From classical to quantum quadratic cost control

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Abstract. Controlling the size of the solution of a (deterministic, stochastic or quantum stochastic) differential equation, by minimizing an appropriate cost functional, is very important in classical and quantum engineering. Of particular importance is the case of linear differential equations and quadratic cost functionals, since in that case the control processes can be explicitly calculated. In this paper we review some basic aspects of the classical theory and we present our results in the quantum case, obtained over the past few years.

1. Classical Linear Control

1.1. **Deterministic Control.** In the classical deterministic case we consider a system whose evolution over a finite time interval is modelled by the solution $x = \{x_t : t \in [0,T]\} \in C([0,T], \mathbb{R}^n)\}$ of an ordinary differential equation of the form [38]

$$dx_t = (A\, x_t + u_t)\, dt \tag{1.1}$$

$$x_0 = x,\ t \in [0,T] \tag{1.2}$$

where $A \in \mathcal{B}(\mathbb{R}^n)$, the space of bounded linear operators on $\mathbb{R}^n$, and $u \in L_\infty([0,T], \mathbb{R}^n)$ or $L_2([0,T], \mathbb{R}^n)$. Although we consider here the finite-dimensional case, the concepts and the results can be extended from $\mathbb{R}^n$ to any Hilbert space $\mathcal{H}$.

We assume that we can interfere with the performance of the model by choosing the "control process" $u = \{u_t : t \in [0,T]\}$ so as to minimize a certain "performance (or cost) functional" $J(u)$. There is a wide variety of such functionals designed for specific models. However, the most computationally accessible one is the "quadratic" performance functional of the form

$$J(u) = \int_0^T (< x_t, Q\, x_t > + < u_t, u_t >)\, dt + < x_T, \Pi\, x_T > \tag{1.3}$$

where $< \cdot, \cdot >$ denotes the usual inner product in the Euclidean space $\mathbb{R}^n$, $\Pi \in \mathcal{B}(\mathbb{R}^n)$, $Q \in \mathcal{B}(\mathbb{R}^n)$, $\Pi \geq 0$, $Q \geq 0$. If the size of $x = \{x_t : t \in [0,T]\}$ is small, the performance functional (1.3) can serve as an approximation to many other functionals which are more adapted to the specific problem considered but also more computationaly complex. Before one looks for the optimal control process, the system to be controlled must be "observable", "controllable", and "stabilizable". The definition of these concepts is as follows [35, 36, 38]:

Observability: Since the state x_t of the system may only be accessible through an observation process $y_t = P\, x_t$, where $P \in \mathcal{B}(\mathbb{R}^n)$, we must be able to recreate x_t (or equivalently

x_0) from y_t. If $P = 1$ we speak of a "completely observable" system. Otherwise the system is only "partially observable".

Controllability: Given an initial state x_0, we should be able to choose the control process $u = \{u_t : t \in [0,T]\}$ so that the system will be steered in a finite time $t_1 \in [0,T]$ to a desired state x_1.

Stabilizability: In order to consider large terminal times T, we need the system to exhibit good long-run behavior i.e to eventually settle down to some steady-state behavior. From the mathematical point of view, this amounts to the asymptotic stability of the initial state of $x = \{x_t : t \in [0,T]\}$ or, equivalently, to the existence of a "feedback" control $u_t = K\, x_t$, where $K \in \mathcal{B}(\mathbb{R}^n)$, for which the system (1.1)-(1.2) is asymptotically stable.

