About the second order moment of the Lagrangian velocity increments

A.S. Lanotte

CNR-ISAC and INFN, Sez. Lecce, Str. Prov. Lecce-Monteroni, 73100 Lecce, Italy

L. Biferale

Dept. Physics and INFN, University of Rome “Tor Vergata”, Italy

G. Boffetta

Dipartimento di Fisica and INFN, University of Torino, via P.Giuria 1, 10125 Torino, Italy

F. Toschi*

Department of Applied Physics and Department of Mathematics & Computer Science, Eindhoven University of Technology, Eindhoven, 5600MB, The Netherlands.
CNR-IAC, Via dei Taurini 19, 00185 Rome, Italy

E-mail: *f.toschi@tue.nl

Abstract. We study the behavior of the second-order Lagrangian structure functions on state-of-the-art numerical data in $3d$ and in $2d$. From a phenomenological connection between Eulerian space-fluctuations and the Lagrangian time-fluctuations, it is possible to rephrase the Kolmogorov 4/5-law into a relation predicting the linear (in time) scaling for the second order Lagrangian structure function. When such a function is directly observed on current experimental or numerical data, it does not clearly display a scaling regime. We introduce a parametrization of the Lagrangian structure functions based on Batchelor suggestion and test it on data. We show that such parametrization supports the idea that both Eulerian and Lagrangian data are consistent with the expected scaling plus finite-Reynolds effects affecting the small- and large-time scales. Furthermore, we are able to make quantitative predictions on the Reynolds number value for which Lagrangian structure functions are expected to display a scaling region.

1. Introduction

The knowledge of the statistical properties of turbulence, and in particular its non-Gaussian statistics, is a key open problem in classical physics with important consequences for applications [1]. The description of a fluid flow can be equally done in the Eulerian frame, where the velocity field at any position and time is known, $\boldsymbol{u}(\boldsymbol{x}(t), t)$, or in the Lagrangian frame where the evolution of fluid tracers, $\boldsymbol{x}(t)$, is followed in time, $\boldsymbol{v}(t) = \boldsymbol{u}(\boldsymbol{x}, t)$ and $\boldsymbol{v}(t) = \dot{\boldsymbol{x}}(t)$. Though the two descriptions are mathematically equivalent, the second bears premises to better shed light into the dynamics of (small) particles dispersed and transported by turbulent flows [2, 3]. One of very few exact results known for homogeneous and isotropic turbulence is the Kolmogorov 4/5-law for inertial range of scales:

$$S_3(r) = \left\langle (\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x}) \cdot \hat{\boldsymbol{r}})^3 \right\rangle = -\frac{4}{5}\,\varepsilon r. \tag{1}$$

This relation connects velocity differences at scale r with the presence of a non vanishing energy flux, ε, which remains constant at increasing the Reynolds number giving rise to the dissipative anomaly of turbulence [1]. The translation of Eqn. 1 to the Lagrangian domain has been suggested long time back [4, 5] but it only relies on phenomenological bases. It connects Eulerian fluctuations at distance r, $\delta_r u = u(x + r) - u(x)$, with Lagrangian temporal velocity difference over a time interval τ, $\delta_\tau v = v(t + \tau) - v(t)$, where space and time are connected through the local eddy turnover time:

$$\delta_\tau v \sim \delta_r u, \qquad \tau \sim r/\delta_r u\,. \tag{2}$$

Here due to the *dimensional* and phenomenological nature of the relation, all geometrical and vectorial properties are neglected. Moreover, it is important to stress that the symbol $\sim$ in (2) is meant as *scale-as* in pure *statistical* sense and not as a deterministic constraint holding point-by-point, as sometimes interpreted [6]. It results that the phenomenological equivalent of the exact law (1) in the Lagrangian domain reads:

$$S_2(\tau) = \langle (\delta_\tau v)^2 \rangle \sim \varepsilon \tau, \tag{3}$$

where the prefactor cannot be exactly controlled. Another important difference with respect to (1) is that the sign of the right hand side is also fixed, implying that (3) cannot be exact, being the energy flux differently sign-defined in $2d$ and in $3d$ turbulence.

This relation is intimately connected with the picture of the Richardson cascade, built in terms of a superposition of eddies at different scales and with different characteristic times (eddy turn over times). The idea is to imagine that Lagrangian fluctuations, $\delta_\tau v$, at a given time-scale, τ, are dominated by those Eulerian eddies, $\delta_r u$, which have a typical decorrelation time of the order of the time lag, τ. Indeed eddies at smaller scales are much less intense, i.e. if $r' \ll r$, then $\delta_{r'} u \ll \delta_r u$, while eddies at larger scales do not contribute to Lagrangian fluctuations being almost frozen on the time lag τ. The *bridge* relation (2) must be considered the zero-*th* order approximation

connecting Lagrangian and Eulerian domains. It cannot be exact and it cannot be applied straightforwardly to all hydrodynamical systems, being strongly based on the hypotheses of locality of the energy transfer process and on the existence of a unique typical eddy turn over time at each scale. Therefore it is not expected that it can explain Lagrangian-Eulerian correlations in conducting flows, as investigated in [7].
Concerning its application in pure hydrodynamical systems in $2d$ and $3d$, the situation is yet ambiguous. On one hand, the bridge relation has been successfully used to predict the probability density function of accelerations and of relative scaling between Lagrangian structure functions [8–10]. On the other hand, when looking at direct scaling *versus* the time lag, unclear results have been obtained. As a consequence, doubts on the validity of such an approach have been raised due the consistently poorer quality of the validation from both numerical and experimental tests [11–13], when compared to the Eulerian counterpart. In this manuscript we specifically address this issue basing our analysis on currently available data. We show that a simple modeling of finite Reynolds effect, affecting the small and the large scales, can be enough to interpret present data, keeping the *bridge* relation valid. This suggestion points in the direction of an enhanced sensitivity to finite-Re corrections in the Lagrangian framework with respect to the Eulerian one.

