Bifurcation results on symplectic manifolds

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To the feminist cause

1. Introduction

Existence and multiplicity of periodic trajectories of Hamiltonian vector fields on symplectic manifolds is a traditional field of research, which found new input from the work on Arnold's conjecture. Fitzpatrick, Pejsachowicz and Recht in [8],[9] studied bifurcation of periodic solutions of one-parameter families of (time dependent) periodic Hamiltonian systems in $\mathbf{R^{2n}}$ relating the spectral flow to the bifurcation of critical points of strongly indefinite functionals.

In [6] we extended their results to families of time dependent Hamiltonian vector fields acting on symplectic manifolds and the related problems of bifurcation of fixed points of one parameter families of symplectomorphisms were discussed. Namely we proved that for a 1-parameter family of time dependent Hamiltonian vector fields, acting on a symplectic manifold M which possesses a known trivial branch u_λ of 1-periodic solutions if the relative Conley Zehnder index of the monodromy path along $u_\lambda(0)$ is defined and does not vanish then any neighborhood of the trivial branch contains 1-periodic solutions not in the branch.

Fixed points of Hamiltonian symplectomorphisms are in one to one correspondence with 1-periodic orbits of the corresponding vector field. Hence as a consequence we obtained, assuming that (M,ω) is a closed symplectic manifold with trivial first De Rham cohomology group, for a path $\phi : [0,1] \to Symp_0(M)$ of symplectomorphisms with a known smooth path $p : [0,1] \to U$ of fixed points, *i.e.*, $p(\lambda)$ is a fixed point of ϕ_λ. If the Conley-Zehnder index $CZ(\phi,p)$ of ϕ along p is defined and does not vanish then there is a bifurcation of fixed points of ϕ from the trivial branch p.

The Arnold conjecture states that a generic Hamiltonian symplectomorphism has more fixed points that could be predicted from the fixed point index. More precisely, by the fixed point theory a diffeomorphism isotopic to the identity with non-degenerate fixed points must have at least as many fixed points as the Euler-Poincaré characteristic of the manifold. But the number of fixed points of a Hamiltonian symplectomorphism verifying the same non-degeneracy assumptions is bounded bellow by the sum of the Betti numbers. Roughly speaking, this can be explained by the presence of a variational structure in the problem. Fixed points viewed as periodic orbits of the corresponding vector field are critical points of the action functional either if the orbits are contractible or when the symplectic form is exact.

Applied to bifurcation of fixed points of one parameter families of Hamiltonian symplectomorphisms our result shows a similar influence on the presence of a variational structure. In order to see the analogy consider a one parameter family of diffeomorphisms $\psi_\lambda; \lambda \in [0,1]$ of an oriented manifold M, assuming for simplicity that $\psi_\lambda(p) = p$ and that p is a non degenerate fixed point of $\psi_i; i = 0,1$. The work of Ize [11] implies that the only homotopy invariant determining the bifurcation of fixed points in terms of the family of linearizations $L \equiv \{T_p\psi_\lambda\}$ at p is given by the parity

$$\pi(L) = \text{sign } \det(T_p\psi_0) \cdot \text{sign } \det(T_p\psi_1) \in \mathbb{Z}_2 = \{1,-1\}.$$

Here det is the determinant of an endomorphism of the oriented vector space T_pM. In other words bifurcation arise whenever the $\det(T_p\psi_\lambda)$ change sign at the end points of the interval. Moreover, any family of diffeomorphisms close enough to ψ in the C^1-topology and having p as fixed point undergoes bifurcation as well. On the contrary if both sign coincide one can find a perturbation as above with no bifurcation points at all. The integer valued Conley-Zehnder index provides a stronger bifurcation invariant for one parameter families of Hamiltonian symplectomorphisms. It forces bifurcation of fixed points whenever the Conley-Zehnder index $\mathcal{CZ}(L)$ is non zero even when $\pi(L) = 1$. The relation between the two invariants is $\pi(L) = (-1)^{\mathcal{CZ}(L)}$.

A natural generalization of the classical Arnold's conjecture estimates the number of intersection points of two Lagrangian submanifolds of a symplectic manifold.

The cause that forces Hamiltonian deformation $L_1 = \phi(L)$ of a compact Lagrangian submanifold L of M to have a huge intersection with L can be explained as follows: by a well known theorem of Weinstein the submanifold L has a neighborhood symplectomorphic to a neighborhood of the zero section in the cotangent bundle $T^*(L)$. If L is simply connected and if L_1 is a Lagrangian submanifold that is C^1 close to L then L_1 is given by the image of the differential $dS : L \to T^*(L)$ of a smooth function $S : L \to \mathbb{R}^{2n}$ and therefore will have as many intersection points with L as critical points has the function S on L. The latter is bounded from below by Lusternik-Schnirelmann inequalities or by Morse inequalities if the critical points are non-degenerate. Of course L_1 need not be C^1-close to L. But when $M = T^*(N)$ using an Hamiltonian isotopy ϕ_λ with $\phi_1 = \phi$ one can still produce a family of generating functions $S : N \times \mathbf{R}^{\mathbf{k}} \to \mathbf{R}$ with k big enough such that critical points of S correspond to intersections of N with L_1. This is a Theorem of Sikorav [18]. Using this theorem one can still get estimates on the number of intersection points but weaker than in the previous case. Functions S as before are usually called generating families.

In [7] we showed that intersections of one parameter families of Lagrangian submanifolds with a given one have stronger bifurcation properties than the intersections of general submanifolds of right codimension essentially for the same reason as above. For families L_λ close enough in the C^1 topology to a given Lagrangian submanifold L_0 bifurcation of intersection points of L_λ with L_0 reduces, by the above described process, to bifurcation of critical points of one parameter families of smooth functions. In this setting bifurcation arises whenever the spectral flow, or what is the same, the difference between the Morse indexes of the end points of the trivial branch is non-zero. This gives a stronger invariant than the usual bifurcation index obtained by comparing the sign of the determinant of the Jacobian matrix of the gradient at the end points of the trivial branch. Via generating functions we showed that the assumption of being C^1 close can be substituted with a more general one without modifying the conclusions.

