On the Fock Representation of the Renormalized Powers of Quantum White Noise

Luigi Accardi
Centro Vito Volterra, Università di Roma Tor Vergata
via Columbia, 2– 00133 Roma, Italy
E-mail: accardi@Volterra.mat.uniroma2.it

Andreas Boukas
Department of Mathematics and Natural Sciences, American College of Greece
Aghia Paraskevi, Athens 15342, Greece
E-mail: andreasboukas@acgmail.gr

Contents

1 Introduction **3**

2 The Boson-Fock Case **4**

3 The q-Deformed Fock Case **7**

Abstract We describe the "no-go" theorems recently obtained by Accardi-Boukas-Franz in [[1]] for the Boson case, and by Accardi-Boukas in [[2]] for the q-deformed case, on the issue of the existence of a common Fock space representation of the renormalized powers of quantum white noise (RPWN).

1 Introduction

Classical (i.e Itô [[5]]) and quantum (i.e Hudson-Parthasarathy [[6, 3]]) stochastic calculi were unified by Accardi, Lu, and Volovich in [[4]] in the framework of Hida's white noise theory by expressing the fundamental noise processes in terms of the Hida white noise functionals a_t and $a_t^\dagger$ defined as follows. Let $L^2_{sym}(\mathbf{R}^n)$ denote the space of square integrable functions on $\mathbf{R}^n$ symmetric under permutation of their arguments, and let $F := \bigoplus_{n=0}^{\infty} L^2_{sym}(\mathbf{R}^n)$ where if $\psi := \{\psi^{(n)}\}_{n=0}^{\infty} \in F$, then $\psi^{(0)} \in \mathbf{C}$, $\psi^{(n)} \in L^2_{sym}(\mathbf{R}^n)$ and

$$\|\psi\|^2 = \|\psi(0)\|^2 + \sum_{n=1}^{\infty} \int_{\mathbf{R}^n} |\psi^{(n)}(s_1, \ldots, s_n)|^2 ds_1 \ldots ds_n$$

The subspace of vectors $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in F$ with $\psi^{(n)} = 0$ for almost all n will be denoted by D_0. Denote by $S \subset L^2(\mathbf{R}^n)$ the Schwartz space of smooth functions decreasing at infinity faster than any polynomial and let $D := \{\psi \in F | \psi^{(n)} \in S, \sum_{n=1}^{\infty} n|\psi^{(n)}|^2 < \infty\}$. For each $t \in \mathbf{R}$ define the linear operator $a_t : D \to F$ by

$$(a_t\psi)^{(n)}(s_1, \ldots, s_n) := \sqrt{n+1}\psi^{(n+1)}(t, s_1, \ldots, s_n)$$

and the operator valued distribution (cf. [[4]] for details) a_t^+ by

$$(a_t^+\psi)^{(n)}(s_1, \ldots, s_n) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta(t - s_i)\psi^{(n-1)}(s_1, \ldots, \hat{s}_i, \ldots, s_n)$$

where ˆ denotes omission of the corresponding variable. The Hida white noise functionals satisfy the Boson commutation relations

$$\begin{aligned} [a_t, a_s^\dagger] &= \delta(t - s) \\ [a_t^\dagger, a_s^\dagger] &= [a_t, a_s] = 0. \end{aligned}$$

In order to consider higher powers of the Hida white noise functionals we will use the renormalization

$$\delta(t)^l \quad = \quad c^{l-1}\,\delta(t), \quad c > 0,\; l = 2, 3, ...$$

A complete analysis of the choice of such a renormalization, as well as a discussion of other possible renormalizations can be found in [[4]].