2. Batchelor parametrization for the Lagrangian second order structure function

We consider the second order moment of velocity increments measured along tracers trajectories in statistically stationary, isotropic and homogeneous (HIT) $3d$ dimensional turbulence:

$$S_2(\tau) \equiv \langle [v_i(t+\tau) - v_i(t)]^2 \rangle \,, \tag{4}$$

where $v_i(t)$ is one component of the turbulent Lagrangian velocity field. As mentioned, the Kolmogorov scaling for the Eulerian velocity increments once translated into the time domain via the *bridge* relation gives -for any velocity component- the linear prediction $S_2(\tau) = C_0\,\varepsilon\tau$ where C_0 is a constant of order unity. Observations suggest that in $3d$ HIT, $C_0 \in [6-7]$ [14]: since even at the largest Reynolds number achieved, both experimental and numerical data do not show a well developed scaling range in $S_2(\tau)$, the value of C_0 measured still displays a weak Re dependence [11–16].

The crucial point that we address is to understand if this poor scaling reflects a real deviation from the linear scaling of Lagrangian turbulence, or if it is just the result of finite Reynolds numbers effects, coming from both ultraviolet (UV) and infrared (IR) cutoffs. In the latter case, one could expect that future DNS and experiments might be able to directly display scaling properties also in the Lagrangian domain, including intermittency. In fact, at the moment intermittency in the Lagrangian domain is studied only using ESS approach [9, 10], hence bypassing the need for well defined power-law behavior in the inertial range.

In order to understand the above issue, it is mandatory to have a control on the effects of viscous and integral scales on the *alleged* inertial range. Due to the lack of control on the analytical side, one possible way is to resort to *phenomenological* models [9, 17, 18], trying to reproduce the behavior of the velocity increments over the entire range of scale/frequency. In particular, over the last decades, a parametrization proposed by Batchelor became quite popular because of its simplicity and capability to include non trivial viscous-effects (such as the intermittency of velocity gradients and acceleration) [9,18], as well as saturation effects happening at the large scales [10,19,20].

In the following, we test the possibility to get a suitable *Batchelor*-like parametrisation able to capture the poor scaling behavior observed on the data. The anticipated success of this goal implies two facts. First, it shows that the absence of a genuine scaling observed at moderate Reynolds *is not in contradiction* with the possibility to have scaling at higher Reynolds. Second, it gives a first hint on how far in Reynolds one needs to go before expecting an observable scaling behavior. Of course the Batchelor parameterization is not based on any analytical result and finds a justification only on its ability to reproduce data. Other parameterizations are very much possible as well, and whether the Batchelor one will agree or not with data at higher Reynolds numbers is an open question for the future.

On a dimensional ground, a parameterization for the time behavior of $S_2(\tau)$ has to reproduce the three following regimes:

$$\begin{cases} S_2(\tau) \sim \tau^2 & \tau \ll \tau_\eta \, , \\ S_2(\tau) \sim \tau^{z_2} & \tau_\eta \ll \tau \ll T_L \, , \\ S_2(\tau) \sim const. & \tau \gg T_L \, , \end{cases} \tag{5}$$

where τ_η is the Kolmogorov time scale and T_L is the large scale Lagrangian eddy turnover time. If we assume a Kolomogorov scaling in the temporal inertial range then $z_2 = 1$, otherwise it can be kept as a free parameter (see also Sec. 3). We recall that by dimensional arguments we have $T_L/\tau_\eta \propto Re_\lambda$. A saturation behaviour for times larger than the eddy turnover one is simply obtained as [18–20]:

$$S_2(\tau) = C_0 \, \frac{\tau^2}{(c_1 \, \tau_\eta^2 + \tau^2)^{\frac{(2-z_2)}{2}}} \, (1 + c_3 \tau / T_L)^{-z_2} \, , \tag{6}$$

where c_1 and c_3 are order one dimensionless constants.
In figure 1, we show the results when we take $T_L/\tau_\eta = 0.1 Re_\lambda$ [14]. It turns out that the effect of finite Reynolds number induced by the large scale saturation are big, since a plateau develops only for very large Reynolds numbers currently unreachable. One can of course play with the parametrization in order to modify the transitions from viscous to inertial, and from inertial to integral ranges. In particular, by changing the functional form of the denominator in eqn. (6) and of the saturation factor, these transitions can be made sharper or smoother [9]. It is thus probable that the observed absence of a clear and well developed plateau in numerical and experimental data is just

