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On the all-order perturbative finiteness of the deformed $\mathcal{N} = 4$ SYM theory

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Abstract

We prove that the chiral propagator of the deformed $\mathcal{N} = 4$ SYM theory can be made finite to all orders in perturbation theory for any complex value of the deformation parameter. For any such value the set of finite deformed theories can be parametrized by a whole complex function of the coupling constant g. We reveal a new protection mechanism for chiral operators of dimension three. These are obtained by differentiating the Lagrangian with respect to the independent coupling constants. A particular combination of them is a CPO involving only chiral matter. Its all-order form is derived directly from the finiteness condition. The procedure is confirmed perturbatively through order g^6.

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1 Introduction

Recently, following [1, 2, 3], there has been a renewed interest in the deformed $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. It has the same field content as $\mathcal{N} = 4$ SYM, namely (in an $\mathcal{N} = 1$ formulation) a gauge superfield V and a set of three chiral matter superfields Φ^I, $I = 1, 2, 3$, all in the adjoint representation of the gauge group $SU(N)$. What distinguishes the deformed theory from $\mathcal{N} = 4$ SYM is the deformed superpotential

$$\mathcal{W} = g\,\kappa\;\mathrm{tr}\left(\Phi^1\;[\Phi^2, \Phi^3]_\omega\right)\,, \tag{1}$$

where g is the $\mathcal{N} = 1$ SYM coupling constant and the deformed commutator is defined as

$$[A, B]_\omega = \omega\;AB - BA\,. \tag{2}$$

The parameter κ can be considered real (its phase can be absorbed into a redefinition of Φ^I), while ω is in general complex. The undeformed $\mathcal{N} = 4$ SYM is recovered when $\kappa = \omega = 1$. Although in principle both κ and ω can be taken as Taylor series expansions in powers of g^2 around $g = 0$, in most of the recent literature [4, 5, 6, 7, 8] the case of constant ω has been commonly considered.

The main feature of the deformed theory is that, despite the breaking of $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 1$, it can be made finite (and thus conformal) by imposing a condition on the parameters g, κ, ω. The search for finite $\mathcal{N} = 1$ theories has a long history [9, 10, 11, 12, 13]. In the most general case one considers a superpotential of the Yukawa type

$$Y_{ijk}\Phi^i\Phi^j\Phi^k\,, \tag{3}$$

where i, j, k are combined color and flavor indices, and Y_{ijk} is a set of complex couplings. These theories are finite if all beta and gamma functions vanish. In the matter sector β_Y and γ_Y are related through the non-renormalization of the chiral vertex [14, 15], so it is sufficient to demand the vanishing of the matrix of gamma functions of the chiral superfields, $\gamma_\Phi(g, Y)^i_j = 0$. This is a set of conditions on the couplings which are to be adjusted order by order in perturbation theory. The existence of a solution in the general case has been studied in [11, 12].

The superpotential (1) is a particular case of (3) with the interesting feature that all matter gamma functions are equal due to the $Z_3 \times Z_3$ symmetry of the potential[1]. So, it is enough to impose a *single finiteness condition*

$$\gamma_\Phi(g, \kappa, \omega) = 0 \tag{4}$$

[1]The most general superpotential with this property was found in [1, 16]; see Section 3.2.4 for more details.

to ensure that the matter propagators and couplings do not undergo infinite renormalization. This feature of the so-called "β–deformed $\mathcal{N} = 4$ theory" with superpotential (1) was essential for finding its gravity dual in [3] and for the subsequent development in the context of the AdS/CFT correspondence [17, 18].

The question about the vanishing of the propagator corrections and of the beta function in the $\mathcal{N} = 1$ gauge sector of theories with the superpotential (3) is more subtle. A three-loop result is available [10], but its generalization to all orders [13, 1] relies on the existence of the so-called "exact β function" [19].

The first steps in the study of the perturbative aspects of the deformed theory with superpotential (1) were made in [4], with a particular accent on the chiral primary operators (CPO) in the theory. Subsequently, the condition for finiteness were established at two loops in [5] and at three loops in [6]. An all-order condition in the large N limit was found in [7]. The set of CPOs ("chiral ring") of the deformed theory was further studied in [5, 6, 7, 8].

In the present paper we concentrate on two particular perturbative issues in the deformed $\mathcal{N} = 4$ SYM theory.

In Section 2 we investigate the finiteness properties of a theory with superpotential $\mathcal{W}$, deformed by a g-dependent deformation parameter. For future convenience we shall write it in the form[2]

$$\begin{aligned}\mathcal{W}_{\kappa,\omega} \;=\;& g\left\{\kappa_\omega(g)\ \mathrm{tr}\left(\Phi^1\ [\Phi^2,\Phi^3]_\omega\right)\ +\ \kappa_\Omega(g)\ \mathrm{tr}\left(\Phi^1\ [\Phi^2,\Phi^3]_\Omega\right)\right\}\\ &+\ g\left\{\kappa_\omega(g)\ \mathrm{tr}\left(\Phi_1^\dagger\ [\Phi_3^\dagger,\Phi_2^\dagger]_{\bar\omega}\right)\ +\ \bar\kappa_\Omega(g)\ \mathrm{tr}\left(\Phi_1^\dagger\ [\Phi_3^\dagger,\Phi_2^\dagger]_{\bar\Omega}\right)\right\}\,, \end{aligned}\tag{5}$$

where ω is a complex constant[3], $\bar\omega$ is its complex conjugate and Ω and $\bar\Omega$ are defined as

$$\Omega = -\frac{N^2 - 2 + 2\,\bar\omega}{(N^2-2)\,\bar\omega + 2}\quad,\qquad \bar\Omega = -\frac{N^2 - 2 + 2\,\omega}{(N^2-2)\,\omega + 2}\,. \tag{6}$$

The main result of Section 2 is the proof that for *any* complex constant ω and *any* complex function $\kappa_\Omega(g)$ satisfying $\kappa_\Omega(0) = 0$, there exists a *unique* function $\kappa_\omega(g)$, such that the chiral propagator is finite to all orders in perturbation theory, with the consequence that the chiral field has a vanishing

[2]The trace is over the color indices of the fundamental representation of the $SU(N)$ gauge group. The generators, T^a, of the fundamental representation are normalized according to $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$.

[3]The g-dependence of the deformation parameter is hidden in the terms proportional to $[\ ,\ ]_\Omega$. Indeed one can rewrite (5) in the form of (1) with a g-dependent deformation parameter $\omega(g)$.

anomalous dimension. To be precise, since we shall always compute the difference between the quantities in the deformed theory and in $\mathcal{N} = 4$ SYM (which corresponds to $\kappa_\omega = \omega = 1$, $\kappa_\Omega = 0$), everywhere in this paper by "finite" we actually mean "as finite as in $\mathcal{N} = 4$ SYM". We derive the explicit form of the finiteness condition at order g^6 for any number of colors N, and at order g^8 in the planar $N \to \infty$ limit. We also briefly discuss the corrections to the three-point vertices.

In Section 3 we study a particular type of CPO of dimension three, namely

$$\mathcal{O}_F = \mathrm{tr}\left(\Phi^1\Phi^2\Phi^3\right) + \alpha\, \mathrm{tr}\left(\Phi^1\Phi^3\Phi^2\right) , \tag{7}$$

which is a mixture of the two terms in the superpotential. Its existence was revealed in [4] where the one-loop value of the mixing coefficient α was determined through a direct two-point function computation. This one-loop result was confirmed in [5, 6] and in [8] it was shown that α is not corrected at two loops. We compute the three-loop correction to the value of α. However, the main purpose of Section 3 is to show that α can in fact be determined *without graph calculations*, but directly from the finiteness condition (4). The key observation is that the quantum corrections to the correlators of composite operators, i.e. their derivatives with respect to each independent coupling, are generated by the insertion of very special CPOs of the type $\mathcal{I} = a\,\mathrm{tr}(W^2) + b\,\mathrm{tr}(\Phi\Phi\Phi)$ (here $\mathrm{tr}(W^2)$ is the $\mathcal{N} = 1$ SYM chiral Lagrangian). The latter are obtained by differentiation of the chiral part of the Lagrangian, taking into account the relation among the couplings. To do this we rewrite the superpotential in the form $\mathcal{W} = f\,\mathrm{tr}\,(\Phi^1[\Phi^2, \Phi^3]) + d\,\mathrm{tr}\,(\Phi^1\{\Phi^2, \Phi^3\})$ and treat the holomorphic couplings f, d as independent, while g is determined from the finiteness condition $\gamma_\Phi(g, f, d, \bar{f}, \bar{d}) = 0$. The derivatives with respect to f, d give rise to two CPOs $\mathcal{I}_{f,d}$. We can say that $\mathcal{I}_{f,d}$ generate quantum corrections along the tangent directions to the surface in the moduli space defined by $\gamma_\Phi = 0$. Then the operator $\mathcal{O}_F$ is simply the linear combination of $\mathcal{I}_f$ and $\mathcal{I}_d$ such that $\mathrm{tr}(W^2)$ drops out. This means that the form of $\mathcal{O}_F$ to *all orders* is directly determined by the corresponding finiteness condition $\gamma_\Phi = 0$. When restricted to three loops, the general formula exactly reproduces the result of our graph calculation. Also, we can immediately explain the observation of [8] that α is not corrected at two loops - it simply follows from the fact that γ_Φ has no two-loop contribution. In Section 3.2.4 we generalize the construction of $\mathcal{O}_F$ to the most general deformed theory which is made finite by a single condition [1, 16].

