

Topological susceptibility in full QCD with Ginsparg-Wilson fermions

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Abstract

We show that, if the formula for the topological charge density operator suggested by fermions obeying the Ginsparg–Wilson relation is employed, it is possible to give a precise and unambiguous definition of the topological susceptibility in full QCD, χ_{tL}^{full} , for finite quark masses on the lattice. The lattice expression of χ_{tL}^{full} looks like the formal continuum one, in the sense that no power divergent subtractions are needed for its proper definition. As a consequence, the small mass behaviour of χ_{tL}^{full} leads directly to a multiplicative renormalizable definition of the chiral condensate that does not require any power divergent subtraction.

1 Introduction

In this paper we discuss the definition and the properties of the topological susceptibility in full QCD with massive quarks extending the results of ref. [1]. Using arguments based on anomalous flavour singlet Ward-Takahashi identities (WTI's), we prove that, if the formula of the topological charge density, Q , suggested by Ginsparg–Wilson (GW) fermions [2, 3, 4] is employed, the full QCD topological susceptibility

$$\chi_{tL}^{\text{full}} = \int d^4x \langle Q(x)Q(0) \rangle \quad (1)$$

does not need any power divergent subtraction at finite non-vanishing values of the quark (pion) mass. Furthermore it vanishes linearly in the quark mass with a coefficient which turns out to be (proportional to) the chiral condensate, as in the formal continuum limit.

The interest of these results lies in the fact that one can exploit the absence of power divergent mixings in the continuum-looking lattice formula (1) to extract the value of the chiral condensate with no need of performing any dangerous power divergent subtraction.

2 Generalities on Ginsparg-Wilson fermions

Regularizing the fermionic part of the QCD action using GW fermions offers the great advantage that global chiral transformations can be defined, which are an exact symmetry of the massless theory, as in the formal continuum theory. This is a consequence of the relation [2]

$$\gamma_5 D + D \gamma_5 = a D \gamma_5 D, \quad (2)$$

where D is the Dirac operator and a is the lattice spacing. Eq. (2) implies the invariance of the massless fermion action under the transformations [5]

$$\delta_A^f \psi = \lambda^f \hat{\gamma}_5 \psi, \quad \delta_A^f \bar{\psi} = \bar{\psi} \gamma_5 \lambda^f, \quad f = 0, 1, \dots, N_f^2 - 1, \quad (3)$$

where $\lambda^0 = \mathbb{1}$ and the λ^f 's, $f \neq 0$, are flavour matrices¹. In the first of the equations above we have introduced the definition

$$\hat{\gamma}_5 = \gamma_5 (1 - aD) \quad (4)$$

with the properties

$$\hat{\gamma}_5^\dagger = \hat{\gamma}_5, \quad \hat{\gamma}_5^2 = 1. \quad (5)$$

Eqs. (3) may be interpreted as the lattice form of the continuum chiral transformations. The Neuberger operator [4] satisfies the GW relation, has the correct continuum limit and is local, though not ultra-local [6]. Another solution of the GW condition (2) is provided by the fixed-point fermionic action of refs. [7, 8].

¹We use the normalization $\text{tr}(\lambda^f \lambda^g) = \delta^{fg}/2$, $[\lambda^f, \lambda^g] = i f^{fgh} \lambda^h$, so that $\{\lambda^f, \lambda^g\} = d^{fgh} \lambda^h + \delta^{fg} \mathbb{1}/N_f$, $f, g, h = 1, \dots, N_f^2 - 1$.

In a GW regularization the lattice QCD fermion action with N_f massive flavour can be written in the form ²

$$S_F = \int d^4x \sum_{r,s=1}^{N_f} \bar{\psi}^r(x) \left[\left(D\delta_{rs} + P_- M_{rs}^\dagger \hat{P}_- + P_+ M_{rs} \hat{P}_+ \right) \psi^s \right](x) \quad (6)$$

where

$$\hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5), \quad P_\pm = \frac{1}{2}(1 \pm \gamma_5), \quad (7)$$