The performance functional (1.3) is particularly useful in the case when a system must operate at or near a particular state, chosen here to be the origin. We can think of the first term of (1.3) as a penalty for being too far away from the origin on $(0,T)$, the second as a penalty for using too much control and the third as a penalty for being too far away from the target at the final time T. The main result in the completely observable, classical case is the following:

Theorem 1. *The performance functional (**??**) associated with the system (**??**)-(**??**) is minimized by the feedback control process*

$$u_t = -\Pi_t\, x_t \tag{1.4}$$

where $\{\Pi_t : t \in [0,T]\}$ *is the solution of the Riccati differential equation*

$$\frac{d}{dt}\Pi_t + A^*\,\Pi_t + \Pi_t\, A + Q - \Pi_t^2 = 0 \tag{1.5}$$

$$\Pi_T = \Pi \tag{1.6}$$

If we restrict to $u_t = -K\, x_t$, i.e to feedback controls with a time-independent coefficient, then equations (**??**)-(**??**) are replaced by the "algebraic" Riccati equation

$$A^*\,\Pi + \Pi\, A + Q - \Pi^2 = 0 \tag{1.7}$$

1.2. **Stochastic Control.** In this case we consider systems whose time evolution is affected by noise. We assume that the noise can be accurately described by Brownian motion. Specifically, we consider systems whose time-evolution is described by the solution $x = \{x_t : t \in [0,T]\}$ of a stochastic differential equation of the form

$$dx_t = (A\, x_t + u_t)\, dt + C\, dB_t \tag{1.8}$$

$$x_0 = x,\ \ t \in [0,T] \tag{1.9}$$

where A and u are as in (**??**)-(**??**) with the added assumtion that u is a stochastic process, $C \in \mathcal{B}(\mathbb{R}^n)$, and $B = \{B_t : t \geq 0\}$ is a vector (in this case n-dimensional) Brownian motion. The performance functional (**??**) takes the form

$$J(u) = E(\int_0^T (< x_t, Q\, x_t > + < u_t, u_t >)\, dt + < x_T, \Pi\, x_T >) \tag{1.10}$$

where E denotes mathematical expectation.

For completely observable systems, Theorem 1 remains true in the stochastic case. For partially observable systems, i.e when $x = \{x_t : t \in [0, T]\}$ is available only through an observation process $y = \{y_t : t \in [0, T]\}$ satisfying

$$dy_t = H\, x_t\, dt + dW_t \tag{1.11}$$

where $H \in \mathcal{B}(\mathbb{R}^n)$, and $W = \{W_t : t \in [0, T]\}$ is a vector (in this case n-dimensional) Brownian motion independent of $B = \{B_t : t \in [0, T]\}$, the main result is as follows:

Theorem 2. *The performance functional (??) associated with the system (??)-(??) and (??) is minimized by the feedback control process*

$$u_t = -\Pi_t\, \hat{x}_t \tag{1.12}$$

where $\{\Pi_t : t \in [0, T]\}$ is the solution of the Riccati equation

$$\frac{d}{dt}\Pi_t + A^*\, \Pi_t + \Pi_t\, A + Q - \Pi_t^2 = 0 \tag{1.13}$$

$$\Pi_T = \Pi \tag{1.14}$$

and $\hat{x} = \{\hat{x}_t : t \in [0, T]\}$ is the minimum mean-square estimate of $x = \{x_t : t \in [0, T]\}$ given $\{y_s : s \leq t\}$, obtained through the Bucy-Kalman filter [36, 38, 39].

The field of classical deterministic and stochastic control is very well developed. For proofs, details and more information we refer to [34, 35, 36, 38, 39].

2. Quantum Evolutions and Langevin Equations

In the Schrödinger picture of quantum mechanics, the state of a quantum system is described (in Dirac's notation) by a time dependent ket vector $|\psi_t\rangle$ evolving in accordance to Schrödinger's equation

$$i\, \hbar\, \frac{d}{dt}\, |\psi_t\rangle = H_t\, |\psi_t\rangle \tag{2.1}$$

where H_t is the Hamiltonian of the system. Quantum mechanical observables are represented by time independent self adjoint operators X_0 acting on the state space. In the Heisenberg picture, the state of the system is represented by a time independent state vector $|\psi_0\rangle$ and it is the observables X_t that vary with time. The connection between the two pictures is provided by the unitary "time evolution operators" U_t that satisfy

