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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 147–152
COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Carleman estimates and null controllability for boundary-degenerate parabolic operators

Piermarco Cannarsa [a], Partick Martinez [b], Judith Vancostenoble [b]

[a] *Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy*
[b] *Institut de mathématiques de Toulouse, U.M.R. C.N.R.S. 5219, Université Paul-Sabatier Toulouse III, 118, route de Narbonne, 31062 Toulouse cedex 4, France*

Received 3 October 2008; accepted after revision 16 December 2008

Available online 23 January 2009

Presented by Gilles Lebeau

Abstract

Motivated by several examples coming from physics, biology, and economics, we consider a class of parabolic operators that degenerate at the boundary of the space domain. We study null controllability by a locally distributed control. For this purpose, a specific Carleman estimate for the solutions of degenerate adjoint problems is proved. ***To cite this article: P. Cannarsa et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Estimations de Carleman et nulle contrôlabilité pour une classe d'opérateurs paraboliques dégénérés. Motivés par de nombreux problèmes venant de la physique, la biologie et de l'économie, nous considérons une classe d'équations paraboliques dont l'opérateur associé dégénère au bord du domaine spatial. Nous étudions la nulle contrôlabilité, établissant en particulier une estimation de Carleman pour l'équation dégénérée adjointe. ***Pour citer cet article : P. Cannarsa et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Les propriétés de contrôlabilité des équations *uniformément* paraboliques ont été largement étudiées depuis les travaux de Lebeau et Robbiano [6], et de Fursikov et Imanuvilov [5], basés sur des estimations de type Carleman. D'un autre côté, il y a peu de résultats concernant les équations paraboliques *dégénérées*, bien que celles-ci apparaissent dans de nombreux domaines : par exemple en aéronautique (l'équation de Crocco), en climatologie (l'équation de Budyko–Sellers), en génétique (l'équation de Fleming–Viot). Le point commun des modèles cités est que la dégénérescence de l'opérateur parabolique est localisée au bord du domaine. Dans un premier temps, nous avons étudié le problème en dimension 1 d'espace ([1]). Dans cette note, nous présentons nos résultats de nulle contrôlabilité en

E-mail addresses: cannarsa@mat.uniroma2.it (P. Cannarsa), martinez@mip.ups-tlse.fr (P. Martinez), vancoste@mip.ups-tlse.fr (J. Vancostenoble).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.12.011

dimension supérieure pour une famille d'opérateurs paraboliques dont la dégénérescence est localisée au bord du domaine.

Soit $\Omega \subset \mathbb{R}^2$ un domaine borné de classe C^4, Γ le bord de Ω et $\omega \Subset \Omega$. Pour $y \in \Gamma$, on note $\nu(y)$ le vecteur unitaire normal sortant au bord ; pour $x \in \Omega$, on note $d(x,\Gamma)$ la distance de x au bord Γ ; pour $\eta > 0$, on note $C(\Gamma,\eta) := \{x \in \Omega, d(x,\Gamma) < \eta\}$. Alors il existe $\eta_1 > 0$ tel que $x \mapsto d(x,\Gamma)$ est de classe $\mathcal{C}^4$ sur $\overline{C(\Gamma,\eta_1)}$; de plus $\forall x \in C(\Gamma,\eta_1)$, $\exists! p_\Gamma(x) \in \Gamma$ such that $d(x,\Gamma) = |x - p_\Gamma(x)|$, et $\nabla d(x,\Gamma) = -\nu(p_\Gamma(x))$.

On étudie la nulle contrôlabilité de (1) sous les hypothèses suivantes :

Hypothèse 0.1. Pour tout $x \in \overline{\Omega}$, $A(x)$ est symétrique positive, à coefficients $\mathcal{C}^0(\overline{\Omega};\mathbb{R}) \cap \mathcal{C}^4(\Omega;\mathbb{R})$; $A(x)$ est définie positive si $x \in \Omega$ et ses valeurs propres $0 < \lambda_1(x) \leqslant \lambda_2(x)$ vérifient : il existe $\eta_0 > 0$, $0 < m < M$ et $\alpha \geqslant 0$ tels que $\forall x \in C(\Gamma,\eta_0)$, $\lambda_1(x) = d(x,\Gamma)^\alpha$ et $\lambda_2(x) \in [m,M]$. Enfin, on suppose que $\forall x \in C(\Gamma,\eta_0)$, $\varepsilon_1(x) = \nu(p_\Gamma(x)) = -\nabla d(x,\Gamma)$ est un vecteur propre de $A(x)$ associé à $\lambda_1(x)$.