ψ ($\bar{\psi}$) is an N_f -dimensional column (row) vector in flavour space and $M = \text{diag}(m_1, \dots, m_{N_f})$. S_F is invariant under the $U_L(N_f) \times U_R(N_f)$ global transformations

$$\begin{aligned} \psi_L &\rightarrow U_L \psi_L & \bar{\psi}_L &\rightarrow \bar{\psi}_L U_L^\dagger \\ \psi_R &\rightarrow U_R \psi_R & \bar{\psi}_R &\rightarrow \bar{\psi}_R U_R^\dagger, \end{aligned} \quad (8)$$

with $U_{L,R} \in U(N_f)_{L,R}$ and

$$\begin{aligned} \psi_L &= \hat{P}_- \psi & \bar{\psi}_L &= \bar{\psi} P_+ \\ \psi_R &= \hat{P}_+ \psi & \bar{\psi}_R &= \bar{\psi} P_-, \end{aligned} \quad (9)$$

if at the same time $M \rightarrow U_L M U_R^\dagger$. In the following we shall restrict ourselves to the flavour vector symmetric case $m_r = m$, with $r = 1, \dots, N_f$. As a consequence of the exact chiral invariance of the massless GW regularization, no additive quark mass renormalization is required. The action is $O(a)$ -improved, since no chiral invariant operators of dimension $d = 5$ can be constructed. In this work we will consider the bilinear scalar and pseudo-scalar quark operators ($f = 0, 1, \dots, N_f^2 - 1$)

$$S^f(x) = \bar{\psi}(x) \lambda^f \left[\left(1 - \frac{a}{2} D \right) \psi \right](x), \quad (10)$$

$$P^f(x) = \bar{\psi}(x) \lambda^f \gamma_5 \left[\left(1 - \frac{a}{2} D \right) \psi \right](x). \quad (11)$$

The ‘‘rotation’’ $(1 - \frac{a}{2} D)$ of the quark field ψ in the above equations leads to definitions of scalar and pseudo-scalar quark densities which have the correct chiral transformation properties, like in the formal continuum theory, and only need a (logarithmically divergent) multiplicative renormalization. Furthermore the operators (10) and (11) are automatically $O(a)$ -improved.

In a GW regularization the gauge anomaly is recovered *à la* Fujikawa [9, 5]. The fermion integration measure is not invariant under $U_A(1)$ transformations (eqs. (3) with $f = 0$), and the topological charge density

$$a^4 Q(x) = -\frac{a}{2} \text{Tr} \left[\gamma_5 D(x, x) \right], \quad (12)$$

originating from the corresponding Jacobian, is related to the index of the lattice Dirac operator, D , by the equation [4, 10, 5]

$$n_+ - n_- = \text{index}(D) = \int d^4x Q(x), \quad (13)$$

²For short we use continuum looking notations with $\int d^4x$ replacing $a^4 \sum_x$.

with n_+ (n_-) the number of zero modes with positive (negative) chirality ³. For a recent review on this subject see [11].

3 The singlet Ward-Takahashi identities

In the chiral limit, the local anomalous flavour-singlet WTI's have the form

$$\partial_\mu \langle \mathcal{A}_\mu^0(x) \widehat{\mathcal{O}}(y) \rangle = 2N_f \langle Q(x) \widehat{\mathcal{O}}(y) \rangle - \langle \delta_A^{0,x} \widehat{\mathcal{O}}(y) \rangle, \quad (14)$$

where $\mathcal{A}_\mu^0(x)$ is the singlet axial current, $\widehat{\mathcal{O}}$ is any renormalized (multi-)local operator, concentrated at points $y \equiv \{y_i, i = 1, \dots, n\}$ and $\delta_A^{0,x} \widehat{\mathcal{O}}$ is its local singlet axial variation. In eq. (14) we have not shown the exponentially suppressed terms coming from the fact that D is not ultra-local [6]. They are of no relevance for the following arguments, as they vanish after integration. Assuming the absence of a $U_A(1)$ massless Goldstone boson, the integrated form of the WTI's (15) reads

$$0 = 2N_f \int d^4x \langle Q(x) \widehat{\mathcal{O}}(y) \rangle - \langle \delta_A^0 \widehat{\mathcal{O}}(y) \rangle. \quad (15)$$

