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A Variational Model for Linearly Elastic Micropolar Plate-Like Bodies

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We consider a micropolar, linearly elastic plate–like body, clamped on its boundary and subject to a system of distance loads. We characterize, by means of Γ–convergence, the limit behavior of the solutions of the equilibrium problem when the thickness of the body vanishes. We show that, for the special case of isotropic mechanical response, the equilibrium problem described by our Γ–limit coincides with a boundary–value problem obtained in a recent deduction of a theory for shearable plates from micropolar elasticity.

Keywords: Micropolar elasticity, plate theories, Γ-convergence

1. Introduction

In the Reissner–Mindlin plate theory a body may be regarded as a continuum collection of *fibers* occupying, in their reference configuration, a flat region of the three–dimensional Euclidean space. Its kinematics is described by a scalar field $\boldsymbol{w}$ (the *transverse displacement*) and a planar vector field $\boldsymbol{\varphi}$ (the *in–plane rotation*). This theory presents some analogies with three–dimensional micropolar elasticity, where bodies are collections of oriented material particles undergoing a *displacement* $\mathbf{u}$ and a *microrotation* $\boldsymbol{\omega}$. This fact motivates recent investigations on the connections between the two theories [1, 7].

In [1] Aganović *et al.* consider a family of equilibrium problems for three–dimensional micropolar bodies whose reference configurations are right cylinders $\Omega^{(h)} = \{(x_1, x_2, \zeta) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega_\perp, \zeta \in (-h, +h)\}$ having cross–section $\Omega_\perp \subset \mathbb{R}^2$ and thickness $2h > 0$. The bodies are all made of the same linearly elastic, isotropic material. Aganović *et al.* show that, as h goes to 0, the solutions $(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})$ of the equilibrium problems converge (with respect to a suitable topology) to the weak solution $(\mathbf{u}, \boldsymbol{\omega})$ of a *limit boundary–value problem.* They also prove that, if the following expansion holds:

$$(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) = (\mathbf{u}, \boldsymbol{\omega}) + h(\mathbf{u}^{(1)}, \boldsymbol{\omega}^{(1)}) + o(h),$$

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then the out–of–plane component of the displacement has the following representation:

$$\mathbf{u}^{(h)}(x_1, x_2, \zeta) = \mathbf{v}(x_1, x_2) + \zeta\boldsymbol{\varphi}(x_1, x_2) + s(x_1, x_2)\zeta\mathbf{x} + o(h), \quad \zeta \in (-h, +h).$$

Moreover they show that, setting $\boldsymbol{w} = \mathbf{v} \cdot \mathbf{z}$, the pair $(\boldsymbol{w}, \boldsymbol{\varphi})$ satisfies a system of partial differential equations similar (but not identical) to the equilibrium equations for a Reissner–Mindlin plate.

In this paper we use the notion of Γ-convergence to offer an alternative derivation of the limit boundary–value problem obtained in [1]. We characterize the solution of this problem as minimizer of a *bi-dimensional* functional[1], obtained as Γ-limit of a family of *three-dimensional* functionals. We provide an explicit formula for the Γ-limit and, differently from Aganović *et al.*, we do not restrict our attention to isotropic materials.

The organization of the paper is the following. We summarize notation in Section 2. In Section 3 we recall some basic tools of the theory of Micropolar Elasticity and we clarify the variational setting for the equilibrium problems of micropolar plate–like bodies. The main result, concerning the asymptotic behavior of minimizers, is stated in Section 4. The proof of the main result is developed in Section 6, after some quick recalls from Γ–convergence in Section 5. In Section 7 we show that for isotropic materials our result agrees with that obtained in [1].

2. Notation

We identify with $\mathbb{R}^3$ the ambient space and we denote by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{z}$ the standard orthonormal basis of $\mathbb{R}^3$. Let $\Omega_\perp$ be a bounded open subset of $\mathbb{R}^2$ and let us denote by $\Gamma_\perp$ its boundary, which is supposed to be Lipschitz. We consider plate–like domains $\Omega^{(h)} \subset \mathbb{R}^3$ having the following form:

$$\Omega^{(h)} = \Omega_\perp \times (-h, +h), \quad h > 0. \tag{1}$$

The *lateral mantle* of $\Omega^{(h)}$ is:

$$\Gamma^{(h)} = \Gamma_\perp \times (-h, +h).$$

Moreover we set:

$$\Omega := \Omega^{(1)}, \quad \Gamma := \Gamma^{(1)} .$$

If $x \equiv (x_1,\ x_2)$ belongs to $\Omega_\perp$, we denote by $(x,\ \tilde{\zeta})$ and (x, ζ) the elements of $\Omega^{(h)}$ and Ω respectively; in particular we regard $\tilde{\zeta}$ as the result of the scaling

$$\zeta = \tilde{\zeta}/h \ , \tag{2}$$

which maps $\Omega^{(h)}$ one–to–one onto Ω.

We denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{a} \cdot \mathbf{b}$ is the Euclidean scalar product of $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{a} \otimes \mathbf{b} \in \mathbb{R}^{3\times3}$ is the matrix whose entries are $(a \otimes b)_{ij} = a_i b_j$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $\mathbf{a} \times \mathbf{b}$ denotes the vector product of $\mathbf{a}$ and $\mathbf{b}$. The spaces of symmetric 3×3 matrices will be denoted by "*Sym*".

[1]With a little abuse of terminology we call "bi-dimensional" a functional defined on fields depending on x_1 and x_2.

Given a matrix $\mathbf{T} \in \mathbb{R}^{3\times 3}$, we denote by $\mathbf{T}_{\perp}$ the $\mathbb{R}^{3\times 2}$ matrix obtained from $\mathbf{T}$ by removing the third column:

$$\mathbf{T}_{\perp} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ T_{31} & T_{32} \end{pmatrix} .$$

With a slight abuse of notation, we also set

$$\mathbf{T}_{\perp} + \mathbf{a} \otimes \mathbf{z} = \begin{pmatrix} T_{11} & T_{12} & a_1 \\ T_{21} & T_{22} & a_2 \\ T_{31} & T_{32} & a_3 \end{pmatrix} . \tag{3}$$

Likewise, if $\mathbf{v}$ is a vector field defined on $\Omega_{\perp}$ or $\Omega^{(h)}$, we define the *surface gradient* of $\mathbf{v}$ as:

$$\nabla_{\perp} \mathbf{v} = \begin{pmatrix} \partial_{x_1} v_1 & \partial_{x_2} v_1 \\ \partial_{x_1} v_2 & \partial_{x_2} v_2 \\ \partial_{x_1} v_3 & \partial_{x_2} v_3 \end{pmatrix} .$$

If $\mathbf{v}$ is a vector field defined on $\Omega^{(h)}$, we define the *scaled gradient* of $\mathbf{v}$ as:

$$\nabla^{(h)} \mathbf{v} = \nabla_{\perp} \mathbf{v} + \frac{1}{h} \partial_{\zeta} \mathbf{v} \otimes \mathbf{z} . \tag{4}$$

Given $\boldsymbol{\omega} \in \mathbb{R}^3$, we denote by $\mathbf{A}^{\omega}$ the unique skew–symmetric matrix such that $\mathbf{A}^{\omega}\, \mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$. Note that

$$|\mathbf{A}^{\omega}| = 2|\boldsymbol{\omega}|. \tag{5}$$

If $\mathcal{U}$ is an open subset of $\mathbb{R}^3$, we denote by $L^2(\mathcal{U};\mathbb{R}^3)$ and $H^1(\mathcal{U};\mathbb{R}^3)$ the usual Hilbert (respectively Lebesgue and Sobolev) spaces of functions defined on $\mathcal{U}$ and taking values on $\mathbb{R}^3$, and we set:

$$\mathbf{L}(\mathcal{U}) := L^2(\mathcal{U};\mathbb{R}^3) \times L^2(\mathcal{U};\mathbb{R}^3) \quad \text{and} \quad \mathbf{H}(\mathcal{U}) := H^1(\mathcal{U};\mathbb{R}^3) \times H^1(\mathcal{U};\mathbb{R}^3) .$$