2 The Boson-Fock Case

In the Boson case the basic commutation relations, the properties of the Fock vacuum vector Φ, and the duality relations are

$$\begin{aligned}
[a_t, a_s^\dagger] &= \delta(t-s) \\
[a_t^\dagger, a_s^\dagger] &= [a_t, a_s] = 0 \\
a_t\,\Phi &= 0 \\
(a_s)^* &= a_s^\dagger \\
\langle \Phi, \Phi \rangle &= 1.
\end{aligned}$$

Let $\mathcal{H}$ be a test function space and for $f \in \mathcal{H}$ and $n, k \in \{0, 1, 2, ...\}$ define the sesquilinear form on D_0

$$B_k^n(f) := \int_{\mathcal{R}^d} f(t)\, a_t^{\dagger^n}\, a_t^k\, dt$$

with involution

$$(B_k^n(f))^* = B_n^k(\bar{f}).$$

More precisely, for ϕ, ψ in D_0 and $k, m \geq 0$,

$$< \psi, B_k^n(f)\,\phi >= \int_{\mathbb{R}^d} f(t)\; < a_t^n\,\psi, a_t^k\,\phi > \; dt.$$

In particular

$$B_0^0(\bar{g}f) = \int_{\mathcal{R}^d} \bar{g}(t)\, f(t)\, dt =< g, f > .$$

In the following we will use the notation

$$B_k^n := B_k^n(\chi_{[0,t]}).$$

It was proved in [[1]] that for all $t, s \in \mathbb{R}_+$ and $n, k, N, K \geq 0$

$$\begin{aligned} [a_t^{\dagger^n} a_t^k, a_s^{\dagger^N} a_s^K] &= \epsilon_{k,0}\epsilon_{N,0} \sum_{l\geq 1} klN^{(l)} c^{l-1}\, a_t^{\dagger^n}\, a_s^{\dagger^{N-L}}\, a_t^{k-L}\, a_s^k\, \delta(t-s) \\ &- \epsilon_{K,0}\epsilon_{n,0} \sum_{L\geq 1} KLn^{(L)} c^{L-1}\, a_s^{\dagger^N}\, a_t^{\dagger^{n-L}}\, a_s^{K-L}\, a_t^k\, \delta(t-s) \end{aligned} \tag{1}$$

Multiplying both sides of (1) by test functions $f(t)\bar{g}(s)$ and formally integrating the resulting identity (i.e. taking $\int\int\, \ldots\, dsdt$), we obtain the commutation relations for the Renormalized Powers of White Noise (RPWN)

$$\begin{aligned} &[B_K^N(\bar{g}), B_k^n(f)] \\ &= \sum_{L\geq 1} b_L(K,n)\, B_{K+k-L}^{N+n-L}(\bar{g}f) - \sum_{l\geq 1} b_l(k,N)\, B_{K+k-l}^{N+n-l}(\bar{g}f) \end{aligned} \tag{2}$$

where $n, k, N, K \in \{0, 1, 2, ...\}$,

$$\epsilon_{n,k} := 1 - \delta_{n,k}$$

where $\delta_{n,k}$ is Kronecker's delta and

$$b_x(y,z) := \epsilon_{y,0}\, \epsilon_{z,0}\, yx\, z^{(x)}\, c_{x-1}$$

where the factorial powers $x^{(y)}$ are defined by

$$x^{(y)} := x(x-1)\cdots(x-y+1)$$

with $x^{(0)} = 1$. In what follows we will use the notation

$$B_k^n := B_k^n(\chi_I)$$

where $I \subset \mathbb{R}$ with $\mu(I) < +\infty$ is fixed. Moreover, to simplify the notations, we will use the same symbol for the generators of the RPWN Lie algebra and for their images in a given representation.

Theorem 1 *(*No-Go Theorem for Boson RPWN*). Let $\mathcal{L}$ be a Lie $*$–sub–algebra of the RPWN Lie algebra with the following properties:*

(i) $\mathcal{L}$ contains B_0^n, and B_0^{2n} where the noise operators are defined on the same interval I and $B_0^0(\chi_I) = \mu(I)$.

(ii) the B_K^N satisfy the commutation relations (2) .