$$X_t = j_t(X_0) = U_t^*\, X_0\, U_t \tag{2.2}$$

and

$$|\psi_t\rangle = U_t\,|\psi_0\rangle \tag{2.3}$$

In the conservative (i.e time independent Hamiltonian) case

$$U_t = e^{-\frac{i}{\hbar}\,t\,H} \tag{2.4}$$

The differential equation satisfied by the U_t's is called an "evolution" equation, while the one satisfied by the X_t's is called a Langevin equation. In general, the Hamiltonian operator is a sum of "creation", "annihilation" and "number" operators and the above mentioned equations contain noise terms defined in terms of such operators. In what follows we will consider systems and noises represented as operators acting on a Hilbert space $\mathcal{H}$ and Boson Fock space $\Gamma(L^2(\mathbb{R}_+,\mathcal{K}))$ respectively. The Boson Fock space $\Gamma = \Gamma(L^2(\mathbb{R}_+,\mathcal{K}))$ can be defined as the Hilbert space completion of the linear span of the "exponential vectors" $\psi(f)$ under the inner product

$$<\psi(f),\psi(g)>= e^{<f,g>} \tag{2.5}$$

where $f,g \in L^2(\mathbb{R}_+,\mathcal{K})$. The set $\mathcal{K}$ will be $\mathbb{C}$ or $l_2(\mathbb{N})$ depending on what kind of noise we consider. Noise will be defined as time dependent operators on the Fock space $\Gamma(L^2(\mathbb{R}_+,\mathcal{K}))$ with differentials defined in terms of the Hida white noise functionals $a_t^\dagger$ and a_t.

For $f \in L^2(\mathbb{R}_+,\mathcal{K})$ and an adjointable linear operator F on $L^2(\mathbb{R}_+,\mathcal{K})$, the "annihilation", "creation" and "conservation" operators $A(f)$, $A^\dagger(f)$ and $\Lambda(F)$ respectively, are defined on the exponential vectors of Γ by

$$A(f)\psi(g) =< f,g > \psi(g) \tag{2.6}$$

$$A^\dagger(f)\psi(g) = \frac{\partial}{\partial\epsilon}|_{\epsilon=0}\psi(g+\epsilon f) \tag{2.7}$$

$$\Lambda(F)\psi(g) = \frac{\partial}{\partial\epsilon}|_{\epsilon=0}\psi(e^{\epsilon F}g) \tag{2.8}$$

where F must be such that the exponential $e^{\epsilon F}$ is defined.

The Itô multiplication table associated with $A(\cdot)$, $A^\dagger(\cdot)$ and $\Lambda(\cdot)$ is

$\cdot$	$dA_t^\dagger(f_1)$	$d\Lambda_t(F_1)$	$dA_t(f_1)$	dt
$dA_t^\dagger(f_2)$	0	0	0	0
$d\Lambda_t(F_2)$	$dA_t^\dagger(F_2f_1)$	$d\Lambda_t(F_2F_1)$	0	0
$dA_t(f_2)$	$< f_2,f_1 > dt$	$dA_t(F_1^*f_2)$	0	0
dt	0	0	0	$0.$

2.1. **First Order White Noise.** The first order (Hudson-Parthasarathy) quantum stochastic differentials dA_t, $dA_t^\dagger$, and $d\Lambda_t$ are defined by

$$dA_t = A(\chi_{[t,t+dt]}) \tag{2.9}$$

$$dA_t^\dagger = A^\dagger(\chi_{[t,t+dt]}) \tag{2.10}$$

$$d\Lambda_t = \Lambda(\chi_{[t,t+dt]}) \tag{2.11}$$

In terms of white noise, the basic noise differentials are

$$dA_t = a_t\, dt,\; dA_t^\dagger = a_t^\dagger\, dt,\; d\Lambda_t = a_t^\dagger\, a_t\, dt. \tag{2.12}$$