We moreover define the spaces of *kinematically admissible fields*:

$$\begin{aligned} \mathcal{K}(\Omega^{(h)}) &:= \{(\mathbf{u}, \boldsymbol{\omega}) \in \mathbf{H}(\Omega^{(h)}) : \mathbf{u} = \boldsymbol{\omega} = \mathbf{0} \text{ on } \Gamma^{(h)}\} , \\ \mathcal{K}_{\perp}(\Omega) &= \{(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega) : \partial_{\zeta} \mathbf{u} = \partial_{\zeta} \boldsymbol{\omega} = \mathbf{0} \text{ in } \Omega\} . \end{aligned}$$

Given a differentiable function $\sigma : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \to \mathbb{R}$, we shall denote by $\partial\sigma : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$ the differential of σ, and by $\partial_{\mathsf{E}}\sigma$ and $\partial_{\mathsf{G}}\sigma$ the matrices of partial derivatives of $\sigma(\mathsf{E}, \mathsf{G})$ with respect to E and G respectively.

3. A family of equilibrium problems for micropolar plate–like bodies

In linear micropolar elasticity the kinematics of a body, occupying in its *reference shape* a region $\mathcal{U} \subset \mathbb{R}^3$, is described by a *displacement field* $\mathbf{u} : \mathcal{U} \to \mathbb{R}^3$ and a *microrotation field* $\boldsymbol{\omega} : \mathcal{U} \to \mathbb{R}^3$ [6]. The *ordinary stress* tensor, denoted by S, is accompanied by a *couple stress* tensor, denoted by C. Moreover, the loads applied to the body consist not only in a distance force and a contact force (as in standard elasticity), but also in a distance couple and a contact couple.

In the sequel in order to simplify notation, we set $\mathbf{W} := \mathbf{A}^{\omega}$.
The *strains* relevant to the theory are the tensor fields defined by:

$$\begin{aligned} \mathbf{E} = \mathbf{E}(\mathbf{u},\boldsymbol{\omega}) &= \nabla\mathbf{u} - \mathbf{W}, \\ \mathbf{G} = \mathbf{G}(\boldsymbol{\omega}) &= \nabla\boldsymbol{\omega}\,. \end{aligned} \tag{6}$$

The stress descriptors $\mathbf{S}$ and $\mathbf{C}$ depend on the strains by means of *constitutive equations* of the form:

$$\mathbf{S} = \mathbb{S}[\mathbf{E}], \quad \mathbf{C} = \mathbb{C}[\mathbf{G}],$$

where $\mathbb{S} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ and $\mathbb{C} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ are linear functions (called *constitutive mappings*) that carry the relevant information on the *composition* of the body. We assume that the constitutive mappings do not depend on the position $x \in \mathcal{U}$. In this case the body is said to be *homogeneous*.
Let $\sigma : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \to \mathbb{R}$ be the function (called *stored–energy density*) defined by

$$\sigma(\mathbf{E},\mathbf{G}) = \frac{1}{2}\mathbb{S}[\mathbf{E}]\cdot\mathbf{E} + \frac{1}{2}\mathbb{C}[\mathbf{G}]\cdot\mathbf{G}. \tag{7}$$

The linear mappings $\mathbb{S}$ and $\mathbb{C}$ satisfy:

$$\mathbb{S}[\mathbf{A}]\cdot\mathbf{B} = \mathbb{S}[\mathbf{B}]\cdot\mathbf{A}, \quad \mathbb{C}[\mathbf{A}]\cdot\mathbf{B} = \mathbb{C}[\mathbf{B}]\cdot\mathbf{A} \quad \text{for all } \mathbf{A},\mathbf{B} \in \mathbb{R}^{3\times3}, \tag{8}$$

and are such that σ is positive definite, i.e.:

$$\sigma(\mathbf{E},\mathbf{G}) \geq C(|\mathbf{E}|^2 + |\mathbf{G}|^2) \quad \text{for all } \mathbf{E},\mathbf{G} \in \mathbb{R}^{3\times3}. \tag{9}$$

3.1. Micropolar plate–like bodies

We restrict our attention to *homogeneous plate–like bodies*, i.e. homogeneous bodies whose undeformed shape is a region $\Omega^{(h)}$ having the form specified in (1). Let σ be a function fulfilling (9) and, for each $h > 0$, let $(\tilde{\mathbf{f}}^{(h)}, \tilde{\mathbf{c}}^{(h)}) \in \mathbf{L}(\Omega^{(h)})$ be a pair of distance loads. We define the *total energy* as:

$$\tilde{\Pi}^{(h)}(\tilde{\mathbf{u}},\tilde{\boldsymbol{\omega}}) = \int_{\Omega^{(h)}} \sigma(\mathbf{E}(\tilde{\mathbf{u}},\tilde{\boldsymbol{\omega}}),\mathbf{G}(\tilde{\boldsymbol{\omega}})) - \int_{\Omega^{(h)}} \{\tilde{\mathbf{f}}^{(h)}\cdot\tilde{\mathbf{u}} + \tilde{\mathbf{c}}^{(h)}\cdot\tilde{\boldsymbol{\omega}}\} \tag{10}$$

and, for each $h > 0$, we look for a solution $(\tilde{\mathbf{u}}^{(h)}, \tilde{\boldsymbol{\omega}}^{(h)})$ of the following *equilibrium problem*:

$$\min_{(\mathbf{u},\boldsymbol{\omega})\in\mathcal{K}(\Omega^{(h)})} \tilde{\Pi}^{(h)}(\mathbf{u},\boldsymbol{\omega}). \tag{11}$$

The problems we here consider are only special instances of the much wider class of variational problems that fall within the scope of micropolar linear elasticity. This is not only because of the peculiar shape of the bodies we consider. In fact, some other assumptions have been made in laying down the functional (10). For example, the linear part of the functional in (10) does not depend on the traces of $(\tilde{\mathbf{u}},\tilde{\boldsymbol{\omega}})$, which means that the *contact loads are null.* Moreover, the restrictions $\tilde{\mathbf{u}}^{(h)} = \mathbf{0}$ and $\tilde{\boldsymbol{\omega}}^{(h)} = \mathbf{0}$ on $\Gamma^{(h)}$ (implicit in the definition of $\mathcal{K}(\Omega^{(h)})$) means that *the body is clamped on the lateral mantle of* $\Omega^{(h)}$.

The convexity of σ implies existence and uniqueness of the solution of the equilibrium problem (11). In fact, convexity ensures the lower semicontinuity of the functional (with respect to the strong topology of $\mathbf{L}(\Omega^{(h)})$) and inequality (9) allows us to get compactness (with respect to the strong topology of $\mathbf{L}(\Omega^{(h)})$) for sequences having bounded energy. Existence and uniqueness follow applying the Direct Methods of Calculus of Variation (see for instance [3]).