Lorsque $\alpha \in [0,1)$, on considère le problème (2) alors que, lorsque $\alpha \in [1,2)$, on considère le problème (3). Dans les deux cas, le problème est bien posé (voir [3]), et on a le résultat suivant :

Théorème 0.1. *Soit $T > 0$. Sous l'Hypothèse* 0.1 *avec $\alpha \in [0,1)$* (*resp. sous l'Hypothèse* 0.1 *avec $\alpha \in [1,2)$*) : *pour tout $u_0 \in L^2(\Omega)$, il existe $h \in L^2(\Omega_T)$ tel que la solution u du problème* (2) (*resp.* (3)) *vérifie $u(T,\cdot) = 0$ in $L^2(\Omega)$.*

Remarques.

(i) L'Hypothèse 0.1 est pratique pour énoncer les résultats mais très restrictive ; toutefois, dans les modèles physiques ayant motivés cette étude, en particulier le modèle de Fleming–Viot, les conditions suivantes sont vérifiées : $\lambda_1(x) \to 0$ comme une puissance de $d(x,\Gamma)$ et $\varepsilon_1(x) - \nu(p_\Gamma(x) \to 0$ quand $x \to \Gamma$; on renvoie le lecteur à [2] pour les résultats sous des hypothèses plus générales : pour des équations paraboliques dégénérées contenant des termes d'ordre inférieur et pour lesquelles $\lambda_1(x) \sim d(x,\Gamma)^\alpha$ et $\varepsilon_1(x) - \nu(p_\Gamma(x)) \to 0$ quand $x \to \Gamma$.

(ii) *Le cas $\alpha \geqslant 2$* : lorsque l'Hypothèse 0.1 est satisfaite avec $\alpha \geqslant 2$, la nulle controllabilité n'est plus vraie (comme dans le cas de la dimension 1, cf. [1]) : ceci découle d'une transformation du problème en une équation parabolique non dégénérée mais posée en domaine non borné, et d'un résultat d'Escauriaza, Seregin, Sverák [4] (voir [2] pour plus de détails).

Comme dans le cas non dégénéré, la nulle contôlabilité énoncée au Théorème 0.1 est équivalente à l'inégalité d'observabilité (4) pour les solutions du problème adjoint associé. Celle-ci est obtenue à partir de l'inégalité de Carleman suivante combinée à une inégalité de type Hardy (voir (9) dans le cas $\alpha \in [0,1)$) :

Théorème 0.2. *On suppose l'Hypothèse* 0.1 *satisfaite avec $\alpha \in [0,2)$. Alors toute solution v du problème adjoint avec donnée initiale $w(T) \in L^2(\Omega)$ satisfait les estimations de Carleman* (7) *et* (8).

1. Introduction

The controllability properties of parabolic equations have been widely studied, in particular after the works by Lebeau and Robbiano [6], and Fursikov and Imanuvilov [5], based on suitable weighted estimates of Carleman type. More recently, controllability theory for parabolic equations has grown in various directions, such as: extension to semilinear parabolic problems, problems in unbounded domains [7], and fluid models, such as the Euler, Stokes, and Navier–Stokes equations. From the controllability viewpoint, the behavior of *uniformly parabolic* equations is by now well understood.

