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SCALING LIMITS OF INTEGRABLE QUANTUM FIELD THEORIES

HENNING BOSTELMANN

Department of Mathematics, University of York,
York YO10 5DD, United Kingdom
henning.bostelmann@york.ac.uk

GANDALF LECHNER

Institute for Theoretical Physics,
Vor dem Hospitaltore 1, 04013 Leipzig, Germany
gandalf.lechner@itp.uni-leipzig.de

GERARDO MORSELLA

Department of Mathematics, University of Roma Tor Vergata,
viale della Ricerca Scientifica 1, I-00133 Roma, Italy
morsella@mat.uniroma2.it

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Short distance scaling limits of a class of integrable models on two-dimensional Minkowski space are considered in the algebraic framework of quantum field theory. Making use of the wedge-local quantum fields generating these models, it is shown that massless scaling limit theories exist, and decompose into (twisted) tensor products of chiral, translation-dilation covariant field theories. On the subspace which is generated from the vacuum by the observables localized in finite light ray intervals, this symmetry can be extended to the Möbius group. The structure of the interval-localized algebras in the chiral models is discussed in two explicit examples.

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1. Introduction

In the analysis of quantum field theories, the information gained by computing the ultraviolet scaling limit and determining its properties is most relevant. Probably the most important application of this principle in physics is perturbative asymptotic freedom of QCD — the key feature which led to its general acceptance as the quantum field theory of strong interactions.

In view of its importance, several approaches to the computation of the scaling limit have been developed, adapted to different descriptions of quantum field theory. In the Lagrangian framework, the requirement that the physical amplitudes are independent of the arbitrary choice of the distance (or energy) scale at which the theory is renormalized, provides an equation for the dependence of renormalized correlation function on coupling constants and the renormalization scale (the Callan–Symanzik equation). Using information from perturbation theory, the coefficients of the equation (the β-functions) can usually be determined, and the scaling limit of correlation functions computed by solving it.

However, these methods are of little help in cases in which perturbation theory is not reliable, or where the theory is not defined in terms of a Lagrangian at all. In order to circumvent these problems, a different approach, based on the algebraic setting of quantum field theory [33], has been proposed by Buchholz and Verch [22] and extended in [12, 13]. In this approach, one considers the *scaling algebra*, i.e. the algebra generated by functions $\lambda \mapsto A_\lambda$ of the scaling parameter with values in the algebra of observables of the theory, satisfying certain specific phase space properties. The scaling limit is then obtained as the GNS representation of the scaling algebra induced by the scaling limit of the vacuum state on the original algebra at finite scales. Relying only on the knowledge of the observables of the theory, this method is completely model-independent, and it proved to be very useful in analyzing the scaling limit of charged sectors, and in providing an intrinsic definition of confined charge [16, 25]. A study of the relations with the Lagrangian approach can be found in [12].

In the present article, we study scaling limits of a certain class of integrable quantum field theories on two-dimensional Minkowski space. It is interesting to note that two-dimensional sigma models, which are integrable field theories — although not directly covered by our results — share with QCD the property of asymptotic freedom, as well as several others (see, e.g., [61] and references therein). We will study a simplified version of these: At finite scale, the models we are interested in describe a single type of scalar neutral Bosons of mass $m > 0$, whose collision theory is governed by a factorizing S-matrix. This means that the particle number is conserved in each scattering process and the n-body S-matrix factorizes into a product of two-body S-matrices, cf. the textbook and review [1, 26] and the references cited therein. A prominent example of a model in the considered class is the Sinh-Gordon model.

We are particularly interested in the connection between the long and short distance regimes of such quantum field theories, represented by the S-matrix on the one hand and the scaling limit on the other hand. For the simplified particle spectrum that we consider here, a factorizing S-matrix can be fully characterized by a single complex-valued function S, the so-called scattering function. It is therefore possible to formulate these models in the spirit of inverse scattering theory, taking a scattering function S and a mass value $m > 0$ as an input. Such a setup is directly related to the long distance regime, and will be more convenient for our analysis than Euclidean perturbation theory (see, e.g., [30]), where the relation to the real time S-matrix is quite indirect.

There exist different, complementary, approaches to the inverse scattering problem. One such approach, known as the form factor program, aims at computing n-point functions of local fields in terms of so-called form factors, i.e. matrix elements of field operators in scattering states [56]. But despite many partial results known in the literature [4], in this approach one usually runs into the problem that the convergence of the appearing infinite series cannot be controlled because of the complicated form of local field operators [3]. Another approach is based on the operator-algebraic framework of quantum field theory and Tomita–Takesaki modular theory, and constructs the models in question by an indirect procedure involving auxiliary field operators with weakened localization properties. Instead of being sharply localized at points in space-time, these fields ("polarization-free generators" [9]) are localized only in infinite wedge-shaped regions ("wedges"). This last approach will be most convenient for our purposes as it is closely connected to the S-matrix and does not rely on series expansions with unknown convergence properties. Starting from a scattering function S and a mass $m > 0$, a solution to the inverse scattering problem has been rigorously constructed in this setting [52, 55, 37, 18, 40]. The main results of this analysis will be recalled in Sec. 2 in a manner adapted to scaling transformations.

The resulting models meet all standard requirements of algebraic quantum field theory, and hence on abstract grounds, a well-defined scaling limit in the sense of Buchholz and Verch exists. In particular, the short distance regime is in principle completely described by the initially chosen S-matrix. We will not fully analyze the Buchholz–Verch limit here, but choose a simplified construction in the same spirit.

For the limit theory, one has natural candidates: massless models with factorizing scattering. These have been described before in a thermodynamical context; see, e.g., [60, 29]. Here, however, we treat them as rigorously constructed quantum field theories on two-dimensional Minkowski space, given in terms of local algebras of observables. These limit theories are interesting in their own right as they provide non-trivial covariant deformations of free field theories (see also [41] for higher-dimensional generalizations), and still depend on the scattering function S one started with. Furthermore, as expected for a scaling limit [13], they are dilation covariant and, as it turns out, (extensions of) chiral nets. This distinguishes them from other massless deformations of quantum field theories that have recently been constructed in the algebraic framework, on two-dimensional Minkowski space [27] and Minkowski half-space [42, 43].

In this paper, we start to explore the relation between the scattering function defining a massive model of the class mentioned above, and the properties of the corresponding scaling limit.

The first step consists of computing the behavior of the scattering function under scaling transformations, and to determine the short distance structure of the n-point functions of the wedge-local generators. This is done in Secs. 2 and 3, respectively. As expected, the mass vanishes in the short distance limit, and we obtain a class of massless (local extensions of) chiral quantum field theories. They are presented in Sec. 4. As we shall explain, their dependence on S is twofold: On the one hand, S determines the decomposition of the two-dimensional massless generators into twisted or untwisted tensor products of chiral fields on the left and right light ray. On the other hand, the chiral components can be generated by massless chiral quantum fields which are localized on half-lines, similar to the massive situation. The commutation relations of these fields directly involve the scattering function S in a manner very similar to the two-dimensional models at finite scale, despite the difference in mass and space-time dimension.

The chiral subtheories always transform covariantly under a representation of the translation-dilation-reflection group of the light ray. Making use of modular theory, we will show that on a subspace of the chiral Hilbert space, one can always extend this affine symmetry by a conformal rotation to the Möbius group (Sec. 5). This conformal subspace is directly related to observables localized in finite intervals on the light ray. But because our construction is based on halfline-local generator fields, such strictly localized observables are derived quantities here, and it is a non-trivial task to characterize them.

We obtain two results in this direction: First, we show that for certain scattering functions, the local chiral observables are fixed points under an additional $\mathbb{Z}_2$-symmetry, which restricts the conformal subspace. Second, we investigate the models given by two simple example scattering functions in full detail in Sec. 6. In these examples, we find conformal nets with central charge $c = 1$ respectively $c = \frac{1}{2}$ in the limit. This analysis also exemplifies that the scaling limit of a Bosonic theory can be generated by the energy-momentum tensor of a Fermi field.

Section 7 contains our conclusions and an account of further work in progress.

2. Two-Dimensional Integrable Models

In this section, we recall the structure of the quantum field theories we are interested in. At finite scale, these models describe a single species of scalar Bosons of mass $m \geq 0$ on two-dimensional Minkowski space. Scattering processes of these particles are governed by a *factorizing S-matrix* [1, 3, 26], i.e. in each collision process the particle number and the momenta are conserved, and the n-particle S-matrix factorizes into a product of two-particle S-matrices. In this situation, the S-matrix is determined by a single function S, called the *scattering function*. Such a restricted form of the collision operator is typical for completely integrable models [34], which provide a rich class of examples for factorizing S-matrices.

The family of model theories we consider is thus parametrized by the two data (m, S), where m is a mass parameter and S a function with a number of properties specified below. Before recalling the construction of these quantum field theories, we define the space of parameters (m, S) and investigate its scaling properties. We will first consider the case $m > 0$, and then obtain the massless case $m = 0$ in a suitable limit.

2.1. *Scaling limits of scattering functions*

The defining properties of a scattering function S can most conveniently be expressed when treating S as a function of the rapidity θ as the momentum space variable, which parametrizes the upper mass shell with mass $m > 0$ according to

$$p_m(\theta) := m \begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix}, \quad \theta \in \mathbb{R}. \tag{2.1}$$

Since Lorentz boosts are translations in the rapidity, the Lorentz invariant scattering function depends only on differences of rapidities. Writing

$$\mathrm{S}(a,b) := \{\zeta \in \mathbb{C} : a < \mathrm{Im}\,\zeta < b\} \tag{2.2}$$

for two real numbers $a < b$, and $\overline{\mathrm{S}(a,b)}$ for the closed strip, the family of all scattering functions and two important subfamilies are defined as follows.

Definition 2.1 (Scattering Functions). (a) A *scattering function* is a bounded and continuous function $S : \overline{\mathrm{S}(0,\pi)} \to \mathbb{C}$ which is analytic in the interior of this strip and satisfies for $\theta \in \mathbb{R}$,

$$\overline{S(\theta)} = S(\theta)^{-1} = S(\theta + i\pi) = S(-\theta). \tag{2.3}$$

The family of all scattering functions is denoted $\mathcal{S}$.

(b) A scattering function $S \in \mathcal{S}$ is called *regular* if there exists $\kappa > 0$ such that S continues to a bounded analytic function in the strip $\mathrm{S}(-\kappa, \pi + \kappa)$. The subfamily of all regular scattering functions is denoted $\mathcal{S}_{\mathrm{reg}} \subset \mathcal{S}$.

(c) A regular scattering function $S \in \mathcal{S}_{\mathrm{reg}}$ is called a *scattering function with limit* if the two limits $\lim_{\theta\to\infty} S(\theta)$ and $\lim_{\theta\to-\infty} S(\theta)$ exist. The family of all scattering functions with limit is denoted $\mathcal{S}_{\mathrm{lim}} \subset \mathcal{S}_{\mathrm{reg}} \subset \mathcal{S}$.

Equations (2.3) express the unitarity, crossing symmetry, and hermitian analyticity of the factorizing S-matrix corresponding to S. For a discussion of these standard properties, we refer to the textbooks and reviews [1, 4, 35, 56, 26]. The regularity assumption in part (b) of Definition 2.1 comes from the fact that for each regular $S \in \mathcal{S}_{\mathrm{reg}}$, a corresponding quantum field theoretic model is known to exist [40], whereas for non-regular scattering functions, this is not known. Particular examples of regular scattering functions (with limit) are the constant functions $S_{\mathrm{free}}(\theta) = 1$ and $S_{\mathrm{Ising}}(\theta) = -1$, corresponding to the interaction-free theory and the Ising model, respectively, and the scattering function of the Sinh-Gordon model with coupling constant $g \in \mathbb{R}$ [2],

$$S_{\mathrm{ShG}}(\theta) := \frac{\sinh\theta - i\sin\dfrac{\pi g^2}{4\pi + g^2}}{\sinh\theta + i\sin\dfrac{\pi g^2}{4\pi + g^2}}. \tag{2.4}$$

The additional assumption in part (c) of the above definition, concerning the existence of limits of scattering functions, is relevant in the context of scaling limits: If distances in Minkowski space are scaled according to $x \to \lambda x$, and Planck's unit of action $\hbar$ is kept fixed, momenta have to be rescaled according to $p \to \lambda^{-1}p$. So rapidities scale like $\theta = \sinh^{-1}\frac{p}{m} \to \sinh^{-1}\frac{p}{\lambda m}$ and converge to $\pm\infty$ for $\lambda \to 0$. Looking at the example of the Sinh-Gordon scattering function (2.4), where the coupling constant is dimensionless and therefore does not scale with λ, we see that the only dependence of $S(\theta_1 - \theta_2)$ on the scale λ is via the scale dependence of θ_1, θ_2. Hence for $S(\theta_1 - \theta_2)$ to have a scaling limit as $\lambda \to 0$, we need to require the existence of the limits as in part (c).

An explicit characterization of such functions is given in the following proposition.

Proposition 2.2 (Scattering Functions with Limits). (a) *The set* $\mathcal{S}_{\lim}$ *of scattering functions with limits consists precisely of the functions*

$$S(\zeta) = \varepsilon \cdot \prod_{k=1}^{N} \frac{\sinh\zeta - \sinh b_k}{\sinh\zeta + \sinh b_k}, \quad \zeta \in \overline{\mathrm{S}(0,\pi)}, \tag{2.5}$$

where $\varepsilon = \pm 1, N \in \mathbb{N}_0$, *and* $\{b_1, \ldots, b_N\}$ *is a set of complex numbers in the strip* $0 < \operatorname{Im} b_1, \ldots, \operatorname{Im} b_N \leq \frac{\pi}{2}$, *such that with each* b_k (*counted according to multiplicity*) *also* $-\overline{b_k}$ *is contained in* $\{b_1, \ldots, b_N\}$.

(b) *For each* $S \in \mathcal{S}_{\lim}$, *the two limits* $S(\infty) := \lim_{\theta\to\infty} S(\theta) = \lim_{\theta\to-\infty} S(\theta)$ *coincide and are equal to* ± 1, *i.e.* $\mathcal{S}_{\lim}$ *is the disjoint union of the sets*

$$\mathcal{S}_{\lim}^{\pm} := \left\{ S \in \mathcal{S}_{\lim} : \lim_{\theta\to\infty} S(\theta) = \lim_{\theta\to-\infty} S(\theta) = \pm 1 \right\}. \tag{2.6}$$

Proof. (a) Each factor $s_{b_k} : \zeta \mapsto \pm(\sinh\zeta - \sinh b_k)(\sinh\zeta + \sinh b_k)^{-1}$ satisfies $s_{b_k}(-\zeta) = s_{b_k}(\zeta + i\pi) = s_{b_k}(\zeta)^{-1} = \overline{s_{-\overline{b_k}}(\zeta)}$ for $\zeta \in \mathbb{R}$. Given any sufficiently small $\delta > 0$, the function s_{b_k} is analytic and bounded in the strip $\mathrm{S}(-\operatorname{Im} b_k + \delta, \pi + \operatorname{Im} b_k - \delta) \supset \mathrm{S}(0,\pi)$. Because the product (2.5) is finite, it follows that S is analytic and bounded in the strip $\mathrm{S}(-\kappa, \pi + \kappa)$ for some $\kappa > 0$. Furthermore, the last two equations in (2.3) hold for S because they hold for each factor s_{b_k}. The first equation in (2.3) holds because of $s_{b_k}(\zeta)^{-1} = \overline{s_{-\overline{b_k}}(\zeta)}$ and the assumed invariance of $\{b_1, \ldots, b_N\}$ under $b_k \to -\overline{b_k}$. Hence each S of the form (2.5) is a regular scattering function. As $\theta \to \pm\infty$, we clearly have $S(\theta) \to \varepsilon$, which shows $S \in \mathcal{S}_{\lim}$.

Now we pick some arbitrary $S \in \mathcal{S}_{\lim}$ and show that it is of the form (2.5). As a regular scattering function, S is bounded and analytic in a strip $\mathrm{S}(-\kappa, \pi + \kappa)$ for some $\kappa > 0$, and since $S \in \mathcal{S}_{\lim}$, we have a limit value $\varepsilon \in \mathbb{C}$ such that $S(\theta) \to \varepsilon$ as $\theta \to \infty$. These properties imply that $S(\theta + i\lambda) \to \varepsilon$ as $\theta \to \infty$, uniformly in $\lambda \in [0, \pi]$ [58, p. 170]. In view of $S(\theta + i\pi) = S(-\theta)$, we also have $S(\theta + i\lambda) \to \varepsilon$ for $\theta \to -\infty$. In particular, the two limits $\lim_{\theta\to\pm\infty} S(\theta)$ along the real line coincide.

Since S has unit modulus on the real line (2.3), we have $|\varepsilon| = 1$, and because of the uniform limit $S(\zeta) \to \varepsilon$ as $\mathrm{Re}(\zeta) \to \pm\infty$, we find $c > 0$ such that $|\mathrm{Re}(\zeta_0)| \leq c$ for all zeros ζ_0 of S. Taking into account that S is continuous on the closed strip $\overline{\mathrm{S}(0,\pi)}$, and of modulus 1 on its boundary, we conclude that it has only finitely many zeros in $\mathrm{S}(0,\pi)$.

Let us denote by $b_1, \ldots, b_N$ those zeros of S whose imaginary parts λ satisfy $0 < \lambda \leq \frac{\pi}{2}$. These zeros come in pairs $\{b_k, -\overline{b_k}\}$ because of (2.3), and there also exist corresponding zeros $i\pi - b_k, i\pi + \overline{b_k}$ in the upper half of the strip. Now consider the product

$$B(\zeta) := \varepsilon \cdot \prod_{k=1}^{N} \frac{\sinh \zeta - \sinh b_k}{\sinh \zeta + \sinh b_k},$$

which is a regular scattering function $B \in \mathcal{S}_{\text{reg}}$, of the form specified in (2.5). Since B has precisely the same zeros as S in $\mathrm{S}(0, \pi)$, and $B(\theta + i\lambda) \to \varepsilon$ for $\theta \to \pm\infty$, also $F := S \cdot B^{-1}$ belongs to $\mathcal{S}$.

By construction, F has no zeros in $\mathrm{S}(0, \pi)$, and $F(\theta + i\lambda)$ converges to 1 for $\theta \to \pm\infty$, uniformly in $0 \leq \lambda \leq \pi$. As F is continuous on $\overline{\mathrm{S}(0, \pi)}$ and of modulus 1 on the boundary of this strip, it is bounded from above and below, i.e. there exists $K > 0$ such that $K < |F(\zeta)| \leq 1$, $\zeta \in \overline{\mathrm{S}(0, \pi)}$. But any scattering function, and in particular F, can be meromorphically continued to $\mathrm{S}(-\pi, \pi)$ by the equations (2.3). In fact, this continuation is given by

$$F(-\zeta) = F(\zeta)^{-1}, \quad \zeta \in \overline{\mathrm{S}(0, \pi)}, \tag{2.7}$$

and as F has no zeros in $\overline{\mathrm{S}(0, \pi)}$, it is actually an analytic continuation for this special scattering function. In view of the boundedness of F on $\overline{\mathrm{S}(0, \pi)}$, there also holds $|F(\zeta)| < K^{-1} < \infty$ for all $\zeta \in \overline{\mathrm{S}(-\pi, \pi)}$. Taking $\zeta = -\theta + i\pi$, $\theta \in \mathbb{R}$, Eqs. (2.7) and (2.3) give

$$F(\theta - i\pi) = F(i\pi - \theta)^{-1} = F(\theta)^{-1} = F(\theta + i\pi), \quad \theta \in \mathbb{R},$$

i.e. F continues to a $(2\pi i)$-periodic, entire function which in view of the above argument is bounded and hence constant. Thus $F(\theta) = \lim_{\theta \to \infty} F(\theta) = 1$, and we arrive at the claimed representation (2.5) for S, namely $S = F \cdot B = B$.