Our goal is to characterize the asymptotic behavior of the minimizers $(\tilde{\mathbf{u}}^{(h)}, \tilde{\boldsymbol{\omega}}^{(h)})$ as h tends to 0. To compare solutions corresponding to different choices of the parameter h, it is convenient to work with functions defined on the same space for all h. Following a standard approach, we blow up the domain $\Omega^{(h)}$ using the change of variable (2). Thus, to each pair $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}) \in \mathcal{K}(\Omega^{(h)})$ we associate the pair $(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega)$ defined by:

$$\mathbf{u}(x,\zeta) := \tilde{\mathbf{u}}(x, h\,\zeta), \quad \boldsymbol{\omega}(x,\zeta) := \tilde{\boldsymbol{\omega}}(x, h\,\zeta), \quad \forall (x,\zeta) \in \Omega. \tag{12}$$

We also replace the energy functional (10) with the functional $\Pi^{(h)} : \mathcal{K}(\Omega) \to \mathbb{R}$ defined by:

$$\Pi^{(h)}(\boldsymbol{\omega}, \mathbf{u}) := \frac{1}{h}\,\tilde{\Pi}^{(h)}(\tilde{\boldsymbol{\omega}}, \tilde{\mathbf{u}}).$$

It is immediate to see that the pair $(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})$, corresponding to the solution $(\tilde{\mathbf{u}}^{(h)}, \tilde{\boldsymbol{\omega}}^{(h)})$ of the equilibrium problem (11), is the unique minimizer of $\Pi^{(h)}$. We now provide the explicit representation of $\Pi^{(h)}$. For every $(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega)$ we set

$$\begin{aligned} \mathbf{E}^{(h)}(\mathbf{u}, \boldsymbol{\omega}) &= \nabla^{(h)}\mathbf{u} - \mathbf{W}, \\ \mathbf{G}^{(h)}(\boldsymbol{\omega}) &= \nabla^{(h)}\boldsymbol{\omega}. \end{aligned} \tag{13}$$

Then (12) implies:

$$\mathbf{E}^{(h)}(\mathbf{u}, \boldsymbol{\omega}) = \mathbf{E}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}), \quad \mathbf{G}^{(h)}(\boldsymbol{\omega}) = \mathbf{G}(\tilde{\boldsymbol{\omega}}).$$

Therefore, by defining the *scaled stored–energy density* as:

$$\sigma^{(h)}(\mathbf{E}, \mathbf{G}) = \sigma(\mathbf{E}_{\perp} + \frac{1}{h}\mathbf{E}\mathbf{z} \otimes \mathbf{z}, \mathbf{G}_{\perp} + \frac{1}{h}\mathbf{G}\mathbf{z} \otimes \mathbf{z}),$$

we have:

$$\sigma(\mathbf{E}^{(h)}(\mathbf{u}, \boldsymbol{\omega}), \mathbf{G}^{(h)}(\boldsymbol{\omega})) = \sigma^{(h)}(\mathbf{E}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}), \mathbf{G}(\tilde{\boldsymbol{\omega}})).$$

It is easy to check that

$$\Pi^{(h)}(\mathbf{u}, \boldsymbol{\omega}) = \Sigma^{(h)}(\mathbf{u}, \boldsymbol{\omega}) - \Lambda^{(h)}(\mathbf{u}, \boldsymbol{\omega}), \tag{14}$$

where

$$\begin{aligned} \Sigma^{(h)}(\mathbf{u}, \boldsymbol{\omega}) &:= \int_\Omega \sigma^{(h)}(\mathbf{E}(\mathbf{u}, \boldsymbol{\omega}), \mathbf{G}(\boldsymbol{\omega})), \\ \Lambda^{(h)}(\mathbf{u}, \boldsymbol{\omega}) &:= \int_\Omega \{\mathbf{f}^{(h)} \cdot \mathbf{u} + \mathbf{c}^{(h)} \cdot \boldsymbol{\omega}\}, \end{aligned} \tag{15}$$

with

$$\mathbf{f}^{(h)}(x_1, x_2, \zeta) := \tilde{\mathbf{f}}^{(h)}(x_1, x_2, h\zeta), \quad \mathbf{c}^{(h)}(x_1, x_2, \zeta) := \tilde{\mathbf{c}}^{(h)}(x_1, x_2, h\zeta), \quad \forall (x,\zeta) \in \Omega, \tag{16}$$

the *scaled loads*.

4. Asymptotic behavior of minimizers

We now study the behavior, as h tends to 0, of the solutions $(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})$ of the variational problems

$$\min_{(\mathbf{u},\boldsymbol{\omega})\in\mathcal{K}(\Omega)} \Pi^{(h)}(\mathbf{u},\boldsymbol{\omega}), \quad h>0. \tag{17}$$

We assume that the family of scaled loads $\{(\mathbf{f}^{(h)}, \mathbf{c}^{(h)})\}$ satisfies the following condition:

$$(\mathbf{f}^{(h)}, \mathbf{c}^{(h)}) \to (\mathbf{f}, \mathbf{c}) \quad \text{in } \mathbf{L}(\Omega). \tag{18}$$

Let us introduce the *limit stored–energy density* $\sigma_{\perp} : \mathbb{R}^{3\times 3}\times\mathbb{R}^{3\times 3}\to\mathbb{R}$ defined by:

$$\sigma_{\perp}(\mathsf{E},\mathsf{G}) := \min_{\mathbf{a},\mathbf{b}\in\mathbb{R}^3} \sigma(\mathsf{E}_{\perp} + \mathbf{a}\otimes\mathbf{z}, \mathsf{G}_{\perp} + \mathbf{b}\otimes\mathbf{z}), \tag{19}$$

and the *limit loads*:

$$\mathbf{f}_{\perp}(x_1,x_2) = \int_{-1}^{+1} \mathbf{f}(x_1,x_2,\zeta)d\zeta, \qquad \mathbf{c}_{\perp}(x_1,x_2) = \int_{-1}^{+1} \mathbf{c}(x_1,x_2,\zeta)d\zeta. \tag{20}$$

Finally we define the *limit functional* $\Pi_{\perp} : \mathcal{K}_{\perp}(\Omega)\to\mathbb{R}$ by setting:

$$\Pi_{\perp}(\mathbf{u},\boldsymbol{\omega}) := \Sigma_{\perp}(\mathbf{u},\boldsymbol{\omega}) - \Lambda_{\perp}(\mathbf{u},\boldsymbol{\omega}) \quad \forall(\mathbf{u},\boldsymbol{\omega})\in\mathcal{K}(\Omega_{\perp}), \tag{21}$$

where

$$\Sigma_{\perp}(\mathbf{u},\boldsymbol{\omega}) := \int_{\Omega_{\perp}} \sigma_{\perp}(\mathsf{E}(\mathbf{u},\boldsymbol{\omega}), \mathsf{G}(\boldsymbol{\omega})), \tag{22}$$

and

$$\Lambda_{\perp}(\mathbf{u},\boldsymbol{\omega}) := \int_{\Omega_{\perp}} \{\mathbf{f}_{\perp}\cdot\mathbf{u} + \mathbf{c}_{\perp}\cdot\boldsymbol{\omega}\}. \tag{23}$$

The functional $\Pi_{\perp}$ captures the limit behavior of minimizers in the sense of the following theorem, which is our main result.