Namely the main result in [7] is as follows. Let N be a closed manifold and let $L = \{L_\lambda\}$ be an exact, compactly supported family of Lagrangian submanifolds of the symplectic manifold $M = T^*(N)$ such that L_0 admits a generating family quadratic at infinity. Let $p : [0,1] \to M$ be a path of intersection points of L_λ with N. Assume that L_λ is transversal to N at $p(\lambda)$ for $\lambda = 0,1$ and that the Maslov intersection index $\mu(L, N; p)$ is different from zero. Then arbitrarily close to the branch p there are intersection points of L_λ with N such that do not belong to p.

The results exposed here were obtained in collaboration with J. Pejsachowicz. Symplectic features nedeed for our purpose are collected in section §2. In section §3 we extend the definitions of the Maslov and Conley-Zehnder indeces to manifolds. This relies on the existence of symplectic trivializations of symplectic vector bundles over an interval. In §4 we outline how the bifurcation results of Fitzpatrick, Pejsachowicz and Recht are applied to the situations described above.

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2. Symplectic features

A symplectic manifold M is a differentiable manifold together with a closed nondegenerate differentiable two form ω, *i.e.*,

$$d\omega = 0 \text{ and } \forall Y \neq 0 \; \exists X : \omega(X, Y) \neq 0, \; X, Y \in T_m M.$$

Hence M must have even dimension and because $\omega^n/n!$ gives the canonical volumen form it is oriented.
The non-degeneracy condition induces an isomorphism between the tangent $T(M)$ and the cotangent space of the manifold $T^*(M)$ that assigns to each vector field X a 1-form $\iota_X \omega = \omega(X, .)$.

A diffeomorphism $\phi : (M, \omega) \to (M, \omega)$ that satisfies $\phi^* \omega = \omega$ is called symplectomorphism. In particular, a simplectomorphism preserves the volumen.
The requirement on the 2-form ω to be closed provides a correspondence between closed 1-forms and conservative vector fields since in this case $\mathcal{L}_X \omega = 0$ if and only if $d(\iota_X \omega) = 0$, such vector fields are called symplectic. The flow generated by a symplectic vector field consist of symplectomorphisms, *i.e.*, $\phi_t^* \omega = \omega \; \forall t$. A vector field is called Hamiltonian if the 1-form $\iota_X \omega$ is exact.
Because on a manifold there are many 1-forms the dimension of the group of symplectomorphisms of M, $Symp(M, \omega)$ is infinity. To the subset of exact 1-forms $\alpha = dH$ corresponds a normal subgroup $Ham(M, \omega)$ of $Symp(M, \omega)$.

In symplectic geometry there are no local invariants like for instance the curvature in Riemannian geometry. Darboux Theorem states that in some neighborhood of a given point one can choose a coordinate system $(U; x_1, \dots x_n, y_1, \dots, y_n)$ such that the restriction of the form to the neighborhood U is $\omega_{|U} = \omega_0 := \sum_{i=1}^n dx_i \wedge dy_i$. Hence the universal local model of a symplectic form is the standard symplectic form ω_0 in $\mathbb{R}^{2n}$. In this case the isomorphism between the tangent and cotangent space is given explicity by $X = \partial/\partial x_j \to \iota_X \omega_0 = dy_j$, $X = \partial/\partial y_j \to \iota_X \omega_0 = -dx_j$.

An important example of symplectic manifolds is the cotangent bundle of any manifold. Let N be an n-dimensional differentiable manifold. Let $T^*(N)$ be the cotangent bundle of N and $\pi : T^*N \to N$ the projection on N. There is a canonical 1-form λ_N on T^*N defined as follows: let ξ be a tangent vector to T^*N at the point $p \in T^*N$ ($\xi \in T_p(T^*(N))$). Since the element p is a cotangent vector on $T_x(N)$ where $x = \pi(p)$ and $\pi_*(\xi) \in T_x(N)$ define $\lambda_N(\xi) := p(\pi_*(\xi))$. In local coordinates $\lambda_N(\xi) = pdq$ and the symplectic 2-form is $\Omega = d\lambda_N$. Being exact it is closed and it is non-degenerate because in local coordinates $\Omega = dp \wedge dq$.

Let W be a vector subspace of a symplectic vector space (V, ω), the symplectic orthogonal to W is the vector subspace $W^\omega := \{v \in V/\omega(v,w) = 0\ \forall v, w \in W\}$. W is said to be isotropic if $W \subset W^\omega$. It is said to be coisotropic if $W \supset W^\omega$. If it is both isotropic and coisotropic it is called Lagrangian.

If W is isotropic, then W^ω is coisotropic and the symplectic form ω induces a symplectic form ϖ on the quotient space W^ω/W defined by $\varpi(v+W, w+W) = \omega(v,w)\ \forall v, w \in W^\omega$. The space $(W^\omega/W, \varpi)$ is called the *isotropic redution*. Moreover if L is a Lagrangian subspace of (V,ω) then $L_W = (L \cap W^\omega)/(L \cap W)$ is a Lagrangian subspace of W^ω/W.

Lagrangian submanifolds of a symplectic manifold (M, ω) are the submanifolds of maximal dimension where the symplectic form vanishes. They are characterized by $TL = (TL)^\omega$. Examples of Lagrangian submanifolds are the vertical fibers of a cotangent bundle T^*N. As for submanifolds transverse to the fibers, any such submanifold is locally the graph of a 1-form $\alpha\colon N \to T^*N$. The graph of a 1-form α is Lagrangian if and only if α is closed. If the 1-form is exact, *i.e.*, if $\alpha = dS$ the funtion S is called a generating function for the corresponding submanifold.

