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SEMICONCAVITY OF THE VALUE FUNCTION FOR A CLASS OF DIFFERENTIAL INCLUSIONS

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Abstract. We provide intrinsic sufficient conditions on a multifunction F and endpoint data φ so that the value function associated to the Mayer problem is semiconcave.

1. **Introduction.**

This paper studies the Mayer problem

$$\min \varphi\big(x(T)\big), \tag{1}$$

where the minimization is over all absolutely continuous arcs $x(\cdot)$ that satisfy the differential inclusion

$$\dot{x}(s) \in F\big(x(s)\big) \quad \text{a.e. } s \in [t, T] \tag{2}$$

and the initial condition

$$x(t) = x. \tag{3}$$

Here $\varphi : \mathbb{R}^n \to \mathbb{R}$ is the endpoint cost, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction that describes the dynamics, and $x \in \mathbb{R}^n$ is the initial state. We denote this optimization problem by $\mathcal{P}(t, x)$, and the value function associated to $\mathcal{P}(t, x)$ is denoted by $V(t, x)$. An optimal solution $\bar{x}(\cdot)$ of $\mathcal{P}(t, x)$ is thus an arc satisfying (2) and (3) with $\varphi\big(\bar{x}(T)\big) = V(t, x)$. Assuming φ is semiconcave, the main goal of this paper is to provide new sufficient conditions on F so that $(t, x) \mapsto V(t, x)$ is semiconcave.

The fact that V is locally Lipschitz is well-known (see e.g. [6, Exercise 4.3.12]). The sought-after semiconcavity conclusion has been obtained previously (see [2] or [3, Chapter 7]) in the case where F is given with a C^{1+} parameterization, which means that F has the form

$$F(x) = \big\{ f(x, u) : u \in U \big\} \tag{4}$$

with $U \subseteq \mathbb{R}^m$ compact and $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ satisfying

- f is continuous in (x, u), and

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- $\nabla_x f(x,u)$ exists and is locally Lipschitz in x independent of $u \in U$: for each $r > 0$, there exists $k > 0$ so that $|x_0|, |x_1| \leq r$ implies
$$\big|\nabla_x f(x_0,u) - \nabla_x f(x_1,u)\big| \leq k|x_0 - x_1|$$
for all $u \in U$.

The well-known Filippov lemma (cf. [3], Theorem 7.1.5) says a multifunction F of the form (4) has (under minimal hypotheses that are in force here) the property that the trajectories of the differential inclusion (2) coincide with the trajectories of the associated control system. That is, an arc $x(\cdot)$ is a solution of (2) if and only if there exists a measurable function $u(\cdot) : [t,T] \to U$ so that $x(\cdot)$ satisfies the ODE $\dot{x}(s) = f\big(x(s), u(s)\big)$ for almost all $s \in [t,T]$. Since the cost function $\varphi(\cdot)$ in the Mayer problem (1) depends only on the endpoints of the trajectories of (2) and not on how F may have been parameterized as in (4), it is natural to look for assumptions directly on F (or, equivalently, its associated Hamiltonian) that imply the semiconcavity of V rather than rely on a particular parameterization. To further illustrate this point, with $n = 1$, the multifunction $F(x) = [-|x|, |x|]$ can be parameterized as
$$F(x) = \{xu : |u| \leq 1\}$$
or as
$$F(x) = \{|x|u : |u| \leq 1\},$$
the former having the C^{1+} parameterization property while the latter does not. Of course the trajectories of the two systems coincide, but the heretofore known theorems asserting semiconcavity of the value function only apply to the former system.

We point out that although Lipschitz multifunctions with convex values always admit parameterizations by Lipschitz functions (see [1] and [7]), it is a much more challenging problem and still an open issue to determine which multifunctions admit parameterizations with smooth functions. If $F(x)$ is originally presented as in (4) by a smooth parameterization, and then the multifunction $F(\cdot)$ is given a reparameterization by one of the known parameterization theorems, the resulting reparameterization would generally not coincide with the original parameterization and would not be smooth (except in the trivial case where $F(x)$ is a singleton for all x). Even worse is that it is not known in general which multifunctions admit a smooth selection - see [8] for the case $n = 1$ - let alone be smoothly parameterized.

Our approach then is not based on finding a smooth parameterization, but rather relies on assumptions stated directly in terms of the multifunction F. Recall the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ associated to F is defined by
$$H(x,p) = \sup_{v \in F(x)} \langle v, p\rangle, \tag{5}$$
and that there is a one-to-one correspondence between Hamiltonian functions that are convex and positively homogeneous in p and multifunctions F with closed and convex values. The relationship is
$$v \in F(x) \quad \Longleftrightarrow \quad \langle v, p\rangle \leq H(x,p) \quad \forall p \in \mathbb{R}^n.$$
Hence assumptions given in terms of H are intrinsic to the trajectories generated by F and are not influenced by any particular parameterization. Moreover, our proof will rely heavily on the maximum principle instead of using a priori estimates on trajectories that ensue from a known smooth parameterization, as is done in [2, 3].

Precise and complete definitions and statements are given in the next two sections, but to give a flavor of our main result, it suffices to say here that in addition to the usual Standing Hypotheses (SH) that are stated precisely in Section 3 below, the new Hypotheses (H) are the following properties which are to hold for each compact convex subset $K \subseteq \mathbb{R}^n$ and $p \neq 0$:

$$(H) \qquad \begin{cases} \mathbf{1}) \quad \text{There exists a constant } c \geq 0 \text{ (depending on } K\text{) so that } x \mapsto H(x,p) \\ \qquad\quad \text{is semiconvex on } K \text{ with constant } c|p|, \\ \text{and} \\ \mathbf{2}) \quad \text{The gradient } \nabla_p H(x,p) \text{ exists and is locally Lipschitz in } x\text{, and} \\ \qquad\quad \text{uniformly so over } p \text{ in compact subsets of } \mathbb{R}^n \setminus \{0\}. \end{cases}$$

Some consequences and equivalent statements of the assumptions (H) are given in Section 3. Somewhat stronger assumptions were introduced by Pliś [9] for another purpose, and we shall compare these later.