2 All-order perturbative finiteness

Before proceeding, let us briefly motivate our conventions. The two deformed commutators $[\ ,\ ]_\omega$ and $[\ ,\ ]_\Omega$ in eq. (5) are just a conventional choice of basis in the two dimensional space of color structures (alternative to f_{abc} and d_{abc}, which correspond to the choices $\omega = 1$ and $\Omega = -1$, respectively). The explicit form of Ω given in eq. (6) is determined by the requirement that $[\ ,\ ]_\Omega$ and $[\ ,\ ]_{\bar\omega}$ are orthogonal in the sense that

$$\sum_{c,d} \operatorname{tr}(T^a\, [T^c, T^d]_\Omega)\, \operatorname{tr}(T^b\, [T^d, T^c]_{\bar\omega}) \;=\; 0\,, \tag{8}$$

and similarly for $\bar\Omega$ and ω. This implies also the vanishing of the color contractions of the ω and $\bar\Omega$ (as well as the $\bar\omega$ and Ω) terms in the superpotential $\mathcal{W}_{\kappa,\,\omega}$ of eq. (5). Note also that if ω is a pure phase $|\,\omega| = 1$, then also $|\,\Omega| = 1$. The real function $\kappa_\omega(g)$ and the complex function $\kappa_\Omega(g)$ have a power series expansion in g^2 that we find useful to cast in the form

$$\kappa_\omega(g) \;=\; \sum_{n=0}^{\infty} \kappa_\omega^{(n)} (g^2 N)^n\,, \tag{9}$$

$$\kappa_\Omega(g) \;=\; \sum_{n=1}^{\infty} \kappa_\Omega^{(n)} (g^2 N)^n\,. \tag{10}$$

Note that by assumption $\kappa_\Omega(0) = 0$, while $\kappa_\omega(0) = \kappa_\omega^{(0)} \neq 0$. At each order in g^2 the general superpotential $\mathcal{W}_{\kappa,\,\omega}$ depends on 3 real parameters. For $n > 0$ they are $\kappa_\omega^{(n)}$ (real) and $\kappa_\Omega^{(n)}$ (complex), while for $n = 0$ we choose them to be $\kappa_\omega^{(0)}$ (real) and ω (complex). As we shall see, this choice allows us to express in a compact form the solution of the condition for the perturbative finiteness of the chiral field propagator to all orders.

2.1 The chiral propagator to all orders

We start by reconsidering the order g^2, g^4 and g^6 conditions for finiteness of the chiral propagator in the theory with the general superpotential $\mathcal{W}_{\kappa,\,\omega}$ of eq. (5) (the order g^6 condition in the case of the superpotential of eq. (1) was found in [6]).

Let us write the action of the deformed theory in the form

$$S_{\kappa,\,\omega} = S_0 + S_v + S_{\mathcal{W}_{\kappa,\,\omega}}\,, \tag{11}$$

where S_0 contains the free kinetic terms and S_v is the part of the standard $\mathcal{N} = 4$ SYM action involving the couplings of the gauge superfield V (including the gauge-fixing term). Finally, $S_{\mathcal{W}_{\kappa,\,\omega}}$ is the part of the action involving the deformed superpotential $\mathcal{W}_{\kappa,\omega}$ given in eq. (5). In this notation the action of the undeformed $\mathcal{N}=4$ SYM theory reads ($\kappa_\omega=\omega=1$, $\kappa_\Omega=0$)

$$S_g = S_0 + S_v + S_{\mathcal{W}_g} \,. \qquad (12)$$

In the deformed theory the lowest θ components of the order g^{2n} correction to the propagator of the chiral superfield[4] $\langle \Phi^1_a(x_1,\theta_1)\Phi^\dagger_1{}_b(x_2,\bar\theta_2)\rangle$, can be compactly written in the form

$$G^{2n}_{\kappa,\omega}(x_1,x_2) = \langle \mathrm{e}^{S_v+S_{\mathcal{W}_{\kappa,\omega}}}\Big|_{g^{2n}}\,\phi^1_a(x_1)\phi^\dagger_1{}_b(x_2)\rangle \,, \qquad (13)$$

where by $\mathrm{e}^{S_v+S_{\mathcal{W}_{\kappa,\omega}}}|_{g^{2n}}$ we denote all terms of order g^{2n} in the expansion of the exponent. The similar computation, which in $\mathcal{N}=4$ SYM reads

$$G^{2n}_{g}(x_1,x_2) = \langle \mathrm{e}^{S_v+S_{\mathcal{W}_g}}\Big|_{g^{2n}}\,\phi^1_a(x_1)\phi^\dagger_1{}_b(x_2)\rangle \,, \qquad (14)$$

is know to give a finite result [20]. Hence, if the computation of the difference

$$\delta G^{2n}(x_1,x_2) = G^{2n}_{\kappa,\omega}(x_1,x_2) - G^{2n}_{g}(x_1,x_2) \,, \qquad (15)$$

gives a finite result, then also $G^{2n}_{\kappa,\omega}(x_1,x_2)$ will be finite. Note that computing the difference is much simpler than each term separately, since most of the vector field contributions cancel out. In particular, as far as the chiral propagator is concerned, up to order g^6, effectively only the superpotential contributes to the difference (for the details see [6]), leaving the quantities

$$\delta G^{2n}(x_1,x_2) = \langle \left(e^{S_{\mathcal{W}_{\kappa,\omega}}}\Big|_{g^{2n}} - e^{S_{\mathcal{W}_g}}\Big|_{g^{2n}} \right) \phi^1_a(x_1)\phi^\dagger_1{}_b(x_2)\rangle \,, \qquad (16)$$

to be evaluated for $n=1,2,3$.

Moreover, at each perturbative order we just have to take into account the primitive divergent superdiagrams (*i.e.* those which do not contain divergent subdiagrams). The only two such contributions to (16) up to order g^6 are shown in Figure 1 and they are both logarithmically divergent [21]. The first has the topology of the one-loop diagram. The second is a genuine three-loop (nonplanar) diagram. It is present only in the deformed theory, because the corresponding color factor in $\mathcal{N}=4$ SYM is zero. Owing to the chiral structure of the superpotential there are no primitive divergent two-loop superdiagrams.

[4]Because of the Z_3 symmetry of the action, all three chiral superfields are on the same footing, so our choice of the flavor index is conventional.

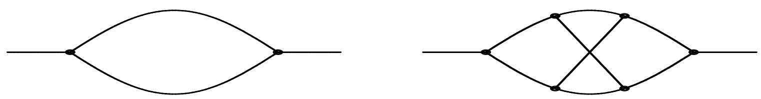

Figure 1: The one-loop and the three-loop diagrams.

As we said, there is a single divergent superdiagram at order g^2. Thus finding the finiteness condition at this order reduces to a simple color contraction problem. The relevant color contractions are

$$\sum_{c,d} \mathrm{tr}(T^a\,[T^c,T^d]_\omega)\ \mathrm{tr}(T^b\,[T^d,T^c]_{\bar\omega}) \;=\; \delta^{ab}\,C_\omega\,, \tag{17}$$

$$\sum_{c,d} \mathrm{tr}(T^a\,[T^c,T^d]_\Omega)\ \mathrm{tr}(T^b\,[T^d,T^c]_{\bar\omega}) \;=\; 0\,, \tag{18}$$

$$\sum_{c,d} \mathrm{tr}(T^a\,[T^c,T^d]_\omega)\ \mathrm{tr}(T^b\,[T^d,T^c]_{\bar\Omega}) \;=\; 0\,, \tag{19}$$

$$\sum_{c,d} \mathrm{tr}(T^a\,[T^c,T^d]_\Omega)\ \mathrm{tr}(T^b\,[T^d,T^c]_{\bar\Omega}) \;=\; \delta^{ab}\,C_\Omega\,. \tag{20}$$

where

$$\begin{aligned} C_\omega &= \frac{(N^2-2)(\omega\bar\omega+1)+2(\omega+\bar\omega)}{8N}\,,\\ C_\Omega &= C_\omega\frac{N^2(N^2-4)}{((N^2-2)\omega+2)((N^2-2)\bar\omega+2)}\,. \end{aligned} \tag{21}$$

The order g^2 (one-loop) finiteness condition then becomes

$$(\kappa_\omega^{(0)})^2\ C_\omega - \frac{N}{4} = 0\,, \tag{22}$$

or equivalently [4, 5, 6] (since $C_\omega > 0$ for any complex ω and integer $N > 2$)

$$(\kappa_\omega^{(0)})^2 \;=\; \frac{2\ N^2}{(N^2-2)(\omega\bar\omega+1)+2(\omega+\bar\omega)}\,, \tag{23}$$

exactly as for the simpler case of the potential in eq. (1). Let us note that since the cancellation is due to the vanishing of the numerical factor in front of one single diagram, both the logarithmically divergent and the finite part contributions vanish.