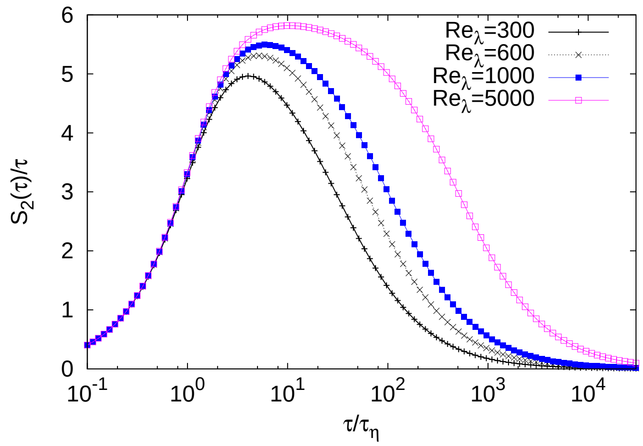

Figure 1. The compensated second order Lagrangian structure function as obtained with the Batchelor parametrization (6), for different Re_λ. The inertial range scaling exponent is fixed to $z_2 = 1$.

a finite Reynolds effect, that in the Lagrangian statisitcs is more pronounced than in the Eulerian case (remind that $L/\eta \propto Re_\lambda^{3/2}$ and $T_L/\tau_\eta \propto Re^{1/2}$). We also note that in order to be consistent with an exponential decay for the velocity correlation function, one can possibly slightly refine the functional form of the saturation factor for large times.

In $2d$ inverse energy cascade regime, both the UV and the IR saturation properties have to be changed to take into account forcing effects acting at small scales, and different viscous dumping at large scales.

In Fig. (2), it is presented an analysis of DNS data of $3d$ HIT at $Re_\lambda = 180, 280, 400, 600$ (see [10, 21, 22]). In particular, we compare the linearly compensated second order Lagrangian structure functions at the four Reynolds numbers (left panel), with curves obtained according to eqs. (6). As one can see the fit is very good. Moreover, in the same figure we also show, to guide the eyes, the result of the Batchelor parametrization for a much higher Reynolds ($Re_\lambda = 5000$), where finally a scaling, i.e. a plateau in the compensated plot, shows up.

It is well known that time correlations along tracers trajectories decay very slowly. Hence, when considering the second order Lagrangian structure function, there is the issue of the long time decaying of the velocity correlation functions. Here we compare the power-law saturation factor $\propto (1 + \tau/T_L)^{-z_2}$ appearing in eqn. (6), with an exponential saturation factor ruling the large times behavior. We used the following interpolation :

$$S_2^*(\tau) = C_0\, T_L\, \frac{\tau}{(c_1\tau_\eta^2 + \tau^2)^{1/2}}(1 - \exp(-c_3\tau/T_L))\,, \qquad (7)$$

where in comparison to expression (6), we have fixed the exponent $z_2 = 1$ and C_0, c_1 and c_3 are free parameters. In the right panel of Fig. (2), we compare the results of the two different functional forms for large time scales. In order to do it properly, we plot

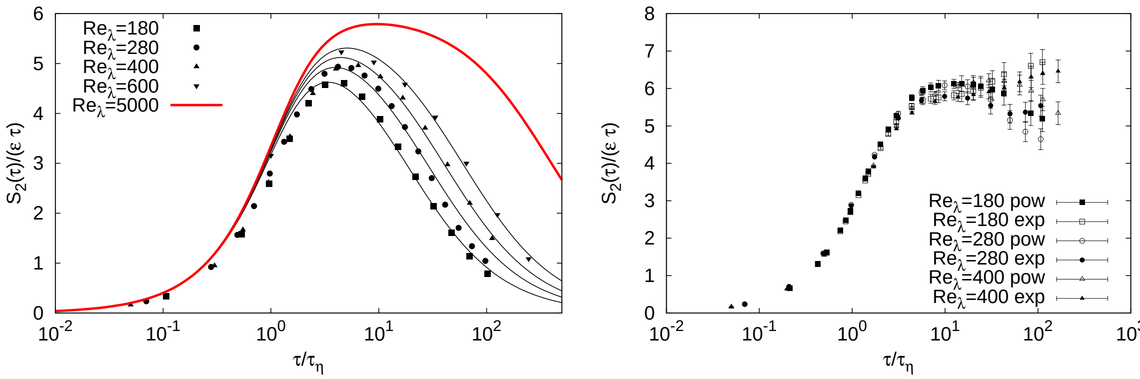

Figure 2. (Left panel) DNS of $3d$ HIT at $Re_\lambda \sim 180, 280, 400, 600$ [10, 21, 22]. $S_2(\tau)$ compensated with $\varepsilon\tau$ versus the Batchelor fit (solid lines) with a power-law large-scale saturation term. (Right panel) Same DNS data of HIT at three different Reynolds numbers, compensated such as to highlight inertial range behavior according to the two Batchelor parametrizations (with large times exponential or power-law behaviour). Fitting parameters are: $c_1 = 2.2$ in the power-law and $c_1 = 2.5$ in the exponential form; $c_3 = 1.0$ in the power-law expression, while $c_3 = 1.5$ in the exponential expression. Error bars are estimated from the anisotropy of velocity components at large scales.

the second order structure function compensated with its whole inertial and integral time scale regime:

$$S_2(\tau)/(\tau/(1 + c_3\tau/T_L)^{-z_2}); \qquad S_2^*(\tau)/(\tau/(1 - \exp(-c_3\tau/T_L)))\,. \tag{8}$$

panel, the exponential fit looks more promising at large scales, even though the power-law and exponential forms are very close. Here clearly we should consider that at large scales we expect to have quite *large* statistical and systematic error bars, due to anisotropy and/or finite size effects (see error bars in the left panel of Fig. 2). It is interesting to note, that once the large scale contamination is removed, compensated data start to show a well-defined plateau already at moderate Reynolds numbers, independently of the fucntional form for the large scale behavior.

