbb{R}^n \colon \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x\rangle}{|y - x|} \leqslant 0 \right\}.$$

Proposition 2.8. *Let u be a semiconcave function on an open set $\mathcal{O} \subset \mathbb{R}^n$, with semiconcavity constant $C > 0$. Assume that, for some $\alpha > 0$, for all $x \in \mathcal{O}$ there exists $p_x \in D^+u(x)$ such that $|p_x| \geqslant \alpha$. For all $\lambda \in \mathbb{R}$, define the level set*

$$U_\lambda := \big\{ x \in \mathcal{O} \colon u(x) \leqslant \lambda \big\}.$$

Then there exists $r > 0$ such that, for all $\lambda \in \mathbb{R}$ and for all $x \in \partial U_\lambda$, there exists a unit vector ν_x such that

$$\overline{B}_r(x - r\nu_x) \cap \mathcal{O} \subseteq U_\lambda. \tag{2.2}$$

Note that, if $\mathcal{O} = \mathbb{R}^n$ or if $U_\lambda \subset \mathcal{O}$ (so that $\partial U_\lambda \cap \partial \mathcal{O} = \emptyset$), the above proposition yields that the level sets U_λ have the interior sphere property of radius r.

We give the proof for the reader's convenience (see also [5,7] for the case of $\mathcal{O} = \mathbb{R}^n$).

Proof. Set $r := \frac{\alpha}{2C}$, fix $\lambda \in \mathbb{R}$, and let $x \in U_\lambda$. By hypothesis, there is vector $p_x \in D^+u(x)$ such that $|p_x| \geqslant \alpha > 0$. Let us show that $\overline{B}_r(x - r\frac{p_x}{|p_x|}) \cap \mathcal{O}$ is contained in U_λ, i.e., for all $v \in \mathbb{R}^n$, $|v| \leqslant 1$,

$$x - r\frac{p_x}{|p_x|} + rv \in \mathcal{O} \quad \Longrightarrow \quad x - r\frac{p_x}{|p_x|} + rv \in U_\lambda.$$

Indeed, from a well-known property of D^+u for semiconcave functions (see [10, Proposition 3.3.1]) it follows that

$$\begin{aligned} u\left(x - r\frac{p_x}{|p_x|} + rv\right) &\leqslant u(x) + \left\langle p_x, rv - r\frac{p_x}{|p_x|}\right\rangle + Cr^2\left|v - \frac{p_x}{|p_x|}\right|^2 \\ &\leqslant \lambda + r\langle p_x, v\rangle - r|p_x| + 2Cr^2\left(1 - \frac{\langle p_x, v\rangle}{|p_x|}\right) \\ &= \lambda + r\big(\langle p_x, v\rangle - |p_x|\big)\left(1 - \frac{2Cr}{|p_x|}\right). \end{aligned}$$

On the other hand,

$$\big(\langle p_x, v\rangle - |p_x|\big)\left(1 - \frac{2Cr}{|p_x|}\right) \leqslant 0$$

by the choice of r and since $|p_x| \geqslant \alpha$. So, if we let $\nu_x := p(x)/|p(x)|$, then $u(x - r\nu_x + rv) \leqslant \lambda$. Thus, the intersection between $\overline{B}_r(x - r\nu_x)$ and $\mathcal{O}$ is contained in U_λ. □

2.2. Control system

Let $\Omega \subseteq \mathbb{R}^n$ be an open domain, and let $\mathcal{K} \subseteq \Omega$ be a closed subset. The set $\mathcal{K}$ is the *target* of our control system, and $\overline{\Omega}$ is the *constraint set*. For simplicity, we denote the signed distance from Ω by

$$d(x) := \bar{d}_\Omega(x).$$

Let $U \subseteq \mathbb{R}^m$ be compact (the *control set*). We consider a control system of the form

$$\begin{cases} \dot{y}(t) = f\big(t, y(t), u(t)\big), & u(t) \in U \text{ for a.e. } t, \\ y(0) = x, \\ y(t) \in \overline{\Omega} & \text{for all } t, \end{cases} \tag{2.3}$$

where $u \in L^1_{\mathrm{loc}}([0,\infty), U)$ is the *control function*, and $f : \mathbb{R}_+ \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is a function such that

$$\begin{cases} \text{(i) } t \mapsto f(t,x,u) \text{ is measurable for every } x \in \mathbb{R}^n, \text{ and for every } u \in U; \\ \text{(ii) } (x,u) \mapsto f(t,x,u) \text{ is continuous for any } t \geqslant 0, \text{ and } \exists L_0 > 0 \text{ such that} \\ \qquad \big|f(t,x,u) - f(t,y,u)\big| \leqslant L_0|x-y|, \quad \forall x,y \in \mathbb{R}^n,\ \forall u \in U,\ \forall t \geqslant 0; \\ \text{(iii) } f \text{ is bounded, i.e.,} \\ \qquad \big|f(t,x,u)\big| \leqslant M_0, \quad \forall x \in \mathbb{R}^n,\ \forall u \in U,\ \forall t \geqslant 0. \end{cases} \tag{2.4}$$

We denote by

$$F(t,x) := f(t,x,U)$$

the set of *admissible velocities*, and by $y^{x,u}(t) = y(t;x,u)$ the unique solution of (2.3) starting from point x, with control $u(\cdot)$. We define the *exit time* of trajectory $y^{x,u}$ by

$$\tau_{\mathcal{K}}(x,u) := \min\{t \geqslant 0:\ y^{x,u}(t) \in \mathcal{K}\},$$

where we set $\tau_{\mathcal{K}}(x,u) := +\infty$ if $y^{x,u}(t) \notin \mathcal{K}$ for all $t \geqslant 0$. We say that $u(\cdot)$ is an *admissible control* (on $[0,\vartheta]$) for a starting point $x \in \overline{\Omega}$ if $y^{x,u}(t) \in \overline{\Omega}$ for all $t \leqslant \tau_{\mathcal{K}}(x,u)$ (for all $t \leqslant \vartheta$), and we define the set of admissible controls as

$$\mathcal{A}(x) := \{u \in L^1_{\mathrm{loc}}:\ y^{x,u}(t) \in \overline{\Omega} \text{ for all } t \leqslant \tau_{\mathcal{K}}(x,u)\}.$$

The *minimum time function* is given by

$$\mathcal{T}_{\mathcal{K}}(x) := \inf_{u(\cdot)\in\mathcal{A}(x)} \tau_{\mathcal{K}}(x,u).$$

Fix a time $\vartheta \geqslant 0$. The *controllable set* to $\mathcal{K}$ in time ϑ is

$$\mathcal{C}(\vartheta) := \{x \in \Omega:\ \mathcal{T}_{\mathcal{K}}(x) \leqslant \vartheta\},$$

and the controllable set to $\mathcal{K}$ is $\mathcal{C} := \bigcup_{\vartheta\geqslant 0} \mathcal{C}(\vartheta)$.