(b) The identity of the limits $\lim_{\theta \to \pm\infty} S(\theta)$ has been shown above, and can also be seen directly from (2.5). Also the fact that these limits can take only the values ± 1 is clear from (2.5). □

As a preparation for the scaling limit of quantum fields, we now compute which effect a space-time scaling $x \to \lambda x$ has on a scattering function with limit. As usual, such a limit involves taking the mass to zero. To keep track of the mass scale, we will use momentum variables with explicit mass dependence instead of the rapidity. For spatial momenta $p = m \sinh \theta$, $q = m \sinh \theta'$, we have

$$\theta - \theta' = \sinh^{-1} \frac{p}{m} - \sinh^{-1} \frac{q}{m} = \sinh^{-1} \left(\frac{p\omega_q^m - q\omega_p^m}{m^2} \right),$$

with the energies $\omega_p^m := (p^2 + m^2)^{1/2}$, $\omega_q^m := (q^2 + m^2)^{1/2}$. Corresponding to any $m > 0$, $S \in \mathcal{S}_{\text{lim}}$, we therefore introduce the function $S_m : \mathbb{R}^2 \to \mathbb{C}$,

$$S_m(p, q) := S \left(\sinh^{-1} \left(\frac{p\omega_q^m - q\omega_p^m}{m^2} \right) \right), \tag{2.8}$$

which shows the mass dependence explicitly. Clearly, S_m inherits many properties from S, see Eq. (2.3). For example, one has the symmetry and scaling relations, for $p, q \in \mathbb{R}$,

$$S_m(q,p) = S_m(p,q)^{-1} = \overline{S_m(p,q)}, \tag{2.9}$$

$$S_m(\lambda^{-1}p, \lambda^{-1}q) = S_{\lambda m}(p,q), \quad \lambda > 0. \tag{2.10}$$

The mass zero limit S_0 of S_m can be computed in a straightforward manner.

Lemma 2.3. *Let $S \in \mathcal{S}^{\pm}_{\lim}$, and $m > 0$. Then, for $p, q \in \mathbb{R}$,*

$$S_0(p,q) := \lim_{\lambda \to 0} S_{\lambda m}(p,q) = \begin{cases} S(\log p - \log q), & p > 0, \ \ q > 0; \\ S(\log(-q) - \log(-p)), & p < 0, \ \ q < 0; \\ S(0), & p = q = 0; \\ S(\infty), & \textit{otherwise}. \end{cases} \tag{2.11}$$

Proof. For any $p, q \in \mathbb{R}$, we have

$$\lim_{\lambda \to 0} (p\omega_q^{\lambda m} - q\omega_p^{\lambda m}) = p|q| - q|p| = \begin{cases} \pm 2pq, & p \cdot q < 0; \\ 0, & p \cdot q \geq 0. \end{cases} \tag{2.12}$$

This implies $(\lambda m)^{-2}(p\omega_q^{\lambda m} - q\omega_p^{\lambda m}) \to \pm\infty$ for $\lambda \to 0$ if $p \cdot q < 0$, and since $S \in \mathcal{S}^{\pm}_{\lim}$, we get $S_{\lambda m}(p,q) \to S(\infty)$ for this configuration of momenta. In the case $p \cdot q \geq 0$, we use l'Hospital's rule to compute the limit,

$$\begin{aligned} \lim_{\lambda \to 0} \frac{p\omega_q^{\lambda m} - q\omega_p^{\lambda m}}{\lambda^2 m^2} &= \lim_{\lambda \to 0} \frac{\dfrac{p\lambda m^2}{\omega_q^{\lambda m}} - \dfrac{q\lambda m^2}{\omega_p^{\lambda m}}}{2\lambda m^2} = \frac{1}{2} \lim_{\lambda \to 0} \left(\frac{p}{\omega_q^{\lambda m}} - \frac{q}{\omega_p^{\lambda m}} \right) \\ &= \begin{cases} 0, & p = q = 0; \\ \varepsilon(p) \cdot \infty, & p \neq 0, \ \ q = 0; \\ -\varepsilon(q) \cdot \infty, & p = 0, \ \ q \neq 0; \\ \dfrac{1}{2}\left(\dfrac{p}{|q|} - \dfrac{q}{|p|} \right), & p \cdot q > 0. \end{cases} \end{aligned}$$

Here $\varepsilon(p)$, $\varepsilon(q)$ denotes the sign of p, q, respectively. Evaluating these expressions in $S \circ \sinh^{-1}$ (2.8) gives the claimed result. □

Note that the limit S_0 is not independent of the scattering function S; in fact, S can be completely recovered from S_0 (2.11). This can be seen as an indication that the short distance behavior of the (m, S)-model will depend on S (but not on m). The limit behavior of the scattering functions will be used in the calculation of the scaling limit of the field theory models discussed in the next section.

2.2. *Massive and massless models with factorizing S-matrices*

We now turn to the description of the family of quantum field theoretic models we are interested in. Each model in this family is specified by two parameters, a mass value $m \geq 0$ and a scattering function $S \in \mathcal{S}_{\lim}$ with limit.

Whereas the most frequently used setting for the discussion of such models is the form factor program [4], their rigorous construction was accomplished only recently with the help of operator-algebraic techniques. The initial idea of this program is due to Schroer [52, 53] and consists in constructing certain auxiliary field operators depending on (m, S). Despite their weaker than usual localization, these fields can be used to define a strictly local, covariant quantum field theory in an indirect manner. The details of this construction, and the passage to algebras of strictly localized observables, was carried out in [37, 18, 39, 40]. In particular, it has been shown that for any choice of (m, S), $m > 0$, $S \in \mathcal{S}_{\rm reg}$, there exists a corresponding quantum field theory with the factorizing S-matrix given by S as its collision operator. In the following, we will outline the structure of these models using a momentum space formulation. For details and proofs, we refer to the articles cited above.

Fixing arbitrary $S \in \mathcal{S}_{\lim}$ and $m \geq 0$, the function S_m is defined via (2.8) for $m > 0$ and via the limit (2.11) for $m = 0$. Note that the zero mass function S_0 (2.11) can be discontinuous at $(0, 0)$ if the signs of $S(0)$ and $S(\infty)$ are different, but still satisfies the symmetry relations (2.9).

Most of the objects introduced below depend on the choice of S, but since we will work with a fixed scattering function in the following, we do not reflect this dependence in our notation. The mass dependence, on the other hand, will always be written down explicitly.

Having fixed (m, S), we first describe the Hilbert space on which the (m, S)-model is constructed. Starting from the single particle space $\mathcal{H}_{m,1} := L^2(\mathbb{R}, dp/\omega_p^m)$, the n-particle spaces $\mathcal{H}_{m,n}$, $n > 1$, are defined as certain S_m-symmetrized subspaces of the n-fold tensor product $\mathcal{H}_{m,1}^{\otimes n}$. To this end, one introduces unitaries $D_n(\tau_j)$, $j = 1, \ldots, n-1$, on $\mathcal{H}_{m,1}^{\otimes n}$,

$$(D_n(\tau_j)\Psi_n)(p_1, \ldots, p_n) := S_m(p_{j+1}, p_j) \cdot \Psi_n(p_1, \ldots, p_{j+1}, p_j, \ldots, p_n). \quad (2.13)$$

Using (2.9), one checks that these operators generate a unitary representation D_n of the group $\mathfrak{S}_n$ of permutations of n letters which represents the transposition exchanging j and $j+1$ by $D_n(\tau_j)$. The n-particle space $\mathcal{H}_{m,n}$ of the (m, S)-model is defined as the subspace of $\mathcal{H}_{m,1}^{\otimes n}$ of vectors invariant under this representation. Explicitly, the orthogonal projection $P_n : \mathcal{H}_{m,1}^{\otimes n} \to \mathcal{H}_{m,n}$ has the form

$$(P_n\Psi_n)(p_1, \ldots, p_n) := \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} S_m^\pi(p_1, \ldots, p_n) \cdot \Psi_n(p_{\pi(1)}, \ldots, p_{\pi(n)}), \quad (2.14)$$

$$S_m^\pi(p_1, \ldots, p_n) := \prod_{\substack{1 \leq l < r \leq n \\ \pi(l) > \pi(r)}} S_m(p_{\pi(l)}, p_{\pi(r)}). \quad (2.15)$$

Setting $\mathcal{H}_{m,0} := \mathbb{C}$, the *$S_m$-symmetric Fock space* over $\mathcal{H}_{m,1}$ is

$$\mathcal{H}_m := \bigoplus_{n=0}^{\infty} \mathcal{H}_{m,n}, \tag{2.16}$$

i.e. its vectors are sequences $\Psi = (\Psi_0, \Psi_1, \Psi_2, \ldots)$, with $\Psi_0 \in \mathbb{C}$, $\Psi_n \in \mathcal{H}_{m,n}$, $n \geq 1$, such that $\|\Psi\|^2 := |\Psi_0|^2 + \sum_{n=1}^{\infty} \int \frac{dp_1}{\omega_1^m} \cdots \frac{dp_n}{\omega_n^m} \,|\, \Psi_n(p_1, \ldots, p_n)|^2 < \infty$. Here and in the following we use the shorthand notation $\omega_k^m = \omega_{p_k}^m = (p_k^2 + m^2)^{1/2}$.

On $\mathcal{H}_m$, there exists a strongly continuous (anti-)unitary positive energy representation U_m of the full Poincaré group $\mathcal{P}$. Denoting by $(x, \theta) \in \mathcal{P}_+^\uparrow$ proper orthochronous transformations consisting of a boost with rapidity θ and a subsequent space-time translation along $x = (x_0, x_1) \in \mathbb{R}^2$, we set

$$(U_m(x,\theta)\Psi)_n(p_1, \ldots, p_n) := e^{i\sum_{j=1}^{n}(\omega_j^m x_0 - p_j x_1)} \cdot \Psi_n(\theta p_1, \ldots, \theta p_n), \tag{2.17}$$

where $\theta p_j := \cosh\theta \cdot p_j - \sinh\theta \cdot \omega_j^m$, $j = 1, \ldots, n$. The space-, time-, and space-time reflections $j_1(x_0, x_1) := (x_0, -x_1)$, $j_0(x_0, x_1) := (-x_0, x_1)$ and $j := j_0 j_1$ are represented as

$$(U_m(j_1)\Psi)_n(p_1, \ldots, p_n) := \Psi_n(-p_n, \ldots, -p_1), \tag{2.18}$$

$$(U_m(j_0)\Psi)_n(p_1, \ldots, p_n) := \overline{\Psi_n(-p_1, \ldots, -p_n)}, \tag{2.19}$$

$$(U_m(j)\Psi)_n(p_1, \ldots, p_n) := \overline{\Psi_n(p_n, \ldots, p_1)}. \tag{2.20}$$

Clearly, all vectors in $\mathcal{H}_{m,1}$ are eigenvectors of the mass operator with eigenvalue m, and the vector $\Omega_m := 1 \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{H}_m$, invariant under U_m, represents the vacuum state. The finite particle number subspace of $\mathcal{H}_m$ is denoted $\mathcal{D}_m$.

On $\mathcal{D}_m$, there act creation and annihilation operators $z_m^\#(\chi)$, $\chi \in \mathcal{H}_{m,1}$, defined as

$$(z_m(\chi)\Psi)_n(p_1, \ldots, p_n) := \sqrt{n+1} \int \frac{dq}{\omega_q^m} \chi(q) \Psi_{n+1}(q, p_1, \ldots, p_n), \tag{2.21}$$

$$z_m^\dagger(\chi) := z_m(\overline{\chi})^* \Leftrightarrow (z_m^\dagger(\chi)\Psi)_n = \sqrt{n} P_n(\chi \otimes \Psi_{n-1}). \tag{2.22}$$

Because of the S_m-symmetrization properties of the vectors in $\mathcal{H}_m$, the distributional kernels $z_m^\#(p)$, $p \in \mathbb{R}$, related to the above operators by the formal integrals $z_m^\#(\chi) = \int \frac{dp}{\omega_p^m} \chi(p) z_m^\#(p)$, satisfy the exchange relations of the Zamolodchikov–Faddeev algebra [59, 28],

$$z_m(p) z_m(q) = S_m(p,q) z_m(q) z_m(p), \tag{2.23}$$

$$z_m^\dagger(p) z_m^\dagger(q) = S_m(p,q) z_m^\dagger(q) z_m^\dagger(p), \tag{2.24}$$

$$z_m(p) z_m^\dagger(q) = S_m(q,p) z_m^\dagger(q) z_m(p) + \omega_p^m \delta(p-q) \cdot \mathbf{1}_{\mathcal{H}_m}. \tag{2.25}$$

Having described the Hilbert space of the (m, S)-model, we now construct field operators on it, and first introduce the necessary test functions.[a] For $f \in \mathscr{S}(\mathbb{R}^2)$

[a]We will use the symbol $\mathscr{S}(\mathbb{R}^n)$ for the Schwartz space on $\mathbb{R}^n$. Given some set $O \subset \mathbb{R}^n$, we also write $\mathscr{S}(O) := \{f \in \mathscr{S}(\mathbb{R}^n) \colon \operatorname{supp} f \subset \overline{O}\}$ for its subspace supported in O.

we write

$$f^{m\pm}(p) := \frac{1}{2\pi} \int d^2x f(x) e^{\pm i(\omega_p^m, p)\cdot x} \tag{2.26}$$

for the restrictions of the Fourier transform of f to the upper and lower mass shell of mass $m \geq 0$. For $m > 0$, we have $f^{m\pm} \in L^2(\mathbb{R}, dp/\omega_p^m)$, and can therefore consider $f^{m\pm} \in \mathcal{H}_{m,1}$ as a single particle vector. For $m = 0$, however, the measure $dp/\omega_p^0 = dp/|p|$ is divergent at $p = 0$, and therefore we can claim $f^{0\pm} \in \mathcal{H}_{0,1}$ only if $f^{0,\pm}(0) = 0$, i.e. if f is the derivative (with respect to x_0 or x_1) of another test function. Bearing this remark in mind, we define a field operator ϕ_m as

$$\phi_m(f) := z_m^\dagger(f^{m+}) + z_m(f^{m-}). \tag{2.27}$$

For general S, this operator is unbounded, but always contains $\mathcal{D}_m$ in its domain and leaves this subspace invariant. Furthermore, one can show that $\phi_m(f)$ is essentially self-adjoint for real-valued f. Regarding its field-theoretical properties, the field ϕ_m is a solution of the Klein-Gordon equation with mass m, has the Reeh–Schlieder property, and transforms covariantly under proper orthochronous Poincaré transformations,

$$U_m(x,\theta)\phi_m(f)U_m(x,\theta)^{-1} = \phi_m(f_{\theta,x}), \quad f_{\theta,x}(y) = f(\Lambda_\theta^{-1}(y-x)). \tag{2.28}$$

Here $\Lambda_\theta = \left(\begin{smallmatrix} \mathrm{ch}\,\theta & \mathrm{sh}\,\theta \\ \mathrm{sh}\,\theta & \mathrm{ch}\,\theta \end{smallmatrix}\right)$ denotes the Lorentz boost with rapidity θ.

For positive mass, these properties have been established in [37]. For $m = 0$, the proof carries over without changes if restricting to derivative test functions, i.e. if $f \in \mathscr{S}(\mathbb{R}^2)$ is assumed to be of the form $f(x) = \partial g(x)/\partial x_k$, $g \in \mathscr{S}(\mathbb{R}^2)$, $k = 0, 1$.

Regarding locality, we first note that in the trivial case $S = 1$, the field ϕ_m coincides with the free scalar field of mass m, which is of course point-local. For $S \neq 1$, however, $\phi_m(x)$ is *not* localized at the space-time point $x \in \mathbb{R}^2$, i.e. in general $[\phi_m(x), \phi_m(x')] \neq 0$ for spacelike separated $x, x' \in \mathbb{R}^2$. Moreover, the covariance property (2.28) does *not* hold for the space-time reflection $U_m(j)$ (2.20) if $S \neq 1$, i.e. the field

$$\phi_m'(f) := U_m(j)\phi_m(f^j)U_m(j)^{-1}, \quad f^j(x) := \overline{f(-x)}, \tag{2.29}$$

is different from ϕ_m in this case. Nonetheless, ϕ_m' shares many properties with ϕ_m, such as the domain and essential self-adjointness, the covariant transformation behavior with respect to proper orthochronous Poincaré transformations (2.28), and ϕ_m' is also a solution of the Klein–Gordon equation with the Reeh–Schlieder property. For the construction of a local quantum field theory with scattering function $S \neq 1$, one has to make use of both fields, ϕ_m and ϕ_m', and exploit their relative localization properties.

For the formulation of this relative localization, we first recall that the *right wedge* is the causally complete region

$$W_R := \{x \in \mathbb{R}^2 : x_1 > |x_0|\}, \tag{2.30}$$

and its causal complement is $W_R' = -W_R =: W_L$, the *left wedge*.

Given $m > 0$ and $S \in \mathcal{S}_{\rm reg}$, it has been shown in [37] that the two fields ϕ_m, ϕ'_m are *relatively wedge-local* to each other in the sense that

$$\begin{aligned}[\phi_m(f), \phi'_m(g)]\Psi = 0, \quad \operatorname{supp} f \subset W_L, \quad \operatorname{supp} g \subset W_R,\\ f, g \in \mathscr{S}(\mathbb{R}^2), \ \ \Psi \in \mathcal{D}_m.\end{aligned} \tag{2.31}$$

The proof of this fact relies on the analytic properties of S_m. In particular, the apparent asymmetry of the above commutation relations in ϕ and ϕ' derives from the fact that the scattering function S associated with ϕ is analytic in the "upper" strip $\mathrm{S}(0,\pi)$, whereas S^{-1}, associated with ϕ', is analytic in the "lower" strip $\mathrm{S}(-\pi, 0)$.

As we saw in Lemma 2.3, S_0 can even be discontinuous, and therefore one cannot directly employ the analyticity arguments in the case $m = 0$. However, using a splitting in chiral components, we will see in Sec. 4.4 that (2.31) is nonetheless still valid in the massless situation.