Hence, at least for $3d$ turbulence, we summarize these indications as follows: (i) the absence of a plateau is probably related to the presnece of strong large-scale and small-scale effects, competing with the inertial range behavior; (ii) as it appears left and right panels of Fig. 2, the Lagrangian inertial range does not coincide with the scaling (or plateau) region, where the second order structure function linearly compensated shows a peak, since the large-scale contamination is still present.

3. Intermittency corrections

It is well known that Lagrangian statistics in $3d$ enjoys intermittent corrections. In particular, acceleration statistics does not obey dimensional scaling: in particular, the normalized acceleration rms, $\langle a^2\rangle\tau_\eta/\varepsilon$ is observed to have a *weak* anomalous dependency on Re_λ:

$$a_{rms}^2 = \langle a^2 \rangle \sim a_0 \frac{\varepsilon}{\tau_\eta} (Re_\lambda)^\gamma \,, \tag{9}$$

with $\gamma \sim 0.2$ (see also Fig. 3). Similarly, the probability density function of the normalized acceleration $P(a/a_{rms})$ possesses strong non-Gaussian and Reynolds-dependent tails [8]. Such intermittent corrections can be explained by invoking again the *bridge* relation previously discussed. section: so doing, it is possible to predict Lagrangian intermittent properties once the Eulerian ones are given, and viceversa [5, 8–10, 21]. In Fig. (3) we report the compilation of data sets at different Reynolds numbers for (9). On these data three curves are superposed: (i) a phenomenological fit proposed in [23] and the prediction obtained by using the bridge relation (2) and two different Multifractal estimates of the Eulerian statistics, based on the longitudinal and on the transverse spatial increments [10]. The numerical data fall well within the two multifractal predictions, confirming the ability of the bridge relation to reproduce Lagrangian properties *without* any additional free parameter. We notice that even considering intermittency, the bridge relation still predicts that (3) holds true, i.e. intermittency is absent for third-order quantities in the Eulerian domain and for second order quantities in the Lagrangian one.

An alternative approach to the multifractal Eulerian/Lagrangian bridge relation can be followed by assuming independent anomalous scaling properties for the two domains. In this case, $S_2(\tau)$ is not constrained to scale linearly and one could assume a pure inertial range intermittent correction:

$$S_2(\tau) \sim \varepsilon\tau (\frac{\tau}{T_L})^{-\gamma} \,, \tag{10}$$

which is not in contradiction with any exact scaling law in Lagrangian turbulence. In [11] the intermittent correction was obtained from a fit of the scaling (9).

In Fig. (4) we apply the intermittent compensation $\tau^{1-\gamma}$ to the DNS data shown in previous sections and observe that the plateau is slightly increased, but it is still very narrow. *Finite Reynolds* effects are overwhelming and the question whether $S_2(\tau)$ scales linearly, or with an intermittent correction or does not scale at all, is still open. More data at higher Reynolds are needed to solve this important puzzle.

4. Inverse cascade in 2d turbulence

In this section we present our results on Lagrangian structure functions for the inverse cascade in $2d$ turbulence. Again, the general question we want to address is whether Lagrangian statistics is compatible with Eulerian statistics, i.e. if a suitable transformation from space to time is able to reproduce Lagrangian statistic given the Eulerian statistics. We remind that, in spite of the fact that the inverse cascade is statistically "simpler" than the direct cascade in $3d$ (since the Eulerian statistics displays Kolmogorov scaling without intermittency corrections [26]), recent works [25] claim that

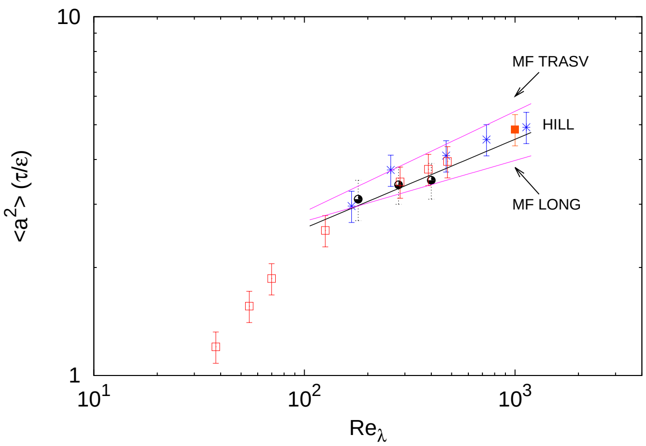

Figure 3. Collection of different numerical data of the scaling of normalized root mean square acceleration as a function of Re_λ. Two lines correspond to the Multifractal prediction using the bridge relation for transverse increments (MF TRASV) leading to $\gamma = 0.17$, or the bridge relation for longitudinal increments (MF LONG) leading to $\gamma = 0.28$ (see [9] for details). These lines can be shifted up or down arbitrarily, being the multifractal prediction valid scaling-wise and not for the prefactors. A third line is a fit proposed by R. Hill in [23], as a superposition of two power laws with $\gamma = 0.25$ and $\gamma = 0.11$. Data are taken from Refs. [10, 21, 22, 28–30]. Error bars are estimated considering a typical 10% uncertainty in the energy dissipation rate.