2.3. Petrov's condition

In the following sections we will assume a controllability assumption "near" the target $\mathcal{K}$. The Petrov condition ensures that if a trajectory $y^{x,u}$ arrives sufficiently close to the target, then you can steer $y^{x,u}$ to $\mathcal{K}$ in a finite time.

Definition 2.9. We say that control system (2.3) and target $\mathcal{K}$ satisfy the *Petrov condition* if there exists $\mu > 0$ such that for any $t \geqslant 0$

$$\min_{u\in U}\langle f(t,x,u), \nu\rangle \leqslant -\mu|\nu|, \quad \forall x \in \partial\mathcal{K},\ \forall \nu \in N_{\mathcal{K}}(x). \tag{2.5}$$

The uniformity needed in Petrov's condition provides the validity of a very useful estimate for the minimum time, in a neighborhood of $\mathcal{K}$.

Proposition 2.10. *If $\mathcal{K} \subset \Omega$ and the control system* (2.3) *satisfies the Petrov condition* (2.5)*, then $\exists \eta, k > 0$ such that*

$$\mathcal{T}_{\mathcal{K}}(x) \leqslant k d_{\mathcal{K}}(x) \quad \forall x \in \mathcal{K} + B_\eta.$$

For a proof we refer the reader to [10].

3. Nonsmooth constraints

In this section we prove the local Lipschitz continuity of the minimum time function in the case of nonsmooth constraints, adapting some techniques from [15]. We consider a dynamics $f(t,x,u)$ satisfying assumptions (2.4), and a Lipschitz condition with respect to the time t. Consider a control system of the form

$$\begin{cases} \dot y(t) = f\big(t, y(t), u(t)\big), \quad u(t) \in U \text{ for a.e. } t, \\ y(0) = x, \\ y(t) \in \overline{\Omega} \quad \text{for all } t. \end{cases}$$

Definition 3.1. A set-valued map $\mathcal{I} : \partial\Omega \rightsquigarrow \mathbb{R}^n$ is called a *Uniformly Hypertangent Conical Field* (or *UHCF*) if there exist $\beta, \delta > 0$ such that, for every $x_0 \in \partial\Omega$, $\mathcal{I}(x_0)$ is a convex closed cone and for any unit vector $v \in \mathcal{I}(x_0)$

$$x + [0,\beta]B(v,\beta) \subseteq \overline{\Omega} \quad \forall x \in B(x_0,\delta) \cap \overline{\Omega}.$$

On the boundary of Ω we impose the following uniform regularity assumption:

$$\begin{cases} \exists \alpha > 0, \text{ and } \exists \mathcal{I}(\cdot) \text{ UHCF s.t. } \forall x \in \partial\Omega \text{ and } \forall t \geqslant 0 \\ F(t,x) \cap \mathcal{I}(x) \cap \big\{v \in \mathbb{R}^n \colon |v| \geqslant \alpha\big\} \neq \emptyset. \end{cases} \tag{3.1}$$

Remark 3.2. We note that assumption (3.1) implies that $\overline{\Omega}$ is *wedged* at all points of its boundary, i.e., $\mathrm{Int}(T_{\overline{\Omega}}(x)) \neq \emptyset$ for all points of $\partial\Omega$ (see [14, p. 166]). Therefore $\partial\Omega$ is Lipschitz by a result due to Rockafellar [17]. Thus, even though no any explicit regularity assumption has been made on $\partial\Omega$ in this paper, we actually need $\partial\Omega$ to be Lipschitz.

Remark 3.3. If Ω is bounded and system (2.3) is autonomous, then (3.1) is satisfied provided that

$$F(x_0) \cap \mathrm{Int}\big(T_{\overline{\Omega}}(x_0)\big) \neq \emptyset \quad \forall x_0 \in \partial\Omega.$$

For a proof, we refer the reader to [17].

Remark 3.4. If $\mathcal{I}(\cdot)$ is a Uniformly Hypertangent Conical Field, then there exists $r > 0$ s.t. for any point $x_0 \in \partial\Omega$

$$\big\langle n(x_0), v\big\rangle \leqslant r|v| \quad \forall v \in \mathcal{I}(x_0)$$

for some unit vector $n(x_0)$.

In addition, for the time dependence of the dynamics f, we assume that

$$\begin{cases} \forall R > 0\ \exists k_R(\cdot) \in L^1(\mathbb{R}_+) \text{ s.t. for any } \vartheta > 0 \\ \big|f(t,x,u) - f(s,x,u)\big| \leqslant k_R(\vartheta)|t-s|, \quad \forall t,s \in [0,\vartheta],\ \forall x \in B_R. \end{cases} \tag{3.2}$$

Our strategy to prove the Lipschitz continuity of the minimum time function, is split in two "big" steps. In the first step, we restrict us to (work with) short times (Lemma 3.6 and Theorem 3.7). In the second step, we get the final result by iterating the process (Theorem 3.8). Aiming at this we define some constants for later use. Let be $0 \leqslant \beta' < \alpha\beta r$, and fix $R > 0$. Consider

$$\hat{\beta} := \alpha\beta r, \tag{3.3}$$

$$\vartheta_0 := \min\left\{\frac{\beta}{M_0}, \frac{R}{4M_0}, \frac{\hat{\beta}-\beta'}{L_0 M_0}, \frac{\hat{\beta}}{2L_0 M_0}\right\}, \tag{3.4}$$

$$\bar{\vartheta} := \sup\left\{t \geqslant 0\colon \int_0^t k_R(\vartheta+\vartheta')\,d\vartheta \leqslant \frac{\hat{\beta}}{4}\ \forall \vartheta' \in [0, \vartheta_0]\right\}, \tag{3.5}$$

$$\hat{\vartheta} := \min\left\{\frac{\delta}{2M_0}, \frac{R}{4M_0}, \frac{\hat{\beta}}{2L_0(2M_0+\hat{\beta})}, \bar{\vartheta}\right\}, \tag{3.6}$$

$$\rho_0 := \min\left\{\frac{\hat{\beta}-\beta'}{2L_0}, \frac{\delta}{2}, \frac{\vartheta_0\hat{\beta}}{4}, \frac{R}{4}\right\}, \tag{3.7}$$

$$k_0 := \frac{\vartheta_0}{\rho_0}. \tag{3.8}$$

Remark 3.5. For any $x_0 \in \partial\Omega$ and $t_0 \geqslant 0$ we have that for all $t \in [t_0, t_0+\vartheta_0]$ there exists $v_t \in \mathcal{I}(x_0)$ such that

$$x + v_t + (t-t_0)\beta B \subseteq \overline{\Omega} \quad \forall x \in B(x_0, \delta) \cap \overline{\Omega}.$$

Proof. By assumption (3.1) there exists a measurable selection $v(s) \in F(s, x_0) \cap \mathcal{I}(x_0)$ for a.e. $s \in [t_0, t_0+\vartheta_0]$ such that $|v(s)| \geqslant \alpha$ for a.e. $s \in [t_0, t_0+\vartheta_0]$. We set

$$v_t := \int_{t_0}^t v(s)\,ds \quad \forall t \in [t_0, t_0+\vartheta_0]. \tag{3.9}$$

Note that $v_t \in \mathcal{I}(x_0)$ by convexity. Then,

$$x + |v_t|\frac{v_t}{|v_t|} + |v_t|\beta B \subseteq \overline{\Omega}.$$

The desired inclusion follows directly, since

$$(t-t_0)\hat{\beta} = (t-t_0)\alpha\beta r \leqslant |v_t|\beta. \qquad \square$$

We recall that β' is a constant arbitrarily chosen, so that $0 \leqslant \beta' < \hat{\beta}$.