Having collected sufficient information about the auxiliary fields ϕ_m, ϕ'_m, one can pass to an operator-algebraic formulation and consider the von Neumann algebras generated by them,

$$\mathcal{M}_m := \{e^{i\phi'_m(f)} : f \in \mathscr{S}(W_R) \text{ real}\}'', \tag{2.32}$$

$$\widehat{\mathcal{M}}_m := \{e^{i\phi_m(f)} \ : \ f \in \mathscr{S}(W_L) \text{ real}\}''. \tag{2.33}$$

Using the relative localization and Reeh–Schlieder property of the fields ϕ_m, ϕ'_m, one can show that $\mathcal{M}_m$ and $\widehat{\mathcal{M}}_m$ commute, and that the vacuum vector Ω_m is cyclic and separating for both of them. The modular data of these algebras act geometrically as expected from the Bisognano–Wichmann theorem [6]. In particular, the modular conjugation J of $(\mathcal{M}_m, \Omega_m)$ coincides with the space-time reflection $U_m(j)$ (2.20), and with this information, it is easy to see that $\mathcal{M}_m$ and $\widehat{\mathcal{M}}_m$ are actually commutants of each other, $\widehat{\mathcal{M}}_m = \mathcal{M}'_m$ [18]. Taking into account the transformation properties of the field ϕ'_m, it also follows that

$$U_m(x,\theta)\mathcal{M}_m U_m(x,\theta)^{-1} \subset \mathcal{M}_m, \quad x \in W_R, \ \ \theta \in \mathbb{R}.$$

In view of these properties, one can consistently define von Neumann algebras of observables localized in double cones (intersections of two opposite wedges). For $y - x \in W_R$, one defines $O_{xy} := (W_R + x) \cap (W_L + y)$ and

$$\mathcal{A}_m(O_{xy}) := U_m(x,0)\mathcal{M}_m U_m(x,0)^{-1} \cap U_m(y,0)\mathcal{M}'_m U_m(y,0)^{-1}. \tag{2.34}$$

Algebras associated to arbitrary regions can then be defined by additivity. The assignment $O \mapsto \mathcal{A}_m(O)$ of space-time regions in $\mathbb{R}^2$ to observable algebras in $\mathcal{B}(\mathcal{H}_m)$ is the definition of the (m, S)-model in the framework of algebraic quantum field theory [33]. Its main properties are summarized in the following theorem.

Theorem 2.4. *Let $m > 0$ and $S \in \mathcal{S}_{\rm reg}$. Then the map $O \mapsto \mathcal{A}_m(O)$ of double cones in $\mathbb{R}^2$ to von Neumann algebras in $\mathcal{B}(\mathcal{H}_m)$ has the following* *properties*:

(a) *Isotony*: $\mathcal{A}_m(O_1) \subset \mathcal{A}_m(O_2)$ *for double cones* $O_1 \subset O_2$.
(b) *Locality*: $\mathcal{A}_m(O_1) \subset \mathcal{A}_m(O_2)'$ *for double cones* $O_1 \subset O_2'$.
(c) *Covariance*: $U_m(g)\mathcal{A}_m(O)U_m(g)^{-1} = \mathcal{A}_m(gO)$ *for each Poincaré transformation* $g \in \mathcal{P}$ *and each double cone* O.
(d) *Reeh–Schlieder property*: *If* $S(0) = 1$, *there exists* $r_0 > 0$ *such that for all double cones* O *which are Poincaré equivalent to* $W_R \cap (W_L + (0, r))$ *with some* $r > r_0$, *there holds* $\overline{\mathcal{A}_m(O)\Omega_m} = \mathcal{H}_m$. *If* $S(0) = -1$, *this cyclicity holds without restriction on the size of* O.
(e) *Additivity*: $\mathcal{M}_m$ *coincides with the smallest von Neumann algebra containing* $\mathcal{A}_m(O)$ *for all double cones* $O \subset W_R$.
(f) *Interaction*: *The collision operator of the quantum field theory defined by the algebras* (2.34) *is the factorizing S-matrix with scattering function* S.

Statements (a)–(c) *also hold if* $m = 0$ *and* $S \in \mathcal{S}_{\lim}$.

The above list shows that the elements of the algebra $\mathcal{A}_m(O)$ (2.34) can consistently be interpreted as the observables localized in O of a local, covariant quantum field theory complying with all standard assumptions. Furthermore, the last item shows that the net $\mathcal{A}_m$ so constructed provides a solution to the inverse scattering problem for the factorizing S-matrix given by S.

Statements (d)–(f) of Theorem 2.4 are only known to hold in the massive case since an important tool for their proof, the split property for wedges [47], and the closely related modular nuclearity condition [17], is not satisfied in the massless case. Thus these properties might or might not be valid in the mass zero limit. In Sec. 6, we will see examples of scattering functions $S \in \mathcal{S}_{\lim}$ for both possibilities.

3. Scaling Limits of Massive Models

As a quantum field theory in the sense of Haag–Kastler [33], the models given by the nets $\mathcal{A}_m$ have a well-defined scaling limit theory [22]. However, for generic scattering function, the local observables of these models are given in a quite indirect manner as elements of an intersection of two wedge algebras. On the other hand, field operators localized in wedges are explicitly known, so that it is not difficult to calculate their behavior under scaling transformations.

To get an idea about the algebraic short distance limit of the (m, S)-models, $m > 0$, we can proceed in the following way, inspired by the results in [12] about the behavior of quantum fields under scaling. We consider rescaled wedge-local field operators of the form

$$N_\lambda \phi_m(\lambda x) \tag{3.1}$$

and let $\lambda \to 0$. The constants N_λ have to be chosen in such a way that the vacuum expectation values of these rescaled fields do not scale to zero or diverge, but approach a finite limit.

The effect of the space-time rescaling $x \mapsto \lambda x$ is easily calculated: Smearing the scaled field with a test function $f \in \mathscr{S}(\mathbb{R}^2)$ amounts to evaluating ϕ_m on a scaled testfunction f_λ,

$$f_\lambda(x) := \lambda^{-2} f(\lambda^{-1}x), \quad \lambda > 0, \ \ x \in \mathbb{R}^2. \tag{3.2}$$

The mass shell restrictions (2.26) of the Fourier transforms of such scaled functions are given by scaling the mass and the momentum,

$$f_\lambda^{m\pm}(p) = f^{\lambda m \pm}(\lambda p). \tag{3.3}$$

As in the analysis of the free field [23], two different choices for the multiplicative renormalization N_λ are possible, namely $N_\lambda = 1$ and $N_\lambda = |\ln \lambda|^{-1/2}$. The latter choice corresponds to an anomalous scaling of the field $\phi_m(f)$ when $f^{0\pm}(0) \neq 0$, which in turn is due to the infrared divergence of the n-point functions of the field in the massless limit. As in the case of free fields, it can be expected that it gives rise to an abelian tensor factor in the scaling limit algebra, at least if $S(0) = 1$. We will however not investigate this possibility any further here.

Choosing therefore $N_\lambda = 1$, we now consider the n-point functions of the rescaled field

$$\mathscr{W}_m^{n,\lambda}(f_1, \ldots, f_n) := \langle \Omega_m, \phi_m(f_{1\lambda}) \cdots \phi_m(f_{n\lambda}) \Omega_m \rangle, \tag{3.4}$$

and study their limit as $\lambda \to 0$.

Proposition 3.1. *Let $m > 0, S \in \mathcal{S}_{\lim}$ and $f_1, \ldots, f_n \in \mathscr{S}(\mathbb{R}^2)$ with $f_j^{0\pm}(0) = 0$, $j = 1, \ldots, n$. Then*

$$\lim_{\lambda \to 0} \mathscr{W}_m^{n,\lambda}(f_1, \ldots, f_n) = \mathscr{W}_0^{n,1}(f_1, \ldots, f_n). \tag{3.5}$$

An analogous statement holds for the expectation values of the fields ϕ'_m, ϕ'_0.

Proof. Both fields, ϕ_m and ϕ_0, are defined as sums of certain creation and annihilation operators, which change the particle number by ± 1. Hence vacuum expectation values of products of an odd number of field operators vanish, i.e. the statement holds trivially if n is odd. We may therefore assume that $n = 2k$ is even, and first consider the vacuum expectation value of a particularly ordered product of creation and annihilation operators. Using (2.21), (2.22) and (2.14), we compute

$$\begin{aligned}
&\langle \Omega_m, z_m(f_{1,\lambda}^{m-}) \cdots z_m(f_{k,\lambda}^{m-}) z_m^\dagger(f_{k+1,\lambda}^{m+}) \cdots z_m^\dagger(f_{n,\lambda}^{m+}) \Omega_m \rangle \\
&\quad = \sum_{\pi \in \mathfrak{S}_k} \langle \overline{f_{k,\lambda}^{m-}} \otimes \cdots \otimes \overline{f_{1,\lambda}^{m-}}, D_k(\pi)(f_{k+1,\lambda}^{m+} \otimes \cdots \otimes f_{n,\lambda}^{m+}) \rangle \\
&\quad = \sum_{\pi \in \mathfrak{S}_k} \int \frac{dp_1}{\omega_1^m} \cdots \frac{dp_k}{\omega_k^m} \prod_{j=1}^k (f_{k-j+1}^{\lambda m-}(\lambda p_j) f_{k+j}^{\lambda m+}(\lambda p_{\pi(j)})) \\
&\qquad \times \prod_{\substack{1 \leq l < r \leq k \\ \pi(l) > \pi(r)}} S_m(p_{\pi(l)}, p_{\pi(r)}),
\end{aligned} \tag{3.6}$$

where we used the scaling relation (3.3) in the last line. Taking into account the scaling relation (2.10) for S_m and $dp/\omega_p^m = d(\lambda p)/\omega_{\lambda p}^{\lambda m}$, the change of variables $p_j \to \lambda p_j$ yields

$$\begin{aligned}
&\langle \Omega_m, z_m(f_{1,\lambda}^{m-}) \cdots z_m(f_{k,\lambda}^{m-}) z_m^\dagger(f_{k+1,\lambda}^{m+}) \cdots z_m^\dagger(f_{n,\lambda}^{m+}) \Omega_m \rangle \\
&\quad = \sum_{\pi \in \mathfrak{S}_k} \int \frac{dp_1}{\omega_1^{\lambda m}} \cdots \frac{dp_k}{\omega_k^{\lambda m}} \prod_{j=1}^{k} (f_{k-j+1}^{\lambda m-}(p_j) f_{k+j}^{\lambda m+}(p_{\pi(j)})) \\
&\qquad \times \prod_{\substack{1 \le l < r \le k \\ \pi(l) > \pi(r)}} S_{\lambda m}(p_{\pi(l)}, p_{\pi(r)}).
\end{aligned} \tag{3.7}$$

In the limit $\lambda \to 0$, the integrand converges pointwise to the corresponding expression with $m = 0$, which is integrable because of our assumption on the test functions $f_j^{0\pm}$. For the Schwartz class functions $f_j^{\lambda m \pm}$, there exist λ-independent integrable bounds, and since $S_{\lambda m}$ has constant modulus 1, and $|\omega_j^{\lambda m}|^{-1} \le |p_j|^{-1}$, we can use dominated convergence to conclude

$$\begin{aligned}
&\lim_{\lambda \to 0} \langle \Omega_m, z_m(f_{1,\lambda}^{m-}) \cdots z_m(f_{k,\lambda}^{m-}) z_m^\dagger(f_{k+1,\lambda}^{m+}) \cdots z_m^\dagger(f_{n,\lambda}^{m+}) \Omega_m \rangle \\
&\quad = \sum_{\pi \in \mathfrak{S}_k} \int \frac{dp_1}{|p_1|} \cdots \frac{dp_k}{|p_k|} \prod_{j=1}^{k} (f_{k-j+1}^{0-}(p_j) f_{k+j}^{0+}(p_{\pi(j)})) \prod_{\substack{1 \le l < r \le k \\ \pi(l) > \pi(r)}} S_0(p_{\pi(l)}, p_{\pi(r)}) \\
&\quad = \langle \Omega_0, z_0(f_1^{0-}) \cdots z_0(f_k^{0-}) z_0^\dagger(f_{k+1}^{0+}) \cdots z_0^\dagger(f_n^{0+}) \Omega_0 \rangle.
\end{aligned} \tag{3.8}$$

After this preparation, we consider the $(2k)$-point function of the field ϕ_m, and expand the fields into creation and annihilation operators,

$$\begin{aligned}
\mathscr{W}_m^{2k,\lambda}(f_1, \ldots, f_{2k}) = \langle \Omega_m, (z_m(f_{1,\lambda}^{m-}) + z_m^\dagger(f_{1,\lambda}^{m+})) \cdots \\
(z_m(f_{2k,\lambda}^{m-}) + z_m^\dagger(f_{2k,\lambda}^{m+})) \Omega_m \rangle,
\end{aligned}$$

and analogously for $m = 0$. These are sums of 2^{2k} terms, each of which is the vacuum expectation value of a $(2k)$-fold product of z_m's and $z_m^\dagger$'s (respectively, z_0's and $z_0^\dagger$'s). Because of the annihilation/creation properties of these operators, all terms in which the number of z's is different from the number of $z^\dagger$'s vanish. So each non-zero term is of the form considered before, up to a reshuffling of creation and annihilation operators.

Picking any one of these terms, we can use the exchange relations of Zamolodchikov's algebra to write the product of creation and annihilation operators as a sum of products of the particular form considered above, where all creation operators stand to the right of all annihilation operators. The only difference to the previous integral expressions is that the reordering may reduce the number of integrations in (3.7) — due to the term $\omega_p^m \delta(p - q)$ in the Zamolodchikov's relation — and introduce various factors of $S_{\lambda m}(p_a, p_b)$, $a, b \in \{1, \ldots, k\}$, in the integrand as well as a permutation of the momenta $p_1, \ldots, p_k$.

But the reorderings are the same for the case $m > 0$ and $m = 0$, and the additional factors $S_{\lambda m}(p_a, p_b)$ converge pointwise and uniformly bounded to their counterparts with $m = 0$ in the limit $\lambda \to 0$. Thus the analogue of the limit (3.8) holds for an arbitrarily ordered product of z's and $z^\dagger$'s, and (3.5) follows. □

According to this result, the scaling limit of n-point functions of the field ϕ_m is given by the n-point functions of the field ϕ_0, with the same scattering function S. We will then regard the $(0, S)$-model as the short distance scaling limit of the (m, S)-model. For a discussion of the relations with the Buchholz–Verch scaling limit, we refer the reader to the conclusions in Sec. 7.

The massless wedge-localized fields obtained in the above limit are actually *chiral*, and split into sums of halfline-localized fields on the two light rays. In the following, we will indicate this split on a formal level; the chiral fields will be defined more precisely in Sec. 4. The localization regions of the various fields appearing in this split can be visualized as in Fig. 1. We will use indices r/ℓ to distinguish between the right/left moving component fields. (To avoid confusion, we notice explicitly that this means that, e.g., the *right* moving field is a function of $x_\mathsf{r} = x_0 - x_1$ only, and therefore lives on the *left* light ray, defined by $x_0 + x_1 = 0$.) Similar to the fields ϕ'_m, ϕ_m on $\mathbb{R}^2$, operators with/without prime are localized on one or the other side of a fixed light ray.

It follows from (2.26), (2.27) that the field ϕ_0 is formally defined by the operator valued distribution

$$\phi_0(x) = \int_{\mathbb{R}} \frac{dp}{2\pi|p|}(e^{i(|p|x_0 - px_1)} z_0^\dagger(p) + e^{-i(|p|x_0 - px_1)} z_0(p)).$$

Splitting now the integration in the sum of an integration over $(-\infty, 0)$ and one over $(0, +\infty)$, and changing variable $p \to -p$ in the former integration, one gets

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}}(\varphi_\mathsf{r}(x_\mathsf{r}) + \varphi'_\ell(x_\ell)).$$

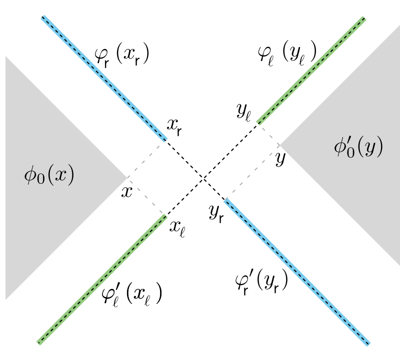

Fig. 1. Massless wedge- and halfline-localized fields with their localization regions.

Here $x_\ell := x_0 + x_1$, $x_\mathsf{r} := x_0 - x_1$ are the left and right light ray components of $x = (x_0, x_1)$, and

$$\varphi_\mathsf{r}(x_\mathsf{r}) := \int_0^{+\infty} \frac{dp}{\sqrt{2\pi p}} (e^{ipx_\mathsf{r}} z_0^\dagger(p) + e^{-ipx_\mathsf{r}} z_0(p)), \tag{3.9}$$

$$\varphi'_\ell(x_\ell) := \int_0^{+\infty} \frac{dp}{\sqrt{2\pi p}} (e^{ipx_\ell} z_0^\dagger(-p) + e^{-ipx_\ell} z_0(-p)), \tag{3.10}$$

are two chiral (one-dimensional) fields living on the left/right light ray of two-dimensional Minkowski space. In order to avoid the infrared divergence which is apparent in the above integrals, one should actually consider the derivatives of these fields. At this formal level this is not really relevant, but we will consistently do so in the following section, where we denote the derivatives of φ_ℓ, φ_r as ϕ_ℓ, ϕ_r, respectively.

Notice also that, according to (2.11) and (2.23)–(2.25), one has, for $p, q > 0$,

$$z_0(-p)z_0(q) = S(\infty)z_0(q)z_0(-p),$$
$$z_0(-p)z_0^\dagger(q) = S(\infty)z_0^\dagger(q)z_0(-p).$$

This implies that φ'_ℓ and φ_r commute, respectively, anticommute, if $S(\infty) = +1$, respectively, $S(\infty) = -1$. Proceeding in the same way for the right-wedge field ϕ'_0, one gets an analogous split into two chiral fields φ'_r, φ_ℓ defined by substituting $z_0^\#(p)$ with $U_0(j)z_0^\#(p)U_0(j)^*$ in formulas (3.9), (3.10). It is then not difficult to see, following the arguments in [37], that

$$[\varphi_\ell(x_\ell), \varphi'_\ell(y_\ell)] = 0 \quad \text{if } x_\ell > y_\ell$$

(see also Proposition 4.2(d) below). This shows that φ_ℓ and φ'_ℓ can be interpreted as being localized in the right and left half-line, respectively. An analogous statement holds of course for φ_r, φ'_r.

The above formal manipulations suggest that the $(0, S)$-model can be written as the (twisted, if $S(\infty) = -1$) tensor product of two chiral models which are again defined in terms of the Zamolodchikov–Faddeev algebra (2.23)–(2.25) with $m = 0$ and $p, q > 0$. These models will be rigorously defined in the next section, and in Sec. 4.4 we will show that such a tensor product decomposition actually holds.

4. Chiral Integrable Models

We saw in the previous section that the wedge-local fields generating the massless $(0, S)$-models factorize into chiral components. To analyze this connection in detail, it turns out to be most convenient to first introduce the chiral fields independently of the previously discussed models on two-dimensional Minkowski space, and discuss the relation to the $(0, S)$-models afterwards.

In this section, we will therefore be concerned with quantum fields on the real line. The development of these models is largely parallel to Sec. 2.2, but has some distinctive differences. Our construction will yield dilation and translation covariant quantum field theory models on $\mathbb{R}$ (thought of as either the right or left light ray), with algebras of observables localized in half lines and intervals.

An important question is whether these models extend to *conformally* covariant theories on the circle. Using results of [32], it turns out that this question is closely related to the size of local algebras associated with bounded intervals. This point will be discussed in detail in Sec. 5.