We next present examples that generate multifunctions satisfying (H) but do not admit a C^1 parameterization. Observe first that if F has the form (4) with f being C^1 in x, then H necessarily has the property

$$H(x,p) = -H(x,-p) \quad \Rightarrow \quad \partial_x H(x,p) = -\partial_x H(x,-p), \tag{6}$$

where ∂_x denotes the Clarke partial subgradient in x. The proof of (6) consists of noting that $H(x,p) = -H(x,-p)$ implies every $u \in U$ both maximizes and minimizes the function $u \mapsto \langle f(x,u), p\rangle$ over U. For such $p \neq 0$, it follows from Theorem 2.8.2 of [4] that

$$\partial_x H(x,p) = \overline{\text{co}} \left\{ \langle \nabla_x f(x,u), p\rangle : u \in U \right\}.$$

Similar reasoning gives

$$\partial_x H(x,-p) = \overline{\text{co}} \left\{ \langle \nabla_x f(x,u), -p\rangle : u \in U \right\}.$$

Thus the necessary condition (6) holds if F has the form (4) with f being C^1 in x.

A simple example of such an $F : \mathbb{R}^1 \rightrightarrows \mathbb{R}^1$ that satisfies (H) but does not admit a C^1 parameterization is $F(x) = [0, |x|]$. Note then that

$$H(x,p) = \begin{cases} |x|p & \text{if } p \geq 0 \\ 0 & \text{if } p < 0, \end{cases}$$

which clearly satisfies (H). (All one-dimensional examples that satisfy (H1) automatically satisfy (H2)). It does not admit a C^1 parameterization since at $x = 0$, one has that

$$\partial_x H(0,1) = [-1,1] \quad \text{and} \quad \partial_x H(0,-1) = \{0\},$$

and so (6) is violated. General one-dimensional examples can be generated by writing $F(x)$ in the form of an interval $[h(x), H(x)]$. If both $-h(\cdot)$ and $H(\cdot)$ are semiconvex, then F satisfies (H), but in view of (6), a C^1 parameterization could exist only if the additional property that $\partial H(x) = -\partial h(x)$ holds whenever $h(x) = H(x)$. Simple higher dimensional examples can also be given by considering

$$F(x) = f(x) + r(x)\overline{I\!\!B}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^2, $r : \mathbb{R}^n \to [0,\infty)$ is semiconvex, and $\overline{I\!\!B} \subset \mathbb{R}^n$ is the closed unit ball. Then

$$H(x,p) = \langle f(x), p\rangle + r(x)|p|$$

satisfies (H), but can only satisfy (6) if $\partial r(x) = -\partial r(x)$ whenever $r(x) = 0$.

The rest of the paper is organized as follows. Basic concepts are reviewed in Section 2, the rest of the hypotheses are precisely stated with some consequences drawn in Section 3, and the main result and its proof form the content of Section 4.

2. **Preliminaries.** We quickly review in this section some basic concepts from nonsmooth analysis. Standard references are [3, 6, 10, 4].

We denote by $I\!B$ the unit ball of $\mathbb{R}^n$ centered at 0 and by $\overline{I\!B}$ its closure.

Let $S \subseteq \mathbb{R}^n$ be a closed set and $x \in S$. A vector $\zeta \in \mathbb{R}^n$ is a *proximal normal* to S at x provided there exists $\sigma > 0$ so that $\langle \zeta, y - x\rangle \leq \sigma \|y - x\|^2$ for all $y \in S$. The *proximal normal cone* to S at x consists of all the proximal normals to S at x and is denoted by $N_S^P(x)$. The Clarke normal cone $N_S(x)$ is defined as

$$\overline{\mathrm{co}}\big\{\zeta : \exists x_i \to x,\, \zeta_i \to \zeta, \zeta_i \in\ N_S^P(x_i)\big\}\,,$$

where $\overline{\mathrm{co}}$ denotes taking the closed convex hull.

Now suppose $f : \mathbb{R}^n \to (-\infty, +\infty]$ is lower semicontinuous and

$$\mathrm{epi}\ f = \{(x, \alpha) : \alpha \geq f(x)\}$$

is its (closed) epigraph. For $x \in \mathrm{dom}\ f := \{x : f(x) < \infty\}$, the set of proximal subgradients of f at x is denoted by $\partial_P f(x)$ and equals

$$\big\{\zeta \in \mathbb{R}^n : (\zeta, -1) \in N^P_{\mathrm{epi}\ f}(x, f(x))\big\}\,.$$

With f locally Lipschitz (i.e. for all compact sets $K \subset \mathbb{R}^n$, there exists a constant k so that $x, y \in K$ implies $|f(x) - f(y)| < k|x - y|$), the Clarke subgradient $\partial f(x)$ equals

$$\overline{\mathrm{co}}\,\big\{\zeta : \exists x_i \to x,\, \zeta_i \to \zeta, \zeta_i \in\ \partial_P f(x_i)\big\}\,,$$

or equivalently, consists of $\zeta \in \mathbb{R}^n$ with $(\zeta, -1) \in N_{\mathrm{epi}\ f}(x, f(x))$. Another characterization is that $\zeta \in \partial f(x)$ if and only if

$$\langle \zeta, v\rangle \leq f^\circ(x; v) := \limsup_{y\to x, h\downarrow 0} \frac{f(y + hv) - f(y)}{h}$$

for all $v \in \mathbb{R}^n$.

The property of *semiconcavity* has several characterizations for both sets and functions, and the following proposition lists some of the functional properties that will be used in the sequel.

Proposition 1. *For a convex set $K \subseteq \mathbb{R}^n$, a function $f : K \to \mathbb{R}$, and a constant $c \geq 0$, the following properties are equivalent:*

(1) *for all x_0, $x_1 \in K$ and $0 \leq \lambda \leq 1$, one has*

$$(1 - \lambda)f(x_0) + \lambda f(x_1) - f(x_\lambda) \leq c\lambda(1 - \lambda)|x_1 - x_0|^2,$$

where $x_\lambda = (1 - \lambda)x_0 + \lambda x_1$,

(2) *f is continuous and, for all $x \in K$ and $z \in \mathbb{R}^n$ with $x \pm z \in K$, one has*

$$f(x + z) + f(x - z) - 2f(x) \leq 2c|z|^2\,,$$

(3) *the map $x \mapsto f(x) - c|x|^2$ is concave.*

Moreover, if K is open and f is locally Lipschitz, then any of the above properties holds true if and only if

(4) *for each $x \in K$ and $\zeta \in \partial f(x)$, one has*

$$f(y) \leq f(x) + \langle \zeta, y - x\rangle + c|y - x|^2$$

for all $y \in K$.

We give the proof for the reader's convenience.