Then $\mathcal{L}$ does not have a Fock representation if the interval I is such that

$$\mu(I) < \frac{1}{c}$$

Proof 1 *If a common Fock representation of the B_k^n existed, one should be able to define inner products of the form*

$$< (a\, B_0^{2n}(\chi_I) + b\, (B_0^n(\chi_I))^2)\Phi, (a\, B_0^{2n}(\chi_I) + b\, (B_0^n(\chi_I))^2)\Phi >$$

where $a, b \in \mathbb{R}$, the noise operators are defined on the same interval I and $B_0^0(I) = \mu(I)$. Using the notation $< x > = < \Phi, x\,\Phi >$ this amounts to the positive semi-definiteness of the quadratic form

$$\begin{aligned} & a^2 < B_{2n}^0(\chi_I) B_0^{2n}(\chi_I) > \\ + & \; 2ab < B_{2n}^0(\chi_I)(B_0^n(\chi_I))^2 > + a^2 < (B_n^0(\chi_I))^2 (B_0^n(\chi_I))^2 > \end{aligned}$$

or equivalently of the matrix

$$A = \left[\begin{array}{cc} < B_{2n}^0(\chi_I)\, B_0^{2n}(\chi_I) > & < B_{2n}^0(\chi_I)\, (B_0^n(\chi_I))^2 > \\ < B_{2n}^0(\chi_I)\, (B_0^n(\chi_I))^2 > & < (B_n^0(\chi_I))^2\, (B_0^n(\chi_I))^2 > \end{array} \right].$$

Using the commutation relations (2) we find that

$$A = \left[\begin{array}{cc} (2n)! c^{2n-1} \mu(I) & (2n)! c^{2n-2} \mu(I) \\ & \\ (2n)! c^{2n-2} \mu(I) & 2(n!)^2 c^{2n-2} \mu(I)^2 + ((2n)! - 2(n!)^2) c^{2n-3} \mu(I) \end{array} \right].$$

A is a symmetric matrix, so it is positive semi-definite if and only if its minors are non-negative. The minor determinants of A are

$$d_1 = (2n)! c^{2n-1} \mu(I) \geq 0$$

and

$$d_2 = 2 c^{4(n-1)} \mu(I)^2 (n!)^2 (2n)! (c\, \mu(I) - 1) \geq 0 \Leftrightarrow \mu(I) \geq \frac{1}{c}.$$

Thus the interval I cannot be arbitrarily small.

3 The q-Deformed Fock Case

In the q-deformed case, where $q \in (-1,1), q \neq 0$, we start with the q-white noise commutation relations

$$a_t\, a_s^\dagger - q\, a_s^\dagger\, a_t \;=\; \delta(t-s)$$

and letting, as in the Boson case,

$$B_k^n(f) := \int_{\mathcal{R}^d} f(t)\, a_t^{\dagger^n}\, a_t^k\, dt$$

we obtain the q-RPWN commutation relations

$$B_k^n(f)\, B_K^N(g) - q^{kN-nK}\, B_K^N(g)\, B_k^n(f) \tag{3}$$
$$= \sum_{\lambda=1}^{k} c^{\lambda-1}\, \phi_\lambda(k,N;q)\, B_{k+K-\lambda}^{n+N-\lambda}(f\,g) - q^{kN-nK} \sum_{\lambda=1}^{K} c^{\lambda-1}\, \phi_\lambda(K,n;q)\, B_{k+K-\lambda}^{n+N-\lambda}(f\,g)$$

where

$$\phi_\lambda(n,k;q) = \begin{cases} q^{(n-\lambda)(k-\lambda)}\, \frac{[k]_q!}{[k-\lambda]_q!}\, \left(\delta_{n,\lambda} + (1-\delta_{n,\lambda})\, n\lambda_q\right) & \text{if } \lambda \leq n \text{ and } \lambda \leq k \\ 0 & \text{if } \lambda > n \text{ or } \lambda > k \end{cases}$$

$$[n]_q := \frac{q^n - 1}{q-1}, \qquad ([0]_q := 0)$$

$$[n]_q! := \prod_{m=1}^{n} [m]_q, \qquad ([0]_q! := 1)$$

$$nk_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \prod_{i=1}^{n-k} \frac{q^{k+i}-1}{q^i-1},$$

and for $k=0$ and/or $K=0$ the corresponding sums on the right hand side of (3) are interpreted as zero. The following theorem was proved in [[2]].