4.1. *S-symmetric Fock space and Zamolodchikov's algebra on the light ray*

As before, we start from a scattering function $S \in \mathcal{S}_{\lim}$. We first define the Hilbert space of the theory. Our "chiral" single particle space is given by

$$\mathcal{H}_1 := L^2(\mathbb{R}, d\beta).$$

The variable β is meant to be related to the momentum p by $p = e^\beta$, as will become clear in (4.1) below. Like in (2.14), we have a unitary action D_n of the permutation group $\mathfrak{S}_n$ on $\mathcal{H}_1^{\otimes n} = L^2(\mathbb{R}^n, d\boldsymbol{\beta})$ which acts on transpositions τ_k by

$$(D_n(\tau_k)\Psi_n)(\beta_1, \ldots, \beta_n) = S(\beta_{k+1} - \beta_k) \cdot \Psi_n(\beta_1, \ldots, \beta_{k+1}, \beta_k, \ldots, \beta_n).$$

Again, our n-particle space $\mathcal{H}_n$ is defined as the D_n-invariant subspace of $\mathcal{H}_1^{\otimes n}$, and the projector onto it is denoted as P_n. We define the Fock space $\mathcal{H} := \bigoplus_{n \geq 0} \mathcal{H}_n$ and its subspace $\mathcal{D} \subset \mathcal{H}$ consisting of vectors of bounded particle number, i.e. of terminating sequences.

We proceed to a representation of the space-time symmetries. On $\mathcal{H}$, we consider a unitary representation U of the affine group G of $\mathbb{R}$, consisting of translations and dilations, $\mathbb{R} \ni \xi \mapsto e^\lambda \xi + \xi'$, and the reflection, $j(\xi) := -\xi$. For translations and dilations, it is defined as, $\xi, \lambda \in \mathbb{R}$,

$$(U(\xi, \lambda)\Psi)_n(\beta_1, \ldots, \beta_n) := e^{i\xi(e^{\beta_1} + \cdots + e^{\beta_n})} \cdot \Psi_n(\beta_1 + \lambda, \ldots, \beta_n + \lambda), \tag{4.1}$$

and the reflection j is represented antiunitarily by

$$\begin{aligned} (U(j)\Psi)_n(\beta_1, \ldots, \beta_n) &:= \overline{\Psi_n(\beta_n, \ldots, \beta_1)} \\ &= \prod_{1 \leq l < r \leq n} S(\beta_r - \beta_l) \cdot \overline{\Psi_n(\beta_1, \ldots, \beta_n)}. \end{aligned} \tag{4.2}$$

Compare this with the two-dimensional case in (2.17), (2.20). We will also use the shorthand notation $U(\xi) := U(\xi, 0)$ for pure translations, and note here that this one parameter group has a positive generator, H. Up to scalar multiples, $\Omega := 1 \oplus 0 \oplus 0 \oplus \cdots$ is the only U-invariant vector in $\mathcal{H}$; it will play the role of the vacuum vector.

As in (2.21), we will make use of "S-symmetrized" annihilation and creation operators, which we label y and $y^\dagger$, in order to distinguish them from $z_m, z_m^\dagger$, since they will take rapidities rather than momenta as arguments. For $\psi \in \mathcal{H}_1$, $\Phi \in \mathcal{D}$, they act by

$$(y^\dagger(\psi)\Phi)_n := \sqrt{n} P_n(\psi \otimes \Phi_{n-1}), \quad y(\psi) := y^\dagger(\overline{\psi})^*. \tag{4.3}$$

Except for the special case $S = -1$, these are unbounded operators containing $\mathcal{D}$ in their domains. Under symmetry transformations, they behave like

$$U(\xi,\lambda)y^\dagger(\psi)U(\xi,\lambda)^{-1} = y^\dagger(U(\xi,\lambda)\psi), \tag{4.4}$$

$$U(\xi,\lambda)y(\psi)U(\xi,\lambda)^{-1} = y(U(-\xi,\lambda)\psi), \tag{4.5}$$

whereas with respect to the reflection j, no such transformation formula holds.

From time to time, we will also work with operator-valued distributions $y(\beta), y^\dagger(\beta)$, $\beta \in \mathbb{R}$, related to the above operators by the formal integrals $y^\#(\psi) = \int d\beta \psi(\beta) y^\#(\beta)$. They satisfy the relations of the Zamolodchikov–Faddeev algebra in the form

$$y(\beta_1)y(\beta_2) = S(\beta_1-\beta_2)y(\beta_2)y(\beta_1), \tag{4.6}$$

$$y^\dagger(\beta_1)y^\dagger(\beta_2) = S(\beta_1-\beta_2)y^\dagger(\beta_2)y^\dagger(\beta_1), \tag{4.7}$$

$$y(\beta_1)y^\dagger(\beta_2) = S(\beta_2-\beta_1)y^\dagger(\beta_2)y(\beta_1) + \delta(\beta_1-\beta_2)\cdot 1. \tag{4.8}$$

It is interesting to note that these are exactly the same relations as used in *massive* two-dimensional models, written in terms of rapidities [40]. We will however see that the interpretation in terms of wedge-local observables must be modified in the chiral case.

4.2. *Half-local quantum fields and observable algebras*

We now set out to construct a pair of quantum fields on $\mathcal{H}$ as sums of Zamolodchikov type creation and annihilation operators, analogous to the two-dimensional case in Sec. 2.2. For the one-dimensional case, these quantum fields will be localized in half-lines, rather than in wedge regions. While we employ largely the same ideas as in the massive two-dimensional case [40], the chiral situation makes some modifications necessary, so that we will need to look into the construction in more detail.

We first introduce the necessary test functions and discuss their properties. For $f \in \mathscr{S}(\mathbb{R}), \psi \in C_0^\infty(\mathbb{R})$, we define their Fourier transforms and the positive/negative frequency components of those with the following conventions.

$$\hat{f}^\pm(\beta) := \pm ie^\beta \tilde{f}(\pm e^\beta) = \pm\frac{ie^\beta}{\sqrt{2\pi}}\int f(\xi)\exp(\pm ie^\beta\xi)d\xi, \tag{4.9}$$

$$\check{\psi}^\pm(\xi) := \mp\frac{i}{\sqrt{2\pi}}\int d\beta\ \psi(\beta)e^{\mp i\xi e^\beta} = \mp\frac{i}{\sqrt{2\pi}}\int_0^\infty dp\frac{\psi(\log p)}{p}e^{\mp ip\xi}. \tag{4.10}$$

Lemma 4.1. *Let $f \in \mathscr{S}(\mathbb{R}), \psi \in C_0^\infty(\mathbb{R})$.*

(a) $\hat{f}^\pm, \check{\psi}^\pm \in \mathscr{S}(\mathbb{R})$. *As maps from $\mathscr{S}(\mathbb{R})$ to $L^2(\mathbb{R}), f \mapsto \hat{f}^\pm$ are continuous.*
(b) *For $\psi \in C_0^\infty(\mathbb{R})$, there holds*

$$(\check{\psi}^\pm)^{\wedge\pm} = \psi, \quad (\check{\psi}^\pm)^{\wedge\mp} = 0.$$

(c) *Let* $f^{\xi,\lambda}(\xi') := f(e^{-\lambda}(\xi'-\xi))$ *and* $f^j(\xi) := \overline{f(-\xi)}$. *Then*

$$(\widehat{f^{\xi,\lambda}})^{\pm}(\beta) = e^{\pm i\xi e^{\beta}}\hat{f}^{\pm}(\beta+\lambda), \quad (\widehat{f^j})^{\pm}(\beta) = -\overline{\hat{f}^{\pm}(\beta)}, \quad \hat{\overline{f}}^{\pm} = \overline{\hat{f}^{\mp}}. \tag{4.11}$$

(d) *Let* $f, g \in \mathscr{S}(\mathbb{R})$, *with* $\operatorname{supp} f \subset \mathbb{R}_+, \operatorname{supp} g \subset \mathbb{R}_-$. *Then* $\hat{f}^+$ *and* $\hat{g}^-$ *have bounded analytic extensions to the strip* $\mathrm{S}(0,\pi)$, *and* $|\hat{f}^+(\beta+i\lambda)|, |\hat{g}^-(\beta+i\lambda)| \to 0$ *as* $\beta \to \pm\infty$, *uniformly in* $\lambda \in [0,\pi]$. *The boundary values at* $\operatorname{Im}\beta = \pi$ *are*

$$\hat{f}^+(\beta+i\pi) = \hat{f}^-(\beta), \quad \hat{g}^-(\beta+i\pi) = \hat{g}^+(\beta), \quad \beta \in \mathbb{R}. \tag{4.12}$$

If $\operatorname{supp} f \subset (r,\infty)$ *and* $\operatorname{supp} g \subset (-\infty,-r)$ *for some* $r > 0$, *then there exist* $c, c' > 0$ *such that*

$$|\hat{f}^+(\beta+i\lambda)| \le ce^{-re^{\beta}\sin\lambda}, \quad |\hat{g}^-(\beta+i\lambda)| \le c'e^{-re^{\beta}\sin\lambda}, \quad 0 \le \lambda \le \pi. \tag{4.13}$$

Proof. (a) It is clear that $\hat{f}^{\pm} \in \mathscr{S}(\mathbb{R})$, and by considering the second formula in (4.10), $\check{\psi}^{\pm}$ is seen to be the Fourier transform of a function in $C_0^\infty(\mathbb{R})$, and hence of Schwartz class, too. Since $\tilde{f} \in \mathscr{S}(\mathbb{R})$, one gets the bound $|\hat{f}^{\pm}(\beta)| \le c_{\pm}(f)\cdot e^{-|\beta|}$ for some Schwartz seminorm $c_{\pm}(f)$, which implies the claimed continuity by estimating the L^2-norm of $\hat{f}^{\pm}$.

(b) By its definition (4.10), $\check{\psi}^{\pm}$ is the inverse Fourier transform of the function $p \mapsto \mp i\theta(\pm p)\psi(\log|p|)/|p|$, where θ denotes the step function. The statement now follows from the Fourier inversion formula.

(c) This is obtained by straightforward calculation.

(d) The analyticity of $\hat{f}^+$ in $\mathrm{S}(0,\pi)$ follows from the analyticity of $\tilde{f}$ in the upper complex half plane (since $\operatorname{supp} f \subset \mathbb{R}_+$), and the fact that the exponential function maps $\mathrm{S}(0,\pi)$ onto the upper half plane. The uniform bound follows from the estimate

$$\begin{aligned} |\hat{f}^+(\beta+i\lambda)| &= \frac{1}{\sqrt{2\pi}}\left|\int_0^\infty d\xi\ \partial_\xi f(\xi)e^{i\xi e^{\beta+i\lambda}}\right| \\ &\le \frac{1}{\sqrt{2\pi}}\int_0^\infty d\xi\,|\partial_\xi f(\xi)|e^{-\xi e^{\beta}\sin\lambda} \\ &\le \frac{\|\partial_\xi f\|_1}{\sqrt{2\pi}}, \end{aligned} \tag{4.14}$$

where in the last step we used $\xi > 0$, $0 \le \lambda \le \pi$.

The claimed boundary value follows directly from the definition of $\hat{f}^+$ in (4.9):

$$\hat{f}^+(\beta+i\pi) = ie^{\beta+i\pi}\tilde{f}(e^{\beta+i\pi}) = -ie^{\beta}\tilde{f}(-e^{\beta}) = \hat{f}^-(\beta). \tag{4.15}$$

So $\hat{f}^+(\beta)$ and $\hat{f}^+(\beta+i\pi)$ converge to zero for $\beta \to \pm\infty$. Since these functions are bounded and analytic in $\mathrm{S}(0,\pi)$, it follows that also $|\hat{f}^+(\beta+i\lambda)| \to 0$ as $\beta \to \pm\infty$, uniformly in $\lambda \in [0,\pi]$ — see, for example, [7, Corollary 1.4.5].

To obtain the sharpened bound (4.13), note that if $\operatorname{supp} f \subset (r,\infty)$, then $f^{-r,0}$ (cf. part (c)) has support in $\mathbb{R}_+$, and $\widehat{f^{-r,0}}^{+}(\beta) = e^{-ire^{\beta}}\hat{f}^{+}(\beta)$ due to (4.11). So there exists $c > 0$ such that for any $\lambda \in [0,\pi]$,

$$c > |e^{-ire^{\beta+i\lambda}}\hat{f}^{+}(\beta+i\lambda)| = e^{re^{\beta}\sin\lambda}|\hat{f}^{+}(\beta+i\lambda)|, \tag{4.16}$$

which implies (4.13).

Finally, given $g \in \mathscr{S}(\mathbb{R})$ with $\operatorname{supp} g \subset \mathbb{R}_-$, respectively $\operatorname{supp} g \subset (-\infty,-r)$, all corresponding statements about $\hat{g}^-$ follow from the previous arguments by considering $f(\xi) := g(-\xi)$, since $\operatorname{supp} f \subset \mathbb{R}_+$, and $\hat{f}^+ = -\hat{g}^-$. □

After these preparations, we define for $f \in \mathscr{S}(\mathbb{R})$ the two field operators,

$$\phi(f) := y^{\dagger}(\hat{f}^{+}) + y(\hat{f}^{-}), \tag{4.17}$$

$$\phi'(f) := U(j)\phi(f^{j})U(j). \tag{4.18}$$

These fields should be thought of as the derivatives of the left/right chiral fields $\varphi^{[\prime]}_{\ell/\mathsf{r}}$ appearing in the decomposition of the massless two-dimensional field ϕ_0, cf. also the figure on page 1130.

For reference, we note the "unsmeared", distributional version of (4.17):

$$\phi(\xi) = i\int \frac{d\beta}{\sqrt{2\pi}} e^{\beta}(e^{ie^{\beta}\xi}y^{\dagger}(\beta) - e^{-ie^{\beta}\xi}y(\beta)). \tag{4.19}$$

The main features of these fields can largely be obtained in the same way as in [37].

Proposition 4.2. *ϕ and ϕ' have the following properties.*

(a) *The map $f \mapsto \phi(f)$ is an operator-valued tempered distribution such that $\mathcal{D}$ is contained in the domain of $\phi(f)$ for all $f \in \mathscr{S}(\mathbb{R})$. For real f, the operator $\phi(f)$ is essentially self-adjoint, with elements from $\mathcal{D}$ as entire analytic vectors.*

(b) *ϕ transforms covariantly under the representation U of the connected component of the affine group, i.e.*

$$U(\xi,\lambda)\phi(f)U(\xi,\lambda)^{-1} = \phi(f^{\xi,\lambda}). \tag{4.20}$$

(c) *The Reeh–Schlieder property holds, i.e. for any non-empty open interval $I \subset \mathbb{R}$, the set*

$$\operatorname{span}\{\phi(f_1)\cdots\phi(f_n)\Omega : f_1,\ldots,f_n \in \mathscr{S}(I), n \in \mathbb{N}_0\} \tag{4.21}$$

is dense in $\mathcal{H}$.

(d) *ϕ and ϕ' are relatively half-local in the following sense: If $f, g \in \mathscr{S}(\mathbb{R})$ satisfy* $\operatorname{supp} f \subset (a,\infty)$, $\operatorname{supp} g \subset (-\infty,a)$ *for some $a \in \mathbb{R}$, then*

$$[\phi(f),\phi'(g)]\Psi = 0 \quad \text{for all } \Psi \in \mathcal{D}. \tag{4.22}$$

Statements (a)–(c) *also hold when ϕ is replaced with ϕ'.*

Proof. (a) It is clear from the definition of $\phi(f)$ that these operators always contain $\mathcal{D}$ in their domains and depend complex linearly on $f \in \mathscr{S}(\mathbb{R})$. Taking into account that the restrictions of the creation/annihilation operators to an n-particle space $\mathcal{H}_n$ are bounded, $\|y^{\#}(\psi)\lceil\mathcal{H}_n\| \leq \sqrt{n+1}\|\psi\|_{\mathcal{H}_1}$, and the continuity of $\mathscr{S}(\mathbb{R}) \ni f \mapsto \hat{f}^{\pm} \in \mathcal{H}_1$ established in Lemma 4.1(a), it follows that ϕ is an operator-valued tempered distribution.

In view of (4.3) and (4.11), we have

$$\phi(f)^* = (y^\dagger(\hat{f}^+) + y(\hat{f}^-))^* \supset y(\overline{\hat{f}^+}) + y^\dagger(\overline{\hat{f}^-}) = y(\widehat{\overline{f}}^-) + y^\dagger(\widehat{\overline{f}}^+) = \phi(\overline{f}).$$

This shows that $\phi(f)$ is hermitian for real f, and the proof of essential self-adjointness can now be completed as in [37, Proposition 1] by showing that any vector in $\mathcal{D}$ is entire analytic for $\phi(f)$.

(b) This is a direct consequence of (4.4) and (4.11).

(c) Let $\mathcal{P}(I)$ denote the algebra generated by all polynomials in the field $\phi(f)$ with $\operatorname{supp} f \subset I$. By standard analyticity arguments making use of the positivity of the generator of $\xi \mapsto U(\xi)$, it follows that $\mathcal{P}(I)\Omega$ is dense in $\mathcal{H}$ if and only if $\mathcal{P}(\mathbb{R})\Omega$ is dense in $\mathcal{H}$. But given any $\psi \in C_0^\infty(\mathbb{R})$, the function $f := \check{\psi}^+ \in \mathscr{S}(\mathbb{R})$ satisfies $\hat{f}^+ = \psi$ and $\hat{f}^- = 0$ (Lemma 4.1(b)), and hence $y^\dagger(\psi) = \phi(f) \in \mathcal{P}(\mathbb{R})$. Since $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{H}_1$, polynomials in the $y^\dagger(\psi)$ create a dense set from Ω.

The proofs of statements (a)–(c) for the field ϕ' are completely analogous.