Proof. For the proof of the fact that (1), (2) and (3) are equivalent see [3, Proposition 1.1.3].

Suppose that K is open and f is locally Lipschitz. Then the fact that (1) implies (4) follows from [3, Proposition 3.3.1], since in this case the generalized gradient coincides with the Fréchet superdifferential (see [3, Theorem 3.3.6]). Conversely, if (4) holds true, then for all x_0, $x_1 \in K$ and $0 \le \lambda \le 1$, one has

$$\begin{aligned}(1-\lambda)f(x_0)+\lambda f(x_1)-f(x_\lambda) &= (1-\lambda)[f(x_0)-f(x_\lambda)]+\lambda[f(x_1)-f(x_\lambda)]\\ &\le (1-\lambda)\big[\langle\zeta, x_0-x_\lambda\rangle + c|x_0-x_\lambda|^2\big]+\lambda\big[\langle\zeta, x_1-x_\lambda\rangle + c|x_1-x_\lambda|^2\big]\end{aligned}$$

where ζ is any fixed element of the nonempty set $\partial f(x_\lambda)$. Since the above right-hand side equals to $c\lambda(1-\lambda)|x_1-x_0|^2$, the conclusion follows. □

A Lipschitz function f satisfying property (1) of Proposition 1 is called *(linearly) semiconcave* on K with constant c. The above "concave" concepts have "convex" counterparts by reversing the inequalities and the signs in the quadratic terms. In other words, f is *semiconvex* on K if and only if $-f$ is semiconcave on K.

The Hausdorff distance between two compact subsets S_1 and S_2 of $\mathbb{R}^n$ is defined as

$$\operatorname{dist}_{\mathcal{H}}(S_1,S_2)=\max\big\{\operatorname{dist}^+_{\mathcal{H}}(S_1,S_2),\operatorname{dist}^+_{\mathcal{H}}(S_2,S_1)\big\},$$

where $\operatorname{dist}^+_{\mathcal{H}}(S,S')=\inf\big\{\varepsilon : S\subseteq S'+\varepsilon I\!B\big\}$.

A multifunction $F:\mathbb{R}^m\rightrightarrows\mathbb{R}^n$ with compact values is Lipschitz on $K\subseteq\mathbb{R}^m$ provided there exists a constant k (called a Lipschitz rank of F) so that $\operatorname{dist}_{\mathcal{H}}\big(F(x),F(y)\big)\le k|x-y|$ for all x, $y\in K$. This holds if and only if for each $p\in\mathbb{R}^n$, $x\mapsto H(x,p)$ is a Lipschitz function on K with constant $k|p|$. The mid-point property for a multifunction F on a convex subset K is that

$$\operatorname{dist}^+_{\mathcal{H}}\Big(2F(x),F(x+z)+F(x-z)\Big)\le c|z|^2 \tag{7}$$

holds whenever x, $x\pm z\in K$. This is equivalent to $x\mapsto H(x,p)$ being semiconvex on K for each $p\in\mathbb{R}^n$ with constant $c|p|$, which is the new assumption (H1) we shall impose in proving our main result. Pliś [9] assumed (7) but with $\operatorname{dist}^+_{\mathcal{H}}$ replaced by $\operatorname{dist}_{\mathcal{H}}$, which is equivalent to $x\mapsto H(x,p)$ being C^{1+} for all $p\in\mathbb{R}^n$.

It should be noted that (7) holds for any F parameterized as in (4) with f being C^{1+} in x. Indeed, every $v\in F(x)$ has the form $v=f(x,u)$ for some $u\in U$. For $z\in\mathbb{R}^n$ with $x\pm z\in K$, then $v_\pm := f(x\pm z,u)\in F(x\pm z)$ satisfies

$$\begin{aligned}\big|v_+ + v_- - 2v\big| &= \big|f(x+z,u)+f(x-z,u)-2f(x,u)\big|\\ &= \Big|\nabla_x f(x_+,u)z+\nabla_x f(x_-,u)(-z)\Big|\\ &\le \big|\nabla_x f(x_+,u)-\nabla_x f(x_-,u)\big|\,|z|\\ &\le k|x_+-x_-|\,|z|,\end{aligned}$$

where $x_\pm$ is a point between $x\pm z$ and k is a Lipschitz constant for $x\mapsto\nabla_x f(x,u)$ on K. Since $|x_+-x_-|\le 2|z|$, the property (7) follows with $c:=2k$.

If the above function or multifunction is defined on all of $\mathbb{R}^n$, then all of the above function and multifunction concepts can be quantified as being *local*, by which is meant that the said property holds in every convex compact neighborhood K of each point in $\mathbb{R}^n$ with the constant depending on K.

We close this section with a lemma concerning partial subgradients under a semiconvexity assumption. Recall that if a Lipschitz function of two variables is convex in one of the variables, then [4, Proposition 2.5.3] says that the component of a subgradient associated to the convex component is contained in the usual convex subgradient of that component. The following result extends this "partial inclusion" by allowing for the weaker semiconvex dependence.

Lemma 2.1. *Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is Lipschitz in (x, y) on a boxed neighborhood $U_x \times U_y = \{(x, y) : \max\{|x - \bar{x}|, |y - \bar{y}|\} < \delta\}$, and that for each $y \in U_y$, the function $x \to f(x, y)$ is semiconvex on U_x with constant independent of y. Then for $\zeta = (\zeta_x, \zeta_y) \in \partial f(\bar{x}, \bar{y})$, one has $\zeta_x \in \partial_x f(\bar{x}, \bar{y})$.*

Proof. Let $\zeta = (\zeta_x, \zeta_y) \in \partial f(\bar{x}, \bar{y})$. We need to show

$$\zeta_x \in \partial_x f(\bar{x}, \bar{y}),$$

which is equivalent to saying

$$\langle \zeta_x, u \rangle \le \limsup_{x \to \bar{x}, h \downarrow 0} \frac{f(x + hu, \bar{y}) - f(x, \bar{y})}{h} \tag{8}$$

for all $u \in \mathbb{R}^n$. Since $\zeta \in \partial f(\bar{x}, \bar{y})$, we have

$$\langle \zeta_x, u \rangle + \langle \zeta_y, v \rangle \le \limsup_{x \to \bar{x}, y \to \bar{y}, h \downarrow 0} \frac{f(x + hu, y + hv) - f(x, y)}{h}$$

for all $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$. Now fix $u \in \mathbb{R}^n$ and set $v = 0$, and so the previous line reduces to

$$\langle \zeta_x, u \rangle \le \limsup_{x \to \bar{x}, y \to \bar{y}, h \downarrow 0} \frac{f(x + hu, y) - f(x, y)}{h}. \tag{9}$$