On the contrary fewer results are known for *degenerate parabolic* equations, even though such a class of operators arise in many theoretical, as well as applied, types of context. Initially, our study was motivated by a boundary layer model appearing in aeronautics (the so-called Crocco equation). Besides, such degenerate problems also appear in climatology (the Budyko–Sellers model), population genetics (the Fleming–Viot model, the Wright–Fisher gene frequency model), economics (the Black–Scholes equation). From a more theoretical viewpoint, the degeneracy (in the normal direction) of a Kolmogorov operator at the boundary of a given space domain, is a necessary condition to guarantee the invariance of the domain itself with respect to the associated stochastic flow. The common point of these problems is that the degeneracy of the parabolic operator occurs at the boundary of the domain. Hence, we studied a general simplified model of this type. After having investigated the one-dimensional case (see [1]), we now concentrate our attention on the case of higher dimension. The goal of this note is to present the results that we obtained concerning the null controllability properties of such problems.

2. Presentation of the problem and main results

In this note, we analyze the null controllability (or, equivalently, the exact controllability to the trajectories) of a class of parabolic equations with degeneracy at the boundary, using a locally distributed control acting on the control region $\emptyset \neq \omega \Subset \Omega$. We assume that Ω is a bounded open set of $\mathbb{R}^2$ whose boundary Γ is of class $\mathcal{C}^4$. We denote by $\nu(y)$ the outward unit normal vector to Ω at any point $y \in \Gamma$, and by $d(x,\Gamma)$ the distance between x and the boundary Γ; given $\eta > 0$, we define $C(\Gamma,\eta) := \{x \in \Omega, d(x,\Gamma) < \eta\}$. Then, there exists $\eta_1 > 0$ such that the function $x \mapsto d(x,\Gamma)$ is of class $\mathcal{C}^4$ in $\overline{C(\Gamma,\eta_1)}$; moreover, $\forall x \in C(\Gamma,\eta_1)$, $\exists! p_\Gamma(x) \in \Gamma$ such that $d(x,\Gamma) = |x - p_\Gamma(x)|$, and $\nabla d(x,\Gamma) = -\nu(p_\Gamma(x))$.

We study the following parabolic equation:

$$u_t - \operatorname{div}\big(A(x)\nabla u\big) = h\chi_\omega, \quad x \in \Omega,\ t > 0, \tag{1}$$

under the following assumption on $A(x)$:

Assumption 2.1. $\forall x \in \overline{\Omega}$, $A(x)$ is a symmetric nonnegative 2×2 matrix, with coefficients of class $\mathcal{C}^0(\overline{\Omega};\mathbb{R}) \cap \mathcal{C}^4(\Omega;\mathbb{R})$; moreover, $\forall x \in \Omega$, $A(x)$ is positive definite and its eigenvalues $0 < \lambda_1(x) \leqslant \lambda_2(x)$ satisfy the following: there exist $\eta_0 > 0$, $0 < m < M$ and $\alpha \geqslant 0$ such that $\forall x \in C(\Gamma,\eta_0)$, $\lambda_1(x) = d(x,\Gamma)^\alpha$ and $\lambda_2(x) \in [m,M]$. Finally we assume that $\forall x \in C(\Gamma,\eta_0)$, $\varepsilon_1(x) = \nu(p_\Gamma(x)) = -\nabla d(x,\Gamma)$ is an eigenvector of $A(x)$ associated to $\lambda_1(x)$.

Observe that the above parabolic operator is degenerate as soon as $\alpha > 0$. As we shall see, the study has to be divided in three parts, according to the value of α: $\alpha \in [0,1)$, $\alpha \in [1,2)$ and $\alpha \geqslant 2$.

2.1. Case $\alpha \in [0,1)$: weakly degenerate diffusion

Assume that Hypothesis 2.1 is satisfied with $\alpha \in [0,1)$. Then we can complete (1) with the usual homogeneous Dirichlet boundary condition:

$$\begin{cases} u_t - \operatorname{div}(A(x)\nabla u) = h\chi_\omega, & (t,x) \in \Omega_T := (0,T) \times \Omega, \\ u(t,x) = 0, & (t,x) \in \Gamma_T := (0,T) \times \Gamma, \\ u(0,x) = u_0(x) \in L^2(\Omega), & x \in \Omega. \end{cases} \tag{2}$$