Any Lagrangian submanifold can be generated locally by a function on the product of N with a paramenter space, in which case it is called generating family.
The definition goes as follows (see [21]). Let V be a finite dimensional vector space. Consider a smooth function $S : N \times V \to \mathbb{R}$ such that the differential dS is transversal to the submanifold

$$N^0 = T^*(N) \times V \times \{0\} \text{ of } T^*(N \times V) \equiv T^*(N) \times V \times V$$

Denote by S_n the function $S_n\colon V \to \mathbb{R}$ defined by $S_n(v) = S(n,v)$ and by S_v the function $S_v\colon N \to \mathbb{R}$ defined by $S_v(n) = S(n,v)$. By the implicit function theorem, the set $C = \{(n,v)/dS_n(v) = 0\}$ of vertical critical points of S is a submanifold of $N \times V$ of the same dimension as N.
Let $e : C \to T^*(N)$ defined by $e(n,v) = dS_v(n)$. The map e is a Lagrangian immersion (but generally not an embedding) of the manifold C into T^*N. Given a Lagrangian submanifold L of $T^*(N)$, S is said to be a *generating family* for L if there is a diffeomorphism h from C onto L such that $e = ih$, where $i\colon L \to T^*(N)$ denotes the inclusion. The generating family S is said to be *quadratic at infinity* if there is a non-degenerate quadratic form Q on V such that $S(n,v) = Q(v)$ for $\|v\|$ big enough.

Diffeomorphisms of a manifold may be identified with their graphs, that is, with submanifolds of $M \times M$ which are mapped diffeomorphically onto M by the projections π_1, π_2. If M carries a symplectic structure ω, the form $\pi_1^*\omega - \pi_2^*\omega$ defines a symplectic structure on the product manifold $M \times M$. A diffeomorphism ϕ of a symplectic manifold (M,ω) is a symplectomorphism if and only if its graph is a Lagrangian submanifold of $(M \times M, \pi_1^*\omega - \pi_2^*\omega)$. Fixed points of ϕ correspond to intersections of the graph with the diagonal Δ of $M \times M$.

On a closed symplectic manifold (M^{2n}, ω) every smooth time dependent (Hamiltonian) function $H\colon \mathbb{R} \times M \to \mathbb{R}$ gives rise to a family of time dependent Hamiltonian vector fields $X\colon \mathbb{R} \times M \to TM$ defined by

$$\omega(X(t,x), \xi) = d_x H(t,x)\xi$$

for $\xi \in T_xM$. If H is periodic in time with period 1, then so is X. By compactness and periodicity the solutions $u(t)$ of the initial value problem for the Hamiltonian differential equation

$$\begin{cases} \frac{d}{dt}u(t) = X(t, u(t)), \\ u(s) = x \end{cases} \tag{2.1}$$

are defined for all times t. The flow (or evolution map) associated to X is the two-parameter family of symplectomorphisms $\psi\colon \mathbb{R}^2 \to Symp(M)$ defined by

$$\psi_{s,t}(x) = u(t)$$

where u is the unique solution of (2.1).
By the uniqueness and smooth dependence on initial value theorems for solutions of differential equations the map $\psi\colon \mathbb{R}^2 \times M \to M$ is smooth. The diffeomorphisms $\psi_{s,t}$ verify the usual cocycle property of an evolution operator *i.e.* ,$\psi_{s,r} \circ \psi_{r,t} = \psi_{s,t}$ and $\psi_{t,t} = \mathrm{Id}$. From this property it follows that for each fixed s, the map sending u into $u(s)$ is a bijection between the set of 1-periodic solutions of the time dependent vector field X and the set of all fixed points of $\psi_{s,s+1}$. Hence in order to find periodic trajectories of (2.1) we can restrict our attention to the fixed points of $P = \psi_{0,1}$. The map $P = \psi_{0,1}$ is called the period or Poincaré map of X.
A 1-periodic trajectory is called non degenerate if $p = u(0)$ is a non degenerate fixed point of P, *i.e.* , if the monodromy operator $S_p \equiv T_pP\colon T_pM \to T_pM$ has no 1 as eigenvalue. Consistently, the eigenvalues of the monodromy operator will be called Floquet multipliers of the periodic trajectory. The particular choice of $s = 0$ is irrelevant to the property of being non degenerate since the Floquet multipliers do not depend on this choice. (see [1])

Every symplectomorphism that can be represented as a time 1-map of such a time dependent Hamiltonian flow is called a Hamiltonian map. If M is simply connected the connected component of the identity map $Symp_0(M, \omega)$ in the space of symplectic diffeomorphisms $Symp(M, \omega)$ consists of Hamiltonian maps (see [12]).

3. The Maslov index and the Conley-Zehnder index

Before going to the manifold setting let us discuss the case of $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ with the standard symplectic form $\omega_0 = \sum dx_i \wedge dy_i$. The group of real $2n \times 2n$ symplectic matrices will be denoted by $\mathbf{Sp}(2n, \mathbb{R})$.

The relative Conley-Zehnder index is a homotopy invariant associated to any path $\psi\colon [0,1] \to \mathbf{Sp}(2n, \mathbb{R})$ of symplectic matrices with no eigenvectors corresponding to the eigenvalue 1 at the end points. This invariant counts algebraically the number of parameters t in the open interval $(0, 1)$ for which $\psi(t)$ has 1 as an eigenvalue. One of the possible constructions uses the Maslov index for non-closed paths. We shall define it along the lines of Arnold [3] for closed paths. For an alternative construction see Robbin and Salamon [15].

The Lagrangian Grassmaniann $\Lambda(n)$ consists of all Lagrangian subspaces of $\mathbb{R}^{2n}$ considered as a topological space with the topology it inherits as a subspace of the ordinary Grassmanian of n-planes. Let J be the selfadjoint endomorphism representing the form ω_0 with respect to the standard scalar product in $\mathbb{R}^{2n}$. Namely, $\omega_0(u, v) \equiv < Ju, v >$. Then J is a complex structure, it is indeed the standard one. It coincides with multiplication by i under the isomorphism sending $(x, y) \in \mathbb{R}^{2n}$

into $x + iy$ in $\mathbb{C}^n$. In terms of this representation, a Lagrangian subspace is characterized by $JL = L^{\perp}$.