The semiconvexity assumption in x says there exists $c > 0$ so that $x \mapsto f(x, y) + c\,|x - \bar{x}|^2$ is convex (Proposition 1(3)), and so the quotients

$$\begin{aligned} &\frac{1}{h}\left\{ f(x + hu, y) + c\,|x - \bar{x} + hu|^2 - f(x, y) - c\,|x - \bar{x}|^2 \right\} \\ &\qquad = \frac{1}{h}\left\{ f(x + hu, y) - f(x, y) \right\} + 2c\,\langle x - \bar{x}, u \rangle + ch|u|^2 \end{aligned} \tag{10}$$

are nondecreasing as functions of h. Now fix $\lambda > 0$, and note that the last two terms in (10) can be added to the limsup in (9) without changing the value. Hence the right hand side of (9) is given by

$$\lim_{\varepsilon \downarrow 0} \sup_{|x - \bar{x}| < \lambda\varepsilon,\, |y - \bar{y}| < \lambda\varepsilon} \frac{f(x + \varepsilon u, y) - f(x, y)}{\varepsilon} \tag{11}$$

This almost gives (8), but we need to replace y by $\bar{y}$. To this end, note that

$$\begin{aligned} \frac{f(x + \varepsilon u, y) - f(x, y)}{\varepsilon} &= \frac{f(x + \varepsilon u, \bar{y}) - f(x, \bar{y})}{\varepsilon} \\ &\quad + \frac{f(x + \varepsilon u, y) - f(x + \varepsilon u, \bar{y})}{\varepsilon} + \frac{f(x, \bar{y}) - f(x, y)}{\varepsilon} \end{aligned} \tag{12}$$

With k the Lipschitz constant of f, and assuming the bound $|y - \bar{y}| < \lambda\varepsilon$ contained in (11), the last two terms in (12) are bounded by λk. Hence the limit in (11) can be bounded above as follows

$$\lim_{\varepsilon\downarrow 0} \sup_{|x-\bar{x}|<\lambda\varepsilon,\, |y-\bar{y}|<\lambda\varepsilon} \frac{f(x+\varepsilon u, y) - f(x,y)}{\varepsilon}$$
$$\leq \lim_{\varepsilon\downarrow 0} \sup_{|x-\bar{x}|<\lambda\varepsilon} \frac{f(x+\varepsilon u, \bar{y}) - f(x,\bar{y})}{\varepsilon} + 2\lambda k$$
$$= \limsup_{x\to\bar{x},\, h\downarrow 0} \frac{f(x+hu, \bar{y}) - f(x,\bar{y})}{h} + 2\lambda k\,.$$

Finally, since u and λ are arbitrary, one concludes that (8) holds to complete the proof. □

Corollary 1. *Suppose H is as in (5) and $x \mapsto H(x,p)$ is locally semiconvex. Then $\partial H(x,p) \subseteq \partial_x H(x,p) \times \partial_p H(x,p)$.*

Proof. The result follows immediately from Lemma 2.1 applied to each component. □

3. Hypotheses and some consequences.

Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction. A well-developed theory for differential inclusions exists under a collection of so-called (strengthened) Standing Hypotheses :

$$(SH) \quad \begin{cases} \mathbf{1})\, F(x) \text{ is nonempty, convex, and compact for each } x \in \mathbb{R}^n, \\ \mathbf{2})\, F \text{ is locally Lipschitz with respect to the Hausdorff metric,} \\ \mathbf{3})\, \text{there exists } r > 0 \text{ so that } \max\{|v| : v \in F(x)\} \leq r(1+|x|). \end{cases}$$

The adjective *strengthened* refers to the Lipschitz assertion in the second assumption (SH2) over the merely closedness of the graph

$$\text{gr } F = \{(x,v) : v \in F(x)\}$$

as given in [6].

Consider the differential inclusion (2) with the initial condition (3). Gronwall's inequality and (SH3) imply that every solution $x(\cdot)$ satisfies

$$|x(s) - x| \leq r(s-t)e^{r(s-t)}(1+|x|) \qquad \forall s \in [t,T].$$

It is well-known that under (SH) and with φ locally Lipschitz, every problem of type $\mathcal{P}(t,x)$ has at least one optimal solution. There are recent and significantly refined necessary conditions for optimality in problem $\mathcal{P}(t,x)$ (cf. [5]), however we only require a somewhat older one.

Theorem 3.1 ([4] Theorem 3.2.6)**.** *Assume that (SH) holds and $\varphi : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz. Suppose $\bar{x}(\cdot)$ is an optimal solution of $\mathcal{P}(t,x)$. Then there exists an arc $\bar{p} : [t,T] \to \mathbb{R}^n$ so that*

$$\big(-\dot{\bar{p}}(s), \dot{\bar{x}}(s)\big) \in \partial H\big(\bar{x}(s), \bar{p}(s)\big) \quad a.e.\ s \in [t,T], \tag{13}$$
$$-\bar{p}(T) \in \partial\varphi\big(\bar{x}(T)\big). \tag{14}$$

Recall the assumption (H1), which says that the Hamiltonian is locally semiconvex in x with constant $c|p|$. This extra property implies that the inclusion in (13) splits component-wise.

Corollary 2. *Under the assumptions and setup of the previous theorem, if assumption (H1) also holds, then the arcs $\bar{x}(\cdot)$ and $\bar{p}(\cdot)$ satisfy*

$$-\dot{\bar{p}}(s) \in \partial_x H\big(\bar{x}(s), \bar{p}(s)\big) \quad \textit{and} \tag{15}$$

$$\dot{\bar{x}}(s) \in \partial_p H\big(\bar{x}(s), \bar{p}(s)\big) \tag{16}$$

for almost all $s \in [t, T]$.

Proof. The result follows immediately from (13) and Corollary 1. □

Remark 1. We note that the (dual) arc $\bar{p}(\cdot)$ has the following property:

- either $\bar{p}(s) \neq 0$ for all $s \in [t, T]$,
- or $\bar{p}(s) = 0$ for all $s \in [t, T]$.