Lemma 3.6. *Assume that* (2.4) *and* (3.1) *hold true. For any point $x_0 \in \partial\Omega$ there exists a control $u^{x_0}(\cdot)$ such that for every $t \in [t_0, t_0+\vartheta_0]$*

$$y^{x,u^{x_0}}(t) + (t-t_0)\beta' B \subseteq \overline{\Omega} \quad \forall x \in B(x_0, \rho_0) \cap \overline{\Omega}. \tag{3.10}$$

Proof. Let us choose $v(s)$ as in Remark 3.5. Then there exists a control $u^{x_0}(s) \in U$ such that $f(s, x_0, u^{x_0}(s)) = v(s)$. We already know, by Remark 3.5, that $x + v_t + (t - t_0)\hat{\beta} B \subseteq \overline{\Omega}$. So, in order to prove inclusion (3.10), we show that $y^{x,u^{x_0}}(t)$ is not too far from $x + v_t$. Indeed,

$$\begin{aligned}
\left|y^{x,u^{x_0}}(t) - (x + v_t)\right| &\leqslant \left| \int_{t_0}^{t} \left[f\left(s, y^{x,u^{x_0}}(s), u^{x_0}(s)\right) - f\left(s, x_0, u^{x_0}(s)\right)\right] ds \right| \\
&\leqslant \int_{t_0}^{t} \left[\left| f\left(s, y^{x,u^{x_0}}(s), u^{x_0}(s)\right) - f\left(s, x, u^{x_0}(s)\right)\right| \right. \\
&\qquad \left. + \left| f\left(s, x, u^{x_0}(s)\right) - f\left(s, x_0, u^{x_0}(s)\right)\right| \right] ds \\
&\leqslant L_0 \int_{0}^{t} \left[\left| y^{x,u^{x_0}}(s) - x\right| + |x - x_0| \right] ds \\
&\leqslant L_0 M_0 \frac{(t - t_0)^2}{2} + L_0 \rho_0 (t - t_0) \\
&= (t - t_0)\left(\frac{L_0 M_0}{2}(t - t_0) + L_0 \rho_0 \right).
\end{aligned}$$

Recalling the definition of ϑ_0 and ρ_0 in (3.4) and (3.7), we have that $\frac{L_0 M_0}{2}(t - t_0) + L_0 \rho_0 \leqslant \hat{\beta} - \beta'$, and then, for all $t \in [t_0, t_0 + \vartheta_0]$,

$$y^{x,u^{x_0}}(t) + (t - t_0)\beta' B \subseteq x + v_t + (t - t_0)\hat{\beta} B \subseteq \overline{\Omega}. \qquad \square$$

Theorem 3.7. *Assume that* (2.4)*,* (3.1) *and* (3.2) *hold true. Take a point $x_0 \in \overline{\Omega} \cap B_{\frac{R}{2}}$ and a time $\vartheta > \hat{\vartheta}$. Fix $t_0 \geqslant 0$ such that $t_0 + \hat{\vartheta} \leqslant \vartheta$ and an admissible control $u_0 \in \mathcal{A}(x_0)$. Then for any trajectory satisfying $y^{x,u}(t_0) \in B_{\rho_0}(y^{x_0,u_0}(t_0)) \cap \overline{\Omega}$ there is an admissible extension of u on $[t_0, t_0 + \hat{\vartheta}]$ such that*

$$\left|y^{x,u}(t) - y^{x_0,u_0}(t)\right| \leqslant C_0 \left|y^{x,u}(t_0) - y^{x_0,u_0}(t_0)\right| \quad \forall t \in [t_0, t_0 + \hat{\vartheta}],$$

where $C_0 = C_0(\vartheta) > 0$.

Proof. We can suppose $y^{x,u}(t_0) \in \partial\Omega$. Otherwise, we set $u(t) = u_0(t)$ until $y^{x,u}(t) \in \overline{\Omega}$, and we have that $|y^{x,u}(t) - y^{x_0,u_0}(t)| \leqslant e^{L_0 \hat{\vartheta}} |y^{x,u}(t_0) - y^{x_0,u_0}(t_0)|$. Moreover, for simplicity, we suppose $t_0 = 0$ so that $y^{x,u}(t_0) = x$ and $y^{x_0,u_0}(t_0) = x_0$.

Now, let $u^x(t) \in U$ be such that $f(t, x, u) = v(t)$ as in Lemma 3.6. Let us set

$$u(t) := \begin{cases} u^x(t) & \text{for } t \leqslant k_0|x - x_0| =: s_x, \\ u_0(t - s_x) & \text{for } t > s_x. \end{cases}$$

For simplicity, we use the notation

$$y_0(t) := y^{x_0,u_0}(t), \qquad y(t) := y^{x,u}(t).$$

Since $s_x \leqslant \vartheta_0$, by Lemma 3.6 we have that

$$y(t) \in \Omega \quad \forall t \leqslant s_x.$$

Moreover, as $\hat{\vartheta} \leqslant \frac{\delta}{2M_0}$ and $s_x \leqslant \vartheta_0$ for all $x \in B(x_0, \rho_0)$, we know that $y_0(t) + v_{s_x} + s_x\hat{\beta}B \subseteq \overline{\Omega}$ in view of Remark 3.5. To ensure that $u(\cdot)$ is admissible on $[0, s_x + \hat{\vartheta}]$, we will bound the distance between $y(t+s_x)$ and $y_0(t) + v_{s_x}$, showing that it is less than $s_x\hat{\beta}$. We set

$$\varphi(t) := \left| y(t+s_x) - \big(y_0(t) + v_{s_x}\big)\right|.$$

Then

$$\begin{aligned}
\varphi(t) - \varphi(0) &= \int_0^t \varphi'(s)\,ds \\
&\leqslant \int_0^t \left| f\big(s+s_x, y(s+s_x), u_0(s)\big) - f\big(s, y_0(s), u_0(s)\big)\right| ds \\
&\leqslant tL_0\big(M_0 s_x + |x_0 - x|\big)e^{L_0 t} + s_x \int_0^t k_R(\vartheta + s_x)\,d\vartheta.
\end{aligned}$$