Using the above description one can identify $\Lambda(n)$ with the homogeneous space $U(n)/O(n)$. This can be done as follows: given any orthonormal basis of a Lagrangian subspace L there exist a unique unitary endomorphism $A \in U(n)$ sending the canonical basis of $L_0 = \mathbb{R}^n \times \{0\}$ into the given one and in particular sending L_0 into L. Moreover the isotropy group of L_0 can be easily identified with $O(n)$. Hence we obtain a diffeomorphism between $U(n)/O(n)$ and $\Lambda(n)$ sending the class $[A]$ into $A(L_0)$. Since the determinant of an element in $O(n)$ is ± 1, the map sending A into the square of the determinant of A factorizes through $\Lambda(n) \equiv U(n)/O(n)$ and hence induces a one form $\Theta \in \Omega^1\big(\Lambda(n)\big)$ given by $\Theta = [det^2]^*\theta$, where $\theta \in \Omega^1(S^1)$ is the standard angular form on the unit circle. This form is called the Keller-Maslov-Arnold form.

The Maslov index of a closed path γ in $\Lambda(n)$ is the integer defined by $\mu(\gamma) = \int_\gamma \Theta$. In other words $\mu(\gamma)$ is the winding number of the closed curve $t \to det^2\big(\gamma(t)\big)$. The Maslov index induces an isomorphism between $\pi_1(\Lambda)$ and Z.
The construction can be extended to non-closed paths as follows: fix $L \in \Lambda(n)$. If L' is any Lagrangian subspace transverse to L then L' can be identified with the graph of a symmetric transformation from JL into itself. It follows from this that the set Λ_L of all Lagrangian subspaces L' transverse to L is an affine space diffeomorphic to the space of all symmetric forms on $\mathbb{R}^n$ and hence it is contractible.
We shall say that a path in $\Lambda(n)$ is admissible with respect to L if the end points of the path are transverse to L. The Maslov index $\mu(\gamma, L)$ of an admissible path γ with respect to L is defined as follows: take any path δ in Λ_L joining the end points of γ and define

$$\mu(\gamma; L) \equiv \mu(\gamma') = \int_{\gamma'} \Theta.$$

where γ' is the path γ followed by δ. The result is independent of the choice of δ. Moreover, since Λ_L is contractible, $\mu(\gamma; L)$ is invariant under homotopies keeping the end points in Λ_L.
Geometricaly, the Maslov index $\mu(\gamma; L)$ can be interpreted as an intersection index of the path γ with the one codimensional analytic set $\Sigma_l = \Lambda(n) - \Lambda_L$ (see [16]). From the definition it follows that the index is additive under concatenation of paths. Namely, given two admissible paths α and β with $\alpha(1) = \beta(0)$

$$\mu(\alpha \star \beta; L) = \mu(\alpha; L) + \mu(\beta; L).$$

Since $\mathbf{Sp}(2n, \mathbb{R})$ is connected it follows from the homotopy invariance that

$$\mu(S\gamma; SL) = \mu(\gamma; L)$$

for any symplectic isomorphism S. This allows to extend the notion of Maslov Index to paths of Lagrangian subspaces in $\Lambda(V)$, where (V, ω) is any finite dimensional symplectic vector space.

Graphs of symplectic endomorphisms are Lagrangian subspaces of the symplectic vector space $V \times V$ endowed with the symplectic form $\omega \times (-\omega)$. The graph of $P \in \mathbf{Sp}(2n, \mathbb{R})$ is transversal to the diagonal $\Delta \subset V \times V$ if and only if 1 is not an eigenvalue of P. A path $\phi\colon [0,1] \to \mathbf{Sp}(2n, \mathbb{R})$ will be called admissible if 1 is not in the spectrum of its end points. For such a path the *relative Conley-Zehnder index* is defined by

$$\mathcal{CZ}(\phi) = \mu(Graph\phi, \Delta). \tag{3.1}$$

From the above discussion it follows that $\mathcal{CZ}(\phi)$ is invariant under admissible homotopies and it is additive with respect to concatenation of paths. If the fixed point space of $\phi(\lambda)$ reduces to $\{0\}$ for all λ then $\mathcal{CZ}(\phi) = 0$.
There is one more property of the Conley-Zehnder index that we use in the sequel. Namely, that for any $\alpha\colon [0,1] \to \mathbf{Sp}(2n, \mathbb{R})$ and any admissible path ϕ

$$\mathcal{CZ}(\alpha^{-1}\phi\alpha) = \mathcal{CZ}(\phi). \tag{3.2}$$

This can be seen as follows. Since the spectrum is invariant by conjugation, the homotopy $(t, s) \to \alpha^{-1}(s)\phi(t)\alpha(s)$ shows that $\mathcal{CZ}(\alpha^{-1}\phi\alpha) = \mathcal{CZ}(\alpha^{-1}(0)\phi\alpha(0))$. Now (3.2) follows by the same argument applied to any path joining $\alpha(0)$ to the identity.
The property (3.2) allows to associate a Conley-Zehnder index to any admissible symplectic automorphism of a symplectic vector-bundle over an interval. Let I be the interval $[0, 1]$, then any symplectic bundle $\pi\colon E \to I$ over I has a symplectic trivialization. If $S\colon E \to E$ is a symplectic endomorphism of E over I well behaved at the end points, then we can define the Conley-Zehnder index of S as follows: if $T\colon E \to I \times \mathbb{R}^{2n}$ is any symplectic trivialization, then $TST^{-1}(\lambda, v)$ has the form $(\lambda, \phi_T(\lambda)v)$ where ϕ_T is an admissible path on $\mathbf{Sp}(2n, \mathbb{R})$. Any change of trivialization induces a change on ϕ_T that has the form of the left hand side in (3.2) and hence $\mathcal{CZ}(\phi_T)$ is independent of the choice of trivialization. Thus the Conley-Zehnder index of S is defined to be $\mathcal{CZ}(S) \equiv \mathcal{CZ}(\phi_T)$.