At the next perturbative order the difference between the superpotentials of eqs. (5) and (1) shows up. Indeed, if we impose the constraint (23) (we always choose the positive square root for the solution) the order g^4 finiteness condition reads

$$\kappa_\omega^{(0)}\ \kappa_\omega^{(1)}\ C_\omega\ =\ 0\,, \tag{24}$$

implying

$$\kappa_\omega^{(1)}\ = 0\,. \tag{25}$$

Note that due to the vanishing of the contractions (18) and (19) the complex coefficient $\kappa_\Omega^{(1)}$ remains undetermined. Hence, contrary to what is said in previous papers the superpotential $\mathcal{W}_{\kappa,\omega}$ is allowed to contain a different from zero order g^3 term. Since the cancellation is again due to the vanishing of numerical factors in front of each diagram, both the divergent and the finite parts are set to zero by eq. (25).

The situation becomes more complicated at order g^6, since at this order both the genuine three-loop diagram and the one-loop diagram (multiplied by order g^4 coefficients) contribute. Thus, if we impose eq. (25), the order g^6 finiteness condition reads

$$\begin{aligned} 2\ \kappa_\omega^{(0)}\ \kappa_\omega^{(2)}\ C_\omega\ &+\ |\kappa_\Omega^{(1)}|^2\ C_\Omega \\ &+\ \frac{3}{256}\frac{\zeta(3)\ (k_\omega^{(0)})^6}{(4\pi^2)^2}\ \frac{(N^2-4)}{N^5}\ (\omega-1)\ (\bar\omega-1)\ P_3(\omega,N) = \ 0\,, \end{aligned} \tag{26}$$

where

$$P_3(\omega,N) = ((\omega^2+\omega+1)(\bar\omega^2+\bar\omega+1) - 9\ \omega\bar\omega)\ N^2 + 5\ (\omega-1)^2\ (\bar\omega-1)^2\,, \tag{27}$$

and C_ω and C_Ω are given in eq. (21). Note that eq. (26) is linear in $\kappa_\omega^{(2)}$, so there is always a unique solution for $\kappa_\omega^{(2)}$ as a function of ω, N and $|\kappa_\Omega^{(1)}|$ for any $\kappa_\Omega^{(1)}$. Let us stress that this is the first case in which there are (nonvanishing) contributions from two different (super)diagrams. In fact the first two terms in eq. (26) come from the diagram with one-loop topology, while the third term comes from the genuine three-loop diagram. Hence, the vanishing of the divergent part does not automatically imply also the cancellation of the (potentially scheme dependent) finite parts. Let us stress that these finite contributions modify only the normalization of the chiral superfield propagator. This finite correction of the normalization, which is present only in the deformed theory (remember that all terms in eq. (26) originate from the deformed theory), will give rise to a logarithmic divergency at the next order (g^8). Hence starting at order g^8 the explicit form of the condition for finiteness of the chiral propagator will in general be scheme dependent.

It is clear that one can proceed iteratively. If we satisfy all the finiteness conditions up to order g^{2n}, then the finiteness condition at order $g^{2(n+1)}$ will schematically read

$$2\ \kappa_\omega^{(0)}\ \kappa_\omega^{(n+1)}\ C_\omega + f_{n+1}(N,\omega,g,\{\kappa_\Omega^{(p)},p=1,\dots,n\}) \ = \ 0\,, \tag{28}$$

where f_{n+1} is a computable function and we have used the lower order equations to express all $\kappa_\omega^{(q)}$, $q=1,\dots,n$ in terms of the coefficients $\kappa_\Omega^{(p)}$, $p=1,\dots,n-1$. Equation (28) is linear in $\kappa_\omega^{(n+1)}$ and thus it has always a unique solution (since both $\kappa_\omega^{(0)}\neq 0$ and $C_\omega\neq 0$).

To summarize we have shown that for any complex constant ω and any complex function $\kappa_\Omega(g)$ satisfying $\kappa_\Omega(0)=0$, there exist a *unique* (possibly scheme dependent) real function $\kappa_\omega(g)$, such that the anomalous dimension of the chiral superfields is zero to all orders in perturbation theory[5].

The question of the finiteness of the vector superfield propagator is more subtle and is beyond the scope of this paper. It is closely related to the coupling constant renormalization which we shall discuss in the next subsection. We note that the finiteness of the vector superfield propagator up to order g^6 follows only from the order g^2 and g^4 conditions in eqs. (23) and (25).

2.2 The three-point vertices

Another important question which we would like to address is whether in the deformed theory there is a coupling constant renormalization. To answer this question, one has to compute the perturbative corrections to the three-point vertex functions. There are four potentially different such vertex functions, namely the triple chiral (or antichiral) vertex, the chiral-antichiral-vector vertex, the triple vector vertex and the ghost-ghost-vector vertex (see Figure 2). All four vertices in the action at tree level are proportional to g, but only the triple chiral vertex at higher orders receives in addition to the standard perturbative corrections also finite corrections from $\kappa_\omega^{(n)}$ and $\kappa_\Omega^{(n)}$. Another peculiar feature of the triple chiral vertex is that its color structure depends on g in a non-trivial way. Indeed, while the three vertices involving the vector field are always proportional to the $SU(N)$ structure constants f^{abc}, the triple chiral vertex is proportional to a linear combination of the two deformed (by ω and Ω) commutators (see eq. (5)).

[5] Since the potential (5) is a special case of the general Yukawa potential (3), our result could in principle be obtained as a particular case of [11]. However, translating the parametrization from (3) to (5) is not an easy task. Therefore we find it useful to give the explicit form of the finiteness condition and its solution adapted to our special case.

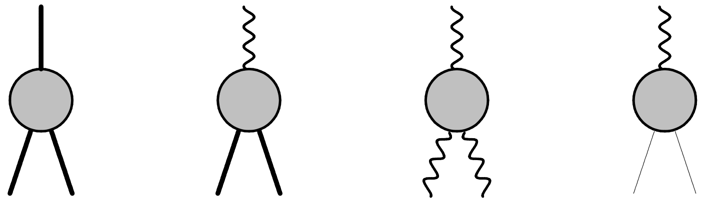

Figure 2: The three-point vertices.

We recall that the triple chiral vertex function obeys a non-renormalization theorem [14, 15] which relates the propagator and the coupling constant renormalization factors. The triple vector vertex is related to the simpler ghost-ghost-vector one by a Slavnov-Taylor identity [22]. A similar identity relates the chiral-antichiral-vector vertex to the matter propagator.

For our purposes it again suffices to compute only the difference between the values of each three-point function in the deformed theory and in $\mathcal{N}=4$ SYM. Our results can be summarized as follows.

At order g^3 and order g^5 all three vertices with external vector lines are exactly equal to the corresponding ones in $\mathcal{N}=4$ SYM. Only the triple chiral vertex receives at order g^5 a finite non-planar correction from the first (super)diagram in Figure 3. It affects both the $[\ ,\ ]_\omega$ and $[\ ,\ ]_\Omega$ structures in the effective superpotential $\mathcal{W}_{\rm eff}$ [8]. In particular, the correction to the $[\ ,\ ]_\omega$ structure is

$$\begin{aligned}\mathcal{W}_{\rm eff}|_{g^5\,,\,\omega} \quad &= \quad \frac{3\,\zeta(3)\;(k_\omega^{(0)})^5}{32\,(4\pi^2)^2}\;\frac{(\omega-1)\;(\bar\omega-1)\;(N^2-4)}{N^2\,((N^2-2)(\omega\bar\omega+1)+2(\omega+\bar\omega))} \\ &\times \quad P_3(\omega,N)\ {\rm tr}\left(\Phi^1\ [\Phi^2,\Phi^3]_\omega\right)\,, \end{aligned} \tag{29}$$

with $P_3(\omega,N)$ defined in eq. (27), while the correction to the $[\ ,\ ]_\Omega$ structure is

$$\begin{aligned}\mathcal{W}_{\rm eff}|_{g^5\,,\,\Omega} \quad &= \quad -\frac{3\,\zeta(3)\;(k_\omega^{(0)})^5}{32\,(4\pi^2)^2}\;\frac{(\omega-1)((N^2-2)\,\bar\omega+2)}{N^2\,((N^2-2)(\omega\bar\omega+1)+2(\omega+\bar\omega))} \\ &\times \quad \Big[N^2((\omega+1)(\bar\omega^2-6\,\omega\bar\omega+\omega^2)+4\omega(\omega\bar\omega^2+1)) \\ &+ \quad (\omega-1)^2\,(\bar\omega-1)\,(7(\omega\bar\omega-1)-3\omega+3\bar\omega)\Big]\ {\rm tr}\left(\Phi^1\ [\Phi^2,\Phi^3]_\Omega\right). \end{aligned} \tag{30}$$

At order g^7 vertices behave quite differently. The ghost-ghost-vector vertex remains equal to its $\mathcal{N}=4$ value. The other three vertices receive corrections from non-planar diagrams. Whether they sum up to zero or not is an open question[6]. Only the triple chiral vertex receives also finite planar corrections coming from the second (super)diagram in Figure 3, which as explained in the next subsection modify the chiral propagator at order g^8.

Figure 3: The two-loop and the only planar three-loop contributions to the triple chiral vertex.

2.3 The order g^8 condition in the planar limit

In this subsection we shall derive the explicit form of the condition for finiteness of the chiral superfield propagator at order g^8. We shall work in the planar limit $N \to \infty$, since this allows us to drastically simplify the necessary calculations. Still, the essential feature, namely the fact that even in the planar limit, unless $|\omega| = 1$, one has to modify the coefficients in the superpotential eq. (5) by higher powers of g, clearly shows up.