Moreover, for $t = 0$, we have that

$$\begin{aligned}
\varphi(0) &= \left| x + \int_0^{s_x} f\big(s, y(s), u^x(s)\big)\,ds - x_0 - \int_0^{s_x} f\big(s, x_0, u^x(s)\big)\,ds \right| \\
&\leqslant |x - x_0| + L_0 \int_0^{s_x} \left| y(s) - x\right| ds \leqslant |x - x_0| + L_0 \int_0^{s_x} M_0 s\,ds \\
&\leqslant |x - x_0| + L_0 M_0 \frac{s_x^2}{2} \leqslant k_0|x - x_0|\left(\frac{1}{k_0} + \frac{L_0 M_0}{2} s_x\right) \\
&\leqslant s_x\left(\frac{\rho_0}{\vartheta_0} + \frac{L_0 M_0 \vartheta_0}{2}\right) \leqslant s_x \frac{\hat{\beta}}{2},
\end{aligned}$$

since $\rho_0 \leqslant \frac{\vartheta_0\hat{\beta}}{4}$ and $\vartheta_0 \leqslant \frac{\hat{\beta}}{2L_0M_0}$. Finally we have an estimate for $\varphi(t)$ on $[0, \hat{\vartheta}]$, i.e.,

$$\begin{aligned}
\varphi(t) &\leqslant \varphi(0) + \big(\varphi(t) - \varphi(0)\big) \\
&\leqslant s_x\left[\frac{\hat{\beta}}{2} + tL_0 e^{L_0\hat{\vartheta}}\left(M_0 + \frac{1}{k_0}\right) + \int_0^t k_R(\vartheta + s_x)\,d\vartheta\right] \\
&\leqslant s_x\hat{\beta},
\end{aligned}$$

since, by definition of $\hat{\vartheta}$, we are sure that

$$\hat{\vartheta} L_0 e^{L_0 \hat{\vartheta}} \left(M_0 + \frac{1}{k_0} \right) + \int_0^{\hat{\vartheta}} k_R(\vartheta + s_x)\, d\vartheta \leqslant \frac{\hat{\beta}}{2}.$$

Forasmuch as

$$y(t + s_x) \in y_0(t) + v_t + t\hat{\beta} B \subseteq \overline{\Omega},$$

$y(\cdot)$ is an admissible trajectory in $[0, \hat{\vartheta}]$. Now, we claim that we have a suitable estimate for the distance between $y(t)$ and $y_0(t)$. Indeed,

$$\begin{aligned} \big|y(t) - y_0(t)\big| &\leqslant \big|y(t+s_x) - y_0(t)\big| + \big|y(t) - y(t+s_x)\big| \\ &\leqslant e^{L_0 \hat{\vartheta}} \left(\big|y(s_x) - x_0\big| + s_x \int_0^{\hat{\vartheta}} k_R(\vartheta + s_x)\, d\vartheta \right) + M_0 s_x \\ &\leqslant |x - x_0| e^{L_0 \hat{\vartheta}} \left(1 + M_0 k_0 + k_0 \int_0^{\hat{\vartheta}} k_R(\vartheta + s_x)\, d\vartheta + M_0 k_0 e^{-L_0 \hat{\vartheta}} \right) \end{aligned}$$

for every $t \leqslant \hat{\vartheta}$. □

Note that all the constants (in particular $\hat{\vartheta}$ and C_0) are independent from the points $x_0, x \in \overline{\Omega}$. This allow us to obtain the following main result.

Theorem 3.8. *Assume that control system* (2.3) *satisfies the Petrov condition, and that assumptions* (2.4), (3.1) *and* (3.2) *hold true. Then the minimum time function $\mathcal{T}_{\mathcal{K}}(x)$ is locally Lipschitz continuous in the controllable set $\mathcal{C}$.*

Proof. Consider a point $x_0 \in \mathcal{C}$. Fix $R \geqslant 2|x_0| + 2M_0 \mathcal{T}_{\mathcal{K}}(x_0)$ and $\vartheta = \mathcal{T}_{\mathcal{K}}(x_0) + \frac{1}{2}$. For any $\frac{1}{2} > \varepsilon > 0$ we can choose an admissible control $u_0 \in \mathcal{A}(x_0)$ such that

$$\tau_0 := \tau_{\mathcal{K}}(x_0, u_0) \leqslant \mathcal{T}_{\mathcal{K}}(x_0) + \varepsilon.$$

Now we want to find $\rho', C' > 0$ such that for any $x \in B(x_0, \rho')$ there is an admissible control $u(\cdot) \in \mathcal{A}(x)$ with $|y^{x,u}(\tau_0) - y^{x_0,u_0}(\tau_0)| \leqslant C'|x - x_0|$.

Note that we can iterate the application of Theorem 3.7 to obtain such an estimate in, at most, $\hat{n} = \max\{n \in \mathbb{N}\colon n \leqslant \frac{\vartheta}{\hat{\vartheta}}\}$ steps. For instance, with $C' = C_0^{\hat{n}}$ and $\rho' = \frac{\rho_0}{C'}$. So, there exists a control $u(\cdot) \in \mathcal{A}(x)$ such that

$$d_{\mathcal{K}}\big(y^{x,u}(\tau_0)\big) \leqslant \big|y^{x,u}(\tau_0) - y^{x_0,u_0}(\tau_0)\big| \leqslant C'|x - x_0|.$$

Recalling that k and η are the constants defined in Proposition 2.10, and setting

$$\rho := \min\left\{\frac{\rho'}{2}, \frac{\eta}{C'}, \frac{M_0 \mathcal{T}_{\mathcal{K}}(x_0)}{C'}\right\},$$

we have that, for all $x \in B_\rho(x_0)$,

$$\mathcal{T}_{\mathcal{K}}\big(y^{x,u}(\tau_0)\big) \leqslant k d_{\mathcal{K}}\big(y^{x,u}(\tau_0)\big) \leqslant kC'|x - x_0|.$$

Since $\mathcal{T}_{\mathcal{K}}(x) \leqslant \tau_0 + \mathcal{T}_{\mathcal{K}}(y^{x,u}(\tau_0))$, we find that for any $x \in B(x_0, \rho)$ there exists a control $u(\cdot) \in \mathcal{A}(x)$ such that

$$\begin{aligned}
\mathcal{T}_{\mathcal{K}}(x) &\leqslant \tau_0 + \mathcal{T}_{\mathcal{K}}\big(y^{x,u}(\tau_0)\big) \leqslant \tau_0 + k d_{\mathcal{K}}\big(y^{x,u}(\tau_0)\big) \\
&\leqslant \tau_0 + k\big|y^{x,u}(\tau_0) - y^{x_0,u_0}(\tau_0)\big| \\
&\leqslant \mathcal{T}_{\mathcal{K}}(x_0) + \varepsilon + kC'|x - x_0|,
\end{aligned}$$

hence

$$\mathcal{T}_{\mathcal{K}}(x) - \mathcal{T}_{\mathcal{K}}(x_0) \leqslant kC'|x - x_0| + \varepsilon.$$

Finally, observe that we can switch the role of x and x_0. So, setting $C := kC'$, using the definition of ρ, and the arbitrary choice of ε, we have that $\mathcal{T}_{\mathcal{K}}(x)$ is Lipschitz continuous of rank C in $B(x_0, \rho)$. $\square$

4. A special class of control systems

In this section we restrict our attention to the class of control systems with admissible velocities of the form $F(t,x) = c(t,x)B$, where c is a scalar function. Clearly, this is a special case of control system (2.3). For these systems it is possible to produce easy proofs for finer results. In spite of its simplicity, this system is interesting for applications, see, for instance, [1,2,11] where the phase field model is applied to dislocation dynamics.