Now let's define the relative Conley-Zehnder index of a path of symplectomorphisms along a path of fixed points: let M be a closed symplectic manifold and let $Symp(M)$ be the group of all symplectomorphisms endowed with the C^1 topology. Let $\phi\colon I \to Symp(M)$ be a smooth path of symplectomorphisms of M. Let $p\colon I \to M$ be a path in M such that $p(\lambda)$ is a fixed point of $\phi(\lambda)$. Floquet multipliers of $\phi(\lambda)$ at $p(\lambda)$ are by definition the eigenvalues of $S_\lambda = T_{p(\lambda)}\phi(\lambda)\colon T_{p(\lambda)}(M) \to T_{p(\lambda)}(M)$. A fixed point will be called non degenerate if none of its Floquet multipliers is one. Consistently, we will call the pair (ϕ, p) admissible whenever $p(i)$ is a non degenerate fixed point of $\phi(i)$ for $i = 0, 1$.
Let $E = p^*[T(M)]$ be the pullback by p of the tangent bundle of M (we use the same notation for the bundle and its total space). The family of tangent maps $S_\lambda = T_{p(\lambda)}\phi(\lambda)$ induces a symplectic automorphism $S\colon E \to E$ over I. Define the *relative Conley-Zehnder index of ϕ along p* by

$$\mathcal{CZ}(\phi; p) \equiv \mathcal{CZ}(S). \tag{3.3}$$

From the properties discussed above it follows that the relative Conley-Zehnder index $\mathcal{CZ}(\phi; p)$ of ϕ along p is invariant by smooth pairs of homotopies $(\phi(s, t), p(s, t))$ such that $\phi(s, t)(p(s, t)) = p(s, t)$ and such that for $i = 0, 1$; $p(s, i)$ is a non degenerate fixed point of $\phi(s, i)$.
The index is additive under concatenation. It follows from (3.2) that it has another interesting property, which for simplicity we state in the case of a constant path $p(t) \equiv p$. If $\phi, \psi\colon I \to Symp(M)$ are two admissible paths in the isotropy subgroup of p then

$$\mathcal{CZ}(\psi \circ \phi, p) = \mathcal{CZ}(\phi \circ \psi, p).$$

In other words $\mathcal{CZ}$ is a trace.

Finally let us define the Maslov intersection index of two families of Lagrangian submanifolds L_λ and N_λ of a symplectic manifold M along a given path $p\colon I \to M$ of intersection points.
Since the interval I is contractible, the pullback $p^*(TM)$ by p of the tangent bundle of M is a trivial bundle whose fiber over λ is the tangent space $T_{p(\lambda)}M$. Taking any trivialization $T\colon p^*(TM) \to I \times \mathbb{R}^{2n}$ of this bundle the images under the trivialization maps $T_\lambda\colon T_{p(\lambda)}M \to \mathbb{R}^{2n}$ of the tangent spaces T_pL_λ and T_pN_λ determine two paths $l(\lambda)$ and $n(\lambda)$ in the space Λ_n of all Lagrangian subspaces of $\mathbb{R}^{2n}$. Assuming that the paths l, n have transverse intersection at the end points, the path $l \times n$ has endpoints transversal to the diagonal Δ in $(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$. Since the space of Lagrangian subspaces transversal to a given one is contractible, if we take any path δ joining the endpoints of $l \times n$ the Maslov index of a close path made by $l \times n$ followed by δ, is independent of the choice of δ. The index of this closed path is by definition the relative Maslov index $\mu(l, n)$ (*cf.* [15]). This index is an integer which counts with appropriate multiplicities the points in $(0, 1)$ where $l(\lambda) \cap n(\lambda) \neq \{0\}$. From the invariance of the Maslov index under the action of the symplectic group it follows that $\mu(l, n)$ is independent of the choice of trivialization. We call it (once more!) the *Maslov intersection index* of the family $L = \{L_\lambda\}$ with $N = \{N_\lambda\}$ along p, and we denote it by $\mu(L, N, p)$.
The last crucial property that we need to mention is the invariance of the Maslov index under isotropic reduction. Consider a Lagrangian subspace $L \subset (V, \omega)$ and a path of Lagrangian subspaces $l : [0, 1] \to \Lambda(V)$ such that the endpoints $l(0)$ and $l(1)$ are transverse to L. If W is an isotropic subspace such that $W \subset L$ which has transverse intersection with $l(t)$ for all $t \in [0, 1]$ then following the lines of Viterbo (*cf.* Proposition 3 of [20]) it can be proved that the path $l_W : [0, 1] \to \Lambda(W^\omega/W)$ defined by $l_W(t) := l(t)/W$ is continuous and that the Maslov index of the path l_W relative to the Lagrangian subspace $L_W := L/W$ of W^ω/W coincide with the Maslov index of the path l relative to the Lagrangian subspace L, that is,

$$\mu_{L_W}(l_W) = \mu_L(l).$$

4. Bifurcations

a) from periodic orbits of 1-parameter families of time dependent Hamiltonian systems

Bifurcation theory deals with the problem of existence of nontrivial solutions arbitrary closed to a known family of solutions. For this purpose one takes into consideration a smooth one parameter family of time dependent Hamiltonian functions $H\colon I \times \mathbb{R} \times M \to \mathbb{R}$, where $I = [0, 1]$ is the parameter set and each $H_\lambda\colon \mathbb{R} \times M \to \mathbb{R}$ is one periodic in time. Let $X \equiv \{X_\lambda\}_{\lambda \in [0,1]}$ be the corresponding one parameter family of Hamiltonian vector fields. Then the flows $\psi_{\lambda,s,t}$ associated to each X_λ depend smoothly on the parameter $\lambda \in I$. Suppose also that the 1-parameter family of Hamiltonian vector fields X_λ possesses a known smooth family of 1-periodic solutions u_λ; $u_\lambda(t) = u_\lambda(t+1)$. Solutions u_λ in this family are called *trivial* and we seek for sufficient conditions in order to find nontrivial solutions arbitrarily close to the given family. Identifying $\mathbb{R}/\mathbb{Z}$ with the circle S^1 we regard the family of trivial solutions either as a path $\tau\colon I \to C^1(S^1; M)$ defined by $\tau(\lambda) = u_\lambda$ or as a smooth map $u\colon I \times S^1 \to M$.