Owing to the properties of the three-point vertices, in the planar limit ($N \to \infty$ and $g^2 N$ fixed), there are significant simplifications. On the one hand, the order g^8 correction (with respect to the $\mathcal{N} = 4$ SYM value) to the vector propagator is zero. Indeed all diagrams which contribute will contain as a subdiagram some lower order vertex with external vector lines which, as we mentioned in the previous subsection, vanish in the planar limit. On the other hand the corrections to the chiral propagator will come only from the diagram with the one loop topology shown in Figure 1 and from the planar three-loop correction to the chiral vertex shown in Figure 3, which leads to an order g^8 planar primitive logarithmically divergent diagram shown in Figure 4.

[6]To answer it one has to carefully compute the numerous three-loop super diagrams which contribute to the various three-point functions. We shall address this issue in a future publication.

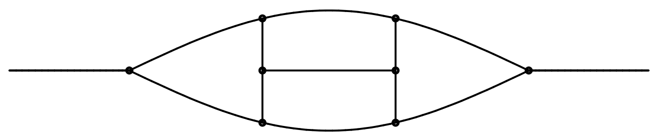

Figure 4: The 4-loop primitive planar diagram.

In the planar limit the finiteness conditions become

$$g^2: \qquad (\kappa_\omega^{(0)})^2 \;=\; \frac{2}{(1+\omega\bar{\omega})}\ , \tag{31}$$

$$g^4: \qquad \kappa_\omega^{(1)} \;=\; 0\ , \tag{32}$$

$$g^6: \qquad 2\ \omega\bar{\omega}\ \kappa_\omega^{(0)}\ \kappa_\omega^{(2)}\ + |\kappa_\Omega^{(1)}|^2 \;=\; 0\ , \tag{33}$$

because $\Omega = -1/\bar{\omega}$, $\bar{\Omega} = -1/\omega$, $C_\omega/C_\Omega = \omega\bar{\omega}$. After imposing these conditions, the cancellation of the order g^8 divergencies coming from the diagram with one-loop topology and the planar 4-loop diagram depicted in Figure 4, leads to the condition

$$\begin{aligned} 2\ \kappa_\omega^{(0)}\ \kappa_\omega^{(3)}\ + \frac{\kappa_\Omega^{(1)}\overline{\kappa_\Omega^{(2)}} + \kappa_\Omega^{(2)}\overline{\kappa_\Omega^{(1)}}}{\omega\bar{\omega}} \quad &= \quad \xi_5\ \frac{8-(\kappa_\omega^{(0)})^8(1+6\,(\omega\bar{\omega})^2+(\omega\bar{\omega})^4)}{(\omega\bar{\omega}+1)} \\ &= \quad -\ 8\ \xi_5\ \frac{(\omega\bar{\omega}-1)^4}{(\omega\bar{\omega}+1)^5}\ , \end{aligned} \tag{34}$$

where ξ_5 is a numerical constant proportional to $\zeta(5)$ (see also [35]), and to obtain the second equality we used the order g^2 condition of eq. (31).

It follows that even in the planar limit, in order to make the theory finite we are obliged to fine-tune order by order in g^2 the coefficients in the superpotential. Indeed, for generic ω the above equation necessarily requires a nonvanishing correction to the superpotential. The only exception is when $|\omega| = 1$, in which case the order g^2 condition alone is sufficient for all order finiteness [7].

Let us stress that eq. (34) is scheme independent, since it follows from requiring the cancellation of the leading logarithmic divergencies. However, as it involves two different (super)diagrams, the vanishing of the divergent part does not automatically imply the cancellation of the (possibly scheme dependent) finite parts. These finite corrections change only the normalization of the chiral superfield propagator and will give rise to logarithmic divergencies at the next order. Hence, even in the planar limit, starting from order g^{10}, the explicit form of the condition for finiteness of the chiral propagator will be scheme dependent.

Let us note also that the 5-loop discrepancy in the planar Maximally Helicity Violating (MHV) amplitudes pointed out in [18], is actually not present thanks to the order g^8 finiteness condition in eq. (34). Indeed the vacuum diagram considered in [18] is equivalent to the 4-loop planar correction to the chiral propagator shown in Figure 4, which, as we have shown, can be cancelled by fine-tuning the parameters in the superpotential.

To conclude, let us briefly comment on the freedom which the conditions for finiteness of the chiral propagator leave. As we already noted, all the coefficients $\kappa_\Omega^{(n)}$ remain arbitrary. Thus one simple choice is $\kappa_\Omega^{(n)} = 0$ for all n. In this case the superpotential in eq. (5) reduces to the simpler expression (1), widely studied in the literature. However, as noted in [8] (see also eq. (30)), even if we start with the simple superpotential proportional only to $[\ ,\ ]_\omega$, the quantum corrections to the effective superpotential $\mathcal{W}_{\text{eff}}$ give rise also to contributions proportional to $[\ ,\ ]_\Omega$. This suggests that we may choose $\kappa_\Omega^{(n)}$ to precisely cancel the quantum corrections to the effective superpotential proportional to $[\ ,\ ]_\Omega$, obtaining

$$\mathcal{W}_{\text{eff}} \sim \ \mathrm{tr}\left(\Phi^1\ [\Phi^2, \Phi^3]_\omega\right)\ , \tag{35}$$

to all orders in perturbation theory. For the first two coefficients one finds in this case $\kappa_\Omega^{(1)} = 0$ and $\kappa_\Omega^{(2)} = -\mathcal{W}_{\text{eff}}|_{g^5,\Omega}/N^2$, where $\mathcal{W}_{\text{eff}}|_{g^5,\Omega}$ is given in eq. (30).

3 The origin of the protected operator $\mathcal{O}_F$

In this section we show that among all CPOs made out of matter superfields the protected operator $\mathcal{O}_F$ (7) occupies a special place. It can be derived directly from the Lagrangian (46) by the so-called insertion procedure, i.e. by exploiting the information that can be obtained in a superconformal theory by taking derivatives of correlation functions with respect to the independent coupling constants. Each such derivative gives rise to the insertion of a CPO which is a combination of the SYM Lagrangian $\mathrm{tr}(W^2)$ and of terms from the superpotential. In this context the protected operator $\mathcal{O}_F$ arises as the particular linear combination of these CPOs which does not contain $\mathrm{tr}(W^2)$. We show that the form of $\mathcal{O}_F$ is directly determined from the finiteness condition $\gamma_\Phi = 0$. We confirm this result by an explicit three-loop calculation. The generalization to a superpotential with cubic terms $\sum_I (\Phi^I)^3$ is straightforward and gives rise to a family of protected operators of this type.

3.1 Quantum corrections through insertions

Here we briefly describe a procedure which provides useful information about the quantum corrections to Euclidean n–point correlation function (for details see [23], Chapter 6.7). Consider the expectation value

$$G \equiv \langle \mathcal{O}(1) \cdots \mathcal{O}(n) \rangle = \int e^{-\int d^4x_0 d^4\theta_0 \mathcal{L}(x_0,\theta_0,\bar{\theta}_0;\, g_i)} \mathcal{O}(1) \cdots \mathcal{O}(n) \tag{36}$$

where it is assumed that the Lagrangian depends of a set of independent coupling constants, g_i. In order to avoid irrelevant (for the present discussion) contact terms we will always take the operators $\mathcal{O}$ at unequal space-time points.

The quantum corrections to the correlator (36) can be obtained by differentiating it with respect to the couplings g_i. Each such derivative leads to the insertion of a derivative of the action into the correlator [7]

$$\begin{aligned} \frac{\partial G}{\partial g_i} &= -\int e^{-\int \mathcal{L}} \left[\int d^4x_0 d^4\theta_0 \, \frac{\partial \mathcal{L}(x_0,\theta_0,\bar{\theta}_0;\, g_j)}{\partial g_i} \right] \mathcal{O}(1) \cdots \mathcal{O}(n) \\ &= -\int d^4x_0 d^4\theta_0 \, \langle \frac{\partial \mathcal{L}(0)}{\partial g_i} \mathcal{O}(1) \cdots \mathcal{O}(n) \rangle \, , \end{aligned} \tag{37}$$

In what follows we assume that the theory is (super)conformal, i.e. all the beta functions β_{g_i} vanish. As we already know, in the deformed theory this is achieved by imposing a constraint on the couplings which should be taken into account when differentiating. We shall come back to this essential point in Section 3.2.

Before discussing the superconformal insertion procedure, let us explain some details about its conformal analog. To be more specific, let us consider the simplest case of scalar operators $\mathcal{O}$ with $n = 2$. After integration over $d^4\theta_0$ in (37) and setting $\theta = \bar{\theta} = 0$, we obtain

$$\frac{\partial}{\partial g_i} \langle \mathcal{O}(1) \mathcal{O}^\dagger(2) \rangle = -\int d^4x_0 \, \langle \frac{\partial L(x_0; g_j)}{\partial g_i} \mathcal{O}(1) \mathcal{O}^\dagger(2) \rangle \, , \tag{38}$$

where $L(x_0)$ is the Lagrangian operator (the top component in the θ expansion of $\mathcal{L}$). The bare operators $\mathcal{O}$ in (38) are in general ill defined. Consequently, they must be renormalized, unless they are protected. Let us start with the case of a single multiplicatively renormalized operator. We will use the notation $[\mathcal{O}](x) = \lim_{\epsilon \to 0} \hat{\mathcal{O}}(x,\epsilon)$ with $\hat{\mathcal{O}}(x,\epsilon) = Z(\epsilon,\mu,g_i)\mathcal{O}(x,\epsilon)$. Here

[7] The functional integral in (36) should be divided by $\int e^{-\int \mathcal{L}}$. Connected correlators are automatically generated in this way by differentiation. To simplify notations we will not explicitly indicate this.