Indeed, let $K \subset \mathbb{R}^n$ be a compact set containing $\bar{x}(s)$ for all $s \in [t, T]$ and let c_K be a Lipschitz constant for F on K, so that $c_K|p|$ turns out to be a Lipschitz constant for $H(\cdot, p)$ on the same set. Then,

$$|\zeta| \leq c_K|p| \qquad \forall \zeta \in \partial_x H(x, p)\,,\ \forall x \in K\,,\ \forall p \in \mathbb{R}^n.$$

Hence, in view of (15),

$$|\dot{\bar{p}}(s)| \leq c_K|\bar{p}(s)| \quad \text{for a.e.} \quad s \in [t, T].$$

Thus, the above alternative follows from the last inequality and Gronwall's lemma.

Two other consequences of (H1) are contained in the following proposition.

Proposition 2. *Suppose F satisfies (SH) and (H1), and let $K \subset \mathbb{R}^n$ be a bounded open convex set. Then there exists a constant $c > 0$ so that*

(1) for each $x \in K$ and $z \in \mathbb{R}^n$ with $x_\pm = x \pm z \in K$, we have

$$H(x_+, p) + H(x_-, p) - 2H(x, p) \geq -c\,|p|\,|z|^2; \quad \textit{and}$$

(2) for each x, $y \in K$, $p \in \mathbb{R}^n$, and $\zeta \in \partial_x H(x, p)$, we have

$$H(y, p) \geq H(x, p) + \langle \zeta, y - x\rangle - c\,|p|\,|y - x|^2.$$

Proof. These follow immediately from the semiconvex versions of Proposition 1(2) and (4), respectively. □

Now recall the assumption (H2), which was stated in terms of H being differentiable in p with the derivative exhibiting Lipschitz dependence in x. The differentiability statement is equivalent to the argmax set of $v \mapsto \langle v, p\rangle$ over $v \in F(x)$ being a singleton $\bar{v}$, which we denote by $F_p(x) = \bar{v}$, and which equals $\nabla_p H(x, p)$. Note that $p \mapsto F_p(x)$ is thus continuous. The Lipschitz statement in (H2) says that $x \mapsto F_p(x)$ is locally Lipschitz. The major utility and impact of (H2) is contained in the following result.

Proposition 3. *Assume that (SH) and (H) hold, and $p(\cdot)$ is an absolutely continuous arc defined on $[t, T]$ with $p(s) \neq 0$ for all $s \in [t, T]$. Then for each $x \in \mathbb{R}^n$, the initial value problem*

$$\textit{(IVP)} \qquad \begin{cases} \dot{x}(s) = F_{p(s)}\big(x(s)\big) & \textit{a.e. } s \in [t, T] \\ x(t) = x \end{cases}$$

has a unique solution $y(\cdot; t, x)$. Moreover, $x \mapsto y(s; t, x)$ is locally Lipschitz on $\mathbb{R}^n$ with constant independent of $s \in [t, T]$.

Proof. Let

$$g(s,x) = F_{p(s)}(x) = \nabla_p H\big(x, p(s)\big)\,.$$

Then $g : [t,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in s (since $p \mapsto F_p(x)$ is continuous), locally Lipschitz in x (by (H2)), and has linear growth in x (by (SH3)). Hence by standard ODE theory, (IVP) has a unique solution $x(\cdot) = y(\cdot; t, x)$ defined on $[t,T]$ with locally Lipschitz dependence on initial data. □

We conclude this section with a result connecting regularity properties of H with geometric properties of the multifunction F. Related results are contained in [10, Proposition 12.60].

Proposition 4. *Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction satisfying (SH), $K \subseteq \mathbb{R}^n$ is compact and convex, and F satisfies (7) on K with constant c.*

(1) Then H satisfies the property (H1) for the same constant c.

(2) Suppose in addition there exists a constant $\theta \geq 1$ such that for all $x \in K$ and $p \in \mathbb{R}^n$, we have

$$v_p \in F_p(x) \quad \Longrightarrow \quad \langle v - v_p, p\rangle \leq -c'\,|p|\,|v - v_p|^\theta \quad \forall v \in F(x), \tag{17}$$

where $c' > 0$ is a constant dependent on K. Then $\nabla_p H(x,p)$ exists for all $x \in K$ and $p \in \mathbb{R}^n \setminus \{0\}$, and is Hölder continuous in x on K with exponent $1/\theta$, and unformly so over p in a compact subset of $\mathbb{R}^n \setminus \{0\}$.

Proof. Let $x \in K$ and $z \in \mathbb{R}^n$ be such that $x_\pm = x \pm z \in K$, and select $v \in F(x)$ so that $H(x,p) = \langle v, p\rangle$. Then, in view of (7), there are vectors $v_\pm \in F(x_\pm)$ such that $|v_+ + v_- - 2v| \leq c|z|^2$, and therefore

$$H(x_+,p) + H(x_-,p) - 2H(x,p) \geq \langle v_+ + v_- - 2v, p\rangle \geq -c\,|p|\,|z|^2$$

whence the conclusion of part (1) follows.

As for part (2), observe (17) yields that $F_p(x)$ is a singleton for all $x \in K$ and $p \neq 0$. As is well-known, this implies $\nabla_p H(x,p)$ exists and $F_p(x) = \{\nabla_p H(x,p)\}$. To prove the continuity assertion, let $x, x' \in K$ and $p \in \mathbb{R}^n \setminus \{0\}$, and set

$$v_p = \nabla_p H(x,p) \quad \text{and} \quad v'_p = \nabla_p H(x',p).$$

Denote by $\bar v_p$ and $\bar v'_p$, respectively, the projections of v_p and v'_p onto the convex sets $F(x')$ and $F(x)$. Then, owing to $(SH2)$,

$$|v_p - \bar v_p| + |v'_p - \bar v'_p| \leq 2k\,|x - x'|\,, \tag{18}$$

where k is the Lipschitz constant of F on K. Moreover, in view of (17)

$$\begin{aligned} \langle \bar v'_p - v_p, p\rangle &\leq -c'\,|p|\,|\bar v'_p - v_p|^\theta \\ \langle \bar v_p - v'_p, p\rangle &\leq -c'\,|p|\,|\bar v_p - v'_p|^\theta, \end{aligned}$$

whence

$$\begin{aligned} |\bar v'_p - v_p|^\theta + |\bar v_p - v'_p|^\theta \quad &\leq \quad -\frac{1}{c'|p|}\Big[\langle \bar v'_p - v'_p, p\rangle + \langle \bar v_p - v_p, p\rangle\Big] \\ &\leq \quad \frac{1}{c'}\Big[|v_p - \bar v_p| + |v'_p - \bar v'_p|\Big] \\ &\leq \quad \frac{2k}{c'}\,|x - x'| \end{aligned}$$

where the last inequality comes from (18). Therefore

$$|\bar v'_p - v_p| + |\bar v_p - v'_p| \leq k'\,|x - x'|^{1/\theta} \tag{19}$$

for $k' = \left(\frac{2k}{c'}\right)^{1/\theta}$, and since $|v_p - v_p'| \le |v_p - \bar v_p| + |\bar v_p - v_p'|$, the conclusion follows from (18) and (19). $\square$

4. **The main result.** We are now able to state and prove our main result.