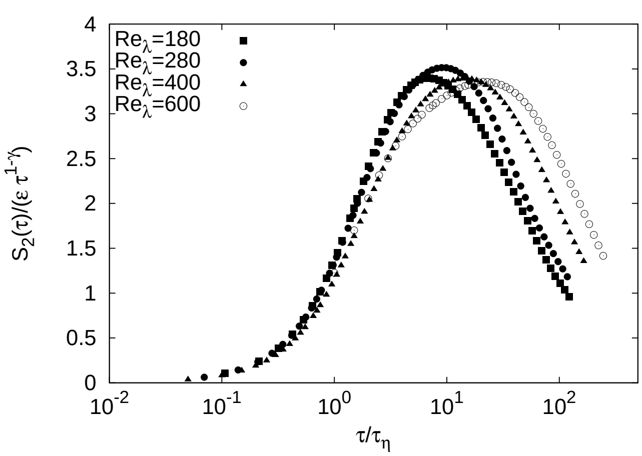

Figure 4. The second order Lagrangian structure function compensated as $S_2(\tau)/\tau^{1-\gamma}$, with $\gamma = 0.22$. The anoumalous correction γ is extracted from acceleration data shown in Fig. (3).

Lagrangian statistics does not reflect this simplicity and cannot be related to Eulerian statistics.

In the following we will consider Eulerian and Lagrangian structure functions obtained from numerical simulations of $2d$ Navier-Stokes equations:

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega - \alpha \omega + f_\omega \,, \tag{11}$$

in the inverse cascade regime at resolutions 2048^2. The forcing f_ω, is active on a range of wavenumbers around $k_f \simeq 256$, is δ-correlated in time and injects energy at a fixed rate ε_I. About one half of the injected energy flows to large scale generating the inverse cascade with a flux ε. The $\alpha\omega$ friction term is necessary to reach a stationary state, and defines the large-scale eddy turnover time $T_L \simeq 1/\alpha$. Different runs correspond to different values of α and therefore to different extension of the inertial range of scales. The smallest characteristic time, the *Kolmogorov time* τ_η is given in the inverse cascade by the time at the forcing scale. The extension of the inertial range in the time domain is thus $T_L/\tau_\eta \propto 1/\alpha$.

In order to compare Eulerian and Lagrangian structure functions, we make use of a simple model, motivated by the cascade model for turbulence. We represent turbulent Eulerian velocity fluctuations $\delta u(r)$ as the superposition of the contributions from different eddies in the cascade [24] : $\delta u(r) = \sum_n u_n f(r/r_n)$, where u_n is the typical fluctuation at the scale r_n. The decorrelation function $f(x)$ is such that $f(x) \sim x$ as $x \ll 1$ and $f(x) \sim 1$ for $x \gg 1$: here, we chose the simple function $f(x) = 1 - exp(-x)$. Within this framework, it is natural to represent the corresponding Lagrangian velocity fluctuation as $\delta v(\tau) = \sum_n v_n f(\tau/\tau_n)$ where $\tau_n \sim r_n^{2/3}$ is the correlation time of the eddies. A minimal realization of this model requires the presence of two scales which govern the crossover from dissipative to inertial scales, η, and from inertial to integral scales, L. We can therefore write, introducing explicitly the scaling behavior in the inertial range,

$$\delta u(r) = U f\left(\frac{r}{L}\right) + U\left[1 - f\left(\frac{r}{L}\right)\right] f\left(\frac{r}{\eta}\right)\left(\frac{r+\eta}{L}\right)^{1/3} \,, \tag{12}$$

which, for Lagrangian increments, translates into

$$\delta v(\tau) = U_L f\left(\frac{\tau}{T_L}\right) + U_L\left[1 - f\left(\frac{\tau}{T_L}\right)\right] f\left(\frac{\tau}{\tau_\eta}\right)\left(\frac{\tau+\tau_\eta}{T_L}\right)^{1/2} \,. \tag{13}$$

In the stationary state we observe an inverse cascade with a Kolmogorov spectrum which extends from the forcing wavenumber $k_f = 256$ to the friction wavenumber $k_\alpha \simeq \varepsilon^{-1/2}\alpha^{3/2}$ [26, 27], as it is shown in Fig. 5.

In Fig. 6 we show the Eulerian second-order structure functions $S_2(r) = \langle(\delta_r u)^2\rangle$, compensated with dimensional scaling $(\varepsilon r)^{2/3}$, for different values of α. An important remark is that, in spite of the clear power-law scaling in the spectra, we do not observe any inertial range scaling for the Eulerian structure functions, even for the most resolved

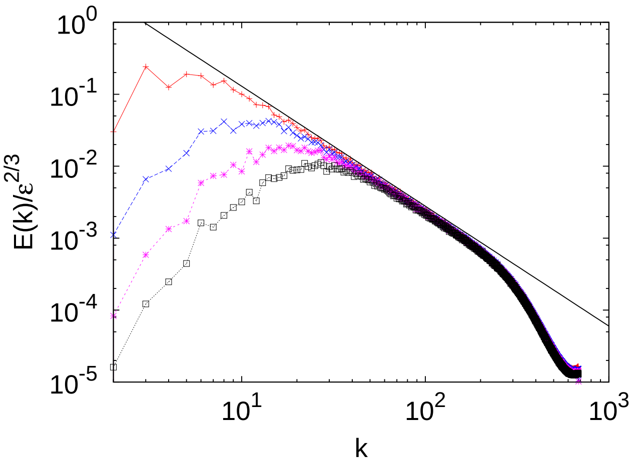

Figure 5. Kinetic energy spectra from direct numerical simulations at resolution 2048^2, with $\alpha = 0.02$ (red +), $\alpha = 0.04$ (blues $\times$), $\alpha = 0.06$ (pink $*$) and $\alpha = 0.08$ (black $\square$). The line represents Kolmogorov spectrum $E(k) = C\varepsilon^{2/3}k^{-5/3}$ with $C = 6$.