$Z(\epsilon, \mu, g_i)$ is a renormalization factor depending on the couplings g_i, on the regulator ϵ (e.g., a four-vector $\vec{\epsilon}$ in point-splitting regularization) and on the renormalization scale (or subtraction point) μ. $\mathcal{O}(x, \epsilon)$ is the regularized version of the bare operator (e.g., with the constituent fields put at distances $\vec{\epsilon}$). Now, suppose that the *renormalized* operator $[\mathcal{O}]$ is a conformal primary of dimension $\Delta = \Delta_0 + \gamma(g_i)$, where Δ_0 is the naive and γ the anomalous dimension. In the point-splitting scheme we have [24]

$$Z(\epsilon, \mu, g_i) = (\epsilon^2 \mu^2)^{-\gamma(g_i)/2} . \tag{39}$$

Repeating the differentiation (38), but this time with the renormalization factors included, we find

$$\begin{aligned} \frac{\partial}{\partial g_i} \langle \hat{\mathcal{O}}(x_1, \epsilon_1) \hat{\mathcal{O}}^\dagger(x_2, \epsilon_2) \rangle &= -\frac{1}{2} \frac{\partial \gamma}{\partial g_i} \ln(\epsilon_1^2 \epsilon_2^2 \mu^4) \ \langle \hat{\mathcal{O}}(x_1, \epsilon_1) \hat{\mathcal{O}}^\dagger(x_2, \epsilon_2) \rangle \\ &- \int d^4 x_0 \ \langle L_{g_i}(x_0) \hat{\mathcal{O}}(x_1, \epsilon_1) \hat{\mathcal{O}}^\dagger(x_2, \epsilon_2) \rangle , \end{aligned} \tag{40}$$

where for short we have set $L_{g_i} = \partial L / \partial g_i$. Since we are taking the derivative of a finite quantity, the apparent logarithmic singularity in the first term of the right-hand side of eq. (40) has to be compensated by a similar singularity in the second term [8].

To show how this comes about we recall that two-point correlator of renormalized scalar operators takes the form predicted by conformal invariance:

$$\langle [\mathcal{O}](x_1) [\mathcal{O}]^\dagger(x_2) \rangle = \lim_{\vec{\epsilon}_{1,2} \to 0} \langle \hat{\mathcal{O}}(x_1, \epsilon_1) \hat{\mathcal{O}}^\dagger(x_2, \epsilon_2) \rangle = \frac{A(g)}{(x_{12}^2)^\Delta} , \tag{41}$$

where $A(g)$ is a (coupling-dependant) normalization constant. We remark that if Δ is an integer ≥ 2, then the distribution $1/x^{2\Delta}$ is singular, with a δ- (or derivatives of δ-) function type singularity. Such contact terms become important in the $n+1$–point correlators with the insertion (37), see below.

The derivatives of the Lagrangian L with respect to the couplings must have the right conformal weight in order to make the integral in (40) conformally invariant (in the limit $\vec{\epsilon}_{1,2} \to 0$). In other words, we assume that the operators $L_{g_i} = \partial L / \partial g_i$ are conformal primaries of dimension four. This assumption can be justified in a superconformal theory such as $\mathcal{N} = 4$ SYM or its deformed $\mathcal{N} = 1$ version (see subsection 3.3).

Now, conformal invariance can also predict the form of the "regular" part of the $2+1$–point correlator in the last term in (40), yielding

$$\langle L_{g_i}(x_0) [\mathcal{O}](x_1) [\mathcal{O}]^\dagger(x_2) \rangle_{\text{regular}} = \frac{B_i(g)}{(x_{01}^2 x_{02}^2)^2 (x_{12}^2)^{\Delta - 2}} . \tag{42}$$

[8]Our discussion about this point is similar to that in [25], except for the regularization scheme used.

where "regular" means $x_0 \neq x_1 \neq x_2$. Inserting (42) into (40) leads to a divergent integral which we regularize by splitting points 1 and 2

$$\frac{B_i(g)}{(x_{12}^2)^{\Delta-2}} \int \frac{d^4x_0}{x_{01}^2 x_{01+\epsilon_1}^2 x_{02}^2 x_{02+\epsilon_2}^2} = -\pi^2 \ln(\epsilon_1^2 \epsilon_2^2 \mu^4) \frac{B_i(g)}{(x_{12}^2)^{\Delta}} + \text{finite part}\,, \quad (43)$$

where we have introduced the subtraction point, μ. We now see that this term can provide the singularity which cancels the logarithm in the first term in the right-hand side of (40), if

$$\frac{\partial \gamma}{\partial g_i} = 2\pi^2 \frac{B_i(g)}{A(g)}\,. \quad (44)$$

The conclusion is that the anomalous dimension $\gamma(g)$ is controlled by the regular part of the $2+1$–point correlator (42). This observation, generalized to the supersymmetric case, explains why CPOs have no anomalous dimension. The reason is that in both the deformed and undeformed theories superconformal invariance forbids the existence of a nonvanishing $\langle L_{g_i}(0)[\mathcal{O}](1)[\mathcal{O}]^\dagger(2)\rangle_{\text{regular}}$ with $[\mathcal{O}]$ being a CPO. Consequently, such operators keep their naive dimension Δ_0, i.e., they are "protected". However, this does not necessarily mean the total absence of quantum corrections to the correlator $\langle[\mathcal{O}][\mathcal{O}]^\dagger\rangle$. Indeed, conformal invariance allows contact term contributions to the $2+1$–point correlator of the form

$$\langle L_{g_i}(x_0)[\mathcal{O}](x_1)[\mathcal{O}]^\dagger(x_2)\rangle_{\text{contact}} = C_i(g)\,\left[\delta^4(x_{01}) + \delta^4(x_{02})\right] \frac{1}{(x_{12}^2)^{\Delta_0}}\,. \quad (45)$$

The appearance of such terms is related to the general fact that the factors $1/x^4$ in (42) are singular distributions with a δ-function type singularity [9]. It is clear that such terms, integrated over the insertion point x_0, will give quantum corrections to the normalization $A(g)$ of the correlator (41). This is precisely what happens to CPOs in the deformed theory, starting at two loops [5, 6]. In the undeformed theory the more powerful extended superconformal symmetry forbids even the contact terms, so there the two-point functions of CPOs are completely protected (for more details see [26]).

We note that the above insertion procedure can be generalized in an obvious way in the presence of mixing, i.e. when the renormalized operators have the form $\hat{\mathcal{O}}_i = Z_{ij}\mathcal{O}_j$.

[9]Note that in the case of CPOs with Δ_0 an integer ≥ 2, an ultralocal contact term, like $\delta^4(x_{01})\delta^4(x_{02})$ for $\Delta_0 = 2$, could be added to (45). We need not consider such terms here since we are assuming $x_{12} \neq 0$.

3.2 CPOs as derivatives of the deformed $\mathcal{N} = 4$ Lagrangian

3.2.1 Holomorphic form of the action. Finiteness condition

For our purposes in this section it is more convenient to rewrite the action of the deformed $\mathcal{N} = 4$ theory as follows:

$$
\begin{aligned}
S \;=\; & \mathrm{tr}\Big\{\int d^4x d^4\theta\, e^{-gV}\Phi^\dagger_I e^{gV}\Phi^I \\
& + \int d^4x_L d^2\theta\, W^\alpha W_\alpha \\
& + \int d^4x_L d^2\theta\, \left(f\,\Phi^1[\Phi^2,\Phi^3] + d\,\Phi^1\{\Phi^2,\Phi^3\}\right) \\
& + \int d^4x_R d^2\bar\theta\, \left(\bar f\,\Phi^\dagger_1[\Phi^\dagger_3,\Phi^\dagger_2] + \bar d\,\Phi^\dagger_1\{\Phi^\dagger_3,\Phi^\dagger_2\}\right)\Big\}\;, \qquad (46)
\end{aligned}
$$

where W_α is the chiral field-strength of the gauge potential V,

$$W_\alpha = -\frac{1}{4g}\bar D^2\left(e^{-2gV}D_\alpha e^{2gV}\right) = -\frac{1}{2}\bar D^2 D_\alpha V + O(g)\,. \qquad (47)$$

Note that in (46) we have written the SYM kinetic term in its chiral form. If needed, it can be rewritten as an antichiral term according to the identity $\int d^2\theta\, W^2 = \int d^2\bar\theta\, \bar W^2$ valid in a topologically trivial background.[10] Unlike Section 2, here we prefer to parametrize the potential by two complex coupling constants, f in front of the commutator (i.e., color tensor f_{abc}) and d in front of the anticommutator (i.e., color tensor d_{abc}). Both of them as well as the gauge coupling g are treated as small perturbative parameters $f \sim d \sim g$. To switch back to the notation of Section 2 we need to perform the change of parametrization

$$
\begin{aligned}
f &= \frac{g}{2}\left[(\omega+1)\kappa_\omega(g) + (\Omega+1)\kappa_\Omega(g)\right] \\
d &= \frac{g}{2}\left[(\omega-1)\kappa_\omega(g) + (\Omega-1)\kappa_\Omega(g)\right]\,. \qquad (48)
\end{aligned}
$$

The deformed theory is finite (and hence conformal) up to three loops [11] if the couplings satisfy the relation

$$\gamma(g,f,d,\bar f,\bar d) = \gamma^{(1)} + \gamma^{(3)} = 0\,. \qquad (49)$$

[10] It is more convenient, although not essential, to work with a real gauge coupling g, therefore we do not introduce an instanton angle.