Quantum evolutions in the tensor product $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{K}))$ have the form

$$dU_t = -((iH + \tfrac{1}{2}\, L^* L)\, dt + L^*\, W\, dA_t - L\, dA_t^\dagger + (1 - W)\, d\Lambda_t)\, U_t \tag{2.13}$$

$$t \in [0, T],\; U_0 = 1 \tag{2.14}$$

where H, L, W are in $\mathcal{B}(\mathcal{H})$, with U unitary and H self-adjoint. The corresponding Langevin equation is

$$\begin{aligned} dj_t(X) &= j_t(i[H, X] - \tfrac{1}{2}(L^* L X + X L^* L - 2L^* X L))\, dt \\ &+ j_t([L^*, X]\, W)\, dA_t + j_t(W^*\, [X, L])\, dA_t^\dagger + j_t(W^*\, X\, W - X)\, d\Lambda_t \end{aligned} \tag{2.15}$$

$$j_0(X) = X,\; t \in [0, T] \tag{2.16}$$

2.2. **Square of White Noise.** The square of white noise (SWN) commutation relations are a functional extension of the $sl(2; \mathbb{R})$ commutation relations

$$[B^-, B^+] = M,\; [M, B^+] = 2B^+,\; [M, B^-] = -2B^- \tag{2.17}$$

where

$$(B^-)^* = B^+,\; M^* = M \tag{2.18}$$

Following "renormalization", the SWN noise differentials are initially defined by

$$dB_t^- = a_t^2\, dt,\; dB_t^+ = {a_t^\dagger}^2\, dt,\; dM_t = a_t^\dagger\, a_t\, dt \tag{2.19}$$

A representation of the $sl(2; \mathbb{R})$ Lie algebra on $l_2(\mathbb{N})$ is defined by

$$\rho^+({B^+}^n M^k {B^-}^l)\, e_m = \theta_{n,k,l,m}\, e_{n+m-l}. \tag{2.20}$$

where e_m, $m = 0, 1, 2, \cdots$ is any orthonormal basis of $l_2(\mathbb{N})$,

$$\theta_{n,k,l,m} := H(n + m - l)\, \sqrt{\frac{m - l + n + 1}{m + 1}}\, 2^k (m - l + 1)_n (m + 1)^{(l)} (m - l + 1)^k, \tag{2.21}$$

$H(x)$ is the Heaviside function ($H(x) = 0$ for $x < 0$; $H(x) = 1$ for $x \geq 0$),

$$0^0 = 1, \;\; (B^+)^n = (B^-)^n = N^n = 0, \quad \text{for } n < 0,$$

and "factorial powers" are defined by

$$\begin{aligned} x^{(n)} &= x(x-1)\cdots(x-n+1) \\ (x)_n &= x(x+1)\cdots(x+n-1) \\ (x)_0 &= x^{(0)} = 1 \end{aligned}$$

Using this representation we obtain

$$dM_t = d\Lambda_t(\rho^+(M)) + dt \tag{2.22}$$

$$dB_t^+ = d\Lambda_t(\rho^+(B^+)) + dA_t^\dagger(e_0) \tag{2.23}$$

$$dB_t^- = d\Lambda_t(\rho^+(B^-)) + dA_t(e_0) \tag{2.24}$$

To obtain a closed Itô multiplication table we use as basic SWN differentials

$$d\Lambda_{n,k,l}(t) = d\Lambda_t(\rho^+({B^+}^n M^k B^{-l})) \tag{2.25}$$

$$dA_m(t) = dA_t(e_m) \tag{2.26}$$

$$dA_m^\dagger(t) = dA_t^\dagger(e_m) \tag{2.27}$$

where $n, k, l, m \in \{0, 1, ...\}$, with Itô multiplication table

$$d\Lambda_{\alpha,\beta,\gamma}(t)\, d\Lambda_{a,b,c}(t) = \sum c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon}\, d\Lambda_{a+\alpha-\gamma+\lambda,\omega+\sigma+\epsilon,\lambda+c}(t) \tag{2.28}$$

$$d\Lambda_{\alpha,\beta,\gamma}(t)\, dA_n^\dagger(t) = \theta_{\alpha,\beta,\gamma,n}\, dA_{\alpha+n-\gamma}^\dagger(t) \tag{2.29}$$