Theorem 4.1. *Let assumptions* (9) *and* (18) *hold. Let* $\Pi^{(h)}$ *be defined as in* (14)–(16) *and let* $\Pi_{\perp}$ *be defined as in* (19)–(23)*. Then, as h tends to* 0*, we have:*

$$\min_{(\mathbf{u},\boldsymbol{\omega})\in\mathcal{K}(\Omega)} \Pi^{(h)}(\mathbf{u},\boldsymbol{\omega}) \to \min_{(\mathbf{u},\boldsymbol{\omega})\in\mathcal{K}_{\perp}(\Omega)} \Pi_{\perp}(\mathbf{u},\boldsymbol{\omega}),$$

and the solutions $(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})$ *of* (17) *converge, strongly in* $\mathbf{H}(\Omega)$*, to the unique solution* $(\mathbf{u},\boldsymbol{\omega})$ *of the limit equilibrium problem:*

$$\min_{(\mathbf{u},\boldsymbol{\omega})\in\mathcal{K}_{\perp}(\Omega)} \Pi_{\perp}(\mathbf{u},\boldsymbol{\omega}). \tag{24}$$

Observe that, using notation (3), we can write:

$$\sigma_{\perp}(\mathsf{E},\mathsf{G}) = \sigma_{\perp}(\mathsf{E}_{\perp} + \mathbf{0}\otimes\mathbf{z}, \mathsf{G}_{\perp} + \mathbf{0}\otimes\mathbf{z}).$$

Observe that $\sigma_\perp$ becomes a positive-definite quadratic function when we restrict it to the first two columns of E and G, that is to say, the elements of $\mathsf{E}_\perp$ and $\mathsf{G}_\perp$ (this ensures existence and uniqueness for the solution of the limit equilibrium problem (24)).

In fact, since σ is quadratic and positive definite, its differential $\partial\sigma$ is an invertible linear map. Hence, the unique solution of the minimization problem in (19) satisfies the linear system:

$$\partial_{\mathsf{E}}\sigma(\mathsf{E}_\perp + \bar{\mathbf{a}}\otimes\mathbf{z}, \mathsf{G}_\perp + \bar{\mathbf{b}}\otimes\mathbf{z})\mathbf{z} = \mathbf{0}, \qquad \partial_{\mathsf{G}}\sigma(\mathsf{E}_\perp + \bar{\mathbf{a}}\otimes\mathbf{z}, \mathsf{G}_\perp + \bar{\mathbf{b}}\otimes\mathbf{z})\mathbf{z} = \mathbf{0},$$

and $(\bar{\mathbf{a}}, \bar{\mathbf{b}})$ depends linearly on $(\mathsf{E}_\perp, \mathsf{G}_\perp)$. This readily implies that $\sigma_\perp$ is quadratic. Moreover, given $\mathsf{E}_\perp, \mathsf{G}_\perp \in \mathbb{R}^{3\times 2}$, by (9) and (19) we get:

$$\begin{aligned} \sigma_\perp(\mathsf{E}_\perp + \mathbf{0}\otimes\mathbf{z}, \mathsf{G}_\perp + \mathbf{0}\otimes\mathbf{z}) &= \min_{\mathbf{a},\mathbf{b}\in\mathbb{R}^3} \sigma(\mathsf{E}_\perp + \mathbf{a}\otimes\mathbf{z}, \mathsf{G}_\perp + \mathbf{b}\otimes\mathbf{z}) \\ \geq \min_{\mathbf{a},\mathbf{b}\in\mathbb{R}^3} C^{-1}(|\mathsf{E}_\perp + \mathbf{a}\otimes\mathbf{z}|^2 + |\mathsf{G}_\perp + \mathbf{b}\otimes\mathbf{z}|^2) &= C^{-1}(|\mathsf{E}_\perp|^2 + |\mathsf{G}_\perp|^2). \end{aligned}$$

5. Γ–convergence

In order to describe the convergence of minimum problems, we will use the notion of Γ-convergence. We recall its definition and the main properties we will use. For more details see for instance [5, 4, 2]).

Definition 5.1. Let X be a metric space[2] and let us denote by d the metric on X. For $n \in \mathbb{N}$ let F_n and F be functionals defined on X and taking values on $\mathbb{R}\cup\{+\infty\}$. Define the Γ- lim inf and the Γ- lim sup of F_n (with respect to the convergence induced by d) as:

$$\Gamma\text{-}\liminf F_n(x) := \inf\left\{\liminf_{n\to\infty} F_n(x_n) : x_n \to x\right\},$$
$$\Gamma\text{-}\limsup F_n(x) := \inf\left\{\limsup_{n\to\infty} F_n(x_n) : x_n \to x\right\}.$$

where we have written $x_n \to x$ to say $d(x_n, x) \to 0$.
We say that F_n Γ-converges to F if for all $x \in X$ we have:

$$\Gamma\text{-}\liminf_{n\to\infty} F_n(x) = \Gamma\text{-}\limsup_{n\to\infty} F_n(x) = F(x).$$

Proposition 5.2. *F_n Γ-converges to F if and only if the two following conditions are satisfied for each $x \in X$:*

(*i*) *for every sequence $\{x_n\}$ converging to x, there holds:*

$$\liminf_{n\to\infty} F(x_n) \geq F(x);$$

(*ii*) *for every $\eta > 0$ there exists $\{x_n\}$ converging to x such that:*

$$\limsup_{n\to\infty} F(x_n) \leq F(x) + \eta\ .$$

[2] Actually this requirement can be weakened taking on X a topology fulfilling the first axiom of countability.

In literature conditions (i) and (ii) are respectively referred to as *Liminf Inequality* and *Approximate Limsup Inequality.* The latter is indeed equivalent to the standard *Limsup Inequality* [2, §1.2].

Definition 5.3. A sequence of functionals $F_n : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *sequentially equi-coercive* if, for any sequence $\{x_n\}$ such that $\sup_n F_n(x_n) < +\infty$, there exists a convergent subsequence.

Theorem 5.4. *Let $\{F_n\}$ be a sequence of sequentially equicoercive functionals defined on X and Γ-converging to F. Then there exists $\min_X F$ and $\min_X F = \lim_{n\to\infty} \inf_X F_n$. Moreover, if x_n is a minimizer of F_n, then every limit of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is a minimizer of F.*

Theorem 5.5. *Let $\{F_n\}$ be a sequence of functionals Γ-converging to F and $\{G_n\}$ be a sequence of continuous (with respect to the same topology for which the Γ-limit is computed) functionals uniformly converging to G. Then the family $\{F_n + G_n\}$ Γ-converges to $F + G$.*

Definition 5.6. Let $\{F_h\}_{h>0}$ be a family of functionals labeled by a continuous parameter h. We say that $\{F_h\}_{h>0}$ Γ-converges to F as h goes to 0, if $\{F_{h_n}\}_{n\in\mathbb{N}}$ Γ-converges to F *for every subsequence* $\{h_n\}_{n\in\mathbb{N}}$ converging to zero as n goes to $+\infty$. In simple way the statements of Proposition 5.2, Theorem 5.4, and Theorem 5.5 can be rephrased in this framework.

6. Proof of the main result

In view of Theorem 5.5 we first study the Γ–limit of $\Sigma^{(h)}$. This is performed in Subsection 6.2, where we also prove an equi-coercivity property for the family of functionals $\Pi^{(h)}$.

6.1. Additional notation

To avoid nested subscripts, we replace h_n with h in the following statement and, as a rule, we do not relabel subsequences. Positive constants are denoted by C or by C_i. For $\{\boldsymbol{\omega}^{(h)}\} \subset H^1(\Omega;\mathbb{R}^3)$ we set $\mathbf{W}^{(h)} := \mathbf{A}^{\boldsymbol{\omega}^{(h)}}$. Moreover we use the shorthand notation: $\mathbf{E}^{(h)} = \mathbf{E}^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})$, $\mathbf{G}^{(h)} = \mathbf{G}^{(h)}(\boldsymbol{\omega}^{(h)})$ (*cf.* (13)). Then, recalling (4), we write:

$$\mathbf{E}^{(h)} = \nabla_{\perp}\mathbf{u}^{(h)} + \frac{1}{h}\partial_\zeta \mathbf{u}^{(h)} \otimes \mathbf{z} - \mathbf{W}^{(h)}, \qquad \mathbf{G}^{(h)} = \nabla_{\perp}\boldsymbol{\omega}^{(h)} + \frac{1}{h}\partial_\zeta \boldsymbol{\omega}^{(h)} \otimes \mathbf{z},$$

and

$$\mathbf{E}^{(h)}_{\perp} = \nabla_{\perp}\mathbf{u}^{(h)} - \mathbf{W}^{(h)}_{\perp}, \qquad \mathbf{G}^{(h)}_{\perp} = \nabla_{\perp}\boldsymbol{\omega}^{(h)}.$$

6.2. Compactness and Γ–convergence

We first give a compactness result for the family of functionals $\Sigma^{(h)}$.