The above problem is well-posed working in suitable weighted Sobolev spaces, namely:

$$H^1_A(\Omega) := \big\{u \in L^2(\Omega) \cap H^1_{\mathrm{loc}}(\Omega) \mid A\nabla u \cdot \nabla u \in L^1(\Omega)\big\}, \qquad H^1_{A,0}(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^1_A(\Omega)},$$
$$H^2_A(\Omega) := \big\{u \in H^1_A(\Omega) \cap H^2_{\mathrm{loc}}(\Omega) \mid \operatorname{div}(A\nabla u) \in L^2(\Omega)\big\},$$

where $\mathcal{D}(\Omega)$ denotes the space of all functions of class $\mathcal{C}^\infty$ that are compactly supported in Ω. In this context, trace and normal trace theories and integration by parts formula (necessary to justify the computations of the Carleman estimates stated below) can be developed. Further useful regularity results concerning the second order derivatives of functions in $H^2_A(\Omega)$ have also been proved in [3]. Finally, the unbounded operator $(\mathcal{A}_1, D(\mathcal{A}_1))$ defined by $D(\mathcal{A}_1) := H^2_A(\Omega) \cap H^1_{A,0}(\Omega)$, and $\forall u \in D(\mathcal{A}_1)$, $\mathcal{A}_1 u = \operatorname{div}(A\nabla u)$, is m-dissipative and self-adjoint, with dense domain in $L^2(\Omega)$. Therefore it generates a strongly C_0-semi-group in $L^2(\Omega)$ that can be proved to be analytic (see [3]). Let us now state our first controllability result:

Theorem 2.2. *Assume that A satisfies Assumption* 2.1 *for some $\alpha \in [0,1)$. Then, for all $T > 0$ and all $u_0 \in L^2(\Omega)$, there exists $h \in L^2(\Omega_T)$ such that the solution u of* (2) *satisfies $u(T,\cdot) = 0$ in $L^2(\Omega)$.*

2.2. Case $\alpha \in [1,2)$: strongly degenerate diffusion

Assume that Hypothesis 2.1 is satisfied with $\alpha \in [1,2)$. Then we associate to (1) the following generalized Neumann boundary condition:

$$\begin{cases} u_t - \operatorname{div}(A(x)\nabla u) = h\chi_\omega, & (t,x) \in \Omega_T := (0,T)\times\Omega, \\ A\nabla u \cdot \nu(t,x) = 0, & (t,x) \in \Gamma_T := (0,T)\times\Gamma, \\ u(0,x) = u_0(x) \in L^2(\Omega), & x \in \Omega. \end{cases} \tag{3}$$

Here again the problem is well-posed working in suitable weighted Sobolev spaces (see [3]). Our second controllability result is the following:

Theorem 2.3. *Assume that A satisfies Assumption* 2.1 *for some $\alpha \in [1,2)$. Then, for all $T > 0$ and $u_0 \in L^2(\Omega)$, there exists $h \in L^2(\Omega_T)$ such that the solution u of* (3) *satisfies $u(T,\cdot) = 0$ in $L^2(\Omega)$.*

2.3. A Carleman estimate for the (degenerate) adjoint problem

As in the nondegenerate case, the null controllability results stated in Theorems 2.2 and 2.3 may be reduced to the following observability inequality

$$\int_\Omega v(x,0)^2\,dx \leqslant C_T \int_{\omega_T} v(x,t)^2\,\mathrm{d}x\,\mathrm{d}t \quad \text{where } \omega_T := (0,T)\times\omega, \tag{4}$$

for the solutions v of the homogeneous adjoint problem with initial condition $v(T) \in L^2(\Omega)$:

$$\begin{cases} v_t + \operatorname{div}(A(x)\nabla v) = 0, & (t,x) \in \Omega_T := (0,T)\times\Omega, \\ v = 0 \quad \text{when } \alpha \in [0,1) \quad \text{or} \quad A\nabla v\cdot\nu(t,x) = 0 \quad \text{when } \alpha \in [1,2), & (t,x) \in \Gamma_T := (0,T)\times\Gamma. \end{cases} \tag{5}$$