(d) Since the Zamolodchikov–Faddeev relations (4.6)–(4.8) are the same as in the massive case in rapidity space, we can establish the following commutation relations in complete analogy to [37, Lemma 4]:

$$\begin{aligned}
[y(\psi_1), U(j)y(\psi_2)U(j)] &= 0,\\
[y^\dagger(\psi_1), U(j)y^\dagger(\psi_2)U(j)] &= 0,\\
([U(j)y(\overline{\psi_1})U(j), y^\dagger(\psi_2)]\Phi)_n(\boldsymbol{\beta}) &= C_n^{\psi_1,\psi_2,+}(\boldsymbol{\beta}) \cdot \Phi_n(\boldsymbol{\beta}),\\
([U(j)y^\dagger(\overline{\psi_1})U(j), y(\psi_2)]\Phi)_n(\boldsymbol{\beta}) &= C_n^{\psi_1,\psi_2,-}(\boldsymbol{\beta}) \cdot \Phi_n(\boldsymbol{\beta}),
\end{aligned} \tag{4.23}$$

where $\psi_1, \psi_2 \in \mathcal{H}_1$, $\Phi \in \mathcal{D}$, and

$$C_n^{\psi_1,\psi_2,\pm}(\boldsymbol{\beta}) = \pm \int d\beta_0 \psi_1(\beta_0)\psi_2(\beta_0) \prod_{k=1}^n S(\pm\beta_0 \mp \beta_k). \tag{4.24}$$

In view of the definition of the fields ϕ and ϕ', the commutator takes the form

$$\begin{aligned}
[\phi(f), \phi'(g)]\Psi_n &= -[y^\dagger(\hat{f}^+) + y(\hat{f}^-), U(j)y^\dagger(\overline{\hat{g}^+})U(j) + U(j)y(\overline{\hat{g}^-})U(j)]\Psi_n\\
&= (C_n^{\hat{f}^+,\hat{g}^-,+} + C_n^{\hat{f}^-,\hat{g}^+,-})\Psi_n \quad \text{for } \Psi_n \in \mathcal{H}_n.
\end{aligned} \tag{4.25}$$

Due to the translational covariance of ϕ and ϕ', it is sufficient to consider the case $a = 0$, i.e. $\operatorname{supp} f \subset \mathbb{R}_+$, $\operatorname{supp} g \subset \mathbb{R}_-$. To show that $C_n^{\hat{f}^+,\hat{g}^-,+} + C_n^{\hat{f}^-,\hat{g}^+,-} = 0$, we note that in the integral

$$C_n^{\hat{f}^+,\hat{g}^-,+}(\boldsymbol{\beta}) = \int d\beta_0 \hat{f}^+(\beta_0)\hat{g}^-(\beta_0) \prod_{k=1}^n S(\beta_0 - \beta_k), \tag{4.26}$$

all three functions, $\hat{f}^+, \hat{g}^-$, and $\beta_0 \mapsto S(\beta_0 - \beta_k)$, have analytic continuations to the strip $\mathrm{S}(0, \pi)$ (Definition 2.1 and Lemma 4.1(d)). According to Definition 2.1, the continuation of S is bounded on this strip, whereas according to Lemma 4.1(d), the functions $\hat{f}^+(\beta_0 + i\lambda), \hat{g}^-(\beta_0 + i\lambda)$ converge to zero for $\beta_0 \to \pm\infty$ uniformly in $\lambda \in [0, \pi]$. This implies that we can shift the contour of integration from $\mathbb{R}$ to $\mathbb{R} + i\pi$ in (4.26). As the boundary values of the integrated functions are given by $\hat{f}^+(\beta_0 + i\pi) = \hat{f}^-(\beta_0)$, $\hat{g}^-(\beta_0 + i\pi) = \hat{g}^+(\beta_0)$, and $S(\beta_0 + i\pi - \beta_k) = S(\beta_k - \beta_0)$, comparison with (4.24) shows $C_n^{\hat{f}^+, \hat{g}^-, +} + C_n^{\hat{f}^-, \hat{g}^+, -} = 0$. □

Proceeding to the algebraic formulation, we denote the self-adjoint closures of $\phi(f)$ and $\phi'(f)$ (with f real) by the same symbols, and introduce the von Neumann algebras generated by them,

$$\mathcal{M} := \{e^{i\phi(f)} : f \in \mathscr{S}(\mathbb{R}_+) \text{ real}\}'', \tag{4.27}$$

$$\widehat{\mathcal{M}} := \{e^{i\phi'(f)} : f \in \mathscr{S}(\mathbb{R}_-) \text{ real}\}''. \tag{4.28}$$

Theorem 4.3. *The algebras $\mathcal{M}$ and $\widehat{\mathcal{M}}$ have the following properties.*

(a) *For $\xi \geq 0, \lambda \in \mathbb{R}$, we have*

$$U(\xi, \lambda)\mathcal{M}U(\xi, \lambda)^{-1} \subset \mathcal{M}. \tag{4.29}$$

(b) *The vector Ω is cyclic and separating for $\mathcal{M}$.*
(c) *The Tomita–Takesaki modular data of $(\mathcal{M}, \Omega)$ are*

$$\Delta^{it} = U(0, -2\pi t), \quad J = U(j). \tag{4.30}$$

(d) $\widehat{\mathcal{M}} = \mathcal{M}'$.

Proof. (a) Given $f \in \mathscr{S}(\mathbb{R}_+)$ and $\xi \geq 0$, $\lambda \in \mathbb{R}$, also $f^{\xi,\lambda}$ lies in $\mathscr{S}(\mathbb{R}_+)$ by (4.11). Since $U(\xi, \lambda)\mathcal{M}U(\xi, \lambda)^{-1}$ is generated by $U(\xi, \lambda)e^{i\phi(f)}U(\xi, \lambda)^{-1} = e^{i\phi(f^{\xi,\lambda})}$, cf. (4.20), the claim follows.

(b) Taking into account that the field operator $\phi(f)$ is self-adjoint, we can use standard arguments (see, e.g., [10]) to derive the cyclicity of Ω for $\mathcal{M}$ from the cyclicity of Ω for ϕ, which was established in Proposition 4.2(c).

Next we note that our fields ϕ, ϕ' have the vacuum Ω as an analytic vector (Proposition 4.2(a)), so that we can apply the results of [11] to conclude that also the unitaries $e^{i\phi(f)}, e^{i\phi'(g)}$ commute for real f, g with $\operatorname{supp} f \subset \mathbb{R}_+$, $\operatorname{supp} g \subset \mathbb{R}_-$. That is, $\widehat{\mathcal{M}} \subset \mathcal{M}'$. But Ω is cyclic for $\widehat{\mathcal{M}}$ by the same argument as above, and hence Ω separates $\mathcal{M}$.

(c) The proof of this claim works precisely as in [18, Proposition 3.1] by exploiting the commutation relations between $U(x)$ and Δ^{it}, J, which are known from a theorem of Borchers [8].

(d) By definition of ϕ', we have $\widehat{\mathcal{M}} = U(j)\mathcal{M}U(j)$. But $U(j)$ coincides with the modular conjugation of $(\mathcal{M}, \Omega)$, and hence, by Tomita's theorem, $\widehat{\mathcal{M}} = U(j)\mathcal{M}U(j) = J\mathcal{M}J = \mathcal{M}'$. □

4.3. *Local operators*

So far we have constructed a Hilbert space $\mathcal{H}$, a representation U of G on $\mathcal{H}$, and a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ associated with a scattering function $S \in \mathcal{S}_{\mathrm{lim}}$, such that these data are compatible in the sense of Theorem 4.3. Given such objects, we now recall how a corresponding local field theory can be constructed. The first step is to define a family of von Neumann algebras associated with intervals, $-\infty < a < b < \infty$, as

$$\mathcal{A}(a,b) := U(a)\mathcal{M}U(a)^{-1} \cap U(b)\mathcal{M}'U(b)^{-1}. \tag{4.31}$$

For general subsets R of $\mathbb{R}$ we set

$$\mathcal{A}(R) := \bigvee_{(a,b)\subset R} \mathcal{A}(a,b). \tag{4.32}$$

This defines in particular the locally generated half line algebras $\mathcal{A}(\mathbb{R}_+) \subset \mathcal{M}$ and $\mathcal{A}(\mathbb{R}_-) \subset \mathcal{M}'$, as well as the global algebra $\mathcal{A} := \mathcal{A}(\mathbb{R})$.

The following properties of the assignment $I \mapsto \mathcal{A}(I)$ are all straightforward consequences of Theorem 4.3, so that we can omit the proof.

Proposition 4.4. *The map $I \mapsto \mathcal{A}(I)$ is an isotonous net of von Neumann algebras on $\mathcal{H}$ which transforms covariantly under the affine group G,*

$$U(g)\mathcal{A}(I)U(g)^{-1} = \mathcal{A}(gI), \quad g \in G. \tag{4.33}$$

This net of algebras is local in the sense that

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2)' \quad \text{whenever } I_1 \cap I_2 = \emptyset. \tag{4.34}$$

Note that no statement regarding the size of the algebras $\mathcal{A}(I)$ is made here. We shall see in Sec. 5 that this question is closely related to the existence of conformal symmetry. However, there is one restriction on the size of $\mathcal{A}(I)$ that we can compute directly: We will show that all local operators commute with $S(\infty)^N$, where $(N\Psi)_n := n \cdot \Psi_n$ is the number operator on $\mathcal{H}$ and $S(\infty)^N$ is defined via spectral calculus. That is, $S(\infty)^N = \sum_{n=0}^{\infty} S(\infty)^n P_n$. In the case $S(\infty) = -1$, this commutation relation limits the size of $\mathcal{A}(I)$. We begin with a preparatory lemma.

Lemma 4.5. *Let $\psi_1, \psi_2 \in \mathscr{S}(\mathbb{R}), n \in \mathbb{N}_0$. The following sequences of bounded operators converge to zero in the weak operator topology as $\lambda \to \infty$.*

$$y(U(0,\lambda)\psi_1)U(j)y(U(0,\lambda)\psi_2)P_n, \tag{4.35}$$

$$y^\dagger(U(0,\lambda)\psi_1)U(j)y^\dagger(U(0,\lambda)\psi_2)P_n, \tag{4.36}$$

$$y^\dagger(U(0,\lambda)\psi_1)U(j)y(U(0,\lambda)\psi_2)P_n, \tag{4.37}$$

$$[y(U(0,\lambda)\psi_1), U(j)y^\dagger(U(0,\lambda)\psi_2)U(j)] - \langle\psi_2,\psi_1\rangle S(\infty)^N. \tag{4.38}$$

Proof. Let $\Psi_n \in \mathcal{H}_n \cap \mathscr{S}(\mathbb{R}^n)$. Expanding the definition (4.3) of the annihilation operator as in (2.21), we find, $k = 1, 2$,

$$\begin{aligned} &\|y(U(0,\lambda)\psi_k)\Psi_n\|^2 \\ &\quad = n \int d^{n-1}\boldsymbol{\beta}\, d\beta_0\, d\beta_0'\, \psi_k(\beta_0 + \lambda)\overline{\psi_k(\beta_0' + \lambda)}\Psi_n(\beta_0, \boldsymbol{\beta})\overline{\Psi_n(\beta_0', \boldsymbol{\beta})}. \end{aligned}$$

As $\lambda \to \infty$, the integrand goes to zero pointwise, and since the functions are all of Schwartz class, we can apply the dominated convergence theorem to prove that $y(U(0,\lambda)\psi_k)\Psi_n \to 0$ in Hilbert space norm. On the other hand, we have the bound $\|y(U(0,\lambda)\psi_k)P_n\| \leq \sqrt{n}\|\psi_k\|$, uniform in λ. Hence

$$\lim_{\lambda\to\infty} y(U(0,\lambda)\psi_k)P_n = 0 \quad \text{in the strong operator topology.} \tag{4.39}$$

By another application of the uniform bound, and using $\|U(j)\| = 1$, the operator (4.35) converges to zero strongly. The adjoint of this operator then vanishes in the weak operator topology. Since this adjoint differs from (4.36) only by trivial redefinitions, the second claim follows. For proving that (4.37) converges weakly to zero in the limit $\lambda \to \infty$, we just need to apply (4.39) on both sides of the scalar product.

For the operator (4.38), we apply (4.23) to obtain, with $\Phi_n, \Psi_n \in \mathcal{H}_n \cap \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned} &\langle \Phi_n, ([y(U(0,\lambda)\psi_1), U(j)y^\dagger(U(0,\lambda)\psi_2)U(j)] - \langle\psi_2,\psi_1\rangle S(\infty)^N)\Psi_n\rangle \\ &\quad = \int d^n\boldsymbol{\beta} \int d\beta_0\, \overline{\Phi_n(\boldsymbol{\beta})}\Psi_n(\boldsymbol{\beta})\psi_1(\beta_0)\overline{\psi_2(\beta_0)} \left(\prod_{l=1}^n S(\beta_l - \beta_0 + \lambda) - S(\infty)^n\right). \end{aligned} \tag{4.40}$$

The integrand tends to zero pointwise, and by another application of the dominated convergence theorem, it follows that the above matrix element vanishes as $\lambda \to \infty$. All matrix elements of (4.38) between vectors of different particle number vanish identically. As (4.38) is bounded in operator norm, uniform in λ, and Φ_n, Ψ_n were chosen from a total set, the operator (4.38) tends to zero in the weak operator topology. □

As a consequence, all local operators are even with respect to the particle number in the case $S(\infty) = -1$.

Proposition 4.6. *If $A \in \mathcal{A}(I)$ for some bounded interval I, then $[A, S(\infty)^N] = 0$.*

Proof. Without loss of generality, let $I = (-1, 1)$. We choose $g \in \mathscr{S}(1,\infty)$ and $g' \in \mathscr{S}(-\infty,-1)$ fixed such that $\langle\widehat{g}^-, \hat{g}'^+\rangle \neq 0$, and set $f^{[\prime]} := g^{[\prime]0,\lambda}$ with $\lambda \geq 0$. For any such λ, the (closed) field operator $\phi(f)$ is affiliated with $U(1)\mathcal{M}U(1)^{-1}$, and $\phi'(f')$ is affiliated with $U(-1)\mathcal{M}'U(-1)^{-1}$. Since both fields contain $\mathcal{D}$ in their domains and leave this subspace invariant, this implies that their product $\phi(f)\phi'(f')$ commutes with $\mathcal{A}(I) = U(1)\mathcal{M}'U(1)^{-1} \cap U(-1)\mathcal{M}U(-1)^{-1}$ on $\mathcal{D}$, cf. (4.31) and

Theorem 4.3(d). Hence we find, $\Phi, \Psi \in \mathcal{D}$,

$$\langle \Phi, [A, \phi(f)\phi'(f')]\Psi \rangle = 0. \tag{4.41}$$

We can write

$$\begin{aligned} -\phi(f)\phi'(f') &= (y^\dagger(\hat{f}^+) + y(\hat{f}^-))U(j)(y^\dagger(\overline{\hat{f}'^+}) + y(\overline{\hat{f}'^-}))U(j) \\ &= \left.\begin{aligned} &y^\dagger(\hat{f}^+)U(j)y^\dagger(\overline{\hat{f}'^+})U(j) + y^\dagger(\hat{f}^+)U(j)y(\overline{\hat{f}'^-})U(j) \\ &+ U(j)y^\dagger(\overline{\hat{f}'^+})U(j)y(\hat{f}^-) + y(\hat{f}^-)U(j)y(\overline{\hat{f}'^-})U(j) \end{aligned}\right\} (*) \\ &\quad + [y(\hat{f}^-), U(j)y^\dagger(\overline{\hat{f}'^+})U(j)]. \end{aligned} \tag{4.42}$$

Inserted into the matrix element (4.41), the expression $(*)$ vanishes as $\lambda \to \infty$ due to Lemma 4.5. In the same way, the remaining commutator converges to $\langle \overline{\hat{g}^-}, \hat{g}'^+ \rangle S(\infty)^N$. Since $\langle \overline{\hat{g}^-}, \hat{g}'^+ \rangle \neq 0$, the claim follows. □

Thus, at least for the class of models with $S(\infty) = -1$, we have some restriction on the size of the local algebras $\mathcal{A}(I)$; in particular, the inclusion $\mathcal{A}(0, \infty) \subset \mathcal{M}$ is proper in these cases.

4.4. *Chiral decomposition of the two-dimensional models*

We now explain the decomposition of the two-dimensional massless $(0, S)$-models described in Sec. 2.2 into chiral components of the form described in Secs. 4.1–4.3.

Given a scattering function $S \in \mathcal{S}_{\lim}$, consider two copies $\mathcal{H}_{\ell/\mathsf{r}}$, $y_{\ell/\mathsf{r}}(\beta), y^\dagger_{\ell/\mathsf{r}}(\beta)$, $U_{\ell/\mathsf{r}}$, $N_{\ell/\mathsf{r}}$ and $\phi_{\ell/\mathsf{r}}$, $\phi'_{\ell/\mathsf{r}}$ of, respectively, the Hilbert space, Zamolodchikov operators, representation of the affine group of $\mathbb{R}$, particle number operators and halfline fields discussed in Secs. 4.1 and 4.2. We will use the notation $y^\#_{\ell/\mathsf{r}}(\psi)' := U_{\ell/\mathsf{r}}(j)y^\#_{\ell/\mathsf{r}}(\bar{\psi})U_{\ell/\mathsf{r}}(j)$.

We also introduce isometries $v_{\ell/\mathsf{r}} : L^2(\mathbb{R}, d\beta) \to L^2(\mathbb{R}, dp/|p|)$ defined by

$$(v_\ell\psi)(p) := \begin{cases} \psi(\log(-p)) & \text{if } p < 0, \\ 0 & \text{if } p \geq 0, \end{cases}$$

$$(v_\mathsf{r}\psi)(p) := \begin{cases} 0 & \text{if } p \leq 0, \\ \psi(\log p) & \text{if } p > 0. \end{cases}$$

It is clear that the map $v : \psi_1 \oplus \psi_2 \in L^2(\mathbb{R}, d\beta) \oplus L^2(\mathbb{R}, d\beta) \mapsto v_\ell\psi_1 + v_\mathsf{r}\psi_2 \in L^2(\mathbb{R}, dp/|p|)$ is unitary. Furthermore, to a given $f \in \mathscr{S}(\mathbb{R}^2)$ we associate functions $f_{\ell/\mathsf{r}} \in \mathscr{S}(\mathbb{R})$ through

$$\begin{aligned} f_\ell(\xi) &:= \frac{1}{2}\int_{\mathbb{R}} d\xi' f\left(\frac{\xi + \xi'}{2}, \frac{\xi - \xi'}{2}\right), \\ f_\mathsf{r}(\xi) &:= \frac{1}{2}\int_{\mathbb{R}} d\xi' f\left(\frac{\xi + \xi'}{2}, \frac{\xi' - \xi}{2}\right). \end{aligned} \tag{4.43}$$

If $f = \partial g/\partial x_k$, with $g \in \mathscr{S}(\mathbb{R}^2)$, $k = 0, 1$, a calculation using (2.26), (4.9) shows that

$$f^{0\pm}(e^\beta) = \frac{(-1)^{k+1}}{\sqrt{2\pi}}\hat{g}_{\mathsf{r}}^{\pm}(\beta), \quad f^{0\pm}(-e^\beta) = -\frac{1}{\sqrt{2\pi}}\hat{g}_\ell^{\pm}(\beta), \quad \beta \in \mathbb{R}, \tag{4.44}$$

or, equivalently,

$$f^{0\pm} = -\frac{1}{\sqrt{2\pi}}(v_\ell \hat{g}_\ell^{\pm} + (-1)^k v_{\mathsf{r}} \hat{g}_{\mathsf{r}}^{\pm}). \tag{4.45}$$

Proposition 4.7. *There exists a unitary operator* $V : \mathcal{H}_\ell \otimes \mathcal{H}_{\mathsf{r}} \to \mathcal{H}_0$ *such that*:

(a) *For all* $\psi \in \mathcal{H}_1$ *there holds, on* $\mathcal{D} \otimes \mathcal{D}$,

$$V^* z_0^\dagger(v_\ell \psi) V = y_\ell^\dagger(\psi)' \otimes 1, \tag{4.46}$$

$$V^* z_0^\dagger(v_{\mathsf{r}} \psi) V = S(\infty)^{N_\ell} \otimes y_{\mathsf{r}}^\dagger(\psi). \tag{4.47}$$

(b) $V^* U_0(x, \theta) V = U_\ell(x_\ell, \theta) \otimes U_{\mathsf{r}}(x_{\mathsf{r}}, -\theta)$, *where* $x_\ell := x_0 + x_1$, $x_{\mathsf{r}} := x_0 - x_1$ *are the left and right light ray components of* $x = (x_0, x_1) \in \mathbb{R}^2$.