[11] We remark that now the counting of "loops" is somewhat different than in the preceding section. There all the couplings were expressed in terms of g, so "n loops" meant perturbative level g^{2n}. Here "n-loop" corrections are homogeneous polynomials of degree $2n$ in the couplings g, f, d.

This is the condition for vanishing anomalous dimension γ_Φ of the matter superfields Φ involving a one-loop

$$16\pi^2\gamma^{(1)} = 2\frac{N^2-4}{N}|d|^2 + 2N|f|^2 - 2Ng^2 \tag{50}$$

and a three-loop

$$(16\pi^2)^3\gamma^{(3)} = -\frac{24\zeta(3)(N^2-4)}{N^3}|d|^2[2(N^2-10)|d|^4 - 3N^2(f^2\bar{d}^2 + d^2\bar{f}^2)] \tag{51}$$

contributions [12] (cf. eq. (26)). The absence of a two-loop contribution is explained by the fact that in carrying out the two- and three-loop calculation of γ_Φ the one-loop condition $\gamma^{(1)} = 0$ has been used, after which no new condition arises at two loops. Alternatively, the calculation may be done without imposing the one-loop condition. In this case one finds [13] a two-loop contribution to γ_Φ of the form

$$\gamma^{(2)} = \gamma^{(1)}\ P^{(1)} \tag{52}$$

where $P^{(1)}$ is some homogeneous polynomial of degree two in the couplings whose explicit form is not important for us. Similarly, $\gamma^{(3)}$ gets modified by a term of the type $\gamma^{(1)}\ P^{(2)}$ with $P^{(2)}$ a polynomial of degree four. Thus, the complete version of the finiteness condition (49) up to three loops has the form

$$\gamma^{(1)}(1 + P^{(1)} + P^{(2)}) + \gamma^{(3)} = 0\,. \tag{53}$$

Clearly, since $P^{(1)},\ P^{(2)} << 1$, we can multiply this equation by $(1 + P^{(1)} + P^{(2)})^{-1}$. In this way we recover (49), up to terms of four-loop order which are beyond the scope of this section.

Following [13], we note that the three-loop condition $\gamma^{(1)} + \gamma^{(3)} = 0$ can be formally reduced to a one-loop condition of the type $\gamma^{(1)} = 0$ by a change of variables (or, equivalently, by a finite coupling renormalization). For instance, one such change is

$$f \ \rightarrow\ f - \frac{9\zeta(3)(N^2-4)}{64\pi^4N^2}\, d^3\bar{d}\bar{f}\,, \qquad d \ \rightarrow\ d + \frac{3\zeta(3)(N^2-10)}{64\pi^4N^2}\, d^3\bar{d}^2\,. \tag{54}$$

Finally, returning to the notation of Section 2, it is easy to see that the condition (49) with $\gamma^{(1)}$ from (50) and $\gamma^{(3)}$ from (51), rewritten in terms of $\kappa_{\omega,\Omega}(g)$ as indicated in (48) and expanded up to g^6, is equivalent to (23) at order g^2, (25) at order g^4 and (26) at order g^6.

[12] This condition is a particular case of the finiteness condition for the most general trilinear superpotential (3) discussed in [13]. It was obtained in its explicit form for the case of the deformed $\mathcal{N} = 4$ SYM potential in [4] at one loop, in [5] at two loops and in [6] at three loops.

3.2.2 Holomorphic derivatives. CPOs as generators of quantum corrections

The general procedure which generates quantum corrections to the n-point correlation functions (36) was described in section 3.1. The derivative with respect to each independent coupling gives rise to an operator insertion into the correlator, as shown in (37). Below we prove that in the specific case of the action (46) this procedure amounts to the insertion of *chiral or antichiral primary operators*.

The condition for finiteness (49) means that in our case the couplings, and hence their variations are not independent, rather they satisfy,

$$\gamma_g \delta g + \gamma_f \delta f + \gamma_d \delta d + \gamma_{\bar{f}} \delta \bar{f} + \gamma_{\bar{d}} \delta \bar{d} = 0\,, \tag{55}$$

which implies for the variation of the n-point correlator G

$$\begin{aligned} \delta G &= G_g \delta g + G_f \delta f + G_d \delta d + G_{\bar{f}} \delta \bar{f} + G_{\bar{d}} \delta \bar{d} \\ &= \left(G_f - \frac{\gamma_f}{\gamma_g} G_g \right) \delta f + \left(G_d - \frac{\gamma_d}{\gamma_g} G_g \right) \delta d + \text{c.c.}\ . \end{aligned} \tag{56}$$

In this equation we have treated the holomorphic couplings f and d as independent, while the gauge coupling is taken as a (real) function of them, $g = g(f, d, \bar{f}, \bar{d})$. This point of view is preferable here because we want to obtain *chiral* operators through differentiation with respect to the *holomorphic* couplings f, d. In Section 2 we adopted the alternative (i.e. *perturbative*) point of view where the gauge coupling is the universal small parameter used in the perturbative calculations.

In order to compute the derivatives of the Lagrangian in (46) (completed with the gauge-fixing term $\xi \int d^4x d^4\theta\, D^2 V \bar{D}^2 V$ and with the ghost term) with respect to the independent holomorphic couplings, we first absorb the gauge coupling g into the gauge potential and the gauge-fixing parameter[13]

$$V \;\to\; g^{-1} V\,, \qquad \xi \;\to\; g^2 \xi\,. \tag{57}$$

The effect of this rescaling is that g now appears only in front of the classical SYM term W^2 in the Lagrangian (recall (47)), while it drops out from the

[13] In [27] it is claimed that this redefinition of V may lead to the so-called "rescaling anomaly", which is used to justify the "exact" NSVZ beta function [19]. However, as mentioned in [27], the rescaling anomaly is not seen in dimensional regularization. Here we adopt the point of view that there exists a scheme free from such anomalies. Note also that the rescaling of the gauge-fixing parameter ξ in (57) has no effect on the correlators (36) of *gauge invariant* composite operators $\mathcal{O}$.

gauge-fixing and ghost terms. So

$$\mathcal{L}_g = -\frac{2}{g}\,\mathrm{tr}(W^2)\,, \tag{58}$$

while the variation with respect to the holomorphic couplings gives

$$\mathcal{L}_f = \mathrm{tr}\left(\Phi^1[\Phi^2,\Phi^3]\right)\,, \qquad \mathcal{L}_d = \mathrm{tr}\left(\Phi^1\{\Phi^2,\Phi^3\}\right)\,. \tag{59}$$

It is now clear that the total derivative of G with respect to each *holomorphic* coupling gives rise to the insertion of a *chiral operator*:

$$\begin{aligned}\frac{\delta G}{\delta f} &= -\int d^4x_L d^2\theta_0\ \langle \mathcal{I}_f(0)\mathcal{O}(1)\cdots\mathcal{O}(n)\rangle \\ \frac{\delta G}{\delta d} &= -\int d^4x_L d^2\theta_0\ \langle \mathcal{I}_d(0)\mathcal{O}(1)\cdots\mathcal{O}(n)\rangle\end{aligned} \tag{60}$$

where

$$\begin{aligned}\mathcal{I}_f &= \frac{2\gamma_f}{g\gamma_g}\,\mathrm{tr}(W^2) + \mathrm{tr}\left(\Phi^1[\Phi^2,\Phi^3]\right) \\ \mathcal{I}_d &= \frac{2\gamma_d}{g\gamma_g}\,\mathrm{tr}(W^2) + \mathrm{tr}\left(\Phi^1\{\Phi^2,\Phi^3\}\right)\,.\end{aligned} \tag{61}$$

In section 3.3 we show that the insertions $\mathcal{I}$ are chiral *primary* operators of protected conformal dimension $\Delta_0 = 3$. Similarly, differentiating with respect to the antiholomorphic couplings amounts to inserting antichiral operators (to this end the SYM kinetic term should be rewritten in its antichiral form).

This result admits the following interpretation[14]. The finiteness condition (49) can be viewed as the equation of a real hypersurface in the complex moduli space of the couplings. In Section 2 we chose to describe it in a parametric fashion, by expressing the matter couplings as functions (perturbative power series) of the gauge coupling g. In this section we use the holomorphic couplings f and d as the independent coordinates on the surface whose tangent space is spanned by the derivatives $\partial/\partial f$ and $\partial/\partial d$. The quantum equivalents of the tangent space vectors are the CPOs (61). Each of them generates quantum corrections to the Green's functions of operators in the theory when moving along the corresponding tangent direction to the surface of couplings.