We provide a self-contained proof of the Lipschitz continuity of $\mathcal{T}_{\mathcal{K}}$, for dynamics that are just measurable in time. For this, the $C^{1,1}$ regularity of $\partial\Omega$ is needed. For more general dynamics $f(t,x,u)$, the Lipschitz continuity of $\mathcal{T}_{\mathcal{K}}$ can also be obtained from a result by Bettiol and Frankowska [6].

Once Lipschitz continuity is obtained, we turn our attention to higher regularity properties. Observe that the minimum time function is not semiconcave in $\overline{\Omega}$, even for very special classes of control systems (see Example 4.4). Nevertheless, we will prove that $\mathcal{T}_{\mathcal{K}}$ is locally semiconcave in Ω.

Let us consider a control system of the form

$$\begin{cases} \dot{y}(t) = c\big(t, y(t)\big)u(t), & u(t) \in B \text{ for a.e. } t, \\ y(0) = x, \\ y(t) \in \overline{\Omega} & \text{for all } t. \end{cases} \tag{4.1}$$

We observe that assuming (2.4) for $f(t,x,u)=c(t,x)u$ is equivalent to require that

$$\begin{cases} c(\cdot,x) \text{ is measurable for all } x\in\mathbb{R}^n \text{ and, for a.e. } t\geqslant 0,\\ \bigl|c(t,x)-c(t,y)\bigr|\leqslant L_0|x-y| \quad \forall x,y\in\mathbb{R}^n,\\ 0\leqslant c(t,x)\leqslant M_0 \quad \forall x\in\mathbb{R}^n. \end{cases} \tag{4.2}$$

For the purpose of prove Lipschitz continuity of $\mathcal{T}_{\mathcal{K}}$, with dynamics just measurable with respect to time t, we require on $\partial\Omega$ more regularity than Section 3, and we shall replace assumption (3.1) with a $C^{1,1}$ regularity of the boundary of Ω, i.e.,

$$\begin{cases} \exists\delta>0 \text{ such that } d(x) \text{ is of class } C^{1,1} \text{ in } \partial\Omega+B_\delta, \text{ with}\\ \bigl|\nabla d(x)-\nabla d(y)\bigr|\leqslant L_1|x-y| \quad \forall x,y\in\partial\Omega+B_\delta, \end{cases} \tag{4.3}$$

for some positive constant L_1.

Remark 4.1. Control system (4.1) satisfies Petrov condition if and only if

$$\exists\mu>0 \quad \text{s.t.} \quad c(t,x)\geqslant\mu \quad \forall x\in\partial\mathcal{K},\ \forall t\geqslant 0. \tag{4.4}$$

Under Petrov condition the minimum time function is locally Lipschitz continuous on the controllable set.

Theorem 4.2. *Suppose that system* (4.1) *satisfies assumptions* (4.2)–(4.4). *Then the minimum time function $\mathcal{T}_{\mathcal{K}}(x)$ is locally Lipschitz continuous in $\mathcal{C}$.*

To prove the above theorem, we use the Lipschitz dependence of solutions to (4.1) with respect to initial data. A proof for a more general context is given in [6]. In the special case of this section, we are able to explicitly construct admissible controls that realize this dependence.

Lemma 4.3. *Under the assumptions of Theorem* 4.2, *fix a time $\vartheta>0$, a point $x\in\overline{\Omega}$, and an admissible control $u_0\in\mathcal{A}(x_0)$. Then there exists a constant $C_0>0$, independent of x_0 and u_0, such that for any point $x\in\overline{\Omega}$ there exists a control $u\in\mathcal{A}(x)$ such that*

$$\bigl|y^{x,u}(t)-y^{x_0,u_0}(t)\bigr|\leqslant C_0|x-x_0| \quad \forall t\leqslant\vartheta. \tag{4.5}$$

Proof. Our strategy is to define an arc $y(\cdot)$ that satisfies inequality (4.5), and then to show that $y=y^{x,u}$ for some admissible control u.

Let us define the function (see Fig. 2)

$$\chi(x):=\begin{cases} 0 & \text{for } d(x)\leqslant-\frac{\delta}{2},\\ \frac{1}{\delta}(\delta+2d(x)) & \text{for } -\frac{\delta}{2}\leqslant d(x)\leqslant 0,\\ 1 & \text{for } 0\leqslant d(x)\leqslant\frac{\delta}{2},\\ \frac{1}{\delta}(2\delta-2d(x)) & \text{for } \frac{\delta}{2}\leqslant d(x)\leqslant\delta,\\ 0 & \text{for } d(x)\geqslant\delta. \end{cases}$$

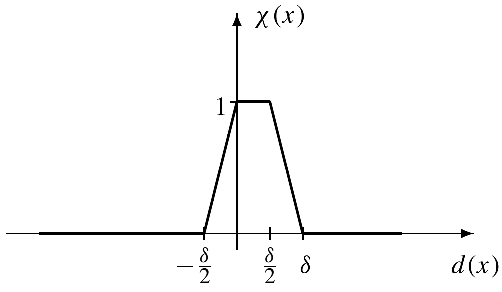

Fig. 2. The graph of χ.

Now, instead of $f(t,x,u) = c(t,x)u$, we consider a "revised" dynamics $\bar f$ defined by

$$\bar f(t,x,u) := c(t,x)\big[u - \langle u, \nabla d(x)\rangle_+ \nabla d(x)\chi(x)\big].$$

Note that $x \mapsto \bar f(t,x,u)$ is Lipschitz continuous. In fact, for $d(x) \leqslant -\frac{\delta}{2}$ and for $d(x) \geqslant \delta$, we have that $\bar f(t,x,u) = f(t,x,u)$. While, for $-\frac{\delta}{2} \leqslant d(x) \leqslant \delta$, all the terms are Lipschitz and bounded.