Since the second term in the r.h.s. of eq. (15) is finite, it follows that $\int d^4x Q(x)$ is also finite, as it has finite insertions with any string of renormalized fundamental fields. Therefore $Q(x)$ can only mix with operators of dimension ≤ 4 and vanishing integral, hence only with $\partial_\mu \mathcal{A}_\mu^0(x)$. No power-divergent subtractions with lower dimensional operators (such as the pseudo-scalar quark density) are required [1]. This is a very distinctive feature of GW fermions with respect to standard Wilson fermions which is directly related to the absence of an additive mass renormalization ⁴. One can define finite operators \widehat{Q} and $\widehat{\mathcal{A}}_\mu^0$ by the equations [14]

$$\widehat{Q}(x) = Q(x) - \frac{Z}{2N_f} \partial_\mu \mathcal{A}_\mu^0(x), \quad (16)$$

$$\widehat{\mathcal{A}}_\mu^0(x) = (1 - Z) \mathcal{A}_\mu^0(x), \quad (17)$$

where Z is the mixing coefficient between Q and $\partial_\mu \mathcal{A}_\mu^0$. The renormalized singlet axial WTI's then become (again up to exponentially small terms)

$$\partial_\mu \langle \widehat{\mathcal{A}}_\mu^0(x) \widehat{\mathcal{O}}(y) \rangle = 2N_f \langle \widehat{Q}(x) \widehat{\mathcal{O}}(y) \rangle - \langle \delta_A^{0,x} \widehat{\mathcal{O}}(y) \rangle. \quad (18)$$

Outside of the chiral limit the integrated singlet axial WTI's read

$$0 = 2m \int d^4x \langle P^0(x) \widehat{\mathcal{O}}(y) \rangle + 2N_f \int d^4x \langle Q(x) \widehat{\mathcal{O}}(y) \rangle - \int d^4x \langle \delta_A^{0,x} \widehat{\mathcal{O}}(y) \rangle. \quad (19)$$

The extra term present in eq. (16) being a total divergence does not contribute to the integrated WTI's (19). If we replace $\widehat{\mathcal{O}}(y)$ with the local operator $Q(0)$, we get

$$0 = 2m \int d^4x \langle P^0(x) Q(0) \rangle + 2N_f \int d^4x \langle Q(x) Q(0) \rangle, \quad (20)$$

³For alternative lattice definitions of Q see the papers of ref. [12].

⁴A discussion of the singlet WTI's for Wilson fermions can be found in ref. [13].

and, similarly, by inserting the multiplicative renormalizable operator $P^0(0)$

$$0 = 2m \int d^4x \langle P^0(x)P^0(0) \rangle + 2N_f \int d^4x \langle Q(x)P^0(0) \rangle - 2\langle S^0(0) \rangle. \quad (21)$$

Putting together eqs. (20) and (21), we obtain

$$\chi_{tL}^{\text{full}} \equiv \int d^4x \langle Q(x)Q(0) \rangle = \frac{(2m)^2}{(2N_f)^2} \int d^4x \langle P^0(x)P^0(0) \rangle - \frac{4m}{(2N_f)^2} \langle S^0(0) \rangle. \quad (22)$$

In the next section we show that the full QCD topological susceptibility, χ_{tL}^{full} , defined above is not affected by power divergences.

3.1 Absence of power divergences in χ_{tL}^{full}

The proof of the absence of power divergences in χ_t^{full} is based on the study of the short distance behaviour of the two terms in the r.h.s. of eq. (22) at small m .

The argument goes through a number of steps. First of all we observe that thanks to the chiral properties of GW fermions, only power divergences of the type m^2/a^2 can possibly be present in eq. (22). The second observation is that the m^2/a^2 divergences separately affecting the two terms in the r.h.s. actually cancel each other. This is the result of the exact compensation between the (quadratically) divergent term arising in $m^2 \int d^4x \langle P^0(x)P^0(0) \rangle$, due to the short distance behaviour of the integrand, and a similar divergent term appearing in $\langle S^0 \rangle$, when one power of the fermionic mass term (brought down from the action) is inserted together with S^0 . The compensation follows from the fact that the short distance (perturbative) behaviour of the two correlators $\langle P^0(x)P^0(0) \rangle$ and $\langle S^0(x)S^0(0) \rangle$ are (in the massless limit) equal (up to a minus sign due to the presence of two extra γ_5 matrices in $\langle P^0(x)P^0(0) \rangle$), leaving behind a finite, computable contribution.