A point $\lambda_* \in I$ is called a *bifurcation point* of periodic solutions from the trivial branch u_λ if every neighborhood of $(\lambda_*, u_{\lambda_*})$ in $I \times C^1(S^1; M)$ contains pairs of the form (λ, v_λ) where v_λ is a nontrivial periodic trajectory of X_λ.
A necessary condition for a point λ_* to be of bifurcation is that 1 is a Floquet multiplier of u_{λ_*}. This condition is not sufficient (See for example [2] Proposition 26.1). Thus non degenerate orbits cannot be bifurcation points of the branch. In what follows we will assume that $u(0)$ and $u(1)$ are non degenerate and we will seek for bifurcation points in the open interval $(0, 1)$.
Consider the path $p\colon I \to M$ given by $p(\lambda) = u_\lambda(0)$. Each $p(\lambda)$ is a fixed point of the symplectomorphism $P_\lambda = \psi_{\lambda,0,1}$. Under our hypothesis, the pair (P, p) is admissible. The number $\mathcal{CZ}(P, p)$ constructed in the previous section will be called the relative Conley-Zehnder index of $X \equiv \{X_\lambda\}_{\lambda\in[0,1]}$ along the trivial family u. We denote it by $\mathcal{CZ}(X, u)$. If this index is not zero one has the following

Theorem A: *Let $X \equiv \{X_\lambda\}_{\lambda\in[0,1]}$ be a one parameter family of 1-periodic Hamiltonian vector fields on a closed symplectic manifold (M, ω). Assume that the family X_λ possesses a known, trivial, branch u_λ of 1-periodic solutions such that $u(0)$ and $u(1)$ are non degenerate. If the relative Conley-Zehnder index $\mathcal{CZ}(X, u) \neq 0$ then the interval I contains at least one bifurcation point for periodic solutions from the trivial branch u.*

For the proof (see [6]) we followed an idea of Salamon and Zehnder [17] (Lemma 9.2.) in the nonparametric case. It consist in using appropiate symplectic trivializations and applying Moser's Method [14] to construct local Darboux coordinates $(V, \psi_{\lambda,t})$ on the manifold M adapted to the λ-parameter family $u_\lambda(t)$ of periodic solutions of the Hamiltonian differential equation

$$\begin{cases} \frac{d}{dt} u_\lambda(t) = X_\lambda(t, u_\lambda(t)), \\ u_\lambda(s) = x \end{cases} \tag{4.1}$$

i.e. , we showed the existence of an open neighborhood V of 0 in $\mathbb{R}^{2n}$ and of a family of symplectomorphisms $\psi_{\lambda,t}\colon V \to M$ that satisfies $\psi_{\lambda,t}(0) = u_\lambda(t)$ and $\psi^*_{\lambda,t}\omega = \omega_0$ on V. The new coordinates allowed us to reduce our problem to the Fitzpatrick, Pejsachowicz and Recht's bifurcation theorem in [9].

b) from intersection points of 1-parameter families of lagrangian submanifolds

Let $T^*(N)$ be the cotangent bundle of a closed manifold N endowed with the standard symplectic structure. We will consider bifurcations of intersections of $N \equiv 0_N$ identified with the zero section of the bundle $T^*(N)$ with an exact one-parameter family of Lagrangian submanifolds $L = \{L_\lambda\}_{\lambda\in[0,1]}$ such that L_λ coincides with L_0 outside of a compact subset of $T^*(N)$. More precisely we consider families $L_\lambda = i_\lambda(L_0)$ where $i_\lambda\colon L_0 \to T^*(N)$ is a smooth family of Lagrangian embeddings with $i_\lambda \equiv i_0$ outside of a compact subset of L_0. Such a family is said to be *compactly supported*. Moreover L is called *exact* if the one-form $i^*\omega(\frac{\partial}{\partial\lambda}, -)$ is exact on $[0, 1] \times L_0$. The natural topology in the space of all Lagrangian submanifolds of a given manifold is discussed in [22]. Remark that a family i_λ as above induces a continuous path in the space $C^\infty(L_0, T^*(N))$ with respect to the fine C^1 topology. Therefore L_r is C^1 close to L_s whenever r is close enough to s.

Let $I = [0,1]$ and let $p\colon I \to T^*(N)$ be a smooth path such that $p(\lambda) \in L_\lambda \cap N$. A point $p(\lambda_*) \in L_{\lambda_*} \cap N$ is called *bifurcation point* from the given path p of intersection points if any neighborhood of $(\lambda_*, p(\lambda_*))$ in $[0,1] \times T^*(N)$ contains points (λ, q) with $q \in L_\lambda \cap N,\ q \neq p(\lambda)$.
It follows from the implicit function theorem that a necessary condition for $p(\lambda_*)$ to be a bifurcation point of intersection is that the manifold L_{λ_*} fails to be transversal to N at $p(\lambda_*)$. This means that for $p_* = p(\lambda_*)$ one has that $T_{p_*}L_{\lambda_*} + T_{p_*}N$ is a proper subset of the tangent space $T_{p_*}(T^*(N))$. Since $\dim T_{p_*}L_{\lambda_*} = \dim T_{p_*}N = \frac{1}{2}\dim T^*(N)$ this turns out to be equivalent to

$$T_{p_*}L_{\lambda_*} \cap T_{p_*}N \neq \{0\}.$$

This condition is not sufficient. Assuming that the manifolds L_0, L_1 are transverse to N, under some extra assumption the nonvanishing of $\mu(L, N, p)$ provides a sufficient condition for the existence of at least one bifurcation point.