So, using control $u_0(\cdot)$, we can define the arc $y(\cdot)$ as the solution of

$$\begin{cases} \dot y(t) = \bar f\big(t, y(t), u_0(t)\big) & \text{a.e. } t, \\ y(0) = x. \end{cases}$$

We can observe that $y(\cdot)$ is also a solution of (4.1), with control strategy

$$u(t) = u_0(t) - \big\langle u_0(t), \nabla d\big(y(t)\big)\big\rangle_+ \nabla d\big(y(t)\big)\chi\big(y(t)\big).$$

Then $y(t) = y^{x,u}(t)$. Now we prove that $u(\cdot)$ is an admissible control. Note that $u(t) \in \overline{B}$ for a.e. $t \geqslant 0$. In fact

$$\begin{aligned} \big|u(t)\big|^2 &= \big|u_0(t)\big|^2 + \chi^2\big(y(t)\big)\big\langle u_0(t), \nabla d\big(y(t)\big)\big\rangle_+^2 - 2\chi\big(y(t)\big)\big\langle u_0(t), \nabla d\big(y(t)\big)\big\rangle_+^2 \\ &\leqslant \big|u_0(t)\big|^2 + \chi\big(y(t)\big)\big\langle u_0(t), \nabla d\big(y(t)\big)\big\rangle_+^2 - 2\chi\big(y(t)\big)\big\langle u_0(t), \nabla d\big(y(t)\big)\big\rangle_+^2 \\ &\leqslant \big|u_0(t)\big|^2 \leqslant 1. \end{aligned}$$

Moreover $y(t) \in \overline{\Omega}$ for all $t \leqslant \vartheta$. If not, arguing by contradiction, we can suppose that there exists $\bar t > 0$ such that

$$d\big(y(\bar t)\big) > 0.$$

Then we set $t_0 := \sup\{t \leqslant \bar t\colon\ d(y(t)) \leqslant 0\}$, and $t_1 := \min\{\bar t, t_0 + \frac{\delta}{2M_0}\}$. By continuity, we have that $d(y(t_0)) = 0$ and

$$d\big(y(t)\big) > 0 \quad \forall t \in (t_0, t_1]. \tag{4.6}$$

Thus,

$$\begin{aligned}
d\big(y(t_1)\big) = d\big(y(t_1)\big) - d\big(y(t_0)\big) &= \int_{t_0}^{t_1} c\big(s, y(s)\big)\big\langle \nabla d\big(y(s)\big), u(s)\big\rangle\, ds \\
&= \int_{t_0}^{t_1} c\big(s, y(s)\big)\Big[\big\langle \nabla d\big(y(s)\big), u_0(s)\big\rangle \\
&\quad - \underbrace{\big\langle \nabla d\big(y(s)\big), \nabla d\big(y(s)\big)\big\rangle}_{\equiv 1}\big\langle \nabla d\big(y(s)\big), u_0(s)\big\rangle_+ \underbrace{\chi\big(y(s)\big)}_{\equiv 1}\Big]\, ds \\
&\leqslant 0,
\end{aligned}$$

in contrast with (4.6).

Finally, using the notation $y_0(t) := y^{x_0,u_0}(t)$, we want to estimate the distance $|y(t) - y_0(t)|$. Let us set

$$\varphi(t) := \frac{1}{2}\big|y(t) - y_0(t)\big|^2.$$

Then, for a.e. $t > 0$ we have

$$\begin{aligned}
\varphi'(t) &= \big\langle y(t) - y_0(t), c\big(t, y(t)\big)u(t) - c\big(t, y_0(t)\big)u_0(t)\big\rangle \\
&\leqslant \big\langle y(t) - y_0(t), \big[c\big(t, y(t)\big) - c\big(t, y_0(t)\big)\big]u_0(t)\big\rangle \\
&\quad - \big\langle y(t) - y_0(t), \nabla d\big(y(t)\big)\big\rangle \underbrace{c\big(t, y(t)\big)}_{\leqslant M_0} \underbrace{\big\langle u_0(t), \nabla d\big(y(t)\big)\big\rangle_+}_{\leqslant 1} \underbrace{\chi\big(y(t)\big)}_{\leqslant 1} \\
&\leqslant L_0\big|y(t) - y_0(t)\big|^2 + \frac{M_0 L_1}{2}\big|y(t) - y_0(t)\big|^2 \\
&\leqslant 2\bigg(L_0 + \frac{M_0 L_1}{2}\bigg)\varphi(t),
\end{aligned}$$

where we used the ESP of Ω (i.e., the ISP of Ω^c; see Proposition 2.5 and Remarks 2.4 and 2.6). Then, setting $C_0 := e^{(L_0 + \frac{M_0 L_1}{2})\vartheta}$, we have $|y(t) - y_0(t)| \leqslant C_0|x - x_0|$. $\square$

Proof of Theorem 4.2. Fix $x_0 \in \mathcal{C}$. We want to find positive constants ρ and C such that

$$\big|\mathcal{T}_{\mathcal{K}}(x) - \mathcal{T}_{\mathcal{K}}(x_0)\big| \leqslant C|x - x_0| \quad \forall x \in B_\rho(x_0). \tag{4.7}$$

Fix $\vartheta = \mathcal{T}_{\mathcal{K}}(x_0) + \frac{1}{2}$. For any $\frac{1}{2} > \varepsilon > 0$ we can choose an admissible control $u_0 \in \mathcal{A}(x_0)$ such that $\tau_0 := \tau_{\mathcal{K}}(x_0, u_0) \leqslant \mathcal{T}_{\mathcal{K}}(x_0) + \varepsilon < \vartheta$.

Owing to Lemma 4.3, there exists a control $u \in \mathcal{A}(x)$ such that

$$d_{\mathcal{K}}\big(y^{x,u}(\tau_0)\big) \leqslant \big|y^{x,u}(\tau_0) - y^{x_0,u_0}(\tau_0)\big| \leqslant C_0|x - x_0|.$$

We can argue as in the proof of Theorem 3.8, and recalling that k and η are the constants defined in Proposition 2.10, if we set

$$\rho := \frac{\eta}{C_0},$$

then we have that, for all $x \in B_\rho(x_0)$,

$$\mathcal{T}_\mathcal{K}\big(y^{x,u}(\tau_0)\big) \leqslant k d_\mathcal{K}\big(y^{x,u}(\tau_0)\big) \leqslant kC_0|x - x_0|.$$

Finally, from this inequality, we obtain that

$$\mathcal{T}_\mathcal{K}(x) \leqslant \tau_0 + \mathcal{T}_\mathcal{K}\big(y^{x,u}(\tau_0)\big) \leqslant \tau_0 + kC_0|x - x_0| \leqslant \mathcal{T}_\mathcal{K}(x_0) + \varepsilon + kC_0|x - x_0|.$$

Since we can switch the role of x and x_0, and for the arbitrary choice of ε, we have (4.7) taking $C = kC_0$. $\square$

Despite of Lipschitz continuity of $\mathcal{T}_\mathcal{K}$ on $\overline{\Omega}$ can be also obtained in more general cases, it is difficult to obtain semiconcavity, even in the case of very simple control systems. Indeed, we give an example with trivial dynamics, inspired by [11], to show that the minimum time function may fail to be semiconcave on $\overline{\Omega}$.