(c) $V^* U_0(j) V = S(\infty)^{N_\ell \otimes N_{\mathsf{r}}}(U_\ell(j) \otimes U_{\mathsf{r}}(j))$.

(d) *For every* $f \in \mathscr{S}(\mathbb{R}^2)$ *such that* $f = \partial g/\partial x_k$ *with* $g \in \mathscr{S}(\mathbb{R}^2)$, $k = 0, 1$, *there holds, on* $\mathcal{D} \otimes \mathcal{D}$,

$$V^* \phi_0(f) V = -\frac{1}{\sqrt{2\pi}}(\phi_\ell'(g_\ell) \otimes 1 + (-1)^k S(\infty)^{N_\ell} \otimes \phi_{\mathsf{r}}(g_{\mathsf{r}})), \tag{4.48}$$

$$V^* \phi_0'(f) V = -\frac{1}{\sqrt{2\pi}}(\phi_\ell(g_\ell) \otimes S(\infty)^{N_{\mathsf{r}}} + (-1)^k 1 \otimes \phi_{\mathsf{r}}'(g_{\mathsf{r}})). \tag{4.49}$$

Proof. (a) Recalling that $S_0(p, q) = S(\infty) = \pm 1$ for $pq < 0$ (2.11), we see that, for $\psi, \psi' \in \mathcal{H}_1$,

$$z_0^\dagger(v_{\mathsf{r}}\psi') z_0^\dagger(v_\ell \psi) = S(\infty) z_0^\dagger(v_\ell \psi) z_0^\dagger(v_{\mathsf{r}}\psi'),$$
$$z_0(v_{\mathsf{r}}\psi') z_0^\dagger(v_\ell \psi) = S(\infty) z_0^\dagger(v_\ell \psi) z_0(v_{\mathsf{r}}\psi').$$

Considering then functions $\psi_1, \ldots, \psi_n, \psi_1', \ldots, \psi_{n'}', \chi_1, \ldots, \chi_m, \chi_1', \ldots, \chi_{m'}' \in \mathcal{H}_1$ with $n + m = n' + m'$, one has

$$\begin{aligned}
&\langle z_0^\dagger(v_\ell\psi_1) \ldots z_0^\dagger(v_\ell\psi_n) z_0^\dagger(v_{\mathsf{r}}\chi_1) \ldots z_0^\dagger(v_{\mathsf{r}}\chi_m)\Omega_0, \\
&\qquad z_0^\dagger(v_\ell\psi_1') \ldots z_0^\dagger(v_\ell\psi_{n'}') z_0^\dagger(v_{\mathsf{r}}\chi_1') \ldots z_0^\dagger(v_{\mathsf{r}}\chi_{m'}')\Omega_0\rangle \\
&\quad = S(\infty)^{(n+n')m} \langle z_0(\overline{v_\ell\psi_{n'}'}) \ldots z_0(\overline{v_\ell\psi_1'}) z_0^\dagger(v_\ell\psi_1) \ldots z_0^\dagger(v_\ell\psi_n)\Omega_0, \\
&\qquad z_0(\overline{v_{\mathsf{r}}\chi_m}) \ldots z_0(\overline{v_{\mathsf{r}}\chi_1}) z_0^\dagger(v_{\mathsf{r}}\chi_1') \ldots z_0^\dagger(v_{\mathsf{r}}\chi_{m'}')\Omega_0\rangle \\
&\quad = \delta_{nn'}\delta_{mm'} \langle z_0^\dagger(v_\ell\psi_1) \ldots z_0^\dagger(v_\ell\psi_n)\Omega_0, z_0^\dagger(v_\ell\psi_1') \ldots z_0^\dagger(v_\ell\psi_n')\Omega_0\rangle \\
&\qquad \times \langle z_0^\dagger(v_{\mathsf{r}}\chi_1) \ldots z_0^\dagger(v_{\mathsf{r}}\chi_m)\Omega_0, z_0^\dagger(v_{\mathsf{r}}\chi_1') \ldots z_0^\dagger(v_{\mathsf{r}}\chi_m')\Omega_0\rangle,
\end{aligned} \tag{4.50}$$

where the second equality follows from the observation that if $n' > n$ (and then $m > m'$), the two vectors in the scalar product vanish, while if $n' < n$, one gets the scalar product of two functions of $n - n' = m' - m$ variables which have supports where all the momenta are positive, respectively, negative. As in (3.6), we have

$$\begin{aligned}
&\langle z_0^\dagger(v_\ell\psi_1)\dots z_0^\dagger(v_\ell\psi_n)\Omega_0, z_0^\dagger(v_\ell\psi'_1)\dots z_0^\dagger(v_\ell\psi'_n)\Omega_0\rangle \\
&\quad = \sum_{\pi\in\mathfrak{S}_n}\int \frac{dp_1}{|p_1|}\cdots\frac{dp_n}{|p_n|}\prod_{j=1}^n(\overline{(v_\ell\psi_j)(p_j)}(v_\ell\psi'_j)(p_{\pi(j)}))\prod_{\substack{1\le a<b\le n\\ \pi(a)>\pi(b)}} S_0(p_{\pi(a)},p_{\pi(b)}) \\
&\quad = \sum_{\pi\in\mathfrak{S}_n}\int d\beta_1\cdots d\beta_n\prod_{j=1}^n(\overline{\psi_j(\beta_j)}\psi'_j(\beta_{\pi(j)}))\prod_{\substack{1\le a<b\le n\\ \pi(a)>\pi(b)}} S(\beta_{\pi(b)}-\beta_{\pi(a)}),
\end{aligned}$$

where the last equality follows by the variable change $p_j = -e^{\beta_j}$ and (2.11). If we now perform the further change of variables $\gamma_j = \beta_{\pi(j)}$ and set $\sigma = \pi^{-1}$ we obtain

$$\begin{aligned}
&\langle z_0^\dagger(v_\ell\psi_1)\dots z_0^\dagger(v_\ell\psi_n)\Omega_0, z_0^\dagger(v_\ell\psi'_1)\dots z_0^\dagger(v_\ell\psi'_n)\Omega_0\rangle \\
&\quad = \sum_{\sigma\in\mathfrak{S}_n}\int d\gamma_1\cdots d\gamma_n\prod_{j=1}^n(\psi'_j(\gamma_j)\overline{\psi_j(\gamma_{\sigma(j)})})\prod_{\substack{1\le a<b\le n\\ \sigma(a)>\sigma(b)}} S(\gamma_{\sigma(a)}-\gamma_{\sigma(b)}) \\
&\quad = \langle y_\ell^\dagger(\overline{\psi'_1})\dots y_\ell^\dagger(\overline{\psi'_n})\Omega_\ell, y_\ell^\dagger(\overline{\psi_1})\dots y_\ell^\dagger(\overline{\psi_n})\Omega_\ell\rangle \\
&\quad = \langle y_\ell^\dagger(\psi_1)'\dots y_\ell^\dagger(\psi_n)'\Omega_\ell, y_\ell^\dagger(\psi'_1)'\dots y_\ell^\dagger(\psi'_n)'\Omega_\ell\rangle.
\end{aligned}$$

A similar (in fact, simpler) calculation shows that

$$\begin{aligned}
&\langle z_0^\dagger(v_\mathsf{r}\chi_1)\dots z_0^\dagger(v_\mathsf{r}\chi_m)\Omega_0, z_0^\dagger(v_\mathsf{r}\chi'_1)\dots z_0^\dagger(v_\mathsf{r}\chi'_m)\Omega_0\rangle \\
&\quad = \langle y_\mathsf{r}^\dagger(\chi_1)\dots y_\mathsf{r}^\dagger(\chi_m)\Omega_\mathsf{r}, y_\mathsf{r}^\dagger(\chi'_1)\dots y_\mathsf{r}^\dagger(\chi'_m)\Omega_\mathsf{r}\rangle.
\end{aligned}$$

Therefore, we see that the scalar product at the beginning of (4.50) equals

$$\begin{aligned}
&= \delta_{nn'}\delta_{mm'}\langle y_\ell^\dagger(\psi_1)'\dots y_\ell^\dagger(\psi_n)'\Omega_\ell, y_\ell^\dagger(\psi'_1)'\dots y_\ell^\dagger(\psi'_n)'\Omega_\ell\rangle \\
&\quad \times \langle y_\mathsf{r}^\dagger(\chi_1)\dots y_\mathsf{r}^\dagger(\chi_m)\Omega_\mathsf{r}, y_\mathsf{r}^\dagger(\chi'_1)\dots y_\mathsf{r}^\dagger(\chi'_m)\Omega_\mathsf{r}\rangle.
\end{aligned}$$

As the sets

$$\begin{aligned}
&\{y_\ell^\dagger(\psi_1)'\dots y_\ell^\dagger(\psi_n)'\Omega_\ell\otimes y_\mathsf{r}^\dagger(\chi_1)\dots y_\mathsf{r}^\dagger(\chi_m)\Omega_\mathsf{r} : \psi_1,\dots,\chi_m\in\mathcal{H}_1, n,m\in\mathbb{N}_0\} \\
&\{z_0^\dagger(v_\ell\psi_1)\dots z_0^\dagger(v_\ell\psi_n)z_0^\dagger(v_\mathsf{r}\chi_1)\dots z_0^\dagger(v_\mathsf{r}\chi_m)\Omega_0 : \psi_1,\dots,\chi_m\in\mathcal{H}_1, n,m\in\mathbb{N}_0\}
\end{aligned}$$

are total in $\mathcal{H}_\ell\otimes\mathcal{H}_\mathsf{r}$ and $\mathcal{H}_0$ respectively, the definition

$$\begin{aligned}
&V y_\ell^\dagger(\psi_1)'\dots y_\ell^\dagger(\psi_n)'\Omega_\ell\otimes y_\mathsf{r}^\dagger(\chi_1)\dots y_\mathsf{r}^\dagger(\chi_m)\Omega_\mathsf{r} \\
&\quad := z_0^\dagger(v_\ell\psi_1)\dots z_0^\dagger(v_\ell\psi_n)z_0^\dagger(v_\mathsf{r}\chi_1)\dots z_0^\dagger(v_\mathsf{r}\chi_m)\Omega_0
\end{aligned}$$

uniquely determines a unitary operator $V : \mathcal{H}_\ell \otimes \mathcal{H}_{\mathsf{r}} \to \mathcal{H}_0$. Equations (4.46) and (4.47) then follow easily.

(b) We recall that U_0 and $U_{\ell/\mathsf{r}}$ are second quantized representations, so that, thanks to the definition of V, it is sufficient to consider the action of $U_0(x,\theta)$ on $v_\ell\psi, v_{\mathsf{r}}\chi$ for $\psi, \chi \in \mathcal{H}_1$. We compute:

$$\begin{aligned}(U_0(x,\theta)v_\ell\psi)(p) &= e^{i(|p|x_0 - px_1)}(v_\ell\psi)(\cosh\theta p - \sinh\theta|p|)\\ &= e^{-ipx_\ell}(v_\ell\psi)(e^\theta p) = (v_\ell U_\ell(x_\ell,\theta)\psi)(p),\end{aligned} \tag{4.51}$$

where the second equality follows from the sign properties of $p \mapsto \cosh\theta p - \sinh\theta|p|$. Similarly $U_0(x,\theta)v_{\mathsf{r}}\chi = v_{\mathsf{r}}U_{\mathsf{r}}(x_{\mathsf{r}}, -\theta)\chi$.

(c) This also follows straightforwardly from (a) thanks to

$$U_0(j)z_0^\dagger(\psi_1)\dots z_0^\dagger(\psi_n)\Omega_0 = z_0^\dagger(\overline{\psi}_n)\dots z_0^\dagger(\overline{\psi}_1)\Omega_0$$

and similar relations for $U_{\ell/\mathsf{r}}(j)$.

(d) Using (a), Eq. (4.48) follows by easy computations from the definition of the fields ϕ_0, $\phi_{\ell/\mathsf{r}}$ and Eq. (4.45), while Eq. (4.49) is a consequence of (4.48), of (c) and of the fact that

$$\begin{aligned}S(\infty)^{N_\ell\otimes N_{\mathsf{r}}}(\phi_\ell(g_\ell)\otimes 1)S(\infty)^{N_\ell\otimes N_{\mathsf{r}}} &= \phi_\ell(g_\ell)\otimes S(\infty)^{N_{\mathsf{r}}},\\ S(\infty)^{N_\ell\otimes N_{\mathsf{r}}}(S(\infty)^{N_\ell}\otimes \phi_{\mathsf{r}}'(g_{\mathsf{r}}))S(\infty)^{N_\ell\otimes N_{\mathsf{r}}} &= 1\otimes \phi_{\mathsf{r}}'(g_{\mathsf{r}}).\end{aligned}$$

□

Since $\mathcal{H}_{0,1}$ is unitarily equivalent to $\mathcal{H}_{\ell,1} \oplus \mathcal{H}_{\mathsf{r},1}$, the above result can be seen as a generalization to the S_0-symmetric Fock space of the classical result on the tensor product decomposition of the symmetric or antisymmetric Fock space built over a direct sum single particle space.

Proposition 4.7(d), together with the halfline-locality of the chiral fields, entails in particular that the fields ϕ_0, ϕ_0' are wedge-local. That is, we have proved the commutation relation (2.31) for the case $m = 0$.

We now come to the decomposition of operators on $\mathcal{H}_0$. For an operator $A \in \mathcal{B}(\mathcal{H}_{\mathsf{r}})$ we define its even/odd parts as

$$A_{\mathsf{e}/\mathsf{o}} := \frac{1}{2}(A \pm S(\infty)^{N_{\mathsf{r}}}AS(\infty)^{N_{\mathsf{r}}}), \tag{4.52}$$

and similarly for $A \in \mathcal{B}(\mathcal{H}_\ell)$. Given then von Neumann algebras $\mathcal{R}_{\ell/\mathsf{r}}$ on $\mathcal{H}_{\ell/\mathsf{r}}$ such that $S(\infty)^{N_{\ell/\mathsf{r}}}\mathcal{R}_{\ell/\mathsf{r}}S(\infty)^{N_{\ell/\mathsf{r}}} = \mathcal{R}_{\ell/\mathsf{r}}$, we consider the following twisted tensor product von Neumann algebras:

$$\mathcal{R}_\ell \,\hat{\otimes}\, \mathcal{R}_{\mathsf{r}} := \mathcal{R}_\ell \otimes \mathcal{R}_{\mathsf{r},\mathsf{e}} + S(\infty)^{N_\ell}\mathcal{R}_\ell \otimes \mathcal{R}_{\mathsf{r},\mathsf{o}}, \tag{4.53}$$

$$\mathcal{R}_\ell \,\check{\otimes}\, \mathcal{R}_{\mathsf{r}} := \mathcal{R}_{\ell,\mathsf{e}} \otimes \mathcal{R}_{\mathsf{r}} + \mathcal{R}_{\ell,\mathsf{o}} \otimes S(\infty)^{N_{\mathsf{r}}}\mathcal{R}_{\mathsf{r}}. \tag{4.54}$$

Of course, if $S(\infty) = 1$, then $\mathcal{R}_\ell \,\hat{\otimes}\, \mathcal{R}_{\mathsf{r}} = \mathcal{R}_\ell \,\check{\otimes}\, \mathcal{R}_{\mathsf{r}} = \mathcal{R}_\ell \otimes \mathcal{R}_{\mathsf{r}}$, the usual tensor product von Neumann algebras. It can be shown [50] that

$$(\mathcal{R}_\ell \,\check{\otimes}\, \mathcal{R}_{\mathsf{r}})' = (\mathcal{R}_\ell)' \,\hat{\otimes}\, (\mathcal{R}_{\mathsf{r}})'. \tag{4.55}$$

The following result will be useful in discussing the splitting of double cone algebras of the two-dimensional theory in the case $S(\infty) = -1$; the analogue for $S(\infty) = 1$ is trivial.

Lemma 4.8. *Let $\mathcal{R}^{(i)}_{\ell/\mathsf{r}} \subset \mathcal{B}(\mathcal{H}_{\ell/\mathsf{r}})$, $i = 1, 2$, be von Neumann algebras such that*

$$(-1)^{N_{\ell/\mathsf{r}}}\mathcal{R}^{(i)}_{\ell/\mathsf{r}}(-1)^{N_{\ell/\mathsf{r}}} = \mathcal{R}^{(i)}_{\ell/\mathsf{r}},$$

and define $\mathcal{R}_{\ell/\mathsf{r}} := \mathcal{R}^{(1)}_{\ell/\mathsf{r}} \cap \mathcal{R}^{(2)}_{\ell/\mathsf{r}}$, $\bar{\mathcal{R}}_\ell := (-1)^{N_\ell}\mathcal{R}^{(1)}_\ell \cap \mathcal{R}^{(2)}_\ell$, $\bar{\mathcal{R}}_\mathsf{r} := \mathcal{R}^{(1)}_\mathsf{r} \cap (-1)^{N_\mathsf{r}}\mathcal{R}^{(2)}_\mathsf{r}$, $\mathcal{R} := (\mathcal{R}^{(1)}_\ell \hat{\otimes} \mathcal{R}^{(1)}_\mathsf{r}) \cap (\mathcal{R}^{(2)}_\ell \check{\otimes} \mathcal{R}^{(2)}_\mathsf{r})$. If $\mathcal{R}_{\ell/\mathsf{r}}$ and $\bar{\mathcal{R}}_{\ell/\mathsf{r}}$ have trivial odd and even parts, respectively, then $\mathcal{R} = \mathcal{R}_\ell \otimes \mathcal{R}_\mathsf{r} + \bar{\mathcal{R}}_\ell \otimes \bar{\mathcal{R}}_\mathsf{r}$.