The above inequality is well-known when dealing with nondegenerate parabolic operators. In the present degenerate context, we need to derive a suitable Carleman estimate to prove it. In order to define the weight functions appearing in the Carleman estimates, we first construct a function with special properties:

Lemma 2.4. *Let $\emptyset \neq \omega_0 \subset \Omega$ be an open set away from Γ and let $\alpha \in [0,2)$. Then there exists a positive number $\eta_2 \leqslant \eta_0$ and a function $\phi \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^4(\Omega)$ such that*

$$\text{(i) } \forall x \in C(\Gamma,\eta_2) \quad \phi(x) = \frac{1}{2-\alpha} d(x,\Gamma)^{2-\alpha}, \qquad \text{(ii) } \{x \in \Omega \mid \nabla\phi(x) = 0\} \subset \omega_0. \tag{6}$$

In particular, by (i)*, ϕ also satisfies*: $\forall x \in C(\Gamma,\eta_2)$, $\nabla\phi(x) = -d(x,\Gamma)^{1-\alpha}\nu(p_\Gamma(x)) = -d(x,\Gamma)^{1-\alpha}\varepsilon_1(x)$.

Now consider the functions θ, σ and ρ defined by

$$\forall t \in (0,T), \quad \theta(t) := \left(\frac{1}{t(T-t)}\right)^4, \quad \sigma(t,x) := \theta(t)\big(\mathrm{e}^{2S\|\phi\|_\infty} - \mathrm{e}^{S\phi(x)}\big), \quad \rho(t,x) := RS\theta(t)\mathrm{e}^{S\phi(x)}.$$

The Carleman estimate for problem (5) reads as follows:

Theorem 2.5. *Let A be a matrix satisfying Assumption* 2.1 *with $\alpha \in [0,2)$. Then there exist constants $\eta > 0$, $C = C(\alpha,\omega,T) > 0$ and $S_0 \geqslant 1$ such that $\forall S \geqslant S_0, \exists r(S) > 0, \forall R \geqslant S^4 + e^{12S\|\phi\|_\infty}$, and for all solution v of* (5) *with initial condition $v(T) \in L^2(\Omega)$, we have the following*:

(i) *Zero order term estimates*:

$$S\int_{\Omega_T} \big|A(x)\nabla\phi\cdot\nabla\phi\big|^2\rho^3v^2\mathrm{e}^{-2R\sigma} + \int_{C(\Gamma,\eta)_T} d(x,\Gamma)^{2-\alpha}\rho^3v^2\mathrm{e}^{-2R\sigma} \leqslant C\int_{\omega_T}\rho^3v^2\mathrm{e}^{-2R\sigma}, \tag{7}$$

(ii) *First order space derivatives estimates*:

$$r(S)\int\limits_{\Omega_T}\frac{\rho}{\theta}\big(A(x)\nabla v\cdot\nabla v\big)\mathrm{e}^{-2R\sigma}+\int\limits_{C(\Gamma,\eta)_T}d(x,\Gamma)^{\alpha}\rho(\nabla v,\varepsilon_1)^2\mathrm{e}^{-2R\sigma}\leqslant C\int\limits_{\omega_T}\rho^3v^2\mathrm{e}^{-2R\sigma}. \tag{8}$$

Let us mention that further estimates on v_t and D^2v can be proved, (see [2]).

2.4. Comments and extensions

(i) *Assumptions on the degeneracy*: in the physical models that motivated our study, in particular in the Fleming–Viot model, the smallest eigenvalue of $A(x)$ goes to zero as $x\to\Gamma$ as some power of $d(x,\Gamma)$, and the associated eigenvector is normal to the boundary for $x\in\Gamma$. It was convenient, here, to state our results under the simple but quite restrictive Assumption 2.1. Nevertheless we refer the reader to [2] for an extension of our results under weaker assumptions, in particular in the case where

$$\lambda_1(x)\sim d(x,\Gamma)^{\alpha}\quad \text{as } x\to\Gamma,$$
$$\varepsilon_1(x)-\nu\big(p_\Gamma(x)\big)\to 0\quad\text{as } x\to\Gamma,\qquad 0<m\leqslant\lambda_2(x)\leqslant M\quad\text{for all } x\in\Omega.$$