Finally, the operator $\mathcal{O}_F$ (7) is nothing but the linear combination of $\mathcal{I}_f$ and $\mathcal{I}_d$ from which the SYM term W^2 drops out:

$$\mathcal{O}_F = \frac{\gamma_d\mathcal{I}_f - \gamma_f\mathcal{I}_d}{\gamma_d - \gamma_f} = \mathrm{tr}\left(\Phi^1\Phi^2\Phi^3\right) + \frac{\gamma_f+\gamma_d}{\gamma_f-\gamma_d}\mathrm{tr}\left(\Phi^1\Phi^3\Phi^2\right)\,. \tag{62}$$

[14] ES is grateful to Ken Intriligator for this remark.

The explicit form of the relative coefficient in (62) is determined by a straightforward calculation. From (49), (50) and (51) we find

$$
\begin{aligned}
\frac{\gamma_f+\gamma_d}{\gamma_f-\gamma_d} &= \frac{N^2\bar{f}+(N^2-4)\bar{d}}{N^2\bar{f}-(N^2-4)\bar{d}} - \frac{9\,\zeta(3)(N^2-4)}{32\pi^4}\,\bar{d}\ \times \qquad (63)\\
&\quad \frac{2(N^2-4)\bar{d}^3 df - 3N^2\bar{f}^3d^2 - N^2\bar{d}^2\bar{f}f^2 + 2(N^2-10)d^2\bar{d}^2\bar{f}}{[N^2\bar{f}-(N^2-4)\bar{d}]^2}\,.
\end{aligned}
$$

The first term coincides with the one-loop result first obtained in [4]. The second term is the three-loop correction; it has been obtained by expanding $(\gamma_f+\gamma_d)/(\gamma_f-\gamma_d)$ in $f\sim d\sim g$ up to g^4. The absence of a two-loop correction in (63) (independently noticed in [8]) is due to the specific form (52) of the two-loop contribution which allows us to rewrite the finiteness condition up to three loops in the form (49).

3.2.3 Perturbative calculation at order g^6

We have checked by an explicit computation that the operator defined in eqs. (62), (63) has vanishing anomalous dimension at order g^6. Note that with the notation introduced in eqs. (2), (6), we can rewrite $\mathcal{O}_F$ (up to an irrelevant rescaling) in the form

$$
\mathcal{O}_F \sim \mathrm{tr}\left(\Phi^1\ [\Phi^2,\Phi^3]_\Omega\right) + (a_2g^2+a_4g^4+\ldots)\ \mathrm{tr}\left(\Phi^1\ [\Phi^2,\Phi^3]_\omega\right)\,. \qquad (64)
$$

We have computed through order g^6 the corrections to the two-point functions of (the lowest θ component of) the operator $\mathcal{O}_F$ with all the operators it can mix with. There are only three such operators, namely

$$
\mathrm{tr}\left(\Phi_1^\dagger\ [\Phi_3^\dagger,\Phi_2^\dagger]_{\bar{\Omega}}\right)\,,\qquad \mathrm{tr}\left(\Phi_1^\dagger\ [\Phi_3^\dagger,\Phi_2^\dagger]_{\bar{\omega}}\right)\quad\text{and}\quad \mathrm{tr}(\bar{W}^2)\,, \qquad (65)
$$

All the corrections to the two-point function of $\mathcal{O}_F$ with $\mathrm{tr}(\bar{W}^2)$, as well as the logarithmically divergent correction to the two-point function with $\mathrm{tr}(\Phi_1^\dagger\ [\Phi_3^\dagger,\Phi_2^\dagger]_{\bar{\Omega}})$ vanish by the color contractions. At order g^6 the logarithmically divergent correction to the two-point function with $\mathrm{tr}(\Phi_1^\dagger\ [\Phi_3^\dagger,\Phi_2^\dagger]_{\bar{\omega}})$ comes only from the three superdiagrams depicted in Figure 5. The first two are genuine order g^6 diagrams, while the last (order g^2) diagram appears multiplied by the order g^4 coefficient a_4 in (64). All three diagrams lead to the same (logarithmically divergent) coordinate space integral. The cancellation of this divergence, i.e. the vanishing of the order g^6 anomalous dimension of $\mathcal{O}_F$, fixes the values of a_2 and a_4. Exactly the same values are obtained from (63) by rewriting it in the notation of Section 2 (see (48)) and expanding in g up to g^4.

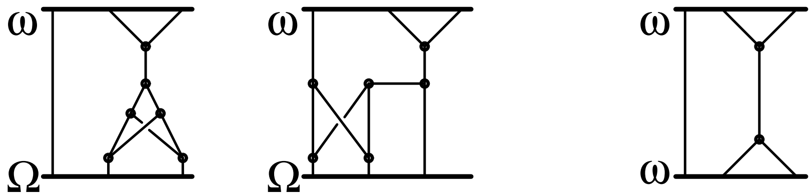

Figure 5: Diagrams contributing to the 2-point functions of $\mathcal{O}_F$ at order g^6.

3.2.4 General potential with a single finiteness condition

The chiral superpotential in eq. (46) has the key feature that it leads to a single finiteness condition. The reason for this is that the matrix of γ functions of the matter superfields Φ^I is proportional to the unit matrix in flavor space,

$$(\gamma_\Phi)^I_J = \gamma\, \delta^I_J\,, \tag{66}$$

so it is enough to demand $\gamma = 0$ to ensure finiteness for all fields. In [1] a generalized superpotential with such a property was proposed in the form

$$f\,\mathrm{tr}\left(\Phi^1[\Phi^2,\Phi^3]\right) + d\,\mathrm{tr}\left(\Phi^1\{\Phi^2,\Phi^3\}\right) + k\,\mathrm{tr}\left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right)\,. \tag{67}$$

Here the holomorphic couplings f, d, k are subject to a single real condition generalizing (49):

$$\gamma(g, f, d, k, \bar{f}, \bar{d}, \bar{k}) = 0\,. \tag{68}$$

Note that for generic values of the couplings the $U(1) \times U(1)$ symmetry of the superpotential is broken, only $U(1)_R$ survives (in addition, there is a discrete symmetry $Z_3 \times Z_3$). It can be shown [16] that (67) is in fact the most general superpotential of this type, up to field redefinitions in the form of $SU(3)$ transformations (recall that the matter kinetic term $\mathrm{tr}(\Phi^\dagger_I \Phi^I)$ is $SU(3)$ invariant).

Just as in section 3.2.2, the derivatives of the Lagrangian with respect to each independent holomorphic coupling (we treat g as dependent and f, d, k as independent) lead to insertion formulae of the type (60). This means that the quantum corrections are now generated by the three CPOs

$$\begin{aligned}
\mathcal{I}_f &= \frac{2\gamma_f}{g\gamma_g}\mathrm{tr}(W^2) + \mathrm{tr}\left(\Phi^1[\Phi^2,\Phi^3]\right) \\
\mathcal{I}_d &= \frac{2\gamma_d}{g\gamma_g}\mathrm{tr}(W^2) + \mathrm{tr}\left(\Phi^1\{\Phi^2,\Phi^3\}\right) \\
\mathcal{I}_k &= \frac{2\gamma_k}{g\gamma_g}\mathrm{tr}(W^2) + \mathrm{tr}\left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right)\,.
\end{aligned} \tag{69}$$

From them we can form a one-parameter family (up to an overall factor) of protected operators analogous to $\mathcal{O}_F$ in eq. (62)

$$h_1\,\mathrm{tr}\left(\Phi^1[\Phi^2,\Phi^3]\right)+h_2\,\mathrm{tr}\left(\Phi^1\{\Phi^2,\Phi^3\}\right)+h_3\,\mathrm{tr}\left((\Phi^1)^3+(\Phi^2)^3+(\Phi^3)^3\right) \quad (70)$$

with the coefficients satisfying the relation

$$h_1\gamma_f+h_2\gamma_d+h_3\gamma_k=0\,. \quad (71)$$

Using the one-loop finiteness condition from [8], we immediately reproduce their form of the dimension three protected operator.

3.3 Operator mixing. Superconformal primaries and descendants

In this subsection we justify our claim that the *chiral* insertions (61) are also superconformal *primaries* and hence they are protected operators (or CPOs). We do this by examining the mixing of all the chiral terms in the Lagrangian (SYM kinetic term and matter superpotential).

Whether an operator is primary or not is a subtle question which can only be answered at the quantum level. In [28] a "rule of thumb" for CPOs made out of matter superfields was proposed, which says that "an operator is primary if it does not contain commutators of superfields under a single color trace". The presence of a commutator is, in fact, a signal that the operator has been obtained from another, lower dimension operator via the field equations, e.g., $\bar{\mathcal{D}}^2\Phi^\dagger \sim g[\Phi,\Phi]$ in the undeformed theory. However, we know that this rule works only in the simplest cases. A counterexample are the 1/4 BPS operators [29] which are mixtures of single and double trace operators, the former containing commutators. This is a typical case of *operator mixing*.