Proof. The von Neumann algebra $\mathcal{R}$ can be decomposed as $\mathcal{R} = \mathcal{R}_{\mathsf{e},\mathsf{e}} + \mathcal{R}_{\mathsf{e},\mathsf{o}} + \mathcal{R}_{\mathsf{o},\mathsf{e}} + \mathcal{R}_{\mathsf{o},\mathsf{o}}$ where, denoting by $[\cdot,\cdot]_\mathsf{e}$ the commutator and by $[\cdot,\cdot]_\mathsf{o}$ the anticommutator,

$$\mathcal{R}_{i,j} = \{A \in \mathcal{R} : [(-1)^{N_\ell} \otimes 1, A]_i = 0 = [1 \otimes (-1)^{N_\mathsf{r}}, A]_j\}, \quad i, j = \mathsf{e}, \mathsf{o}. \tag{4.56}$$

Similarly, defining $\mathcal{R}^{(1)} := \mathcal{R}^{(1)}_\ell \hat{\otimes} \mathcal{R}^{(1)}_\mathsf{r}$, $\mathcal{R}^{(2)} := \mathcal{R}^{(2)}_\ell \check{\otimes} \mathcal{R}^{(2)}_\mathsf{r}$, one has $\mathcal{R}^{(1)} = \mathcal{R}^{(1)}_\mathsf{e} + \mathcal{R}^{(1)}_\mathsf{o}$, $\mathcal{R}^{(2)} = \mathcal{R}^{(2)}_\mathsf{e} + \mathcal{R}^{(2)}_\mathsf{o}$ with respect to the action of $1 \otimes (-1)^{N_\mathsf{r}}$ and $(-1)^{N_\ell} \otimes 1$, respectively. It is then clear that $\mathcal{R}_{i,j} = \mathcal{R}^{(1)}_j \cap \mathcal{R}^{(2)}_i$ for $i, j = \mathsf{e}, \mathsf{o}$. In particular,

$$\mathcal{R}_{\mathsf{e},\mathsf{e}} = (\mathcal{R}^{(1)}_\ell \otimes \mathcal{R}^{(1)}_{\mathsf{r},\mathsf{e}}) \cap (\mathcal{R}^{(2)}_{\ell,\mathsf{e}} \otimes \mathcal{R}^{(2)}_\mathsf{r}) = \mathcal{R}_{\ell,\mathsf{e}} \otimes \mathcal{R}_{\mathsf{r},\mathsf{e}} = \mathcal{R}_\ell \otimes \mathcal{R}_\mathsf{r}.$$

Similarly, $\mathcal{R}_{\mathsf{o},\mathsf{o}} = \bar{\mathcal{R}}_\ell \otimes \bar{\mathcal{R}}_\mathsf{r}$. In order to get the statement, it is therefore sufficient to show that $\mathcal{R}_{\mathsf{e},\mathsf{o}} = \emptyset = \mathcal{R}_{\mathsf{o},\mathsf{e}}$. To this end, consider the Tomiyama slice map $E^\omega_\ell : \mathcal{B}(\mathcal{H}_\ell \otimes \mathcal{H}_\mathsf{r}) \to \mathcal{B}(\mathcal{H}_\ell)$, $\omega \in \mathcal{B}(\mathcal{H}_\mathsf{r})_*$, defined by the fact that $\varphi(E^\omega_\ell(A)) = (\varphi \otimes \omega)(A)$ for all $\varphi \in \mathcal{B}(\mathcal{H}_\ell)_*$, $A \in \mathcal{B}(\mathcal{H}_\ell \otimes \mathcal{H}_\mathsf{r})$ [51]. It is then easy to see that if $A \in \mathcal{R}_{\mathsf{o},\mathsf{e}} = (\mathcal{R}^{(1)}_\ell \otimes \mathcal{R}^{(1)}_{\mathsf{r},\mathsf{e}}) \cap (\mathcal{R}^{(2)}_{\ell,\mathsf{o}} \otimes S(\infty)^{N_\mathsf{r}}\mathcal{R}^{(2)}_\mathsf{r})$ then $E^\omega_\ell(A) \in \mathcal{R}_{\ell,\mathsf{o}}$ and therefore $\mathcal{R}_{\mathsf{o},\mathsf{e}} = \emptyset$ by hypothesis. Similarly one shows that $\mathcal{R}_{\mathsf{e},\mathsf{o}} = \emptyset$. □

Given bounded open intervals $I, J \subset \mathbb{R}$ we introduce the double cone

$$O_{I,J} := \{x \in \mathbb{R}^2 : x_\ell \in I, x_\mathsf{r} \in J\}. \tag{4.57}$$

Proposition 4.9. *With V the unitary of Proposition 4.7, and with $\mathcal{M}_{\ell/\mathsf{r}}, \mathcal{M}'_{\ell/\mathsf{r}}$ the von Neumann algebras generated by the fields $\phi_{\ell/\mathsf{r}}(f), \phi'_{\ell/\mathsf{r}}(g)$ with* $\operatorname{supp} f \subset \mathbb{R}_+$, $\operatorname{supp} g \subset \mathbb{R}_-$ *respectively, there holds*:

$$V^*\mathcal{M}'_0 V = \mathcal{M}'_\ell \hat{\otimes} \mathcal{M}_\mathsf{r}, \tag{4.58}$$

$$V^*\mathcal{M}_0 V = \mathcal{M}_\ell \check{\otimes} \mathcal{M}'_\mathsf{r}, \tag{4.59}$$

$$V^*\mathcal{A}_0(O_{I,J})V = \mathcal{A}_\ell(I) \otimes \mathcal{A}_\mathsf{r}(J) + \bar{\mathcal{A}}_\ell(I) \otimes \bar{\mathcal{A}}_\mathsf{r}(J), \tag{4.60}$$

where $\bar{\mathcal{A}}_{\ell/\mathsf{r}}(a,b) := \alpha^{\ell/\mathsf{r}}_a(\mathcal{M}_{\ell/\mathsf{r}}) \cap S(\infty)^{N_{\ell/\mathsf{r}}}\alpha^{\ell/\mathsf{r}}_b(\mathcal{M}'_{\ell/\mathsf{r}})$, and $\alpha^{\ell/\mathsf{r}}_\xi = \operatorname{Ad} U_{\ell/\mathsf{r}}(\xi)$.

Proof. We start by showing $V^*\mathcal{M}'_0 V \subset \mathcal{M}'_\ell \hat{\otimes} \mathcal{M}_\mathsf{r}$. First, observe that if $f = \partial g/\partial x_1$ with $g \in \mathscr{S}(\mathbb{R}^2)$, thanks to the fact that the spaces of finite particle vectors $\mathcal{D}_{\ell/\mathsf{r}} \subset \mathcal{H}_{\ell/\mathsf{r}}$ are cores for $\phi'_\ell(g_\ell)$ and $\phi_\mathsf{r}(g_\mathsf{r})$, $\mathcal{D}_0$ is a core for $\phi_0(f)$ and

$V\mathcal{D}_\ell \otimes \mathcal{D}_\mathsf{r} = \mathcal{D}_0$, it follows from (4.48) that

$$V^* e^{i\sqrt{2\pi}\phi_0(f)} V = e^{i(S(\infty)^{N_\ell}\otimes\phi_\mathsf{r}(g_\mathsf{r}) - \phi'_\ell(g_\ell)\otimes 1)}.$$

If now $\operatorname{supp} g \subset W_L$, one has $\operatorname{supp} g_{\ell/\mathsf{r}} \subset \mathbb{R}_\mp$ and then $e^{-i\phi'_\ell(g_\ell)\otimes 1} = e^{-i\phi'_\ell(g_\ell)} \otimes 1 \in \mathcal{M}'_\ell \hat{\otimes} \mathcal{M}_\mathsf{r}$. Moreover the identity

$$e^{iS(\infty)^{N_\ell}\otimes\phi_\mathsf{r}(g_\mathsf{r})} = 1 \otimes (e^{i\phi_\mathsf{r}(g_\mathsf{r})})_\mathsf{e} + S(\infty)^{N_\ell} \otimes (e^{i\phi_\mathsf{r}(g_\mathsf{r})})_\mathsf{o} \tag{4.61}$$

is easily verified on finite particle vectors and entails $e^{iS(\infty)^{N_\ell}\otimes\phi_\mathsf{r}(g_\mathsf{r})} \in \mathcal{M}'_\ell \hat{\otimes} \mathcal{M}_\mathsf{r}$. The desired inclusion is then obtained with the help of the Trotter formula [48, Theorem VIII.31]

$$e^{i(S(\infty)^{N_\ell}\otimes\phi_\mathsf{r}(g_\mathsf{r}) - \phi'_\ell(g_\ell)\otimes 1)} = \text{s-}\lim_{n\to\infty} (e^{i(S(\infty)^{N_\ell}\otimes\phi_\mathsf{r}(g_\mathsf{r}))/n} e^{-i(\phi'_\ell(g_\ell)\otimes 1)/n})^n, \tag{4.62}$$

and by analogous considerations in the case $f = \partial g/\partial x_0$. Similarly, one gets $V^*\mathcal{M}_0 V \subset \mathcal{M}_\ell \check{\otimes} \mathcal{M}'_\mathsf{r}$, but then thanks to (4.55) there holds

$$V^*\mathcal{M}_0 V \subset \mathcal{M}_\ell \check{\otimes} \mathcal{M}'_\mathsf{r} = (\mathcal{M}'_\ell \hat{\otimes} \mathcal{M}_\mathsf{r})' \subset V^*\mathcal{M}_0 V, \tag{4.63}$$

which proves (4.58) and (4.59).

In order to show (4.60), we first observe that, thanks to Poincaré covariance, it is sufficient to consider $I = (-a, 0)$, $J = (0, a)$, $a > 0$, so that,

$$\begin{aligned}
\mathcal{A}_\ell(I) &= \alpha^\ell_{-a}(\mathcal{M}_\ell) \cap \mathcal{M}'_\ell,\\
\mathcal{A}_\mathsf{r}(J) &= \mathcal{M}_\mathsf{r} \cap \alpha^\mathsf{r}_a(\mathcal{M}'_\mathsf{r}),\\
V^*\mathcal{A}_0(O_{I,J})V &= (\alpha^\ell_{-a} \otimes \alpha^\mathsf{r}_a)(V^*\mathcal{M}_0 V) \cap V^*\mathcal{M}'_0 V\\
&= (\alpha^\ell_{-a}(\mathcal{M}_\ell) \check{\otimes} \alpha^\mathsf{r}_a(\mathcal{M}'_\mathsf{r})) \cap (\mathcal{M}'_\ell \hat{\otimes} \mathcal{M}_\mathsf{r}),
\end{aligned}$$

where we used Proposition 4.7(b) and formulas (4.58), (4.59). According to Proposition 4.6, the algebras $\mathcal{A}_\ell(I)$, $\mathcal{A}_\mathsf{r}(J)$ have trivial odd parts and if $S(\infty) = -1$, by an analogous statement for the anticommutator, $\bar{\mathcal{A}}_\ell(I), \bar{\mathcal{A}}_\mathsf{r}(J)$ have trivial even parts. They thus satisfy the assumptions of Lemma 4.8, which yields (4.60). □

This result completely clarifies the split of the massless two-dimensional models into chiral theories, and the influence of the scattering function on this decomposition. We will therefore restrict attention to the chiral theories on the light ray from now on.

5. Local Observables and Conformal Symmetry

The local net $\mathcal{A}$ on the real line, as constructed in Sec. 4, is covariant under the affine group G, containing translations and dilations of the light ray. It is a natural question to ask whether this model can be extended to a conformal field theory; that is, whether the net $\mathcal{A}$ can be extended to the one-point compactification of $\mathbb{R}$ (the circle), covariant under an extension of the representation U to the Möbius group $\mathrm{PSL}(2, \mathbb{R}) \supset G$.

The existence of such a conformal extension is a nontrivial question. In the physics literature, conformal symmetry is usually derived from translation-dilation symmetry under the additional (and sometimes implicit) assumption of existence of a local energy density. In our context, however, the energy density is not at our disposal. Without such additional data, dilation symmetry does in general *not* imply conformal symmetry; counterexamples have been constructed [20]. Thus, we need to exploit other specific properties of the situation at hand in order to obtain conformal extensions.

To that end, we first construct a subspace $\mathcal{H}_{\mathsf{loc}} \subset \mathcal{H}$ on which the vacuum is cyclic for the local algebras $\mathcal{A}(I)$.

Lemma 5.1. *The subspace* $\mathcal{H}_{\mathsf{loc}} := \overline{\mathcal{A}(a,b)\Omega} \subset \mathcal{H}$ *is independent of* $-\infty \leq a < b \leq \infty$, *and invariant under* U.

Proof. Given $0 < b < \infty$, we will first show $\overline{\mathcal{A}(0,b)\Omega} = \overline{\mathcal{A}(\mathbb{R})\Omega}$. Let $\Psi \perp \mathcal{A}(0,b)\Omega$. For any $A \in \mathcal{A}(0,b) \subset \mathcal{M}$, we know that $A\Omega \in \operatorname{dom}\Delta^{1/2}$, where Δ^{it} is the modular group of $(\mathcal{M},\Omega)$ as before. Thus the function $t \mapsto \langle \Psi, \Delta^{it} A\Omega\rangle$ has an analytic continuation to the strip $\mathrm{S}(-\frac{1}{2},0)$. But since Δ^{it} acts as a dilation (Theorem 4.3(c)), the function vanishes on the boundary for $t < 0$, and hence everywhere. This shows $\Psi \perp \mathcal{A}(0,b')\Omega$ for any $0 < b' < \infty$. Applying a similar Reeh–Schlieder type argument to the function $\xi \mapsto \langle \Psi, U(\xi)A\Omega\rangle$, using the positivity of the generator of the translation group, we see that $\Psi \perp \mathcal{A}(I)\Omega$ for any finite interval $I \subset \mathbb{R}$. Hence $\Psi \perp \mathcal{A}(\mathbb{R})\Omega$, and we arrive at $(\mathcal{A}(0,b)\Omega)^\perp \subset (\mathcal{A}(\mathbb{R})\Omega)^\perp$. But since $(\mathcal{A}(\mathbb{R})\Omega)^\perp \subset (\mathcal{A}(0,b)\Omega)^\perp$ by isotony, $\overline{\mathcal{A}(0,b)\Omega} = \overline{\mathcal{A}(\mathbb{R})\Omega}$ follows. The latter space is invariant under U by construction of $\mathcal{A}(\mathbb{R})$, which implies the lemma. □

The reason for considering $\mathcal{H}_{\mathsf{loc}}$ is that it is the largest space on which we can expect an extension of $\mathcal{A}$ to a net on the circle, and consequently of U to the Möbius group. Namely, if the $\mathcal{A}(I)$ are covariant under such an extension of U, one shows by the same methods as above that $\mathcal{H}_{\mathsf{loc}}$ is invariant under the extended representation as well; thus the extended net $\mathcal{A}$ acts on $\mathcal{H}_{\mathsf{loc}}$ with cyclic vacuum vector.

It is a noteworthy fact that, after restriction of our net to $\mathcal{H}_{\mathsf{loc}}$, such a conformal extension *always* exists, as we shall show now.

Theorem 5.2. *The representation* $U\lceil\mathcal{H}_{\mathsf{loc}}$ *extends to a strongly continuous unitary representation of* $\mathrm{PSL}(2,\mathbb{R})$ *on* $\mathcal{H}_{\mathsf{loc}}$, *and* $I \mapsto \mathcal{A}(I)\lceil\mathcal{H}_{\mathsf{loc}}$ *extends to a local net on the circle, conformally covariant under this representation.*

Proof. By construction, Ω is cyclic and separating for $\mathcal{A}(\mathbb{R}_+)\lceil\mathcal{H}_{\mathsf{loc}}$. In fact, the modular group associated with this pair is $\Delta^{it}\lceil\mathcal{H}_{\mathsf{loc}}$, since the modular KMS condition is preserved under the restriction. Hence the translation-dilation covariant net $\mathcal{A}\lceil\mathcal{H}_{\mathsf{loc}}$ has the Bisognano–Wichmann property. Making use of the modular group of the interval algebra $(\mathcal{A}(0,1)\lceil\mathcal{H}_{\mathsf{loc}},\Omega)$, the extensions of the net and symmetry group now follow from [32, Theorem 1.4]. □

Thus, the questions of conformal symmetry and the size of the local algebras are intimately connected: The algebras $\mathcal{A}(a, b)$ are large if, and only if, the domain $\mathcal{H}_{\mathsf{loc}}$ of the extended representation U is large. In particular, for the case $S(\infty) = -1$, Proposition 4.6 already gives us a restriction: All local operators are even. This directly implies:

Proposition 5.3. *If $S(\infty) = -1$, then $\mathcal{H}_{\mathsf{loc}} \subset \mathcal{H}_{\mathsf{e}}$, where $\mathcal{H}_{\mathsf{e}}$ is the space of even particle number vectors.*

At this point, it is unknown (for general scattering function S) what the actual size of $\mathcal{H}_{\mathsf{loc}}$ is; we cannot exclude that it contains just multiples of Ω, and consequently $\mathcal{A}(I) = \mathbb{C}\mathbf{1}$. In Sec. 6, we will however determine $\mathcal{H}_{\mathsf{loc}}$ and the local algebras $\mathcal{A}(I)$ explicitly in simple examples of S.

6. Conformal Scaling Limits for Constant Scattering Functions

In this section, we illustrate the structure of the local algebras $\mathcal{A}(I)$, and of their extension to a conformally covariant theory on $\mathcal{H}_{\mathsf{loc}}$, in the examples of a constant scattering function: $S = \pm 1$.

The simplest possible case is $S = 1$. In this case, the Zamolodchikov–Faddeev relations (4.6)–(4.8) are the usual canonical commutation relations for annihilators and creators. In fact, one checks from the definitions that the field ϕ is nothing else than the free $U(1)$ current, and $\mathcal{A}(I)$ the associated local algebras; see, e.g., [19]. It is well known that the vacuum is cyclic for these algebras; thus $\mathcal{H}_{\mathsf{loc}} = \mathcal{H}$. In fact, the representation U extends to the well-known representation of the conformal group with central charge $c = 1$.

The first non-trivial example is $S = -1$. A Euclidean version of the associated massive two-dimensional quantum field theory can be obtained by considering the scaling limit of the spin correlation functions of the two-dimensional Ising model off the critical point [46]. In the context of factorizing S-matrices, this quantum field theoretic model, and in particular its formulation on Minkowski space, is often just referred to as "the Ising model".

This model has been investigated from a number of different perspectives. In [54] and previous work cited therein, Schroer and Truong give formulas for associated quantum field operators. In [5], the form factors of one of these fields are calculated, see also [4] for the calculation of the scaling dimension of field operators in the short distance limit. In [38], the existence of local observables in the two-dimensional model, as formulated here in terms of wedge algebras, was proven, and in [21], the model was generalized to higher dimensions, and its local and non-local aspects were discussed.

In our context, we are dealing with (a chiral component of) the massless limit of this system, which should hence be related to the Ising model *at* the critical point. On the field theoretical side, one expects this to be described by a chiral Fermi field, covariant under a representation of the Möbius group with central charge $c = \frac{1}{2}$

[45]. In our context, it is not immediately evident that the algebras $\mathcal{A}(I)$ consist of the observables related to a Fermi field. However, we shall show now that this is indeed the case.

On the technical side, in the case $S = -1$, our relations (4.6)–(4.8) are canonical *anti*commutation relations. As a consequence, the "smeared" creation and annihilation operators $y^\dagger(\psi)$, $y(\psi)$ are bounded, namely [14, Proposition 5.2.2]

$$\|y^\dagger(\psi)\| = \|y(\psi)\| = \|\psi\|_{\mathcal{H}_1}. \tag{6.1}$$

This will simplify our arguments considerably.

Proposition 4.6 gives only an "upper estimate" for the size of the local algebras $\mathcal{A}(I)$. We now want to determine the size of $\mathcal{A}(I)$ explicitly. In fact, we will show in detail how these algebras are generated by the energy density of a chiral Fermi field.

To that end, it is very helpful to introduce a new field operator ψ by

$$\psi(\xi) := \frac{1}{\sqrt{2\pi}} \int d\beta\, e^{\beta/2} \left(\sqrt{i} e^{ie^\beta \xi} y^\dagger(\beta) + \frac{1}{\sqrt{i}} e^{-ie^\beta \xi} y(\beta) \right). \tag{6.2}$$

The smeared field $\psi(f)$ is a well-defined bounded operator for any test function $f \in \mathscr{S}(\mathbb{R})$, since the functions $\beta \mapsto e^{-\beta/2} \hat{f}^\pm(\beta)$ belong to $L^2(\mathbb{R})$. One readily checks that $\psi(f)$ is selfadjoint for real f, and transforms covariantly under translations and dilations according to

$$U(\xi', \lambda)\psi(\xi)U(\xi', \lambda)^{-1} = e^{\lambda/2}\psi(e^\lambda(\xi + \xi')).$$

In particular, ψ has scaling dimension $\frac{1}{2}$.