We now make explicit and precise the line of arguments sketched above.

1. Each term in eq. (22) is even under the (non-anomalous) spurionic symmetry [15]

$$\mathcal{R}_5^{\text{sp}} \equiv \mathcal{R}_5 \times [m \rightarrow -m] \quad \mathcal{R}_5 : \begin{cases} \psi & \rightarrow \psi' = \hat{\gamma}_5 \psi \\ \bar{\psi} & \rightarrow \bar{\psi}' = -\bar{\psi} \gamma_5 \end{cases} \quad (23)$$

where \mathcal{R}_5 is an element of the chiral group. Since only the identity operator, which is even under $\mathcal{R}_5^{\text{sp}}$, can contribute a divergent term in the two terms in the r.h.s. of eq. (22), we conclude that only m^2/a^2 power divergences can be present, as they are even under $m \rightarrow -m$. In other words chiral invariance forbids power divergences like m/a^3 and m^3/a .

2. If we order the terms contributing to eq. (22) in powers of m , we get

$$4N_f^2 \chi_{tL}^{\text{full}} = -4m \langle S^0(0) \rangle \Big|_{m=0} + 4m^2 \int d^4x \left[\langle S^0(x)S^0(0) \rangle + \langle P^0(x)P^0(0) \rangle \right]_{m=0} + \quad (24)$$

$$+ \left[\text{O}(m^3/m^4) \text{ with/without } \text{S}\chi\text{SB} \right],$$

where odd powers of m can be present in the expansion only if chiral symmetry is spontaneously broken. For the purpose of studying the structure of power divergences, we only need to examine terms $O(m)$ and $O(m^2)$. Higher order terms are at most logarithmically divergent ⁵.

3. The first term in the expansion (24) is finite. In fact, i) owing to the exact chiral symmetry of the massless GW fermionic action, there cannot be any mixing between the identity and the operator S^0 (with a a^{-3} divergent coefficient), because they transform in the opposite way under $\mathcal{R}_5^{\text{SP}}$; ii) the quantity $m\langle S^0(0)\rangle|_{m=0}$ is not logarithmically divergent, as a consequence of the non-singlet WTI's.
4. The sum of the next two terms is finite. To prove this result it is convenient to consider the set of WTI's ($f = 1, \dots, N_f^2 - 1$; $g, h = 0, \dots, N_f^2 - 1$)

$$\begin{aligned}
0 &= \int d^4z \int d^4x \partial_\mu \langle \mathcal{A}_\mu^f(z) P^g(x) S^h(0) \rangle = \\
&= 2m \int d^4z \int d^4x \langle P^f(z) P^g(x) S^h(0) \rangle + \\
&\quad - \int d^4x \langle \delta_A^f P^g(x) S^h(0) \rangle - \int d^4x \langle P^g(x) \delta_A^f S^h(0) \rangle, \tag{25}
\end{aligned}$$

where δ_A^f represents the operation of taking the axial variation with flavour index f .

5. Combining the above WTI's, one gets in the chiral limit the soft-pion theorem (no sum over repeated indices, $d^{fgh} \neq 0$)

$$\begin{aligned}
&\int d^4x \left[\langle S^0(x) S^0(0) \rangle + \langle P^0(x) P^0(0) \rangle \right]_{m=0} = \\
&= F_\pi N_f \left[\int d^4x \langle \pi^f | T_E(P^f(x) S^0(0) + S^f(x) P^0(0)) | 0 \rangle \Big|_{m=0} + \right. \\
&\quad \left. - \frac{2}{d^{fgh}} \int d^4x \langle \pi^f | T_E(P^g(x) S^h(0)) | 0 \rangle \Big|_{m=0} \right], \tag{26}
\end{aligned}$$

where T_E means Euclidean time-ordering ⁶ and we have used the definition

$$\langle 0 | \partial_\mu \mathcal{A}_\mu^f | \pi^g \rangle = \delta^{fg} F_\pi m_\pi^2. \tag{27}$$