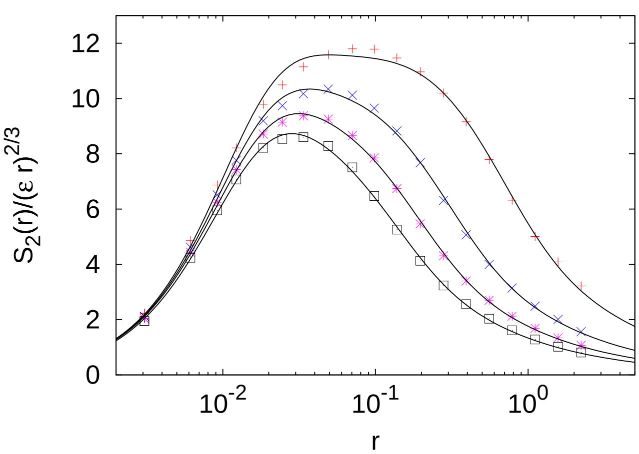

Figure 6. Eulerian second-order structure function $S_2(r)$ in the inverse energy cascade, compensated with Kolmogorov scaling $(\varepsilon r)^{2/3}$. Colors and symbols as in Fig. 5. Lines represent the fit with $(\delta u(r))^2$ given by eq. (12), which gives the ratio $L/\eta = 12$ ($\alpha = 0.08$), $L/\eta = 16$ ($\alpha = 0.06$), $L/\eta = 25$ ($\alpha = 0.04$) and $L/\eta = 54$ ($\alpha = 0.02$).

simulation. Nonetheless, the simple two-scale model (12) and (13) is able to reproduce quite accurately the crossovers from dissipative and to integral scales, with parameters $(\eta/L, \tau_\eta/T_L, U, U_L)$ which change according to dimensional predictions (cfr. caption of Figs. 6 and 7). Lagrangian structure functions $S_2(\tau)$ compensated with $\varepsilon\tau$ are shown in Fig. 7 together with the prediction obtained from model (13). The model fits the data quite well at least at small and intermediate times and with parameters which change with the extension of the inertial range. We remark that the model parametrization (as the Batchelor or any other parametrization) is not supposed to give a perfect fit of Eulerian or Lagrangian structure functions. The point we want to make here is that,

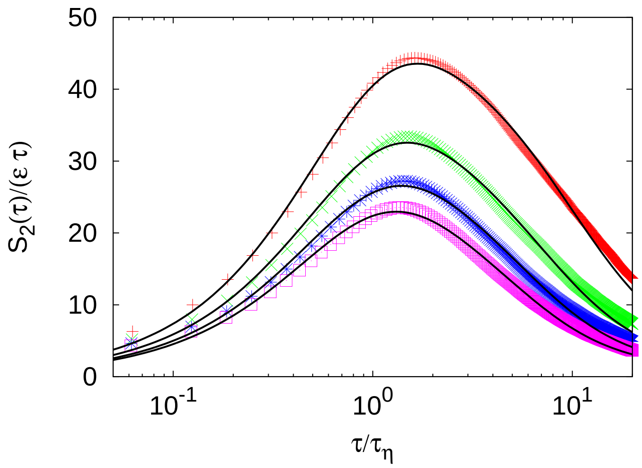

Figure 7. Second order Lagrangian structure function $S_2(\tau)$ in the inverse energy cascade, compensated with $\varepsilon\tau$. Colors and symbols as in Fig 5. Lines represent the fit with $(\delta v(\tau))^2$ given by eq. (13), which gives the ratio of times $T_L/\tau_\eta = 7.8$ ($\alpha = 0.08$), $T_L/\tau_\eta = 8.9$ ($\alpha = 0.06$), $T_L/\tau_\eta = 11.2$ ($\alpha = 0.04$) and $T_L/\tau_\eta = 16.1$ ($\alpha = 0.02$).

within the approximation model, the quality of data fit is comparable for Eulerian and Lagrangian statistics. More sophisticated models, e.g. based on the superposition of a hierarchy of characteristic scales and times would probably give a better fit of the numerical data.

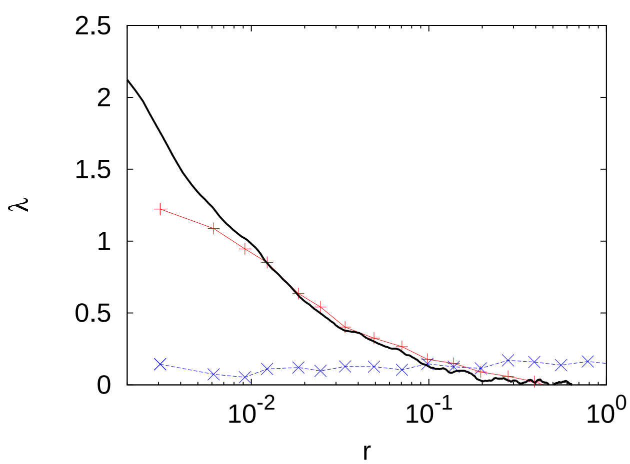

Figure 8. Excess kurtosis $\lambda(r) = S_4/S_2^2 - 3$ for Eulerian longitudinal structure function ($\times$ blue), for the Eulerian x-component structure function (+ red), and Lagrangian structure function (black line) with time rescaled as $r = 0.035\tau^{3/2}$. Data refer to the run with $\alpha = 0.02$.