Example 4.4. Let ν be a unit vector. Consider $\mathcal{K} = \overline{B(0, \frac{1}{2})}$ and $\Omega = B(0, R) \setminus \overline{B(2\nu, 1)}$, with $R > 4$. Set $c(t, x) \equiv 1$. Then arguing as in [11], one can show that, for $\sqrt{3} - \frac{1}{2} < \vartheta < 3$, the curvature of $\partial\mathcal{C}(\vartheta)$ blows up near $\partial B(2\nu, 1)$.

We claim that the minimum time function is not semiconcave in Ω. For suppose $\mathcal{T}_\mathcal{K}$ is semiconcave with constant $C > 0$. Then, since $\mathcal{T}_\mathcal{K}$ is a viscosity solution of the eikonal equation $|\nabla\mathcal{T}_\mathcal{K}(x)| = 1$ in Ω, we have that, for all $x \in \Omega$, at least one vector $p \in D^+\mathcal{T}_\mathcal{K}(x)$ must satisfy $|p| = 1$. Therefore the assumptions of Proposition 2.8 are satisfied with $\alpha = 1$. Choose $r = \frac{1}{2C}$, fix $\vartheta \in (\sqrt{3} - \frac{1}{2}, 3)$ and let

$$x \in \partial\mathcal{C}(\vartheta) \cap \partial B(2\nu, 1).$$

Take a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \Omega$ converging to x, such that $\mathcal{T}_\mathcal{K}$ is differentiable in x_k for all $k \in \mathbb{N}$. Set $\vartheta_k = \mathcal{T}_\mathcal{K}(x_k)$. Thanks to (2.2) we have that

$$\overline{B}_r\big(x_k - r\nabla\mathcal{T}_\mathcal{K}(x_k)\big) \cap \Omega \subseteq \mathcal{C}(\vartheta_k) \quad \forall k \in \mathbb{N}.$$

But this is impossible since $1/r$ would represent a bound for the curvature of $\partial\mathcal{C}(\vartheta_k)$ arbitrarily close to x.

In the above example it is no coincidence that semiconcavity brakes down at the boundary of Ω. Indeed, we will show that $\mathcal{T}_\mathcal{K}$ is locally semiconcave in Ω. For this purpose, we shall restrict the analysis to autonomous systems, i.e.,

$$c(t, x) = c(x) \quad \forall t.$$

Like for unconstrained problems, we will assume that, for some $L_2, m_0 > 0$,

$$\begin{cases} \left|\nabla c(x) - \nabla c(y)\right| \leqslant L_2|x-y| & \forall x, y \in \mathbb{R}^n, \\ c(x) \geqslant m_0 & \forall x \in \mathbb{R}^n. \end{cases} \tag{4.8}$$

Theorem 4.5. *Assume that* (4.2) *holds true. Let $c(t,x) = c(x)$, and let* (4.8) *be satisfied. If $\mathcal{T}_{\mathcal{K}}(\cdot)$ is locally Lipschitz continuous in Ω, then $\mathcal{T}_{\mathcal{K}} \in \mathrm{SC}_{\mathrm{loc}}(\Omega \setminus \mathcal{K})$.*

To prove the above theorem, let us consider the unconstrained control system

$$\begin{cases} \dot{y}(t) = c\big(y(t)\big)u(t), \quad u(t) \in B \text{ for a.e. } t, \\ y(0) = x \end{cases}$$

and the associated exit time

$$\tilde{\tau}_{\mathcal{K}}(x,u) := \min\big\{t \geqslant 0\colon\ y^{x,u}(t) \in \mathcal{K}\big\}.$$

The (unconstrained) minimum time function is defined by

$$\widetilde{\mathcal{T}}_{\mathcal{K}}(x) := \inf_{u(\cdot)\in L^1_{\mathrm{loc}}} \tilde{\tau}_{\mathcal{K}}(x,u).$$

Then, as shown in [8] under milder assumptions than those of Theorem 4.5, $\widetilde{\mathcal{T}}_{\mathcal{K}}(x)$ is locally semiconcave in $\mathbb{R}^n \setminus \mathcal{K}$.

Proof of Theorem 4.5. Let $x_0 \in \Omega \setminus \mathcal{K}$ and define the constants

$$\begin{aligned} \eta &:= \left|d(x_0)\right|, \\ \tau_0 &:= \mathcal{T}_{\mathcal{K}}(x_0), \\ \vartheta &:= \min\left\{\frac{\eta}{4M_0}, \tau_0\right\}. \end{aligned}$$

Since $\vartheta > 0$ and controllable sets are closed, there exists $\delta_\eta > 0$ such that

$$\overline{B(x_0,\delta_\eta)} \cap \mathcal{C}(\tau_0 - \vartheta) = \emptyset.$$

Now, consider the optimal time problem with target $\mathcal{C}(\tau_0 - \vartheta)$. As recalled above, the unconstrained minimum time function $\widetilde{\mathcal{T}}_{\mathcal{C}(\tau_0-\vartheta)}(\cdot)$ is locally semiconcave in $\mathbb{R}^n \setminus \mathcal{C}(\tau_0 - \vartheta)$. Moreover, $\mathcal{T}_{\mathcal{K}}(x) = \mathcal{T}_{\mathcal{C}(\tau_0-\vartheta)}(x) + \tau_0 - \vartheta$ for all $x \in B(x_0,\delta_\eta)$. Therefore, it suffices to prove that

$$\mathcal{T}_{\mathcal{C}(\tau_0-\vartheta)}(x) = \widetilde{\mathcal{T}}_{\mathcal{C}(\tau_0-\vartheta)}(x) \quad \forall x \in B(x_0,\varepsilon), \tag{4.9}$$

where

$$\varepsilon := \min\left\{\delta_\eta, \frac{\eta}{4M_0L}\right\},$$

and L is a Lipschitz constant for $\mathcal{T}_{\mathcal{K}}$ in $\overline{B(x_0,\eta)}$.