$$dA_m(t)\, d\Lambda_{a,b,c}(t) = \theta_{c,b,a,m}\, dA_{c+m-a}(t) \tag{2.30}$$

$$dA_m(t)\, dA_n^\dagger(t) = \delta_{m,n}\, dt \tag{2.31}$$

where

$$c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon} = \tag{2.32}$$
$$\binom{\gamma}{\lambda}\binom{\gamma-\lambda}{\rho}\binom{\beta}{\omega}\binom{b}{\epsilon} 2^{\beta+b-\omega-\epsilon} S_{\gamma-\lambda-\rho,\sigma}\, a^{(\gamma-\lambda)} (a+\lambda-1)^{(\rho)} (a-\gamma+\lambda)^{\beta-\omega} \lambda^{b-\epsilon},$$

$S_{\gamma-\lambda-\rho,\sigma}$ are the Stirling numbers of the first kind and $\sum$ in (2.28) denotes the finite sum

$$\sum_{\lambda=0}^{\gamma} \sum_{\rho=0}^{\gamma-\lambda} \sum_{\sigma=0}^{\gamma-\lambda-\rho} \sum_{\omega=0}^{\beta} \sum_{\epsilon=0}^{b}$$

All other products of differentials are equal to zero.

Quantum evolutions are of the form

$$dU_t = ((-\tfrac{1}{2}\,(D_-^*|D_-^*) + iH)\,dt + d\mathcal{A}_t(D_-) \qquad (2.33)$$
$$+d\mathcal{A}_t^\dagger(-r(W)D_-^*) + d\mathcal{L}_t(W-I))\,U_t$$
$$U_0 = 1 \qquad (2.34)$$

while Langevin equations are of the form

$$dj_t(X) = \qquad (2.35)$$
$$j_t(i\,[X,H] - \tfrac{1}{2}\{(D_-^*|D_-^*)\,X\} + (r(W)D_-^*|X\,r(W)D_-^*))\,dt$$
$$+j_t(d\mathcal{A}_t^\dagger(D_-^*\,X - r(W^*\,X\circ W)D_-^*))$$
$$+j_t(d\mathcal{A}_t(X\,D_-l(W^*\circ X\,W)D_-))$$
$$+j_t(d\mathcal{L}_t(W^*\,X\circ W - X))$$
$$j_0(X) = X,\ t\in[0,T]. \qquad (2.36)$$

where H is a bounded self-adjoint system operator, W is a $\circ$-product (see (2.44) for the definition of the $\circ$-product) unitary operator and $D_- = \sum_m D_{-,m}\otimes e_m$, where the D_m's are bounded system operators.

In equations (2.33)-(2.34) and (2.35)-(2.36) we have used

(i) evolution coefficients:

$$D_+ = \textstyle\sum_n D_{+,n}\otimes e_n \qquad (2.37)$$
$$D_- = \textstyle\sum_m D_{-,m}\otimes e_m \qquad (2.38)$$
$$D_1 = \textstyle\sum_{\alpha,\beta,\gamma} D_{1,\alpha,\beta,\gamma}\otimes\rho^+(B^{+^\alpha}M^\beta B^{-^\gamma}) \qquad (2.39)$$
$$E_1 = \textstyle\sum_{a,b,c} E_{1,a,b,c}\otimes\rho^+(B^{+^a}M^b B^{-^c}) \qquad (2.40)$$

where the left hand sides of the tensor products corespond to bounded system operators