One interesting result discussed in [25] is that Lagrangian statistics in two dimensions is not Gaussian even if Eulerian statistics is very close to Gaussian in the inverse cascade. Our simulations confirm this result but suggest that this is a delicate point as the statistics may depend on the observable. Figure 8 shows the excess kurtosis for Lagrangian structure function $\lambda(\tau) = S_4(\tau)/S_2(\tau)^2 - 3$ and for Eulerian structure function $\lambda(r) = S_4(r)/S_2(r)^2 - 3$, both are shown for x-component velocity increments and longitudinal velocity increments. The kurtosis of longitudinal velocity increments is constant and close to Gaussian value at all scales but this is not the case for Eulerian increments of a component of the velocity. We do not have a simple explanation for this observation but we think that this is a possible origin of the deviation from Gaussianity observed in Lagrangian statistics. Indeed, Fig. 8 also shows that the Lagrangian excess kurtosis $\lambda(\tau)$ is very close to $\lambda(r)$, when one replot time using the *bridge* relation $\tau = cr^{2/3}$. Of course this rescaling can work only in the inertial range and therefore we observe deviations at small r.

5. Conclusions

The Lagrangian and Eulerian description of the velocity field of a fluid are of course correlated and it must be possible to rephrase some statistical properties of Eulerian turbulence in terms of Lagrangian counterparts. A simple phenomenological description has been proposed to connect velocity fluctuations in time to velocity fluctuation in space, $\delta_r u \sim \delta_\tau v$, where the time-lag, τ, and space separation, r, are connected by the relation $\tau \sim r/\delta_r u$. From such a phenomenological connection one can obtain the prediction that $S_2(\tau) \sim \varepsilon\tau$, independently of the Eulerian intermittency, which is the Lagrangian rephrasing of the Kolmogorov 4/5-law.

Recently doubts were raised on the validity of the $S_2(\tau)$ scaling relation and more generally on the phenomenological Eulerian vs. Lagrangian mapping. Advocated reasons for questioning the validity of such a relation are: (i) the fact that such a relation, contrarily to the 4/5-law, is not rigorously derived; (ii) the fact that the scaling of the $S_2(\tau)$ appears to be of poorer quality than its Eulerian counterpart.

In the present manuscript we have addressed the issue of the consistency of present state-of-the-art numerical data with the linear dimensional scaling for the $S_2(\tau)$, both in $3d$ and $2d$ turbulence. More specifically we have tried to answer the question whether or not the present data are consistent with the linear scaling for the $S_2(\tau)$ plus finite Reynolds effects.

Eulerian and Lagrangian data, both $3d$ and $2d$, appear to agree equally well with a Batchelor-like parametrization, which takes into account dissipative and integral effects in a phenomenological way.

This indicates that present $3d$ and $2d$ Lagrangian data are to be considered consistent with the idea of the phenomenological Eulerian vs. Lagrangian connection once finite Reynolds number are kept into account. The use of the Batchelor parametrization allows to make prediction on the values of Reynolds number for which a given window of direct scaling is expected to appear.

In $3d$ turbulence, different scaling exponents for transverse and longitudinal spatial increments are observed [10, 31], something not fully understood. Along a Lagrangian trajectory, both longitudinal and transverse Eulerian fluctuations are naturally mixed and entangled, introducing some uncertainties in the *bridge* relation as discussed here. In the $2d$ inverse cascade regime, Eulerian longitudinal increments do not show any deviation from Gaussianity, while the excess kurtosis measured on a mixed longitudinal and transverse Eulerian increments is different from zero. The Lagrangian equivalent of the latter Eulerian measurements is also non Gaussian and in agreement with the *bridge* relation. Therefore, there are still many open points that must be further clarified.

We acknowledge useful discussions with E. Bodenchatz, M. Cencini, G. Falkovich, S. Musacchio, A. Pumir, P.K. Yeung and H. Xu. We thank P.K. Yeung for sharing with us some unpublished data at $Re_\lambda = 1000$ on the acceleration variance, shown in Fig. 3. We thank the Kavli Institute for Theoretical Physics four our stay in the framework of the 2011 *The Nature of turbulence* program. This research was supported in part by the National Science Foundation under Grant No. PHY05-51164. Also, this research was partly supported by the Project of Knowledge Innovation Program (PKIP) of Chinese Academy of Sciences, grant No. KJCX2.YW10 and by the Cost Action MP0806. We also acknowledge support from CINECA and Caspur.

References

[1] U. Frisch, *Turbulence: the legacy of A.N. Kolmogorov* (Cambridge University Press, 1995).

[2] F. Toschi and E. Bodenschatz, "Lagrangian Properties of Particles in Turbulence", *Ann. Rev. Fluid Mech* **41** 375 (2008).

[3] P.K. Yeung, "Lagrangian investigation of Turubulence", *Annu. Rev. Fluid Mech.* **34** 115 (2002).

[4] M.S. Borgas, "The Multifractal Lagrangian Nature of Turbulence", *Phil. Trans. R. Soc. London* A **342**, 379 (1993).

[5] G. Boffetta, F. De Lillo & S. Musacchio, "Lagrangian statistics and temporal intermittency in a shell model of turbulence", *Phys. Rev. E* **66**, 066307 (2002).

[6] O. Kamps, R. Friedrich & R. Grauer, "Exact relation between Eulerian and Lagrangian velocity increment statistics", *Phys. Rev. E* **79** 066301 (2009).