Lemma 6.1 (Compactness for the scaled stored energies). *Let $\{(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\} \subset \mathcal{K}(\Omega)$ be a sequence such that*

$$\sup_{h>0} \Sigma^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) < +\infty.$$

Then there exist a subsequence of $\{(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})\}$ *(not relabeled) and* $(\mathbf{u},\boldsymbol{\omega})\in\mathbf{H}(\Omega)$ *such that* $(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})$ *converges to* $(\mathbf{u},\boldsymbol{\omega})$ *with respect to the strong topology of* $\mathbf{L}(\Omega)$ *as* h *tends to* 0.

Proof. By the coercivity condition (9) we have

$$\sup_{h>0}\left(||\mathsf{E}^{(h)}||^2_{L^2(\Omega;\mathbb{R}^3)}+||\mathsf{G}^{(h)}||^2_{L^2(\Omega;\mathbb{R}^3)}\right)<+\infty.$$

Hence $\nabla_\perp\boldsymbol{\omega}^{(h)}$ and $\frac{1}{h}\partial_\zeta\boldsymbol{\omega}^{(h)}$ are bounded in $L^2(\Omega;\mathbb{R}^{3\times 2})$ and $L^2(\Omega;\mathbb{R}^3)$ respectively; thus, by extracting a subsequence (not relabeled), we get:

$$\partial_\zeta\boldsymbol{\omega}^{(h)}\rightharpoonup\mathbf{0},\quad \frac{1}{h}\partial_\zeta\boldsymbol{\omega}^{(h)}\rightharpoonup\mathbf{b}\quad\text{in } L^2(\Omega;\mathbb{R}^3).$$

By Poincaré's inequality, $\boldsymbol{\omega}^{(h)}$ is bounded in $H^1(\Omega;\mathbb{R}^3)$ and, by extracting a further subsequence, it follows:

$$\boldsymbol{\omega}^{(h)}\rightharpoonup\boldsymbol{\omega}\quad\text{in } H^1(\Omega;\mathbb{R}^3)\text{ with }\partial_\zeta\boldsymbol{\omega}=\mathbf{0}. \tag{25}$$

In the identity

$$\nabla_\perp\mathbf{u}^{(h)}+\frac{1}{h}\partial_\zeta\mathbf{u}^{(h)}\otimes\mathbf{z}=\mathsf{E}^{(h)}+\mathsf{W}^{(h)}$$

the right–hand side is a bounded sequence in $L^2(\Omega;\mathbb{R}^{3\times 3})$ and, arguing as in the derivation of (25), we have:

$$\mathbf{u}^{(h)}\rightharpoonup\mathbf{u}\quad\text{in } H^1(\Omega;\mathbb{R}^3)\text{ with }\partial_\zeta\mathbf{u}=\mathbf{0}.$$

Finally we observe that, by well known results of functional analysis, the selected subsequence converges with respect to the strong topology of $\mathbf{L}(\Omega)$. □

Remark 6.2. Since $\frac{1}{h}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})$ is bounded in $\mathbf{L}(\Omega)$, we have that $\partial_\zeta(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})\to 0$ in $\mathbf{L}(\Omega)$.

Lemma 6.3 (Compactness for the total energies). *Let* $\{(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})\}\subset\mathcal{K}(\Omega)$ *be a sequence such that*

$$\sup_{h>0}\Pi^{(h)}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})<+\infty.$$

Then there exist a subsequence of $\{(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})\}$ *(not relabeled) and* $(\mathbf{u},\boldsymbol{\omega})\in\mathbf{H}(\Omega)$ *such that* $(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})$ *converges to* $(\mathbf{u},\boldsymbol{\omega})$ *with respect to the strong topology of* $\mathbf{L}(\Omega)$.

Proof. As a consequence of Lemma 6.1 it is enough to show that, if $\{(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})\}$ is such that $\sup_{h>0}\Pi^{(h)}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})<+\infty$, then it holds:

$$\sup_{h>0}\Sigma^{(h)}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})<+\infty. \tag{26}$$

For this purpose we note that:

$$\Pi^{(h)}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})\le\Pi^{(h)}(\mathbf{0},\mathbf{0})=0, \tag{27}$$

and therefore recalling (14), (15) and using Hölder's inequality, we get:

$$\Sigma^{(h)}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})\le\|(\mathbf{f}^{(h)},\mathbf{c}^{(h)})\|_{\mathbf{L}(\Omega)}\|(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})\|_{\mathbf{L}(\Omega)}\le C\|(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})\|_{\mathbf{L}(\Omega)}, \tag{28}$$

where the last inequality follows from assumption (18). Moreover, using Poincaré's inequality, equation (5) and the inequality $|\mathbf{A}-\mathbf{B}|^2+|\mathbf{B}|^2 \geq C(|\mathbf{A}|^2+|\mathbf{B}|^2)$ (holding for all 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$), we get:

$$\begin{aligned}\Sigma^{(h)}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)}) &\geq C_1\int_\Omega |\nabla^{(h)}\mathbf{u}^{(h)}-\mathbf{W}^{(h)}|^2+|\nabla^{(h)}\boldsymbol{\omega}^{(h)}|^2\\ &\geq C_2\int_\Omega |\nabla^{(h)}\mathbf{u}^{(h)}-\mathbf{W}^{(h)}|^2+|\mathbf{W}^{(h)}|^2\\ &\geq C_3\int_\Omega |\nabla^{(h)}\mathbf{u}^{(h)}|^2+|\mathbf{W}^{(h)}|^2\\ &\geq C_4||(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})||^2_{\mathbf{L}(\Omega)}.\end{aligned}$$

Hence, using the so–called Young's inequality with ε, we obtain:

$$C_5||(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})||_{\mathbf{L}(\Omega)}-\frac{1}{2\varepsilon}||(\mathbf{f}^{(h)},\mathbf{c}^{(h)})||_{\mathbf{L}(\Omega)}-\frac{\varepsilon}{2}||(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})||^2_{\mathbf{L}(\Omega)}\leq \Pi^{(h)}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)}).$$

Choosing $\varepsilon>0$ small enough and using (27), we have:

$$\|(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})\|_{\mathbf{L}(\Omega)}\leq C.$$

The last inequality combined with (28) yields (26). □

Proposition 6.4 (Γ-convergence for the scaled stored energies). *Let* $\overline{\Sigma}^{(h)}$, $\Sigma_\perp$: $\mathbf{L}(\Omega)\to\mathbb{R}\cup\{+\infty\}$ *be defined as:*

$$\overline{\Sigma}^{(h)}(\mathbf{u},\boldsymbol{\omega}):=\begin{cases}\Sigma^{(h)}(\mathbf{u},\boldsymbol{\omega}) & \text{if } (\mathbf{u},\boldsymbol{\omega})\in\mathcal{K}(\Omega)\\ +\infty & \text{otherwise}\end{cases}$$

$$\overline{\Sigma}_\perp(\mathbf{u},\boldsymbol{\omega}):=\begin{cases}2\displaystyle\int_{\Omega_\perp}\sigma_\perp(\mathbf{E}_\perp(\mathbf{u},\boldsymbol{\omega}),\mathbf{G}_\perp(\boldsymbol{\omega}))dx_1dx_2 & \text{if } (\mathbf{u},\boldsymbol{\omega})\in\mathcal{K}_\perp(\Omega),\\ +\infty & \text{otherwise.}\end{cases}$$

Then the family $\{\overline{\Sigma}^{(h)}\}$ *Γ-converges to* $\Sigma_\perp$ *with respect to the strong topology of* $\mathbf{L}(\Omega)$.