(ii) module operators $\mathcal{A}$, $\mathcal{A}^\dagger$ and $\mathcal{L}$ genericaly defined by:

$$\mathcal{A}(a\otimes\xi) = a\otimes A(\xi) \qquad (2.41)$$
$$\mathcal{A}^\dagger(a\otimes\xi) = a\otimes A^\dagger(\xi) \qquad (2.42)$$
$$\mathcal{L}(a\otimes T) = a\otimes\Lambda(T) \qquad (2.43)$$

and

(iii) basic operations:

$$(2.44)\quad D_1 \circ E_1 = \sum_{\alpha,\beta,\gamma,a,b,c} \sum c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,a,b} D_{1,\alpha,\beta,\gamma} E_{1,a,b,c} \otimes \rho^+(B^{+a+\alpha-\gamma+\lambda} M^{\omega+\sigma+\epsilon} B^{-\lambda+c})$$

$$(D_-^*|D_+) = \sum_n D_{-,n}\, D_{+,n} \otimes 1 \tag{2.45}$$

$$r(D_1)D_+ = \sum_{n,\alpha,\beta,\gamma} D_{1,\alpha,\beta,\gamma} \theta_{\alpha,\beta,\gamma,n-\alpha+\gamma} D_{+,n-\alpha+\gamma} \otimes e_n \tag{2.46}$$

$$l(E_1)D_- = \sum_{n,\alpha,\beta,\gamma} D_{,n+\alpha-\gamma} \theta_{\gamma,\beta,\alpha,n+\alpha-\gamma} E_{1,\alpha,\beta,\gamma} \otimes e_n \tag{2.47}$$

where $\sum$ is as in (**??**). The SWN Ito table can be concisely written as

$$d\mathcal{A}_t(D_-)\, d\mathcal{A}_t^\dagger(D_+) = (D_-^*|D_+)\, dt \tag{2.48}$$

$$d\mathcal{L}_t(D_1)\, d\mathcal{L}_t(E_1) = d\mathcal{L}_t(D_1 \circ E_1) \tag{2.49}$$

$$d\mathcal{L}_t(D_1)\, d\mathcal{A}_t^\dagger(D_+) = d\mathcal{A}_t^\dagger(r(D_1)D_+) \tag{2.50}$$

$$d\mathcal{A}_t(D_-)\, d\mathcal{L}_t(E_1) = d\mathcal{A}_t(l(E_1)D_-) \tag{2.51}$$

For details we refer to [2]-[10], [12], [15]-[24],[26]-[28], [33], [37], [39].

3. Quantum Control

3.1. **First order white noise.** We consider Langevin equations of the form (**??**)-(**??**) associated with evolution equations of the form (**??**)-(**??**). Generalizing from the clasical case, we consider quadratic cost functionals of the form

$$J_{\xi,T}(L,W) = \int_0^T [\, \|j_t(X)\,\xi\|^2 + \tfrac{1}{4}\|j_t(L^*L)\,\xi\|^2\,]\, dt + \tfrac{1}{2}\|j_T(L)\,\xi\|^2 \tag{3.1}$$

where T is an arbitrary terminal time, $\xi = u \otimes \psi(f) \in \mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+,\mathcal{K}))$ is arbitrary, and the system operators L, W are viewed as controls, chosen to minimize $J_{\xi,T}(L,W)$. This cost functional is obtained from the cost functional

$$Q_{\xi,T}(u) = \int_0^T [< U_t\,\xi, X^2\, U_t\,\xi > + < u_t\,\xi, u_t\,\xi >]\, dt - < u_T\,\xi, U_T\,\xi > \tag{3.2}$$

associated with (**??**)-(**??**) as in the classical case, by restricting to feedback controls $u_t = -\frac{1}{2}\, L^* L\, U_t$. The main result is as follows:

Theorem 3. *Let X be a system space observable such that the pair $(i\,H,\, X)$ is stabilizable (i.e $\exists\ K \in \mathcal{B}(\mathcal{H})$ such that $i\,H + KX$ is the generator of an asymptotically stable semigroup $\mathcal{F}_t$ i.e $\exists\ M > 0$ and $\omega < 0$ such that $||\mathcal{F}_t|| \le M\, e^{\omega\, t}$). The quadratic performance functional $J_{\xi,T}(L,W)$ is minimized by*

$$L = \sqrt{2}\, \Pi^{1/2}\, W_1 \ \ (\textit{polar decomposition of } L) \tag{3.3}$$

and

$$W = W_2 \tag{3.4}$$

where Π *is the positive self-adjoint solution of the algebraic Riccati equation*

$$i\,[H,\Pi]+\Pi^2+X^2=0 \tag{3.5}$$

and W_1, W_2 *are bounded unitary system operators commuting with* Π. *Moreover*

$$\min_{L,W}\ J_{\xi,T}(L,W)=<\xi,\Pi\,\xi> \tag{3.6}$$

independent of T.