Theorem 4.1. *Assume F satisfies (SH) and (H), and $\varphi(\cdot)$ is locally semiconcave. Then V is locally semiconcave on $(-\infty, T] \times \mathbb{R}^n$.*

Proof. The fact that $V(\cdot,\cdot)$ is locally Lipschitz on $(-\infty, T] \times \mathbb{R}^n$ is contained in Exercise 3.12(b) in Chapter 4 of [6]. Alternatively, it can be verified by combining the Lipschitz parametrization theorem for multifunctions (see, either [1, Theorem 9.6.2] or [7, Theorem 1]) with the Lipschitz regularity result for the value function of a Mayer problem in the parametrized case ([3, Theorem 7.2.3]).

So, it suffices to show $V(\cdot,\cdot)$ is semiconcave on every bounded subset of $(-\infty, T] \times \mathbb{R}^n$. Let $t \le T$ and $M > 0$. We will show V is semiconcave on $[t, T] \times M\mathbb{B}$ by verifying the mid-point property stated in Proposition 1(2). Let

$$K := \{y \in \mathbb{R}^n : |y| < e^{r(T-t)}(1+M) - 1\},$$

and note that $|x| < M$ implies that any solution $x(\cdot)$ of (2),(3) satisfies $x(s) \in K$ for all $s \in [t, T]$. Let c' denote the semiconvexity constant of φ, and k_1 the Lipschitz rank of F, both on K.

With t fixed, we first consider the semiconcavity of $x \mapsto V(t,x)$. Let $|\bar x| < M$, and suppose $z \in \mathbb{R}^n$ satisfies $|x_\pm| < M$ where $x_\pm := \bar x \pm z$. Let $\bar x(\cdot)$ be an optimal solution to $\mathcal{P}(t, \bar x)$, and let $\bar p(\cdot)$ be an associated adjoint arc. Thus $\big(\bar x(\cdot), \bar p(\cdot)\big)$ satisfies the transversality condition (14), and by Corollary 2, also satisfies the inclusions (15) and (16). Moreover, on account of Remark 1, only two cases may occur: either $\bar p(s) \ne 0$ for every $s \in [t, T]$, or $\bar p(s) = 0$ for all such s.

Suppose $\bar p(s) \ne 0$ for all $s \in [t, T]$. With $p(\cdot) = \bar p(\cdot)$ in Proposition 3, let k_2 be the Lipschitz constant of $x \mapsto y(s; t, x)$ on $\{x : |x| \le M\}$, which is independent of $s \in [t, T]$. Now let $x_\pm(\cdot) = y(\cdot; t, x_\pm)$ be the solutions of (IVP) with the initial data $x_\pm$. Then the Lipschitz assertion in Proposition 3 says that $x_\pm(s) \in K$ for all $s \in [t, T]$, and so $x_\pm(\cdot)$ satisfy

$$|x_+(s) - x_-(s)| \le k_2 |x_+ - x_-| = 2k_2|z| \qquad \forall s \in [t, T]. \tag{20}$$

Recall from (16) and (H2) that $\dot{\bar x}(s) \in \partial_p H\big(\bar x(s), \bar p(s)\big) = \big\{F_{\bar p(s)}\big(\bar x(s)\big)\big\}$, which implies $\bar x(\cdot) = y(\cdot; t, x)$ is the unique solution of (IVP). Again by the Lipschitz assertion in Proposition 3, for all $s \in [t, T]$ we have

$$\begin{aligned} |x_\pm(s) - \bar x(s)| &= \big|y(s; t, x_\pm) - y(s; t, \bar x)\big| \\ &\le k_2 |x_\pm - \bar x| = k_2 |z|. \end{aligned} \tag{21}$$

Next, suppose $\bar p(s) = 0$ for all $s \in [t, T]$, denote by E the subset of $[t, T]$ at which $\dot{\bar x}$ exists, and define $f : [t, T] \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(s,x) = \begin{cases} \mathrm{proj}_{F(x)}(\dot{\bar x}(s)) & s \in E \\ \mathrm{proj}_{F(x)}(0) & s \in [t, T] \setminus E, \end{cases}$$

where $\mathrm{proj}_{F(x)}(y)$ stands for the orthogonal projection of y onto $F(x)$. Recalling (SH), it is easy to see that

- $x \mapsto f(s,x)$ is continuous for all $s \in [t, T]$,
- $s \mapsto f(s,x)$ is measurable for all $x \in \mathbb{R}^n$,
- $|f(s,x)| \le r(1 + |x|)$ for all $(s,x) \in [t, T] \times \mathbb{R}^n$.

This time, let $x_\pm(\cdot)$ be the solutions of the initial value problems

$$\begin{cases} \dot{x}(s) = f(s, x(s)) & \text{a.e. } s \in [t, T] \\ x(t) = x_\pm. \end{cases}$$

Notice that $x_\pm(\cdot)$ are, in particular, solutions of (2). Since E has full Lebesgue measure in $[t, T]$, for a. e. $s \in [t, T]$ we have

$$|\dot{x}_\pm(s) - \dot{\bar{x}}(s)| = |\dot{\bar{x}}(s) - \mathrm{proj}_{F(x_\pm(s))}(\dot{\bar{x}}(s))| \leq k_1 |x_\pm(s) - \bar{x}(s)|.$$

Therefore, applying Gronwall's lemma we obtain once again Lipschitz estimates of the form (20) and (21) for suitable Lipschitz constants. Hereafter, we will denote by $x_\pm(\cdot)$ the trajectories provided by either one of the above constructions, assuming that estimates (20) and (21) hold true.