From the O.P.E.'s

$$P^f(x) S^0(0) \simeq S^f(x) P^0(0) \simeq \frac{1}{x^3} P^f(0), \tag{28}$$

$$P^f(x) S^g(0) \simeq \sum_h d^{fgh} \frac{1}{x^3} P^h(0), \tag{29}$$

one concludes that the integrals in the r.h.s. of eq. (26) are indeed finite.

⁵Whether the logarithmically divergent terms proportional to m^4 might be reabsorbed by renormalizing m is a question that can be decided by a perturbative calculation.

⁶For simplicity we do not employ a different notation for the operators in the matrix elements appearing in the r.h.s. of eq. (26).

4 Final considerations

A number of interesting consequences follow from the formulæ (18) and (22).

1. In the full theory $m_{\eta'}^2 \neq 0$ and there is no massless particle that can couple to P^0 . So it is immediately seen that χ_{tL}^{full} vanishes in the chiral limit ($m \rightarrow 0$).
2. A formula for the (quenched) η' mass [16, 17] can be obtained starting from the Fourier transform of the WTI (18) at zero quark mass, if one chooses $\widehat{O} = \widehat{Q}$. For completeness we recall here the standard argument which goes as follows. First we observe that the $U_A(1)$ variation of \widehat{Q} is zero. Taking the Fourier transform of eq. (18) with \widehat{O} replaced by \widehat{Q} , one gets (in the chiral limit)

$$i \int d^4x e^{-ipx} p_\mu \langle \widehat{\mathcal{A}}_\mu^0(x) \widehat{Q}(0) \rangle = 2N_f \int d^4x e^{-ipx} \langle \widehat{Q}(x) \widehat{Q}(0) \rangle. \quad (30)$$

In the limit $N_f/N_c \rightarrow 0$, where the η' mass vanishes, the l.h.s. of eq. (30) is dominated at small p by the η' pole, leading to the expansion

$$i \int d^4x e^{-ipx} p_\mu \langle \widehat{\mathcal{A}}_\mu^0(x) \widehat{Q}(0) \rangle \Big|_{N_f/N_c=0} = \lim_{N_f/N_c \rightarrow 0} m_{\eta'}^2 F_{\eta'}^2 \frac{p^2}{p^2 + m_{\eta'}^2} + \mathcal{O}(p^2). \quad (31)$$

If, as indicated in the above formula, the limit $p \rightarrow 0$ is taken after the limit $N_f/N_c \rightarrow 0$, one ends up with the relation

$$\frac{m_{\eta'}^2 F_{\eta'}^2}{2N_f} \Big|_{N_f/N_c=0} = \lim_{p \rightarrow 0} \lim_{N_f/N_c \rightarrow 0} \int d^4x e^{-ipx} \langle \widehat{Q}(x) \widehat{Q}(0) \rangle, \quad (32)$$

where standard counting arguments [19] ensures that the l.h.s. of eq. (32) has a finite limit as $N_f/N_c \rightarrow 0$. At this point if it is assumed that, taking the limit $N_f/N_c \rightarrow 0$ in the r.h.s. of (31) is equivalent to drop the fermion determinant, one arrives at the famous WV formula⁷

$$\frac{m_{\eta'}^2 F_\pi^2}{2N_f} \Big|_{N_f/N_c=0} = \int d^4x \langle Q(x) Q(0) \rangle \Big|_{\text{YM}}. \quad (33)$$

Notice that in the limit $N_f/N_c \rightarrow 0$, the mixing coefficient, Z , in eq. (16) vanishes and $F_{\eta'}$ becomes equal to F_π . Thus in eq. (33) we have replaced $F_{\eta'}$ with F_π and at the same time \widehat{Q} with Q , although in this limit the integral of the divergence of the singlet axial current does not vanish. Recalling the expression (12) and the index theorem (13), one can equivalently write for the (quenched) η' mass the suggestive formula

$$\frac{m_{\eta'}^2 F_\pi^2}{2N_f} \Big|_{N_f/N_c=0} = \lim_{V \rightarrow \infty} \frac{\langle (n_+ - n_-)^2 \rangle_V}{V}. \quad (34)$$

where V is the space-time volume of the lattice.