(ii) *The case of stronger degeneracies i.e.* $\alpha\geqslant 2$: when Assumption 2.1 holds with $\alpha\geqslant 2$, null controllability does not hold true anymore. Indeed, as already noted for the 1-dimensional case [1], such problems may be transformed into nondegenerate parabolic problems in unbounded domains. Then, using results by Micu, Zuazua [7] and Escauriaza, Seregin, Sverák [4], one easily checks that null controllability fails for the transformed problem when an unbounded region is left without control. See [2] for more details.

(iii) The above results can be extended to dimension N for parabolic operators with lower order terms, provided similar structural assumptions are satisfied, see [2].

3. Technical tools and idea of the proofs

3.1. Hardy type inequalities

When dealing with degenerate problems, Hardy type inequalities are very useful tools. In the present context, such inequalities turn out to be fundamental for the proof of Theorem 2.5. In particular, in order to treat the case of $\alpha\in[0,1)$, we use the following

Lemma 3.1. *For all* $\alpha\in[0,1)$*, there exists a positive constant* $C_H=C_H(\alpha)$ *such that, for all* $\eta\in(0,\eta_0)$*, and all functions* $z\in H^1_{A,0}(\Omega)$*, we have*

$$\int\limits_{C(\Gamma,\eta)}d(x,\Gamma)^{\alpha-2}z(x)^2\,\mathrm{d}x\leqslant C_H\int\limits_{C(\Gamma,\eta)}d(x,\Gamma)^{\alpha}\big(\nabla z(x)\cdot\varepsilon_1(x)\big)^2\,\mathrm{d}x. \tag{9}$$

(9) naturally extends the well-known 1-dimensional Hardy inequality. Usually, N-dimensional Hardy inequalities involve the full gradient term $|\nabla z(x)|$ (see, e.g., [8]), rather than its "normal" component $\nabla z(x)\cdot\varepsilon_1(x)$, which is essential for our purposes. The idea of the proof of (9) is very simple: using the natural C^1-diffeomorphism between $C(\Gamma,\eta)$ and $\Gamma\times(0,\eta)$, we apply the 1-d Hardy inequality on all the segments that are normal to the boundary; then we integrate along the boundary (as in [9]).

3.2. Idea of the proof of Theorem 2.5

First we need to prove Lemma 2.4. The construction of ϕ is simple when the domain Ω is convex. In the general case, the idea is, first, to construct a Morse function satisfying (6)(i) and then to move its critical points (that are in finite number) into the control region ω_0 using a suitable flow.

For the proof of Theorem 2.5, we first proceed as usual: we write the equation satisfied by $z:=v\mathrm{e}^{-R\sigma}$ under the form $P_R^+z+P_R^-z=0$ where P_R^+z is the symmetric part of the operator. Then an important step to provide a Carleman estimate is to get a suitable lower bound for the scalar product $\langle P_R^+ z, P_R^- z\rangle$, provided by integrations by parts. One difficulty, here, is that integration by parts with respect to the space variable over Ω is not obviously justified due to the fact that the solution is not $H^2(\Omega)$. To avoid this difficulty, we initially make the computations over smaller domains $\{x \in \Omega,\ d(x, \Gamma) > \delta\}$ and then we recover the result over Ω as $\delta \to 0$ using properties of $W^{1,1}(\Omega)$ functions and the regularity results proved in [3]. Then to estimate the expression of $\langle P_R^+ z, P_R^- z\rangle$, we need to use weight functions adapted to degeneracy, whence the necessity of Lemma 2.4. Once the choice of the weight functions is made, we absorb the nonpositive terms into the dominant positive terms, using in an essential way the Hardy type inequality of Lemma 3.1. (Note that such strategy, that is studying what happens far from the boundary and then close to the boundary using Hardy type inequalities, is classical; see, e.g., [9].)

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