[7] H. Homann, O. Kamps, R. Friedrich & R. Grauer, "Bridging from Eulerian to Lagrangian statistics in 3D hydrodynamical and magnetohydrodynamical turbulent flows", *New J. Phys.* **11** 073020 (2009).

[8] L. Biferale, G. Boffetta, A. Celani, B. Devenish, A.S. Lanotte & F. Toschi, "Multifractal Statistics of Lagrangian velcoity and acceleration in turbulence", *Phys. Rev. Lett.* **93** 064502 (2004).

[9] A. Arneodo et al., "Universal Intermittent Properties of Particle Trajectories in Highly Turbulent Flows", *Phys. Rev. Lett.* **100**, 254504 (2008).

[10] R. Benzi, L. Biferale, R. Fisher, D.Q. Lamb & F. Toschi, "Inertial range Eulerian and Lagrangian statistics from numerical simulations of isotropic turbulence", *Journ. Fluid Mech.* **653**, 221 (2010).

[11] G. Falkovich, H. Xu, A. Pumir, E. Bodenshatz, L. Biferale, G. Boffetta, A.S. Lanotte & F. Toschi, "On Lagrangian single particle statistics", *Phys. Fluids* **24**, 055102 (2012).

[12] L. Biferale, E. Bodenschatz, M. Cencini, A.S. Lanotte, N.T. Ouellette, F. Toschi & H. Xu, " Lagrangian Structure Functions in Turbulence: A Quantitative Comparison between Experiment and Direct Numerical Simulation", *Phys Fluids* **20**, 065103 (2008).

[13] B.L.Sawford and P.K. Yeung "Kolmogorov similarity scaling for one-particle Lagrangian statistics", *Phys. Fluids* **23** 091704 (2011).

[14] P.K. Yeung, S.B. Pope & B. L. Sawford, “Reynolds number dependence of Lagrangian statistics in large numerical simulations of isotropic turbulence,” *J. Turb.* **7**, 58 (2006).

[15] N. Mordant, J. Delour, E. Léveque, O. Michel, A. Arnéodo & J.-F. Pinton, “Lagrangian Velocity Fluctuations in Fully Developed Turbulence: Scaling, Intermittency, and Dynamics”, *J. Stat. Phys.* **113**, 5/6 (2003).

[16] R-C. Lien & E.A. DAsaro, “The Kolmogorov constant for the Lagrangian velocity spectrum and structure function”, *Phys. Fluids*, **14**, 4456 (2001).

[17] B.L. Sawford,”Reynolds number effects in Lagrangian stochastic models of turbulent dispersion”, *Phys. Fluids A* **3** 1577 (1991).

[18] C. Meneveau,”Transition between viscous and inertial-range scaling of turbulence structure functions”, *Phys. Rev. E* **54**, 3657-3663 (1996).

[19] L. Biferale, F. Mantovani, M. Sbragaglia, A. Scagliarini, F. Toschi & R. Tripiccione, “High resolution numerical study of Rayleigh-Taylor turbulence using a thermal lattice Boltzmann scheme”, *Phys. Fluids*, **22**, 115112 (2010). arXiv:1009.5483.

[20] S. Kurien and K. R. Sreenivasan,”Anisotropic scaling contributions to high-order structure functions in high-Reynolds-number turbulence”, *Phys. Rev. E* **62**, 22062212 (2000).

[21] L. Biferale, G. Boffetta, A. Celani, A. Lanotte & F. Toschi, “Particle trapping in three-dimensional fully developed turbulence”, *Phys. Fluids* **17**, 021701 (2005).

[22] J. Bec, L. Biferale, A.S. Lanotte, A. Scagliarini & F. Toschi,”Turbulent pair dispersion of inertial particles” *Journ. Fluid Mech.* **645**, 497528 (2010).

[23] R.J. Hill, “Scaling of acceleration in locally isotropic turbulence”, *Jour. Fluid Mech.* **452** 361 (2002).

[24] L. Biferale, G.Boffetta, A. Celani, A. Crisanti & A. Vulpiani, “Mimicking a turbulent signal: sequential multiaffine processes”, *Phys Rev. E* **57** R6261 (1998)

[25] O. Kamps and R. Friedrich, “Lagrangian statistics in forced two-dimensional turbulence”, *Phys. Rev. E* **78**, 036321 (2008).

[26] G. Boffetta and R.E. Ecke, “Two-Dimensional Turbulence”, *Ann. Rev. Fluid Mech.* **44**, 427 (2012).

[27] G. Boffetta and S. Musacchio, “Evidence for the double cascade scenario in two-dimensional turbulence”, *Phys. Rev. E* **82** 016307 (2010).

[28] P. Vedula and P.K. Yeung “Similarity scaling of acceleration and pressure statistics in numerical simulations of isotropic turbulence”, *Phys. Fluids* **11** 1208 (1999).

[29] T. Gotoh and D. Fukayama “Pressure spectrum in homogeneous turbulence”, *Phys. Rev. Lett.* **86** 3775 (2001).

[30] T. Ishihara et al. “Small-scale statistics in high-resolution direct numerical simulation of turbulence: Reynolds number dependence of one-point velocity gradients”, *J. Fluid Mech.* **592** 335 (2007).

[31] T. Gotoh, D. Fukayama & T. Nakano, “Velocity field statistics in homogeneous steady turbulence obtained using a high-resolution direct numerical simulations”, *Phys. Fluids* **14**, 1065 (2002).