⁷For a discussion of several subtleties on the derivation of this formula see ref. [18].

3. Paying due care to flavour matrix normalization, one can combine the non-singlet WTI (written for a given fixed flavour, h)

$$0 = 2m \int d^4x \langle P^h(x)P^h(0) \rangle - \frac{1}{N_f} \langle S^0(0) \rangle \quad (35)$$

with eq. (22), obtaining

$$(2N_f)^2 \int d^4x \langle Q(x)Q(0) \rangle = (2m)^2 \int d^4x \langle P^0(x)P^0(0) \rangle \Big|^{ZV}, \quad (36)$$

where the superscript ZV means that only Zweig-Violating (hairpin) diagrams should be included in the r.h.s. of the equation in carrying out the fermion functional integral. For GW fermions the relation (36) is an algebraic identity that can be directly proved, using eq. (2), the definition (12) and the explicit expressions of P^0 and S^0 (eqs. (10) and (11)). A few observations are in order here.

- An equation like (36) holds also if the fermion determinant is neglected (quenching). This follows immediately from the GW relation, after the fermion integration is performed, as shown in the Appendix.
- In the quenched limit the r.h.s. of eq. (36) possesses a double pole at vanishing quark (pion) mass with a residue related to $m_{\eta'}^2$ [16, 17]. To be precise its residue is in our normalization $m_{\eta'}^2 m_\pi^4 F_\pi^2 / 2N_f$ ⁸. This observation was the basis of the many quenched simulations carried out in the years in lattice QCD aimed at extracting the mass of the pseudo-scalar flavour singlet, starting from the seminal work of Hamber, Marinari, Parisi and Rebbi (see refs. [20, 21, 22]).
- With the idea of trying to set up a formula for the η' mass which would not depend on the details of the lattice definition of the topological charge density operator, in ref. [23] for Wilson and few years later for staggered fermions [24], the equation

$$\frac{m_{\eta'}^2 F_\pi^2}{2N_f} \Big|_{N_f/N_c=0} = \lim_{m \rightarrow 0} \frac{(2m)^2}{(2N_f)^2} \int d^4x \langle P^0(x)P^0(0) \rangle \Big|_{\text{quenched}}^{ZV} \quad (37)$$

was argued to hold in the limit $N_f/N_c \rightarrow 0$ ⁹. Eq. (37) can be regarded as an expression of the residue of the $1/m^2$ double pole present in the $\int d^4x \langle P^0(x)P^0(0) \rangle$ correlator at $N_f/N_c = 0$ (i.e. in the absence of the fermion determinant). In this way no use of the identity (36) is actually made, though eq. (37) is obviously consistent with (36) for GW fermions.

4. As in the formal continuum theory [26], the chiral condensate can be extracted from the small m expansion of the (lattice) full topological susceptibility, as defined by eqs. (12) and (22)

$$4N_f^2 \chi_{tL}^{\text{full}} = -4m \langle S^0(0) \rangle \Big|_{m=0} + \mathcal{O}(m^2), \quad (38)$$

⁸We are using the definition $\langle 0 | \partial_\mu \hat{A}_\mu^0 | \eta' \rangle = \sqrt{2N_f} F_{\eta'} m_{\eta'}^2$, with the identification of $F_{\eta'}$ with F_π in the quenched limit. We are also assuming that the pion bound state exists even in the quenched theory.

⁹Actually the way the η' mass formula was written in ref. [23] is wrong by the finite normalization factor $1 - \partial \bar{M}^0(M) / \partial M |_{M=M_{cr}}$, see ref. [25] for notations.

with no need of power divergent subtractions.