Theorem B: *Let N be a closed manifold and let $L = \{L_\lambda\}$ be an exact, compactly supported family of Lagrangian submanifolds of $T^*(N)$ such that L_0 admits a generating family quadratic at infinity. Let $p : [0,1] \to T^*(N)$ be a path of intersection points of L_λ with N. Assume L_λ is transverse to N at $p(\lambda)$ for $\lambda = 0,1$ and that the Maslov intersection index $\mu(L, N, p) \neq 0$, then there exist a $\lambda_* \in (0,1)$ such that $p(\lambda_*)$ is a point of bifurcation for intersection points of L_λ with N from the trivial branch p.*
If $L_0 = N$ then the first assumption of the theorem holds by taking $S = 0$.

The basic idea of the proof of Theorem B is to convert our problem to that of finding bifurcations of critical points of one parameter families of functionals. We used a result of Sikorav which guarantees the existence of generating families for deformations of Lagrangian submanifolds under Hamiltonian isotopies (see proposition 1.2 and Remark 1.7 in [18]). More precisely, if ϕ_λ is a Hamiltonian isotopy of $T^*(N)$ and if $L_0 \subset T^*(N)$ is generated by a family quadratic at infinity then there exists a smooth family of functions $S_\lambda\colon N \times \mathbb{R}^k \to \mathbb{R}$ quadratic at infinity such that $\phi_\lambda(L_0)$ is generated by the family S_λ. On the other hand Chaperon [4] [5] proved that any one parameter exact compactly supported family of Lagrangian embeddings $L_\lambda = i_\lambda(L_0)$ can be extended to a Hamiltonian isotopy of the ambient manifold. Putting both results toghether we have that for any smooth family L_λ of Lagrangian submanifolds of $T^*(N)$ there exists a smooth family

$$S\colon [0,1] \times N \times \mathbb{R}^k \to \mathbb{R}$$

quadratic at infinity such that S_λ generates L_λ, where $S_\lambda(n,v) = S(\lambda, n, v)$.
Thus each $L_\lambda = e_\lambda(C_\lambda)$ where $C_\lambda = \{(u,v)/v \text{ is critical of } S_{\lambda,n}\}$, the functions $S_{\lambda,n}\colon \mathbb{R}^k \to \mathbb{R}$ and $S_{\lambda,v}\colon N \to \mathbb{R}$ are given by $S_{\lambda,n}(v) = S_\lambda(n,v)$ and $S_{\lambda,v}(n) = S_\lambda(n,v)$ and $e_\lambda\colon C_\lambda \to T^*(N)$ is defined by $e_\lambda(n,v) = dS_{\lambda,v}(n)$.
Since here each e_λ is an embedding it induces a bijection between critical points of $S_\lambda\colon N \times \mathbb{R}^k \to \mathbb{R}$ and intersection points in $L_\lambda \cap N$. Therefore the path of intersection points p has a corresponding path $\tau\colon I \to N \times \mathbb{R}^k$ of critical points of S_λ. Because L_0, L_1 are transversal to N at $p(0), p(1)$ it follows that $\tau(0)$ and $\tau(1)$ are non-degenerate critical points. This is a direct consequence of the linear algebra of symplectic reductions. Indeed, let $N' = N \times \mathbb{R}^k$ and consider the symplectic manifold $T^*(N') = T^*(N) \times \mathbb{R}^{2k}$. The manifold $\{0\} \times \mathbb{R}^k$ is an isotropic submanifold of $T^*(N')$ and $T^*(N)$ is the symplectic reduction of $T^*(N')$ modulo the isotropic

submanifold $\{0\}\times\mathbb{R}^k$. On the other hand N' and dS_λ are lagrangian submanifolds of $T^*(N')$ whose symplectic reductions are N and L_λ respectively. Since L_λ intersects transversally N at $p(\lambda)$, for $\lambda = 0, 1$, then dS_λ intersects transversally N'. But this is equivalent to the non-degeneracy of the critical point $\tau(\lambda)$ for $\lambda = 0, 1$.

At any critical point the Hessian $H(S_\lambda, \tau(\lambda))$ of S_λ at $\tau(\lambda)$ is a well defined symmetric bilinear form. The Morse index $m(S, x)$ of S at a nondegenerate critical point is the dimension of the negative eigenspace of $H(S, x)$. From Morse theory the inequality $m(S_1, \tau(1)) \neq m(S_0, \tau(0))$ guarantees the existence of bifurcation critical points [13]. Since L_λ is the image of $dS_{\lambda,v}\colon N \to T^*(N)$, identifying N with the zero section we have that L_λ is transversal to N for $\lambda = 0, 1$ and by the localization properties of the relative Maslov index (Theorem 2.3 in [16]) it equals the difference of the Morse indeces at the endpoints of the path, that is,

$$\mu(dS, N', \tau) = m(S_1, \tau(1)) - m(S_0, \tau(0)).$$

But the Maslov index is invariant under isotropic reduction thus

$$\mu(dS, N', \tau) = \mu(L, N, p).$$

Hence the hypothesis of Theorem B implies that it is possible to find a sequence of critical points of S_λ bifurcating from the trivial branch. Via e_λ those critical points correspond to nontrivial intersections of L_λ with N

c) FROM FIXED POINTS OF A ONE PARAMETER FAMILY OF SYMPLECTOMORPHISMS
We discusse now bifurcations of a path of fixed points of a one parameter family of symplectomorphisms. Consider a closed symplectic manifold (M, ω). We assume here that the first Betti number $\beta_1(M)$ of M vanishes, since in this case any symplectic diffeomorphism belonging to the connected component of the identity $Symp_0(M)$ of the group of all symplectic diffeomorphisms can be realized as the time one map of a 1-periodic Hamiltonian vector field. The following result can be obtain as a consequence either of Theorem A or of Theorem B.