To show (4.9), fix $x \in B(x_0, \varepsilon)$ and define $\tau_x := \widetilde{\mathcal{T}}_{\mathcal{C}(\tau_0-\vartheta)}(x)$. First, let us prove the inequality $\mathcal{T}_{\mathcal{C}(\tau_0-\vartheta)}(x) \leqslant \widetilde{\mathcal{T}}_{\mathcal{C}(\tau_0-\vartheta)}(x)$. Let $y^x(\cdot)$ be an optimal trajectory for x, i.e., a trajectory of the unconstrained problem such that $y^x(0) = x$ and $y^x(\tau_x) \in \mathcal{C}(\tau_0 - \vartheta)$. Since $x \in B(x_0, \varepsilon)$, we have $|d(x)| \geqslant \frac{3}{4}\eta$ (notice that it is not restrictive to assume $M_0 L \geqslant 1$) and $\tau_x \leqslant \vartheta + L\varepsilon$. Therefore, for all $t \in [0, \tau_x]$,

$$\begin{aligned}\left|d\left(y^x(t)\right)\right| &\geqslant \frac{3}{4}\eta - M_0\tau_x \geqslant \frac{3}{4}\eta - M_0(\vartheta - L\varepsilon)\\ &\geqslant \frac{\eta}{4}.\end{aligned}$$

This proves that $y^x(t) \in \Omega$ for all $t \in [0, \tau_x]$, and then

$$\mathcal{T}_{\mathcal{C}(\tau_0-\vartheta)}(x) \leqslant \widetilde{\mathcal{T}}_{\mathcal{C}(\tau_0-\vartheta)}(x).$$

The opposite inequality is trivial, since any trajectory of (4.1) also solves the unconstrained system. Hence $\mathcal{T}_{\mathcal{K}}(x)$ is semiconcave in $B(x_0, \varepsilon)$. $\square$

Note that, in Theorem 4.5, we made no assumptions on the regularity of $\partial\Omega$ (we only need Ω to be an open domain). So, this proposition applies to all contexts in which Lipschitz continuity of $\mathcal{T}_{\mathcal{K}}(x)$ holds true.

Remark 4.6. The result of Theorem 4.5 can be extended to more general dynamics $f(x, u)$ (using similar arguments). In order to apply the semiconcavity results of [8] for unconstrained systems, we just need to assume that

- $F(x)$ is a convex set and has the ISP of radius r for all $x \in \mathbb{R}^n$;
- $B_\delta \subseteq F(x)$ for all $x \in \mathbb{R}^n$;
- $x \rightsquigarrow \partial F(x)$ is a Lipschitz boundary map (see [8] for the definition);
- $x \mapsto \nabla_x f(x, u)$ is Lipschitz continuous of rank L_2 for all $u \in U$

for some positive constants r, δ, L_2.

5. Optimal exit time problems

The set-up of this section is similar to Section 3, but we consider the autonomous system

$$\begin{cases} \dot{y}(t) = f\big(y(t), u(t)\big), & u(t) \in U \text{ for a.e. } t,\\ y(0) = x, & \\ y(t) \in \overline{\Omega} & \text{for all } t, \end{cases} \tag{5.1}$$

and we add a *running cost* $L : \mathbb{R}^n \times U \to \mathbb{R}$ that satisfies the following assumptions

$$\begin{cases} \text{(i) there exist } M_1, \alpha_1 > 0 \text{ such that}\\ \quad M_1 \geqslant L(x, u) \geqslant \alpha_1 > 0 \quad \forall x \in \overline{\Omega};\\ \text{(ii) } L \text{ is continuous, and } \exists L_1 > 0 \text{ such that}\\ \quad \big|L(x, u) - L(y, u)\big| \leqslant L_1|x - y| \quad \forall x, y \in \mathbb{R}^n. \end{cases} \tag{5.2}$$

For any $x \in \Omega$ and $u \in \mathcal{A}(x)$, if $\tau_{\mathcal{K}}(x,u) < +\infty$ we set the *cost functional*

$$J(x,u) := \int_0^{\tau_{\mathcal{K}}(x,u)} L\big(y^{x,u}(t), u(t)\big)\,dt$$

or $J(x,u) := +\infty$ otherwise. Finally we define the *value function*

$$\mathcal{V}_{\mathcal{K}}(x) := \inf_{u \in \mathcal{A}(x)} J(x,u).$$

To obtain a local Lipschitz continuity also for this value function, we analyze the minimum time function of an equivalent problem. Let us consider

$$\begin{cases} \dot{y}(t) = \bar{f}\big(y(t), u(t)\big), \quad u(t) \in U \text{ for a.e. } t, \\ y(0) = x, \\ y(t) \in \overline{\Omega} \quad \text{for all } t, \end{cases} \tag{5.3}$$

where

$$\bar{f}(x,u) := \frac{1}{L(x,u)} f(x,u),$$

and denote by $\overline{\mathcal{T}}_{\mathcal{K}}(\cdot)$ the minimum time function for the control system (5.3). Now, call

$$\mathcal{C}(\lambda) := \big\{x \in \Omega\colon\ \mathcal{V}_{\mathcal{K}}(x) \leqslant \lambda\big\}$$

the controllable set with cost λ (for system (5.1)), and

$$\overline{\mathcal{C}}(\vartheta) := \big\{x \in \Omega\colon\ \overline{\mathcal{T}}_{\mathcal{K}}(x) \leqslant \vartheta\big\}$$

the controllable set in time ϑ (for system (5.3)). Then, $\overline{\mathcal{T}}_{\mathcal{K}}(\cdot)$ can be identified as the value function of the original exit time problem.

Proposition 5.1. *Assume hypotheses* (2.4) *and* (5.2). *Then*

$$\mathcal{V}_{\mathcal{K}}(x) = \overline{\mathcal{T}}_{\mathcal{K}}(x) \quad \forall x \in \overline{\Omega}.$$

Moreover for any $\lambda > 0$ we have $\mathcal{C}(\lambda) = \overline{\mathcal{C}}(\lambda)$.

For the proof we refer the reader to [12]. This allows us to use the results of the previous section for the minimum time function.

Proposition 5.2. *Assume that control system* (5.1) *satisfies Petrov's condition, and that assumptions* (2.4), (3.1) *and* (5.2) *hold true. Then the value function $\mathcal{V}_{\mathcal{K}}(x)$ is locally Lipschitz in the controllable set $\mathcal{C}$.*

Proof. On account of Proposition 5.1, we only have to prove that control system (5.3) satisfies hypothesis of Theorem 3.8.

It is easy to check that $\bar{f}$ satisfies Petrov's condition and (3.1), with $\bar{\mu} = \frac{\mu}{M_1}$ and $\bar{\alpha} = \frac{\alpha}{M_1}$.

To check hypothesis (2.4), observe that $|\bar{f}|$ is bounded by $\frac{M_0}{\alpha_1}$, and that

$$\begin{aligned}\left|\bar{f}(x,u) - \bar{f}(y,u)\right| &\leqslant \left|\frac{L(y,u)f(x,u) - L(x,u)f(y,u)}{L(x,u)L(y,u)}\right| \\ &\leqslant \frac{L(y,u)|f(x,u) - f(y,u)| + |f(y,u)||L(y,u) - L(x,u)|}{L(y,u)L(x,u)} \\ &\leqslant \left(\frac{L_0}{\alpha_1} + \frac{M_0 L_1}{\alpha_1^2}\right)|x-y|. \qquad \square\end{aligned}$$

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