5 Conclusions

In this paper we have shown that there are no m^2/a^2 power divergences in χ_{tL}^{full} , as defined in eq. (22). Thus, if GW fermions are used, the topological susceptibility of the full theory

$$\chi_{tL}^{\text{full}} = a^{-6} \int d^4x \langle \frac{1}{2} \text{Tr} [\gamma_5 D(x, x)] \frac{1}{2} \text{Tr} [\gamma_5 D(0, 0)] \rangle \quad (39)$$

can be employed to extract the physical value of the chiral condensate using the eq. (38), without the need of performing any power subtraction. The reason is that the a^{-2} power divergence in $\langle S^0 \rangle$, outside the chiral limit, is exactly compensated by a similar divergence in the $\langle P^0 P^0 \rangle$ correlator in the r.h.s. of eq. (22).

Actually we have proved more than that. We have proved that all a^{-2} power divergences in the $\langle P^0 P^0 \rangle$ correlator arise from the ZC contributions. This conclusion follows from the non-singlet WTI (35), where we see that only ZC diagrams contribute.

This observation may be of some interest in view of the formula (37), where the (quenched) η' mass is expressed in terms of the ZV quenched correlator of two singlet pseudo-scalar quark densities. In this case, in order to conclude that power divergences are also absent from the r.h.s. of eq. (37), one must assume that taking the limit $N_f/N_c \rightarrow 0$ in eq. (36) is equivalent to dropping the fermion determinant and does not introduce unexpected $1/a^2$ -power divergences. If this is the case, one can imagine to check the validity of the formula (33) and the assumptions underlying it by comparing the value of the YM topological susceptibility (r.h.s. of eq. (34)) with the residue of the double pole that arises at zero quark mass when the fermion determinant is dropped from eq. (36) (thus ending up with eq. (37)). In this context it is interesting to mention that the YM topological susceptibility has been recently computed at several values of the lattice spacing by counting the number of zero modes of the Neuberger-Dirac operator [27, 28]. Data are compatible with the scaling behaviour expected for a quantity of dimension $d = 4$ and no sign of power divergences (within errors).

Appendix

In this appendix we want to prove that the formula

$$\frac{N_f}{2a^3} \int d^4x \langle \langle \text{Tr} [\gamma_5 D(x, x)] \rangle \rangle = m \int d^4x \langle \langle P^0(x, x) \rangle \rangle, \quad (40)$$

where $\langle \langle \dots \rangle \rangle$ means fermion field contraction only, holds configuration by configuration in the massive theory. To this end we recall that the expression of the massive GW operator in eq. (6) is

$$D_m = (1 - \frac{am}{2})D + m. \quad (41)$$

In terms of D_m the GW relation (2) takes the form

$$\gamma_5 D_m + D_m \gamma_5 - 2m\gamma_5 = \frac{a}{(1 - \frac{am}{2})} (D_m \gamma_5 D_m - m(\gamma_5 D_m + D_m \gamma_5) + m^2 \gamma_5). \quad (42)$$

Putting similar terms together and multiplying by the inverse of the massive GW operator, D_m^{-1} , we get from (42), after taking the trace and integrating over space-time

$$-m \int d^4x \text{Tr} [\gamma_5 D_m^{-1}] = \frac{a}{2} \int d^4x \text{Tr} [\gamma_5 D_m]. \quad (43)$$

At this point it is enough to observe that

1) contracting the fermion fields in the operator P^0 appearing in eq. (40) gives

$$-\frac{1}{N_f} \int d^4x \langle\langle P^0(x, x) \rangle\rangle = \frac{1}{a^4(1 - \frac{am}{2})} \int d^4x \text{Tr} [\gamma_5 D_m^{-1}], \quad (44)$$

2) tracing D_m with γ_5 yields

$$\text{Tr} [\gamma_5 D_m] = (1 - \frac{am}{2}) \text{Tr} [\gamma_5 D]. \quad (45)$$

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References

- [1] L. Giusti, G.C. Rossi, M. Testa and G. Veneziano, Nucl. Phys. B 628 (2002) 234.
- [2] P.H. Ginsparg and K.G. Wilson, Phys. Rev. D25 (1982) 2649.
- [3] R. Narayanan and H. Neuberger, Phys. Lett. B302 (1993) 62; Phys. Rev. Lett. 71 (1993) 3251; Nucl. Phys. B412 (1994) 574 and B443 (1995) 305.
- [4] H. Neuberger, Phys. Lett. B417 (1998) 141; Phys. Rev. D57 (1998) 5417.
- [5] M. Lüscher, Phys. Lett. B428 (1998) 342.
- [6] P. Hernández, K. Jansen and M. Lüscher, Nucl. Phys. B552 (1999) 363.
- [7] P. Hasenfratz and F. Niedermayer, Nucl. Phys. B414 (1994) 785; T. DeGrand et al., Nucl. Phys. B454 (1995) 587.