We are now prepared to estimate the value function. One notes that

$$\begin{aligned} &V(t, \bar{x} + z) + V(t, \bar{x} - z) - 2V(t, \bar{x}) \\ &\qquad = V(t, x_+) + V(t, x_-) - 2V(t, \bar{x}) \\ &\qquad \leq \varphi\big(x_+(T)\big) + \varphi\big(x_-(T)\big) - 2\varphi\big(\bar{x}(T)\big) \\ &\qquad = \varphi\big(x_+(T)\big) + \varphi\big(x_-(T)\big) - 2\varphi\left(\frac{x_+(T) + x_-(T)}{2}\right) \\ &\qquad\qquad + 2\left[\varphi\left(\frac{x_+(T) + x_-(T)}{2}\right) - \varphi\big(\bar{x}(T)\big)\right]. \end{aligned}$$

Since $x_\pm(T) \in K$, the semiconcavity of $\varphi(\cdot)$ implies that

$$\begin{aligned} \varphi\big(x_+(T)\big) + \varphi\big(x_-(T)\big) - 2\varphi\left(\frac{x_+(T) + x_-(T)}{2}\right) &\leq c' \big|x_+(T) - x_-(T)\big|^2 \\ &\leq 4c' k_2^2 |z|^2, \end{aligned} \tag{22}$$

where the last inequality holds by (20).

The transversality condition (14) says that $-\bar{p}(T) \in \partial\varphi\big(\bar{x}(T)\big)$, and since φ is semiconcave, one has $-\partial\varphi(x) = \partial_P(-\varphi)(x)$. Therefore $\bar{p}(T) \in \partial_P(-\varphi)\big(\bar{x}(T)\big)$, and by Proposition 1(4) we have

$$\begin{aligned} &\varphi\left(\frac{x_+(T) + x_-(T)}{2}\right) - \varphi\big(\bar{x}(T)\big) \\ &\qquad \leq \left\langle -\bar{p}(T), \frac{x_+(T) + x_-(T)}{2} - \bar{x}(T) \right\rangle + c\left|\frac{x_+(T) + x_-(T)}{2} - \bar{x}(T)\right|^2. \end{aligned} \tag{23}$$

Now (21) implies the second term on the right hand side in (23) is bounded above by $ck_2^2|z|^2$. We still need to treat the first term in (23), which is the most difficult to estimate. It can be rewritten as

$$\begin{aligned} &\left\langle -\bar{p}(T), \frac{x_+(T) + x_-(T)}{2} - \bar{x}(T) \right\rangle \\ &\qquad = \frac{1}{2}\int_t^T \frac{d}{ds}\big\langle -\bar{p}(s), x_+(s) + x_-(s) - 2\bar{x}(s) \big\rangle\, ds \\ &\qquad = \frac{1}{2}\int_t^T \Big[\big\langle -\dot{\bar{p}}(s), x_+(s) + x_-(s) - 2\bar{x}(s) \big\rangle \qquad (24) \\ &\qquad\qquad + \big\langle \bar{p}(s), 2\dot{\bar{x}}(s) - \dot{x}_+(s) - \dot{x}_-(s) \big\rangle\Big]\, ds. \qquad (25) \end{aligned}$$

Now, each of $\bar{x}(\cdot)$, $x_\pm(\cdot)$ are solutions to (IVP) with $p(t)=\bar{p}(t)$, and so, for almost all $s\in[t,T]$, one has $\langle \bar{p}(s),\dot{x}(s)\rangle = H\big(x(s),\bar{p}(s)\big)$ for $x(\cdot)=\bar{x}(\cdot)$, $x_\pm(\cdot)$. Therefore the last term (25) can be rewritten as

$$\begin{aligned}
&\frac{1}{2}\int_t^T \big\langle \bar{p}(s), 2\dot{\bar{x}}(s)-\dot{x}_+(s)-\dot{x}_-(s)\big\rangle\,ds\\
&\quad=\int_t^T\left\{H\big(\bar{x}(s),\bar{p}(s)\big)-\frac{H\big(x_+(s),\bar{p}(s)\big)+H\big(x_-(s),\bar{p}(s)\big)}{2}\right\}ds\\
&\quad=\int_t^T\left\{H\big(\bar{x}(s),\bar{p}(s)\big)-H\left(\frac{x_+(s)+x_-(s)}{2},\bar{p}(s)\right)\right\}ds\\
&\qquad+\int_t^T\left\{H\left(\frac{x_+(s)+x_-(s)}{2},\bar{p}(s)\right)\right.\\
&\qquad\qquad\left.-\frac{H\big(x_+(s),\bar{p}(s)\big)+H\big(x_-(s),\bar{p}(s)\big)}{2}\right\}ds
\end{aligned}\tag{26}$$

We now use Proposition 2 to bound the two integrals in (26). The second integrand is bounded above for each $s\in[t,T]$ by

$$c\,|\bar{p}(s)|\,|x_+(s)-x_-(s)|^2\,,$$

which is a consequence of Proposition 2(1), and this in turn is bounded by (20) by $4\,c\,k_2^2\,\|\bar{p}(\cdot)\|_\infty|z|^2$. As for the first one, recall that

$$-\dot{\bar{p}}(s)\in\partial_x H\big(\bar{x}(s),\bar{p}(s)\big)$$

by (15), and so by Proposition 2(2), the first integrand in (26) is bounded above by

$$\left\langle \dot{\bar{p}}(s),\frac{x_+(s)+x_-(s)}{2}-\bar{x}(s)\right\rangle + c\,|\bar{p}(s)|\,\left|\frac{x_+(s)+x_-(s)}{2}-\bar{x}(s)\right|^2 \tag{27}$$

As we integrate (27) from t to T, the result of the first term in (27) is precisely the negative of (24), and so these terms cancel. The integral of the second term in (27) is bounded above by $(T-t)c\|\bar{p}(\cdot)\|_\infty k_2^2|z|^2$.

We now collect all the bounds together, and conclude

$$V(t,\bar{x}+z)+V(t,\bar{x}-z)-2V(t,\bar{x})\le\bar{c}|z|^2$$

where $\bar{c}=4c'k_2^2+2ck_2^2+8ck_2^2\|\bar{p}(\cdot)\|_\infty+2(T-t)c\|\bar{p}(\cdot)\|_\infty k_2^2$.