3.3.1 Deformation with f, d terms

In a quantum theory operators can mix if they have the same quantum numbers. For instance, the chiral terms in the Lagrangian (46)

$$F=\mathrm{tr}(W^2)\,,\qquad B_1=\mathrm{tr}\left(\Phi^1[\Phi^2,\Phi^3]\right)\,,\qquad B_2=\mathrm{tr}\left(\Phi^1\{\Phi^2,\Phi^3\}\right) \quad (72)$$

are scalars of dimension 3 and of R charge 2/3 (in units in which θ has charge 1). They also have vanishing $U(1)\times U(1)$ charges generated by the currents [15]

$$\begin{aligned}\mathcal{V}_X &= \mathrm{tr}\left(2\,\Phi_1^\dagger\Phi^1 - \Phi_2^\dagger\Phi^2 - \Phi_3^\dagger\Phi^3\right)\\ \mathcal{V}_Y &= \mathrm{tr}\left(\Phi_2^\dagger\Phi^2 - \Phi_3^\dagger\Phi^3\right)\ .\end{aligned} \tag{73}$$

The conservation of these currents

$$D^2\mathcal{V}_{X,Y} = \bar{D}^2\mathcal{V}_{X,Y} = 0\,, \tag{74}$$

follows from the field equations of the Lagrangian (46). The conclusion of this analysis is that the operators (72) can mix among themselves. The combinations $\mathcal{I}_{f,d}$ in eq. (61) are two such mixtures. Notice that $F = O(g^0)$ (recall (47)), while $B_{1,2}$ appear in (61) multiplied by the couplings $f \sim d \sim g$. This means that the leading term in (61), i.e. the one that survives in the free theory ($g = f = d = 0$), is F, while the appearance of $B_{1,2}$ is a quantum effect.

We can construct a third mixture of the same three operators $B_{1,2}$ and F, where their roles are exchanged: $B_{1,2}$ are the leading terms and F comes about because of quantum corrections. This mixing pattern can also be seen as originating from the so-called "Konishi anomaly" [30]. The Konishi operator is the sum of the three matter kinetic terms in the Lagrangian (46), $\mathcal{K} = \mathrm{tr}(\Phi_I^\dagger\Phi^I)$. Hitting it with $\bar{D}^2$ and using the *classical* field equations, we obtain

$$\bar{D}^2\mathcal{K} = 3(f\,B_1 + d\,B_2)\,. \tag{75}$$

Unlike the currents (73) the Konishi operator is not conserved and hence not protected.

In the quantum theory the classical (non-conservation) equation (75) has to be corrected by an anomaly term. For example, at one loop one finds

$$\bar{D}^2\mathcal{K} = 3(f\,B_1 + d\,B_2) + ag^2\,F \equiv g\mathcal{K}'\,, \tag{76}$$

where a is some computable numerical coefficient [16]. In fact, the quantum corrected equation defines the superdescendant of the Konishi multiplet $\mathcal{K}'$. Although not a superconformal primary, $\mathcal{K}'$ is an operator with well-defined conformal properties (conformal primary).

[15] For brevity the covariantizing factors $e^{\pm 2gV}$ are suppressed.

[16] In the literature there are claims that the Konishi anomaly is one-loop exact (see, e.g. [31]). Such claims fail to take into account the presence of the (classical) B terms and their non-trivial mixing at the quantum level with the operator F. This is most clearly seen if one repeats the two-loop calculation of [32] *in the presence of a matter self-coupling* (ES thanks Marc Grisaru for discussions on this point). See also [34] for a general discussion of the Konishi anomaly in the context of $\mathcal{N} = 4$ SYM.

Returning to the first two combinations (61), we can now argue that they are *superconformal primaries*. Indeed, if they were descendants, we should be able to find some lower dimensional primaries from which we could obtain (61) through supersymmetry transformations (or, equivalently, through spinor derivatives, as in (76)). Since the leading F terms in (61) are fermion bilinears, we can only use two supersymmetries on a scalar operator of dimension two, or one supersymmetry on a spinor of dimension 5/2. Given the $U(1)\times U(1)\times U(1)_R$ charges of $B_{1,2}$, the only scalar candidates are the three matter kinetic terms. Two combinations are the $U(1)\times U(1)$ currents (73), which have no descendants obtained acting with $\bar{D}^2$. The third one is the Konishi operator $\mathcal{K}$ which gives rise to the descendant $\mathcal{K}'$. Similar arguments rule out a fermionic primary. Thus, the operators (61) must be primary, and since they are also chiral, they are protected (or CPOs).

We remark that the protected operators are orthogonal to the Konishi descendant (the latter has a non-vanishing anomalous dimension, the former do not), implying

$$\langle \mathcal{K}'\ \mathcal{I}^\dagger_{f,d}\rangle = 0\,. \tag{77}$$

This condition can be efficiently used for determining the right mixture in (76) not only at one loop, but also beyond (see [33, 34])[17].

3.3.2 General deformation with f, d, k terms

In the case of the superpotential (67) the $U(1)\times U(1)$ symmetry is broken and only $U(1)_R$ survives. This means that we can extend the set of chiral operators (72) by adding the new terms

$$B_3 = \mathrm{tr}\left((\Phi^1)^3\right)\,, \qquad B_4 = \mathrm{tr}\left((\Phi^2)^3\right)\,, \qquad B_5 = \mathrm{tr}\left((\Phi^3)^3\right)\,. \tag{78}$$

appearing in (67). Out of the set of six operators (72) and (78) we can form the three protected combinations of eq. (69) and three new, unprotected combinations. One of the latter is, as before, the Konishi descendant

$$\bar{D}^2\mathcal{K} = 3\,(f\,B_1 + d\,B_2 + 3k(B_3+B_4+B_5)) + ag^2\,F \equiv g\mathcal{K}'\,. \tag{79}$$

The two new ones are descendants of the former $U(1)\times U(1)$ currents (73) which are not conserved anymore

$$\bar{D}^2\mathcal{V}_X = 3k(2B_3 - B_4 - B_5)\,, \qquad \bar{D}^2\mathcal{V}_Y = 3k(B_4 - B_5)\,. \tag{80}$$

[17]In the recent paper [8] the logic has been inverted, using the orthogonality of the protected operator $\mathcal{O}_F$ to the Konishi descendant as a tool for determining the former from the known form of the latter. However, experience shows that the direct determination of the Konishi anomaly beyond one loop is not an easy task.

If we now switch off the new deformation by setting $k = 0$, conservation of currents is restored, while at the same time the chiral operators $2B_3 - B_4 - B_5$ and $B_4 - B_5$ become primary and thus protected.

3.3.3 Undeformed theory

In the undeformed $\mathcal{N} = 4$ theory we have $f = g$ and $d = k = 0$ which restores the full $SU(3) \times U(1)_R \subset SU(4)$ symmetry. The Konishi operator has a scalar descendant $\mathcal{K}_{10}$ of dimension three in the $\underline{10}$ of $SU(4)$. Its $SU(3)$ singlet projection $\mathcal{K}_{10/1}$ [18] in the $\mathcal{N} = 1$ formulation of the theory is the Konishi anomaly, a mixture of B_1 and F, which is the analog of $\mathcal{K}'$. The same $SU(3)$ invariant operators B_1 and F form another combination,

$$\mathcal{O}_{10/1} = 2F - gB_1 = -g\mathcal{I}_f \, . \tag{81}$$

We observe that $\mathcal{O}_{10/1}$ is on the one hand proportional to $\mathcal{I}_f$ of eq. (61) computed with $\gamma = 2N(|f|^2 - g^2) = 0$, $d = 0$. On the other hand, it is a descendant *in the* $\mathcal{N} = 4$ *sense* of the protected stress-tensor multiplet $\mathcal{O}_{20'}$. It can be obtained by making two on-shell $\mathcal{N} = 4$ supersymmetry transformations on the highest-weight chiral projection $\mathcal{O}_{20'/6} = \mathrm{tr}(\Phi^1\Phi^1)$ of the 1/2 BPS operator $\mathcal{O}_{20'}$,

$$\mathcal{O}_{10/1} = (\bar{Q}_{\mathcal{N}=4})^2 \mathcal{O}_{20'/6} \, . \tag{82}$$

It is important to realize that $\mathcal{O}_{10/1}$ is not a descendant of $\mathcal{O}_{20'}$ in the $\mathcal{N} = 1$ sense. This confirms that $\mathcal{I}_f$ is a superconformal primary from the $\mathcal{N} = 1$ point of view.

Finally, in the undeformed case the operator $\mathcal{O}_F$ eq. (62) is a particular state in the $SU(3)$ 10-plet projection $\mathcal{O}_{50/10}$ of the protected 1/2 BPS operator $\mathcal{O}_{50}$ whose highest weight is $\mathrm{tr}((\Phi^1)^3)$.

4 Conclusions

We have shown that for *any* complex value of the deformation parameter ω there exists a whole family (parametrized by the complex function $\kappa_\Omega(g)$) of conformally invariant $\mathcal{N} = 1$ deformations of $\mathcal{N} = 4$ SYM. In each such theory is present a special CPO $\mathcal{O}_F$, of dimension three, whose explicit form to all orders can be determined directly from the finiteness condition $\gamma_\Phi = 0$.

[18] We denote by m/n the n-dimensional $SU(3)$ projection of an m–dimensional $SU(4)$ multiplet.

In the recent paper [35] the planar limit of the deformed $\mathcal{N}=4$ SYM theory has been investigated in detail up to ten loops. Our, order g^8 planar, calculation is in agreement with the four-loop formula obtained there. However, we disagree with the conclusions of [35], where it is claimed that the deformed $\mathcal{N}=4$ SYM can be made conformally invariant only if the deformation parameter β is real (i.e. for $|\omega|=1$ in our notation). We believe that the contradiction they find at the five-loop level is an artefact of the use of dimensional regularization and that the solution to this problem is given in [11, 12].

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