- [8] P. Hasenfratz, Nucl. Phys. B (Proc. Suppl.) 63 (1998) 53.
- [9] K. Fujikawa, Phys. Rev. Lett. 42 (1979) 1195; Phys. Rev. D21 (1980) 2848 [Erratum, *ibidem* D22 (1980) 1499].
- [10] P. Hasenfratz, V. Laliena and F. Niedermayer, Phys. Lett. B427 (1998) 125.
- [11] L. Giusti, Nucl. Phys. B (Proc. Suppl.) 119 (2003) 149.
- [12] P. Di Vecchia, K. Fabricius, G.C. Rossi and G. Veneziano, Nucl. Phys. B192 (1981) 392.
M. Lüscher, Commun. Math. Phys. 85 (1982) 39.
K. Fabricius and G.C. Rossi, Phys. Lett. B127 (1983) 229.
A. Di Giacomo, H. Panagopoulos and E. Vicari, Nucl. Phys. B338 (1990) 294 and references therein.
- [13] D. Guadagnoli and S. Simula, Nucl. Phys. B670 (2003) 264.
- [14] W.A. Bardeen, Phys. Rev. 184 (1969) 1848 and Nucl. Phys. B75 (1974) 246.
- [15] R. Frezzotti and G.C. Rossi, hep-lat/0306014 and hep-lat/0309157.
- [16] E. Witten, Nucl. Phys. B156 (1979) 269.
- [17] G. Veneziano, Nucl. Phys. B159 (1979) 213 and Phys. Lett. B95 (1980) 90.
- [18] E. Seiler and I.O. Stamatescu, MPI-PAE/PTh 10/87;
E. Seiler, Phys. Lett. B525 (2002) 355.
- [19] G. Veneziano, Nucl. Phys. B117 (1976) 519;
E. Witten, Nucl. Phys. B160 (1979) 57.
- [20] H. Hamber, E. Marinari, G. Parisi and C. Rebbi, Nucl. Phys. B225 (1983) 475.
- [21] Y. Kuramashi, M. Fukugita, H. Mino, M. Okawa and A. Ukawa, Phys. Rev. Lett. 72 (1994) 3448;
W.A. Bardeen, A. Duncan, E. Eichten and H. Thacker, Nucl. Phys. B (Proc. Suppl.) 83 (2000) 215 and Phys. Rev. D62 (2000) 114505.
- [22] G. Kilcup, J. Grandy and L. Venkataraman, Nucl. Phys. B (Proc. Suppl.) 47 (1996) 358 and *ibidem* 53 (1997) 259;
Ch. Michael *et al.* (UKQCD Coll.), Nucl. Phys. B (Proc. Suppl.) 83 (2000) 185 and Phys. Lett. B491 (2000) 123.
- [23] M. Bochicchio, G.C. Rossi, M. Testa and K. Yoshida, Phys. Lett. B149 (1984) 487.
- [24] J. Smit and J.C. Vink, Nucl. Phys. B284 (1987) 234; *ibidem* B286 (1987) 485 and B298 (1988) 557.

- [25] M. Bochicchio, L. Maiani, G. Martinelli, G.C. Rossi and M. Testa, Nucl. Phys. B262 (1985) 331.
- [26] R.J. Crewther, Phys. Lett. 70B (1977) 349;
H. Leutwyler and A. Smilga, Phys. Rev. D46 (1992) 5607.
- [27] L. Giusti, M. Lüscher, P. Weisz and H. Wittig, JHEP 0311 (2003) 023.
- [28] L. Del Debbio and C. Pica, hep-lat/0309145.