Next, we consider the joint semiconvexity of $V(\cdot,\cdot)$ in (t,x). Suppose $\tau>0$ is such that $t+\tau\le T$, and let $\bar{x},z\in\mathbb{R}^n$ with $|x_\pm|<M$ where $x_\pm=\bar{x}\pm z$. We need to show there exists a constant c so that

$$V(t+\tau,x_+)+V(t-\tau,x_-)-2V(t,\bar{x})\le c\big(|\tau|^2+|z|^2\big). \tag{28}$$

Let $\bar{x}(\cdot)$ be an optimal solution to $\mathcal{P}(t,\bar{x})$, let $\bar{p}(\cdot)$ be an associated adjoint arc, and suppose $\bar{p}(s)\ne 0$ for every $s\in[t,T]$. Let $x_-(\cdot)$ be the solution to (IVP) as in Proposition 3 that is defined on the interval $[t-\tau,t+\tau]$ with

$$p(s)=\bar{p}\left(\frac{t+\tau+s}{2}\right) \tag{29}$$

and initial condition $x_-(t-\tau)=\bar{x}-z$. By the Principle of Optimality, we have

$$V(t-\tau,\bar{x}-z)\le V\big(t+\tau,x_-(t+\tau)\big)$$

and

$$V(t,\bar{x})=V\big(t+\tau,\bar{x}(t+\tau)\big)\,.$$

Therefore

$$\begin{aligned}
&V(t+\tau,x_+)+V(t-\tau,x_-)-2V(t,\bar{x})\\
&\quad\le\ V\big(t+\tau,x_+\big)+V\big(t+\tau,x_-(t+\tau)\big)-2V\big(t+\tau,\bar{x}(t+\tau)\big)\\
&\quad=\ V\big(t+\tau,x_+\big)+V\big(t+\tau,x_-(t+\tau)\big)-2V\left(t+\tau,\frac{x_++x_-(t+\tau)}{2}\right)\\
&\qquad\qquad+2V\left(t+\tau,\frac{x_++x_-(t+\tau)}{2}\right)-2V\big(t+\tau,\bar{x}(t+\tau)\big)\\
&\quad\le\ c\big|x_+-x_-(t+\tau)\big|^2\\
&\qquad\qquad+2\left[V\left(t+\tau,\frac{x_++x_-(t+\tau)}{2}\right)-V\big(t+\tau,\bar{x}(t+\tau)\big)\right],
\end{aligned}$$

where in the last inequality we used the semiconcavity property of $x\mapsto V(t+\tau,x)$ that was proven above. Observe that

$$\big|x_+-x_-(t+\tau)\big|=\Big|2z-\int_{t-\tau}^{t+\tau}\dot{x}_-(s)\,ds\Big|\le c_1\big(|z|+\tau\big) \tag{30}$$

for some constant c_1. Moreover,

$$\begin{aligned}
x_++x_-(t+\tau)-2\bar{x}(t+\tau)\ &=\ \int_{t-\tau}^{t+\tau}\dot{x}_-(s')\,ds'-2\int_t^{t+\tau}\dot{\bar{x}}(s)\,ds\\
&=\ 2\int_t^{t+\tau}\big[\dot{x}_-(2s-t-\tau)-\dot{\bar{x}}(s)\big]\,ds,
\end{aligned}$$

where in the first integral term we used the change of variables

$$\frac{t+\tau+s'}{2}=s\,.$$

Now,

$$\dot{x}_-(2s-t-\tau)\in F_{\bar{p}(s)}\big(x_-(2s-t-\tau)\big)$$

owing to (29), and

$$\dot{\bar{x}}(s)\in F_{\bar{p}(s)}\big(\bar{x}(s)\big)\,.$$

Since $x\mapsto F_{\bar{p}(s)}(x)$ is locally Lipschitz by (H2), there exists a constant c_2 so that for every $s\in[t,t+\tau]$

$$\begin{aligned}
\big|\dot{x}_-(2s-t-\tau)-\dot{\bar{x}}(s)\big|\ &\le\ c_2\big|x_-(2s-t-\tau)-\bar{x}(s)\big|\\
&\le\ c_3\big(|z|+\tau\big).
\end{aligned}$$

Therefore,

$$\big|x_++x_-(t+\tau)-2\bar{x}(t+\tau)\big|\le c_3\tau(|z|+\tau)\le 2c_3\big(|z|^2+\tau^2\big). \tag{31}$$

At this point, let us observe that estimates (30) and (31) remain true when $\bar{p}(s)=0$ for all $s\in[t,T]$. Indeed, one can proceed as above defining $x_-(\cdot)$ by

$$\begin{cases}\dot{x}(s)=f_-(s,x(s)) & \text{a.e. } s\in[t-\tau,t+\tau]\\ x(t)=x_-.\end{cases}$$

where

$$f_-(s,x):=f\Big(\frac{t+\tau+s}{2},x\Big)\qquad (s,x)\in[t-\tau,t+\tau]\times\mathbb{R}^n.$$

Finally, using the Lipschitz property of $x\mapsto V(t,x)$, we have by (30) and (31) that

$$V(t+\tau,x_+)+V(t-\tau,x_-)-2V(t,\bar{x})\le\big(cc_1^2+2c_3\big)\big(|z|^2+\tau^2\big),$$

which finishes the proof. □

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REFERENCES

[1] J.-P. Aubin and H. Frankowska, "Set-Valued Analysis," Birkhäuser, Boston, 1990.
[2] P. Cannarsa and H. Frankowska, *Some characterizations of optimal trajectories in control theory*, SIAM J. Control Optim., **29** (1991), 1322–1347.
[3] P. Cannarsa and C. Sinestrari, "Semiconcave Functions, Hamilton–Jacobi Equations, and Optimal Control," Birkhäuser, Boston, 2004.
[4] F. H. Clarke, "Optimization and Nonsmooth Analysis," Wiley, New York, 1983.
[5] F. H. Clarke, "Necessary Conditions in Dynamic Optimization," Memoir of the American Mathematical Society, **816**, 2005.
[6] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski, "Nonsmooth Analysis and Control Theory," Springer, New York, 1998.
[7] A. Ornelas, *Parametrization of Carathéodory multifunctions*, Rend. Sem. Mat. Univ. Padova, **83** (1990), 33–44.
[8] C. Knuckles and P. R. Wolenski, C^1 *selections of multifunctions in one dimension*, Real Analysis Exchange, **22** (1997), 655–676.
[9] A. Pliś, *Accessible sets in control theory*, in "International Conference on Differential Equations" (Los Angeles, 1974), Academic Press, (1975), 646–650.
[10] R. T. Rockafellar and R. Wets, "Variational Analysis," Springer-Verlag, Berlin Heidelberg, 1998.

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