Proof. In view of Proposition 5.2 we first prove the *Liminf Inequality*, which now takes the following form:

Given $(\mathbf{u},\boldsymbol{\omega})\in\mathcal{K}(\Omega)$ *and* $\{(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})\}\subset\mathcal{K}(\Omega)$ *converging to* $(\mathbf{u},\boldsymbol{\omega})$ *in* $\mathbf{L}(\Omega)$*, then:*

$$\Sigma_\perp(\mathbf{u},\boldsymbol{\omega})\leq\liminf_{h\to 0}\Sigma^{(h)}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)}).$$

By (19) we have:

$$\Sigma_\perp(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)})=\int_\Omega\sigma_\perp(\mathbf{E}^{(h)}_\perp,\mathbf{G}^{(h)}_\perp)\leq\int_\Omega\sigma(\mathbf{E}^{(h)},\mathbf{G}^{(h)})=\Sigma^{(h)}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)}).$$

Hence, to prove the Liminf Inequality, it is enough to show that

$$\Sigma_\perp(\mathbf{u},\boldsymbol{\omega})\leq\liminf_{h\to 0}\Sigma_\perp(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)}). \tag{29}$$

Let us prove (29). For $\mathbf{E}_\perp = \mathbf{E}_\perp(\mathbf{u},\boldsymbol{\omega})$ and $\mathbf{G}_\perp = \nabla_\perp \boldsymbol{\omega}$, the convexity of $\sigma_\perp$ implies:

$$\sigma_\perp(\mathbf{E}_\perp^{(h)}, \mathbf{G}_\perp^{(h)}) \geq \sigma_\perp(\mathbf{E}_\perp, \mathbf{G}_\perp) + \partial\sigma_\perp(\mathbf{E}_\perp, \mathbf{G}_\perp) \cdot (\mathbf{E}_\perp^{(h)} - \mathbf{E}_\perp, \mathbf{G}_\perp^{(h)} - \mathbf{G}_\perp). \tag{30}$$

Let us consider a subsequence of $\{(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\}$ (not relabeled) such that $\lim_{h\to 0} \Sigma^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) = \liminf_{h\to 0} \Sigma^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})$. By Lemma 6.1 we can extract a further subsequence such that $(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) \rightharpoonup (\mathbf{u}, \boldsymbol{\omega})$ in $\mathbf{H}(\Omega)$. By Rellich's Theorem we know that $\boldsymbol{\omega}^{(h)}$ converges to $\boldsymbol{\omega}$ in $L^2(\Omega;\mathbb{R}^3)$ and hence we get:

$$\mathbf{E}_\perp^{(h)} \rightharpoonup \mathbf{E}_\perp \quad \text{and} \quad \mathbf{G}_\perp^{(h)} \rightharpoonup \mathbf{G}_\perp \quad \text{in } L^2(\Omega;\mathbb{R}^{3\times 2}).$$

Integrating (30) over Ω, and passing to the limit we obtain (29).

We now prove the *Approximate Limsup Inequality*, which now takes the form:

For every $(\mathbf{u},\boldsymbol{\omega}) \in \mathcal{K}_\perp(\Omega)$ and for every $\eta > 0$ there exists a sequence $\{(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\} \subset \mathcal{K}(\Omega)$ such that

$$(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) \to (\mathbf{u}, \boldsymbol{\omega}) \quad \textit{in } \mathbf{L}(\Omega)$$

and

$$\limsup_{h\to 0} \Sigma^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) \leq \Sigma_\perp(\mathbf{u},\boldsymbol{\omega}) + \eta.$$

Let $(\mathbf{u},\boldsymbol{\omega}) \in \mathcal{K}_\perp(\Omega)$ and $\eta > 0$ be fixed. We set $\mathbf{E}_\perp = \nabla_\perp \mathbf{u} - \mathbf{W}_\perp$, and $\mathbf{G}_\perp = \nabla_\perp \boldsymbol{\omega}$. For every $x \in \Omega_\perp$, let $(\mathbf{a}(x), \mathbf{b}(x))$ be the unique pair such that the minimum in (19) is attained. Then

$$\Sigma_\perp(\mathbf{u},\boldsymbol{\omega}) = \sigma(\mathbf{E}_\perp + \mathbf{a}\otimes\mathbf{z}, \mathbf{G}_\perp + \mathbf{b}\otimes\mathbf{z}). \tag{31}$$

For every $\varepsilon > 0$, let $(\mathbf{a}_\varepsilon, \mathbf{b}_\varepsilon) \in \mathbf{H}(\Omega)$ be such that

$$||(\mathbf{a}_\varepsilon - \mathbf{a}, \mathbf{b}_\varepsilon - \mathbf{b})||_{\mathbf{L}(\Omega)} \leq \varepsilon \tag{32}$$

and define

$$\mathbf{u}_\varepsilon^{(h)}(x,\zeta) := \mathbf{u}(x) + h\zeta(\mathbf{a}_\varepsilon(x) + \boldsymbol{\omega}(x)\times\mathbf{z}) + \frac{1}{2}h^2\zeta^2\mathbf{b}_\varepsilon(x)\times\mathbf{z}$$

and

$$\boldsymbol{\omega}_\varepsilon^{(h)}(x,\zeta) := \boldsymbol{\omega}(x) + h\zeta\mathbf{b}_\varepsilon(x), \quad \forall (x,\zeta)\in\Omega.$$

We claim that for ε small enough the sequence $\{\mathbf{u}_\varepsilon^{(h)}, \boldsymbol{\omega}_\varepsilon^{(h)}\}$ satisfies the Approximate Limsup Inequality. Clearly, this sequence converges to $(\mathbf{u},\boldsymbol{\omega})$. Moreover it holds

$$\begin{aligned}\mathbf{E}^{(h)}(\mathbf{u}_\varepsilon^{(h)}, \boldsymbol{\omega}_\varepsilon^{(h)}) &= \nabla_\perp \mathbf{u}_\varepsilon^{(h)} + \frac{1}{h}\partial_\zeta \mathbf{u}_\varepsilon^{(h)} \otimes \mathbf{z} - \mathbf{A}^{\omega_\varepsilon^{(h)}} \\ &= \mathbf{E}_\perp + \mathbf{a}_\varepsilon\otimes\mathbf{z} + h\mathbf{E}_1 + h^2\mathbf{E}_2,\end{aligned}$$

and

$$\mathbf{G}^{(h)}(\boldsymbol{\omega}_\varepsilon^{(h)}) = \mathbf{G}_\perp + \mathbf{b}_\varepsilon\otimes\mathbf{z},$$

for suitable $\mathbf{E}_1, \mathbf{E}_2$ belonging to $L^2(\Omega;\mathbb{R}^{3\times 3})$. Since σ is a quadratic function, it immediately follows:

$$\Sigma^{(h)}(\mathbf{u}_\varepsilon^{(h)}, \boldsymbol{\omega}_\varepsilon^{(h)}) = \int_\Omega \sigma(\mathbf{E}_\perp + \mathbf{a}_\varepsilon\otimes\mathbf{z}, \mathbf{G}_\perp + \mathbf{b}_\varepsilon\otimes\mathbf{z}) + o_h(1), \tag{33}$$

where $\lim_{h\to 0} o_h(1) = 0$. Moreover, we have

$$\int_\Omega \sigma(\mathbf{E}_\perp + \mathbf{a}_\varepsilon \otimes \mathbf{z}, \mathbf{G}_\perp + \mathbf{b}_\varepsilon \otimes \mathbf{z}) \leq \int_\Omega \sigma(\mathbf{E}_\perp + \mathbf{a} \otimes \mathbf{z}, \mathbf{G}_\perp + \mathbf{b} \otimes \mathbf{z}) + \frac{\eta}{2} + C_\eta ||(\mathbf{a}_\varepsilon - \mathbf{a}, \mathbf{b}_\varepsilon - \mathbf{b})||_{\mathbf{L}(\Omega)}$$

for $C_\eta > 0$ large enough. Therefore, selecting $\varepsilon \leq C_\eta^{-1}\eta/2$ and passing to the limit in (33), we obtain:

$$\lim_{h\to 0} \Sigma^{(h)}(\mathbf{u}_\varepsilon^{(h)}, \boldsymbol{\omega}_\varepsilon^{(h)}) = \int_\Omega \sigma(\mathbf{E}_\perp + \mathbf{a} \otimes \mathbf{z}, \mathbf{G}_\perp + \mathbf{b} \otimes \mathbf{z}) + \eta,$$

and the thesis follows from (31). □

6.3. Proof of Theorem 4.1

Let us introduce the *extended functionals* $\overline{\Pi}^{(h)}$, $\overline{\Pi}_\perp : \mathbf{L}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined as:

$$\overline{\Pi}^{(h)}(\mathbf{u}, \boldsymbol{\omega}) := \begin{cases} \Pi^{(h)}(\mathbf{u}, \boldsymbol{\omega}) & \text{if } (\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

$$\overline{\Pi}_\perp(\mathbf{u}, \boldsymbol{\omega}) := \begin{cases} \Pi_\perp(\mathbf{u}, \boldsymbol{\omega}) & \text{if } (\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}_\perp(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

By (18) we have that the sequence $\Lambda^{(h)}$ converges to $\Lambda_\perp$ uniformly with respect to the strong topology of $\mathbf{L}(\Omega)$. Therefore by Theorem 5.5 and Proposition 6.4 it immediately follows that $\overline{\Pi}^{(h)}$ Γ-converges to $\overline{\Pi}_\perp$ with respect to the strong topology of $\mathbf{L}(\Omega)$. By Theorem 5.4 and Lemma 6.3 we conclude that , as h tends to 0,

$$\min_{(\mathbf{u},\boldsymbol{\omega})\in\mathcal{K}(\Omega)} \Pi^{(h)}(\mathbf{u}, \boldsymbol{\omega}) \to \min_{(\mathbf{u},\boldsymbol{\omega})\in\mathcal{K}_\perp(\Omega)} \Pi_\perp(\mathbf{u}, \boldsymbol{\omega})$$

and the family $\{(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\}$ of solutions of problem (17), up to a subsequence, converges to the unique minimizer of $\Pi_\perp$ in $\mathcal{K}_\perp(\Omega)$ with respect to the strong topology of $\mathbf{L}(\Omega)$.

Now we show that the family of minimizers converges also with respect to the strong topology of $\mathbf{H}(\Omega)$. Let us define

$$\mathbf{E}_\perp = \nabla_\perp \mathbf{u} - \mathbf{W}, \qquad \mathbf{G}_\perp = \nabla_\perp \boldsymbol{\omega}.$$

Moreover, since $\boldsymbol{\omega}^{(h)} \to \boldsymbol{\omega}$ in $L^2(\Omega; \mathbb{R}^3)$, we have:

$$\nabla_\perp \mathbf{u}^{(h)} - \mathbf{W}_\perp^{(h)} \rightharpoonup \nabla_\perp \mathbf{u} - \mathbf{W} \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times 2}).$$

Therefore it holds:

$$\mathbf{E}_\perp^{(h)} \rightharpoonup \mathbf{E}_\perp, \quad \mathbf{G}_\perp^{(h)} \rightharpoonup \mathbf{G}_\perp \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times 2}). \tag{34}$$

We pass to the limit in the chain of inequalities

$$\begin{aligned}
\Pi^{(h)}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)}) - \Pi_{\perp}(\mathbf{u},\boldsymbol{\omega}) \;\ge\;& \Sigma_{\perp}(\mathbf{u}^{(h)},\boldsymbol{\omega}^{(h)}) - \Sigma_{\perp}(\mathbf{u},\boldsymbol{\omega}) \\
&+ \int_{\Omega}\{\mathbf{f}^{(h)}\cdot\mathbf{u}^{(h)} + \mathbf{c}^{(h)}\cdot\boldsymbol{\omega}^{(h)} - \cdot\mathbf{u} - \mathbf{c}\cdot\boldsymbol{\omega}\} \\
\ge\;& \int_{\Omega}\partial\sigma_{\perp}(\mathbf{E}_{\perp},\mathbf{G}_{\perp})\cdot(\mathbf{E}_{\perp}^{(h)}-\mathbf{E}_{\perp},\mathbf{G}_{\perp}^{(h)}-\mathbf{G}_{\perp}) \\
&+C\left(||\mathbf{E}_{\perp}^{(h)}-\mathbf{E}_{\perp}||^2 + ||\mathbf{G}_{\perp}^{(h)}-\mathbf{G}_{\perp}||^2\right) \\
&+ \int_{\Omega}\{\mathbf{f}^{(h)}\cdot\mathbf{u}^{(h)} + \mathbf{c}^{(h)}\cdot\boldsymbol{\omega}^{(h)} - \mathbf{f}\cdot\mathbf{u} - \mathbf{c}\cdot\boldsymbol{\omega}\},
\end{aligned}$$

and, using (34) and (18), we conclude:

$$\nabla_{\perp}\mathbf{u}^{(h)} = \mathbf{E}^{(h)} - \mathbf{W}_{\perp}^{(h)} \to \mathbf{E} - \mathbf{W} = \nabla_{\perp}\mathbf{u} \quad \text{in } L^2(\Omega;\mathbb{R}^{3\times 2}). \tag{35}$$

The thesis follows using (35) and recalling that $(\mathbf{u},\boldsymbol{\omega}) \in \mathcal{K}_{\perp}(\Omega)$ and that by Remark 6.2 we know that:

$$\partial_{\zeta}\mathbf{u}^{(h)} \to \mathbf{0}, \quad \partial_{\zeta}\boldsymbol{\omega}^{(h)} \to \mathbf{0} \quad \text{in } L^2(\Omega;\mathbb{R}^3).$$

7. The isotropic case

In this section we verify that our limit problem agrees with that obtained in [1] for the case of isotropic materials. If we denote by $\mathbf{1} \in \mathbb{R}^{3\times 3}$ the identity matrix, for these materials the constitutive mappings take the form:

$$\begin{aligned}
\mathbb{S}[\mathbf{E}] &= \lambda(\mathrm{tr}\mathbf{E})\mathbf{1} + (\mu+\nu)\mathbf{E} + \mu\mathbf{E}^T, \\
\mathbb{C}[\mathbf{G}] &= \rho(\mathrm{tr}\mathbf{G})\mathbf{1} + (\sigma+\tau)\mathbf{G} + \sigma\mathbf{G}^{\mathsf{T}},
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
&3\lambda + 2\mu + \nu > 0, \quad 2\mu + \nu > 0, \quad \nu > 0, \\
&3\alpha + 2\beta + \gamma > 0, \quad 2\beta + \gamma > 0, \quad \gamma > 0.
\end{aligned}$$