Theorem 2 *(*No-Go Theorem for q-RPWN*). Let $q \in (-1,1), q \neq 0$ and for a fixed interval $I \subset \mathbb{R}$ and $n, k \geq 0$ let $B^n_k := B^n_k(\chi_I)$ with $B^0_0 = \mu(I) \cdot 1$, the measure of I. Let also the "vacuum vector" Φ be such that $B^n_k \, \Phi = 0$ whenever $k \neq 0$ and let $\langle\, x \,\rangle := \langle \Phi, x \, \Phi \rangle$ denote the "vacuum expectation" of an operator x. We assume that $\langle \Phi, \Phi \rangle = 1$. Define*

$$A(n,q;I) := \begin{bmatrix} < B^0_{2n} \, B^{2n}_0 > & < B^0_{2n} \, (B^n_0)^2 > \\ \\ < B^0_{2n} \, (B^n_0)^2 > & < (B^0_n)^2 \, (B^n_0)^2 > \end{bmatrix}.$$

For any choice of n and q the matrix $A(n,q;I)$ cannot be positive semi-definite for all $I \subset \mathbb{R}$.

Proof 2 *Using commutation relations (3) we find*

$$A(n,q;I) = \begin{bmatrix} \mu(I)\, c^{2n-1} \, [2n]_q! & c^{2n-2} \mu(I) \, [2n]_q! \\ \\ c^{2n-2} \mu(I) \, [2n]_q! & \mu(I)^2 \, c^{2n-2} \, (1 + q^{n^2}) \, ([n]_q!)^2 \\ & + \mu(I) \, c^{2n-3} \, ([n]_q!)^2 \sum_{\lambda=1}^{n-1} q^{(n-\lambda)^2} \, n\lambda_q^2 \end{bmatrix}.$$

$A(n,q;I)$ is a symmetric matrix, so it is positive semi-definite if and only if its minors are non-negative. The minor determinants of $A(n,q;I)$ are

$$d_1 = \mu(I) \, c^{2n-1} \, [2n]_q!$$

which is non-negative for all I and

$$d_2 = \mu(I)^2 \, c^{4n-4} \, [2n]_q! \cdot$$
$$\cdot \left(c\, \mu(I) \, (1 + q^{n^2}) \, ([n]_q!)^2 + ([n]_q!)^2 \sum_{\lambda=1}^{n-1} q^{(n-\lambda)^2} \, n\lambda_q^2 - [2n]_q! \right)$$

which, as in the Boson case, is bigger or equal to zero if and only if

$$\mu(I) \geq \frac{1}{c}$$

which cannot be true for arbitrarily small I.

References

[1] Accardi L., Boukas A., Franz U. *Renormalized powers of quantum white noise*, to appear in Infinite Dimensional Analysis, Quantum Probability, and Related Topics (2005).

[2] Accardi L., Boukas A., Franz U. *Higher Powers of q-deformed White Noise* , to appear in Methods of Functional and Topology (2005).

[3] L. Accardi, *A mathematical theory of quantum noise*, Proceedings of the first world congress of the Bernoulli Society, Ed. Prohorov and Sazonov, vol.1 (1987)

[4] L. Accardi, Y.G. Lu, I.V. Volovich, *White noise approach to classical and quantum stochastic calculi*, Lecture Notes of the Volterra International School of the same title, Trento, Italy, 1999, Volterra Center preprint 375.

[5] Ito K., *On stochastic differential equations*, Memoirs Amer. Math. Soc. 4 (1951).

[6] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhauser Boston Inc., 1992.