Some early work on the quadratic cost control of quantum processes was done by V.P. Belavkin [24] who considered evolutions with scalar coefficients, and classical noise without jumps i.e W was 1, and the coefficents of dA_t and $dA_t^\dagger$ were equal, thus amounting to stochastic differential equations driven by classical Brownian motion $A_t+A_t^\dagger$.

3.2. **Square of white noise.** In this case, corresponding to (**??**)-(**??**) and (**??**)-(**??**), we consider the cost functional

$$J_{\xi,T}(D_-,W)=\int_0^T [\,\|j_t(X)\,\xi\|^2+\tfrac{1}{4}\|j_t((D_-^*|D_-^*))\,\xi\|^2\,]\,dt+\tfrac{1}{2}<\xi,j_T((D_-^*|D_-^*))\,\xi> \tag{3.7}$$

where D_-, W are the controls, to be chosen so as to minimize $J_{\xi,T}(D_-,W)$. The main result has as follows:

Theorem 4. *Let* X *be a bounded self-adjoint system operator such that the pair* $(i\,H,\ X)$ *is stabilizable. The performance functional* $J_{\xi,T}(D_-,W)$ *is minimized by choosing*

$$D_-=\sum_n D_{-,n}\otimes e_n \tag{3.8}$$

and

$$W=\sum_{\alpha,\beta,\gamma} W_{\alpha,\beta,\gamma}\otimes\rho^+(B^{+^\alpha}M^\beta B^{-\gamma}) \tag{3.9}$$

such that

$$\tfrac{1}{2}\,(D_-^*|D_-^*)=(\tfrac{1}{2}\sum_n D_{-,n}\,D_{-,n}^*)\otimes 1=\Pi, \tag{3.10}$$

and

$$[D_{-,n},D_{-,m}]=[D_{-,n},D_{-,m}^*]=0 \tag{3.11}$$

$$[D_{-,n},W_{\alpha,\beta,\gamma}]=[D_{-,n},W_{\alpha,\beta,\gamma}^*]=0 \tag{3.12}$$

for all n,m,α,β,γ, *where* Π *is the positive self-adjoint solution of the algebraic Riccati equation*

(3.13) $$i\,[H,\Pi]+\Pi^2+X^2=0$$

Moreover

(3.14) $$\min_{D_-,W}\ J_{\xi,T}(D_-,W)=<\xi,\Pi\,\xi>$$

independent of T.

For proofs and details on the material presented in this section we refer to [1, 3, 11, 13, 14, 25, 29, 30, 31, 32].

4. Appendix: *Mathematica* algorithms

The calculation of the product of two stochastic differentials, in the case of the square of white noise, can be made easy with the use of symbolic programming. Following is a *Mathematica* algorithm for computing (**??**)-(**??**). The algorithm also computes the product of the Hudson-Parthasarathy differentials (**??**)-(**??**) as a special case (rf. [7]).

Algorithm 1. *This algorithm computes the, in general, noncommutative products of the generalized SWN stochastic differentials* $d\Lambda_{n,k,l}(t)$, $dA_m(t)$ *and* $dA^\dagger_m(t)$, *where* $n,k,l,m=0,1,...,$ *and "time"* dt. *Each sentence corresponds to a new input. Inputs are separated by space.*

$p[x_-,y_-]=\textbf{If}\ [x==y==0,1,x\hat{\ }y]$

$u[x_-,n_-]=\textbf{Product}[x-i+1,\{i,1,n\}]$

$v[x_-,n_-]=\textbf{Product}[x+i-1,\{i,1,n\}]$

$\theta[n_-,k_-,l_-,m_-]=\textbf{If}\ [n+m-1<0,0,Sqrt[(m-l+n+1)/(m+1)]2\hat{\ }k\,v[m-l+1,n]\,u[m+1,l]\,p[m-l+1,k]]$