The differential of σ is given by:

$$\partial\sigma(\mathbf{E},\mathbf{G}) = (\partial_{\mathbf{E}}\sigma, \partial_{\mathbf{G}}\sigma) = (\mathbb{S},\mathbb{C}).$$

Given $\mathbf{E}_{\perp} \in \mathbb{R}^{3\times 2}$ and $\mathbf{G}_{\perp} \in \mathbb{R}^{3\times 2}$, let $\bar{\mathbf{a}} \equiv \bar{\mathbf{a}}(\mathbf{E}_{\perp}) \in \mathbb{R}^3$ and $\bar{\mathbf{b}} \equiv \bar{\mathbf{b}}(\mathbf{G}_{\perp}) \in \mathbb{R}^3$ be the solutions of the minimum problem (19). It is easy to check that $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ satisfy:

$$\begin{aligned}
\partial_{\mathbf{E}}\sigma(\mathbf{E}_{\perp} + \bar{\mathbf{a}}\otimes\mathbf{z})\cdot\mathbf{a}\otimes\mathbf{z} = 0 \quad \forall\mathbf{a}\in\mathbb{R}^3, \\
\partial_{\mathbf{G}}\sigma(\mathbf{G}_{\perp} + \bar{\mathbf{b}}\otimes\mathbf{z})\cdot\mathbf{b}\otimes\mathbf{z} = 0 \quad \forall\mathbf{b}\in\mathbb{R}^3.
\end{aligned}$$

This means that:

$$\mathbb{S}[\mathbf{E}_{\perp} + \bar{\mathbf{a}}\otimes\mathbf{z}]\mathbf{z} = \mathbf{0}, \qquad \mathbb{C}[\mathbf{G}_{\perp} + \bar{\mathbf{b}}\otimes\mathbf{z}]\mathbf{z} = \mathbf{0}. \tag{37}$$

This implies that

$$\sigma_{\perp}(\mathbf{E}_{\perp},\mathbf{G}_{\perp}) = \frac{1}{2}\mathbb{S}_{\perp}[\mathbf{E}_{\perp}]\cdot\mathbf{E}_{\perp} + \frac{1}{2}\mathbb{C}_{\perp}[\mathbf{G}_{\perp}]\cdot\mathbf{G}_{\perp},$$

where we have set:

$$\mathbb{S}_{\perp}[\mathbf{E}_{\perp}] := \mathbb{S}[\mathbf{E}_{\perp} + \bar{\mathbf{a}}(\mathbf{E}_{\perp}) \otimes \mathbf{z}], \qquad \mathbb{C}_{\perp}[\mathbf{G}_{\perp}] := \mathbb{C}[\mathbf{G}_{\perp} + \bar{\mathbf{b}}(\mathbf{G}_{\perp}) \otimes \mathbf{z}].$$

Let us denote by $\mathbf{1}_{\square}$ the matrix *plane projector*, defined as:

$$\mathbf{1}_{\square} = \mathbf{1} - \mathbf{z} \otimes \mathbf{z}.$$

We set:

$$\mathbf{E}_{\square} := \mathbf{1}_{\square}\mathbf{E}_{\perp}, \qquad \mathbf{c} := (\mathbf{E}_{\perp})^{\mathsf{T}}\mathbf{z}$$

so that $\mathbf{E}_{\perp}$ is decomposed as:

$$\mathbf{E}_{\perp} = \mathbf{E}_{\square} + \mathbf{z} \otimes \mathbf{c}.$$

Now, the constitutive equation $(36)_1$ becomes

$$\mathbf{S} = \lambda(\mathrm{tr}\mathbf{E}_{\square} + \mathbf{z} \cdot \bar{\mathbf{a}})\mathbf{1}_{\square} + (\mu + \nu)\mathbf{E}_{\square} + \mu\mathbf{E}_{\square}^{\mathsf{T}} + \mathbf{z} \otimes ((\mu + \nu)\mathbf{c} + \mathbf{1}_{\square}\mathbf{z}), \tag{38}$$

and the first equation of (37) can be rewritten as:

$$\lambda(\mathrm{tr}\mathbf{E}_{\square} + \mathbf{z} \cdot \bar{\mathbf{a}})\mathbf{z} + (\mu + \nu)\mathbf{a} + \mu((\mathbf{z} \cdot \bar{\mathbf{a}})\mathbf{z} + \mathbf{c}) = \mathbf{0}. \tag{39}$$

Taking the scalar product of both sides of (39) with $\mathbf{z}$, we obtain:

$$\mathbf{z} \cdot \bar{\mathbf{a}} = -\frac{\lambda}{\lambda + 2\mu + \nu}\mathrm{tr}\mathbf{E}_{\square}. \tag{40}$$

Applying $\mathbf{1}_{\square}$ to both sides of (39), we get the plane components of $\bar{\mathbf{a}}$:

$$\mathbf{1}_{\square}\bar{\mathbf{a}} = -\frac{\mu}{\mu + \nu}\mathbf{c}. \tag{41}$$

Substituting (40) and (41) in (38), we have:

$$\mathbb{S}_{\perp}[\mathbf{E}_{\perp}] = \lambda_{\perp}\mathrm{tr}\mathbf{E}_{\square} + (\mu + \nu)\mathbf{E}_{\square} + \mu\mathbf{E}_{\square}^{\mathsf{T}} + \nu_{\perp}\mathbf{z} \otimes (\mathbf{E}_{\perp}^{\mathsf{T}}\mathbf{z}),$$

where

$$\lambda_{\perp} := \lambda\frac{2\mu + \nu}{\lambda + 2\mu + \nu}, \qquad \nu_{\perp} := \nu\frac{2\mu + \nu}{\mu + \nu}.$$

Likewise, we easily obtain

$$\mathbb{C}_{\perp}[\mathbf{G}_{\perp}] = \alpha_{\perp}\mathrm{tr}\mathbf{G}_{\square} + (\beta + \gamma)\mathbf{G}_{\square} + \beta\mathbf{G}_{\square}^{\mathsf{T}} + \gamma_{\perp}\mathbf{z} \otimes (\mathbf{G}_{\perp}^{\mathsf{T}}\mathbf{z}),$$

where

$$\alpha_{\perp} := \alpha\frac{2\beta + \gamma}{\alpha + 2\beta + \gamma}, \qquad \gamma_{\perp} := \gamma\frac{2\beta + \gamma}{\beta + \gamma}.$$

The stationarity condition for $\Pi_{\perp}$ is:

$$\int_{\Omega_{\perp}} \mathbb{S}_{\perp}[\mathbf{E}_{\perp}(\mathbf{u}, \boldsymbol{\omega})] \cdot \mathbf{E}_{\perp}(\delta\mathbf{u}, \delta\boldsymbol{\omega}) + \mathbb{C}_{\perp}[\mathbf{G}_{\perp}(\boldsymbol{\omega})] \cdot \mathbf{G}_{\perp}(\delta\boldsymbol{\omega}) - \Lambda_{\perp}^{(h)}(\delta\mathbf{u}, \delta\boldsymbol{\omega}) = 0$$

for all test functions $(\delta\mathbf{u}, \delta\boldsymbol{\omega}) \in \mathcal{K}_{\perp}(\Omega)$. This coincides with the weak formulation of the limit problem obtained in [1] (equation (22)).

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