Using techniques similar to those in [21, Lemma 6.1], we can clarify the relation between ψ and the halfline-local fields ϕ, ϕ'.

Proposition 6.1. *Let $a < b$, and consider test functions $f \in \mathscr{S}(a, b)$, $g \in \mathscr{S}(b, \infty)$, $h \in \mathscr{S}(-\infty, a)$. Then*

$$\{\psi(f), \phi(g)\} = 0, \quad [\psi(f), \phi'(h)] = 0. \tag{6.3}$$

The algebra $\mathcal{P}_{\mathsf{e}}(a, b)$ of even polynomials in ψ, smeared with test functions having support in (a, b), is a subalgebra of $\mathcal{A}(a, b)$, and we have $\overline{\mathcal{P}_{\mathsf{e}}(a, b)\Omega} = \mathcal{H}_{\mathsf{e}} = \mathcal{H}_{\mathsf{loc}}$. The algebra $\mathcal{P}_{\mathsf{e}}(\mathbb{R})$ acts irreducibly on $\mathcal{H}_{\mathsf{e}}$.

Proof. From the definitions (4.17) and (6.2), we see that we have $\psi(f) = \phi(k)$ if the function k fulfills $\hat{k}^\pm(\beta) = i^{\mp 1/2} e^{-\beta/2} \hat{f}^\pm(\beta)$. A short computation shows that such a function k can in fact be found, namely $k = K * f$, where K is the inverse Fourier transform of the distribution $p \mapsto 1/\sqrt{i(p + i0)}$.

Due to its analyticity and boundedness properties in Fourier space, K has support in the right half line ([49, Theorem IX.16], see also [21, Lemma 6.1]; note that we use different conventions for the Fourier transform). Thus $\operatorname{supp} k \subset (a, \infty)$. From the relative locality of ϕ and ϕ', see Proposition 4.2(d), it follows that $[\psi(f), \phi'(h)] = [\phi(k), \phi'(h)] = 0$.

To establish the first relation in (6.3), we compute in the sense of distributions,

$$\begin{aligned}\{\psi(\xi),\phi(\xi')\} &= \frac{i}{2\pi}\int d\beta e^{3\beta/2}\left(-\sqrt{i}e^{ie^{\beta}(\xi-\xi')}+\frac{1}{\sqrt{i}}e^{-ie^{\beta}(\xi-\xi')}\right)\\ &= -\frac{i^{3/2}}{2\pi}\int_{-\infty}^{\infty}dp\sqrt{p+i0}\;e^{-ip(\xi'-\xi)}.\end{aligned}$$

As before, this distribution has support only for $\xi'-\xi>0$, as desired.

Now due to (6.3), even polynomials in the field ψ, smeared with test functions having support in the interval (a,b), *commute* with both $\phi(g)$ and $\phi'(h)$. Since all fields involved are bounded operators, this directly implies that any such polynomial is an element of $\mathcal{A}(a,b)$; see Eq. (4.31).

For the proof of the cyclicity statement, let $\mathcal{P}(a,b)$ denote the algebra of all (even and odd) polynomials in ψ, smeared with test functions supported in (a,b). Then Ω is cyclic for $\mathcal{P}(a,b)$. (This follows with arguments as in Proposition 4.2(c); the extra factor $e^{\beta/2}$ in (6.2) can be absorbed in the test functions.) Applying the projector $E_{\mathsf{e}}=\frac{1}{2}(1+(-1)^N)$ onto $\mathcal{H}_{\mathsf{e}}$, we obtain

$$\mathcal{H}_{\mathsf{e}}=E_{\mathsf{e}}\mathcal{H}=E_{\mathsf{e}}\overline{\mathcal{P}(a,b)\Omega}=\overline{E_{\mathsf{e}}\mathcal{P}(a,b)E_{\mathsf{e}}\Omega}=\overline{\mathcal{P}_{\mathsf{e}}(a,b)\Omega}.$$

Further,

$$\overline{\mathcal{P}_{\mathsf{e}}(a,b)\Omega}\subset\overline{\mathcal{A}(a,b)\Omega}=\mathcal{H}_{\mathsf{loc}}.$$

But from Proposition 5.3, we know that $\mathcal{H}_{\mathsf{loc}}\subset\mathcal{H}_{\mathsf{e}}$. Hence $\mathcal{H}_{\mathsf{e}}=\mathcal{H}_{\mathsf{loc}}=\overline{\mathcal{P}_{\mathsf{e}}(a,b)\Omega}$.

Irreducibility of $\mathcal{P}_{\mathsf{e}}(\mathbb{R})$ now follows from cyclicity of Ω and from the spectrum condition for translations by standard arguments [57, Theorem 4-5]. □

Having seen that the even local algebras $\mathcal{A}(I)$ are non-trivial, we now want to understand their structure more explicitly in terms of local field operators. To this end, we first note that ψ satisfies the anticommutation relation of a free Fermi field,

$$\{\psi(\xi),\psi(\xi')\}=\frac{1}{2\pi}\int d\beta\;e^{\beta}(e^{ie^{\beta}(\xi-\xi')}+e^{-ie^{\beta}(\xi-\xi')})=\delta(\xi-\xi'). \tag{6.4}$$

This observation suggests to introduce a normal ordered even field,

$$T(\xi):=\frac{i}{2}:\psi(\xi)\partial_{\xi}\psi(\xi):=\frac{i}{2}\lim_{\xi'\to\xi}(\psi(\xi)\partial_{\xi'}\psi(\xi')-\langle\Omega,\psi(\xi)\partial_{\xi'}\psi(\xi')\Omega\rangle), \tag{6.5}$$

as a candidate for a local energy density. This limit exists in the sense of matrix elements between vectors from $\mathcal{D}_0$, where $\mathcal{D}_0\subset\mathcal{D}$ denotes those vectors in which each n-particle component is smooth and of compact support. Expressing $T(\xi)$ in terms of creation and annihilation operators, see Eq. (6.8) below, it is also easy to see that T is an operator-valued distribution.

Proposition 6.2. *The field T is point-local, relatively local to the algebras $\mathcal{P}_{\mathsf{e}}(a,b)$, transforms covariantly under U, and integrates to the generator H of* *translations,*

$$\int_{-\infty}^{\infty} d\xi \; T(\xi) = H, \tag{6.6}$$

where the integral is understood in the sense of matrix elements between vectors from $\mathcal{D}_0$. *With central charge* $c = \frac{1}{2}$, *we have the Lüscher–Mack commutation relations,*

$$i[T(\xi), T(\xi')] = -\delta'(\xi - \xi')(T(\xi) + T(\xi')) + \frac{c}{24\pi}\delta'''(\xi - \xi'). \tag{6.7}$$

Proof. In view of the anticommutation relation (6.4), we also have $\{\psi(\xi), \partial_{\xi'}\psi(\xi')\} = -\delta'(\xi - \xi') \cdot 1$ and $\{\partial_\xi\psi(\xi), \partial_{\xi'}\psi(\xi')\} = -\delta''(\xi - \xi') \cdot 1$, which implies that T is a point-local field. Relative locality to $\mathcal{P}_{\mathsf{e}}$ follows from (6.4) as well. The covariance of T under translations and dilations is clear from its definition; note that T has scaling dimension two.

To establish (6.6), we write down the normal ordered product (6.5) in terms of creation and annihilation operators,

$$\begin{aligned} T(\xi) = -\frac{1}{4\pi}\int d\beta \int d\gamma \; e^{(\beta+3\gamma)/2}(ie^{i(e^\beta+e^\gamma)\xi}y^\dagger(\beta)y^\dagger(\gamma) + ie^{-i(e^\beta+e^\gamma)\xi}y(\beta)y(\gamma) \\ - e^{i(e^\beta-e^\gamma)\xi}y^\dagger(\beta)y(\gamma) - e^{-i(e^\beta-e^\gamma)\xi}y^\dagger(\gamma)y(\beta)). \end{aligned} \tag{6.8}$$

(We read this in the sense of sesquilinear forms on $\mathcal{D}_0 \times \mathcal{D}_0$.) The first two terms, containing two creators and annihilators, respectively, vanish after integration over ξ, because they involve exponentials of $\pm i(e^\beta + e^\gamma)\xi$ and the factor $(e^\beta + e^\gamma)$ is strictly positive. Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} d\xi \; T(\xi) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} d\xi \int d\beta \; d\gamma \; e^{(\beta+3\gamma)/2} \\ &\quad \times (e^{i(e^\beta-e^\gamma)\xi}y^\dagger(\beta)y(\gamma) + e^{-i(e^\beta-e^\gamma)\xi}y^\dagger(\gamma)y(\beta)) \\ &= \int d\beta \; e^\beta y^\dagger(\beta)y(\beta) = H. \end{aligned}$$

In summary, T is a local, translation and dilation covariant field of scaling dimension two that integrates up to H and is relatively local to the net $\mathcal{P}_{\mathsf{e}}$, which acts irreducibly on $\mathcal{H}_{\mathsf{e}}$ (Proposition 6.1). Hence the hypotheses of the Lüscher–Mack theorem [44] are fulfilled (see [31, Theorem 3.1]), and the commutation relation (6.7) follows. The value of the central charge c can then be computed from the vacuum two-point function.

$$\begin{aligned} \langle\Omega, T(\xi)T(\xi')\Omega\rangle &= -\frac{1}{16\pi^2}\int d\beta_1 \int d\beta_2 \int d\gamma_1 \int d\gamma_2 \; e^{(\beta_1+\beta_2+3\gamma_1+3\gamma_2)/2} \\ &\quad \times \exp(-i(e^{\beta_1} + e^{\gamma_1})\xi + i(e^{\beta_2} + e^{\gamma_2})\xi') \\ &\quad \times \langle\Omega, y(\beta_1)y(\gamma_1)y^\dagger(\beta_2)y^\dagger(\gamma_2)\Omega\rangle \\ &= \frac{1}{2} \cdot \frac{1}{24\pi} \cdot \frac{1}{2\pi}\int_0^\infty dk \; k^3 e^{-ik(\xi-\xi')}. \end{aligned}$$

Taking the antisymmetric part of this distribution and comparing it with (6.7), we read off $c = \frac{1}{2}$. □

This identifies the field T as the energy density of a real free chiral Fermi field. Expanding T into its Fourier modes L_n, $n \in \mathbb{Z}$, we therefore have a representation of the Virasoro algebra with central charge $\frac{1}{2}$ in our chiral net $\mathcal{A}$, and a corresponding subnet $I \mapsto \mathrm{Vir}_{1/2}(I)$ on $\mathcal{H}_{\mathsf{e}}$. This net transforms covariantly under a unitary representation of the Möbius group, with the generator K of the special conformal transformations given by $K = \int d\xi \xi^2 T(\xi)$ [31].

The local algebras are now completely fixed by the following theorem.

Theorem 6.3. *In the chiral model with scattering function* $S = -1$, *the net of local von Neumann algebras is the Virasoro net with central charge* $\frac{1}{2}$, *i.e.* $\mathcal{A}(I) = \mathrm{Vir}_{1/2}(I)$ *for any interval* $I \subset \mathbb{R}$.

Proof. Both nets, $\mathcal{A}$ and $\mathrm{Vir}_{1/2}$, can be restricted to the even subspace $\mathcal{H}_{\mathsf{loc}} = \mathcal{H}_{\mathsf{e}} \subset \mathcal{H}$, and both have the vacuum as a cyclic vector on this space — see Proposition 6.1 regarding cyclicity for $\mathcal{P}_{\mathsf{e}}(I) \subset \mathrm{Vir}_{1/2}(I)$. For every interval I, we have $\mathrm{Vir}_{1/2}(I) \subset \mathcal{A}(I)$ by construction, and the same then follows for any subset $I \subset \mathbb{R}$, cf. (4.32). But the Virasoro net on the real line is Haag-dual [36]. Hence

$$\mathrm{Vir}_{1/2}(I) \subset \mathcal{A}(I) \subset \mathcal{A}(I')' \subset \mathrm{Vir}_{1/2}(I')' = \mathrm{Vir}_{1/2}(I),$$

which implies $\mathrm{Vir}_{1/2}(I) = \mathcal{A}(I)$. □

7. Conclusions

In this paper, we have investigated the short-distance scaling limit of $(1+1)$-dimensional models of quantum field theory with a factorizing scattering matrix, for a certain class of two-particle scattering functions S. At finite scale, these models are generated by wedge-local field operators depending on S in an explicit manner. Proceeding to scale zero, we showed that this feature is also maintained in the limit, and investigated the limit theories in terms of their generators.

As might heuristically be expected, the limit turned out to be a massless, dilation covariant theory which extends (trivially, if $S(\infty) = 1$) a chiral theory. We were able to establish this fact on the level of local von Neumann algebras: The observable algebras $\mathcal{A}(O)$ associated with double cones contain the tensor products of local interval algebras $\mathcal{A}(I)$ of the chiral components. For algebras associated with unbounded regions (wedges and half-lines), one obtains a tensor product as well, but with a grading in the case $S(\infty) = -1$.

We then investigated in more detail the individual chiral components of the limit theory, which are of interest in their own right. They are translation-dilation-reflection covariant models on the real line; and while they are massless, they are formally very similar to the massive two-dimensional models, viewed in rapidity space. These theories can be defined on the level of half-line algebras or, by considering intersections, on the level of interval algebras.

Our particular approach to the scaling limit via the wedge-local fields has the merit that the computation of the limit is rather easy on the level of the generators, but this comes at the price of an indirect characterization of the local fields and observables of the limit theory. In particular, the nontriviality of the interval algebras $\mathcal{A}(I)$ is not guaranteed by our construction. The analysis presented here is thus complementary to other approaches to the short-distance behavior of the models considered, and it is interesting to compare the different procedures.

In case the point-local quantum fields contained in the models at finite scale are sufficiently explicit, one might base the scaling limit analysis on these quantities. However, as the S-matrix is taken here as the main input into the construction, for most of the models no Lagrangian formulation or local fields are known. Moreover, even if point-local fields can be constructed, for example by Euclidean perturbation theory, their relation to the real-time S-matrix is very indirect. One can therefore expect a rigorous analysis of the connection between the collision operator on the one hand and the short-distance limit on the other hand to be quite involved with this method. For example, in the Ising model explicit formulas for local fields are available, but have a rather complicated form [46]. By comparison, the S-matrix and wedge-local generators of this model are extremely simple. As we have shown, it is possible to circumvent the construction of the local fields at finite scale, and still completely analyze the corresponding scaling limit theory.

Because observables localized in bounded space-time regions are only characterized indirectly in our approach, a detailed comparison with techniques based on local observables is difficult. One can expect however that the limit of double-cone-local objects would possibly yield *less* (but in no case more) limit points than those obtained when working with wedge-local objects, in some analogy to the scaling limit of charge sectors [25, 24]. In this sense, the limit theory that we compute is maximally large.

Another approach to the scaling limit is that of Buchholz and Verch [22]. Here one defines the limit in terms of bounded local operators, and in this sense of more general objects, since unitaries $\exp i\phi(f)$ and their weak limit points would be included. This might yield a larger limit theory than ours, and indeed, one expects [16] a large center to occur in the limit algebras. Due to technical difficulties in fully describing this central part of the algebras, we did not yet proceed in this direction. These problems are present even in the free field case, and their complete clarification will probably require a modification of the Buchholz–Verch framework. We hope to return to this point elsewhere. It is not excluded that such a more general approach would yield additional "quantum" observables as well, not only "classical" observables in the center of the algebras. Nevertheless, let us remark that the wedge algebras $\mathcal{M}_0, \widehat{\mathcal{M}}_0$ that we constructed in the limit theory are Haag-dual, and to this degree maximal; any additional local observables could only be accommodated on an extended Hilbert space.

In the approach chosen here, a central question turned out to be whether the chiral models constituting the scaling limit extend to conformal quantum field theories on the circle, covariant under the Möbius group. We showed that there is indeed always such an extension, namely on the subspace $\mathcal{H}_{\mathsf{loc}} \subset \mathcal{H}$ generated from the vacuum by the local algebras. In this sense, a conformal extension exists if and only if the local algebras are large. As a general feature, we showed that in the case $S(\infty) = -1$, the local subspace $\mathcal{H}_{\mathsf{loc}}$ contains only even particle number vectors, and all local operators must be even with respect to the particle number as well.

This effect is illustrated by the models with the constant scattering functions $S = \pm 1$. In these two cases, we explicitly computed the local algebras of the chiral components. For the free field ($S = 1$), one obtains the minimal model with conformal charge $c = 1$, and for the Ising model ($S = -1$), one obtains the minimal model with $c = \frac{1}{2}$. However, in the case of non-constant S, the exact size of the local algebras remains an open question. In fact, our present results do not rule out the possibility that they are trivial in the sense $\mathcal{H}_{\mathsf{loc}} = \mathbb{C}\Omega$.

Another related problem is to clarify the significance of the function S entering in the definition of our chiral models. At finite scale, S directly corresponds to the S-matrix, and its physical interpretation is clear [40]. From a more mathematical point of view, S is an invariant (under unitary equivalence) of the two-dimensional massive nets, and in particular, two models with different scattering function are never equivalent. By comparison, the significance of S is much less understood in the scaling limit, despite it formally being equal to a two-dimensional scattering function. For $S(\infty) = 1$, it seems clear that any formulation of scattering theory of massless two-dimensional models (cf. [15]) yields a trivial scattering matrix, due to the tensor product structure of the local algebras. In the terminology of [29], we deal with models with trivial left-right scattering, while our scattering function S determines the left-left and right-right scattering. However, on a single light ray or in a single chiral component, scattering theory in the usual sense cannot be formulated, and is not physically meaningful.

Furthermore, S is not known to be an invariant of the chiral models. Therefore it is possible that models with different scattering functions, inequivalent at finite scale, become equivalent in the scaling limit. Such an effect would actually be expected for asymptotically free theories, and is supported by results obtained in another approach to massless factorizing scattering: In [60], a model similar to ours — yet with a richer particle spectrum — is analyzed by means of a Thermodynamic Bethe Ansatz. Assuming for a moment that the results of [60] can be transferred to our situation by analogy, one is lead to the conjecture that our local net $I \mapsto \mathcal{A}(I)$ is actually unitarily equivalent to the minimal conformal model with $c = 1$ (for $S(\infty) = 1$) or $c = \frac{1}{2}$ (for $S(\infty) = -1$), irrespective of the details of the function S. This would mean that the interaction described by S vanishes in the scaling limit, and that the limit models are actually complicated reparametrizations of the free Bose or Fermi field. However, the technical arguments of [60] are largely based on thermodynamical considerations and do not directly apply in our context.

This situation would be compatible with our present results as well. A rigorous answer to the question which of the possible scenarios, ranging from trivial local algebras to asymptotically free theories, is realized for which scattering function, would deepen our understanding of the short distance structure of quantum field theory. Further results in this direction will be presented elsewhere.

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