$c[\beta_-,\gamma_-,a_-,b_-,\lambda_-,\rho_-,\sigma_-,\omega_-,\epsilon_-]=\textbf{Binomial}[\gamma,\lambda]\,\textbf{Binomial}[\gamma-\lambda,\rho]\,\textbf{Binomial}[\beta,\omega]\,\textbf{Binomial}[b,\epsilon]\,2\hat{\ }(\beta+b-\omega-\epsilon)\,\textbf{StirlingS1}[\gamma-\lambda-\rho,\sigma]\,u[a,\gamma-\lambda]\,u[a+\lambda-1,\rho]\,p[a-\gamma+\lambda,\beta-\omega]\,p[\lambda,b-\epsilon]$

$\textbf{NCM}[d\Lambda[\alpha_-,\beta_-,\gamma_-],d\Lambda[a_-,b_-,s_-]]=\textbf{Sum}[c[\beta,\gamma,a,b,\lambda,\rho,\sigma,\omega,\epsilon]\,d\Lambda[a+\alpha-\gamma+\lambda,\omega+\sigma+\epsilon,\lambda+s],\{\lambda,0,\gamma\},\{\rho,0,\gamma-\lambda\},\{\sigma,0,\gamma-\lambda-\rho\},\{\omega,0,\beta\},\{\epsilon,0,b\}]$

$\textbf{NCM}[d\Lambda[a_-,b_-,c_-],dA^\dagger[m_-]]=\theta[a,b,c,m]dA^\dagger[a+m-c]$

$\textbf{NCM}[dA[m_-],d\Lambda[a_-,b_-,c_-]]=\theta[c,b,a,m]dA[c+m-a]$

$\textbf{NCM}[dA[m_-],dA^\dagger[n_-]]=\textbf{KroneckerDelta}[m,n]dt$

$\textbf{NCM}[dA[m_-],dA[n_-]]=0$

$\textbf{NCM}[dA^\dagger[m_-],dA^\dagger[n_-]]=0$

$\textbf{NCM}[dA^\dagger[m_-],dA[n_-]]=0$

$\mathbf{NCM}[dA^\dagger[m_-], d\Lambda[\alpha_-, \beta_-, \gamma_-]] = 0$

$\mathbf{NCM}[d\Lambda[\alpha_-, \beta_-, \gamma_-], dA[m_-]] = 0$

$\mathbf{NCM}[d\Lambda[\alpha_-, \beta_-, \gamma_-], dt] = 0$

$\mathbf{NCM}[dt, d\Lambda[\alpha_-, \beta_-, \gamma_-]] = 0$

$\mathbf{NCM}[dA[m_-], dt] = 0$

$\mathbf{NCM}[dt, dA[m_-]] = 0$

$\mathbf{NCM}[dA^\dagger[m_-], dt] = 0$

$\mathbf{NCM}[dt, dA^\dagger[m_-]] = 0$

$\mathbf{NCM}[dt, dt] = 0$

For example, using the above algorithm to compute $d\Lambda_{4,1,2}(t)\, d\Lambda_{1,2,1}(t)$ we obtain

$\mathrm{NCM}[d\Lambda[4,1,2], d\Lambda[1,2,1]] = 8d\Lambda[4,1,2] + 16d\Lambda[4,2,2] + 10d\Lambda[4,3,2] + 2d\Lambda[4,4,2] + 32d\Lambda[5,0,3] + 32d\Lambda[5,1,3] + 10d\Lambda[5,2,3] + d\Lambda[5,3,3]$

while for $d\Lambda_{4,2,1}(t)\, dA^\dagger_2(t)$ we obtain

$\mathrm{NCM}[d\Lambda[4,2,1], dA^\dagger[2]] = 48\sqrt{5}\, dA^\dagger[5]$

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