COROLLARY: *Assume that $\beta_1(M) = 0$. Let ϕ_λ be a path in $Symp_0(M)$ such that $\phi_\lambda(p) = p$ for all λ and such that as fixed point of ϕ_0 and ϕ_1, p is non degenerate. Then if $\mathcal{CZ}(\phi, p) \neq 0$, there exist a $\lambda_* \in (0, 1)$ such that any neighborhood of (λ_*, p) in $I \times M$ contains a point (λ, q) such that q is a fixed point of ϕ_λ different from p (i.e λ_* is a bifurcation point for fixed points of ϕ_λ from the trivial branch p).*
Moreover the same is true for any close enough path in the C^1-topology lying in the isotropy group of p.

To each symplectomorphism ϕ_λ there corresponds a time dependent family of vector fields X_λ, and to each of this it corresponds a time dependent family of Hamiltonian function H_λ. In [6] we proved that there exist a family of time dependent hamiltonian functions $H'\colon I \times I \times V \to \mathbb{R}$ that depends smoothly on the parameter λ such that ϕ_λ is the time-one map of the corresponding time-dependent Hamiltonian vector field $X'_\lambda : I \times M \to TM$. Then because of the one to one correspondence between 1-periodic orbits of the Hamiltonian vector field with fixed points of the period map we can apply Theorem A.

Let us discuss now the relationship with intersection points of lagrangian submanifolds. Consider $M \times M$ with the symplectic form $\pi_1^*\omega - \pi_2^*\omega$. Given a path of symplectomorphisms ϕ_λ and a path of fixed points $p(\lambda)$ of ϕ_λ having non-degenerate end points (*i.e.*, such that $T_{p(\lambda)}\phi_\lambda$ is nonsingular for $\lambda = 0, 1$), the path of fixed points corresponds to a path of intersection points of the graph of ϕ_λ with the diagonal Δ and the Maslov intersection index $\mu(\text{Graph } \phi, \Delta, p \times p)$ along the intersection path is well defined and coincides with the relative Conley-Zehnder index of ϕ along p.

By Weinstein's theorem [22] any Lagrangian submanifold of a symplectic manifold has a neighborhood that is symplectomorphic to a neighborhood of the zero section of its own cotangent bundle. We apply Weinstein's theorem to the diagonal Δ in $M \times M$ and then modify the Hamiltonian and the flow $\phi_{\lambda,t}$ outside of a neighborhood of p in such a way that the new flow equals the identity outside of a compact neighborhood of p. There the graph of ϕ_λ coincide with Δ and thus it can be viewed as a one parameter family of Lagrangian submanifolds of $T^*\Delta$ with compact support.

Since $H^1(M, \mathbb{R}) = 0$ we get that $L \equiv L_\lambda$ is exact. Moreover by Sikorav's theorem L_0 possesses a generating family being ϕ_0 isotopic by a Hamiltonian isotopy to the identity map of T_*N. Hence we can apply Theorem B to the family L and Δ.

We close this section with a formula that allows to compute the individual contribution of a regular point in the trivial branch to the Conley-Zehnder index and give an example where bifurcation cannot be detected using the parity.

Assume that λ_0 is an isolated point in the set

$$\Sigma = \{\lambda / p(\lambda) \text{is a degenerate fixed point of } \ \phi(\lambda)\}.$$

Define $\mathcal{CZ}_{\lambda_0}(\phi) \equiv \lim_{\epsilon \to 0} \mathcal{CZ}(\phi; p_{|[-\epsilon,\epsilon]})$. The point λ_0 is called regular (cf. [16]) if the quadratic form Q_{λ_0} on the eigenspace $E_1(S_{\lambda_0}) = Ker(S_{\lambda_0} - Id)$ corresponding to the eigenvalue 1 defined by $Q_{\lambda_0}(v) = \omega(\dot{S}_{\lambda_0} v, v)$ is nondegenerate.
Here $S_\lambda = T_{p(\lambda)}\phi(\lambda)$ as before and $\dot{S}_{\lambda_0}$ denotes the intrinsic derivative of the vector bundle endomorphism S (See [10] chap 1 sect 5). If t_0 is a regular point then it is an isolated point in Σ and

$$\mathcal{CZ}_{\lambda_0}(\phi) = -\sigma(Q_{\lambda_0}) \tag{4.2}$$

where σ denotes the signature of a quadratic form. This formula follows from the definition of the intrinsic derivative and formula (2.8) in [9].

Example: Let M be the symplectic manifold $S^2 = \mathbb{C} \cup \{\infty\}$. Consider the closed path of symplectic maps $\phi_\theta : S^2 \to S^2; \theta \in [0,1]$ defined by

$$\phi_\theta(z) = \begin{cases} e^{i2\pi(\theta - 1/2)} \cdot z & \text{if } z \in \mathbb{C}, \\ \infty & \text{if } z = \infty \end{cases}$$

ϕ_θ is a rotation of angle $\theta - 1/2$ so it leaves fixed only the points $z = 0$ and $z = \infty$ except for $\theta = 1/2$, in which case the fixed point set is the sphere S^2. For each θ the tangent map $T_0\phi_\theta$ of ϕ_θ at the fixed point $z = 0$ equals ϕ_θ. The only value of θ for which 1 is an eigenvalue of the tangent map $T_0\phi_\theta$ is $\theta = 1/2$ for which the corresponding eigenspace is $\mathbb{C}$. Moreover 0 is a regular degenerate fixed point of $\phi_{1/2}$. The relative Conley-Zehnder index $\mathcal{CZ}_0(\phi; 0)$ of the symplectic isotopy ϕ along the constant path of fixed points $p = 0$ coincides with the signature of the quadratic form $Q_{1/2} = \omega(\dot{\phi}_{1/2} -, -)$ that is non degenerate on the eigenspace $E_1(\phi_{1/2})$. Then since

$$\dot{\phi}(1/2) = i2\pi Id$$

it follows from (4.2) that

$$\mathcal{CZ}_0(\phi;0) = -\sigma[v \to \omega(\dot{\phi}(1/2)v, v)] = \sigma[v \to 2\pi < v, v >] = 2.$$

Therefore any closed path of symplectomorphisms on the sphere keeping 0 fixed and homotopic to ϕ has nontrivial fixed points close to zero.

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