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de FINETTI'S THEOREM, SUFFICIENCY, AND DOBRUSHIN'S THEORY

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We show that an extension of Dobrushin's theory provides a natural tool to handle the "inverse problem of sufficiency". The structure of the paper is the following: 0-Introduction. 1-Algebraic characterisation of conditional expectation. 2-de Finetti's theorem. 3-Sufficient statistics. 4-Markovianity and prediction statistics. 5-Dobrushin's theory. 6-Conditional martingales. 7-Quantum de Finetti's theorem.

0. INTRODUCTION

The main mathematical problem arising in the neo-bayesian approach to the foundations of statistical inference, originated by de Finetti's ideas, can be formulated as follows:
describe all the probability measures compatible with some a priori assumptions of statistical type; isolate, among these ones, those with the most elementary structure, find an explicit parametrization for them, and express all the remaining ones as functions (usually convex combinations) of them. Frequently the set of probability measures compatible with the given statistical assumptions is a Choquet simplex and the "elementary measures" are the extremal points of this simplex. The most delicate problem in this connection is to find an explicit parametrization for the extremal measures.

The a priori statistical assumptions which have been considered up to now can be subdivided into two main classes:

(i) assumptions defined by symmetry conditions.

(ii) assumptions defined by the assignment of classes of sufficient statistics.

de Finetti's theorem is the prototype of results concerning the first class, and investigations based on the second class of assumptions have been carried out recently by many authors, among which: S.L.Lauritzen [17], L.Accardi [1], E.B.Dynkin [13], M.Campanino and F. Spizzichino [4], D.Cifarelli and E.Regazzini [5], [6] and several others.

Purpose of the present paper is to show that Dobrushin's theory, in the generalized form proposed in [2], provides a natural tool to handle both classes of problems from an unified point of view. This unification does not rest on a

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formal abstract level, but provides a new insight into the problems considered, as well as a very handy and matematically precise notation. Moreover, since Dobrushin's theory was motivated by the analysis of mathematical models considered in statistical mechanics and quantum field theory, there is the concrete hope that the knowledge of these connections will stimulate a fruitful exchange of ideas and tecniques between these discriplines and the more traditional lines of thought of mathematical statistics.

The starting point of the present paper consists in the remark that both types of a priori statistical assumptions, mentioned above, can be reduced to the a priori assigment of a family of conditional probability measures and to the requirement that the probability measures, admissible for the problem, be such that the family of conditional probabilities, naturally associated with each of them, coincides with the given one. The basic problem is therefore the analysis of the structure of the set of probability measures compatible with a given family of conditional probabilities, and the set of tecniques and results developed to answer this question will be referred to as "Dobrushin's theory" (cf. § (5) for a precise formulation). For technical reasons, as well as for notational convenience it is preferable to formulate Dobrushin's theory in terms of conditonal expectations, rather than probabilities (cfr. [2]). In doing so we take advantage of Doob-Moy's algebraic characterization (cfr. [23]) of the conditional expectations as an operator on function spaces of measurable functions (in the present paper we will limit ourselves to L^∞ - spaces, or occasionally to L^2-spaces).

In §(1) we give a precise formulation of this characterization and introduce some notations which will be used throughout the paper. In §(2) we illustrate in some detail how our ideas apply to the particular case of de Finetti's theorem.

In §(3) we formulate, following [1], the <u>inverse problem of sufficiency</u> and in §(4), after a digression on the algebraic formulations of the Markov property, we show how Lauritzen's "extreme models" can be included in the framework of the inverse problem of sufficiency, and consequently of Dobrushin's theory. The general form of this theory is outlined in §(5) and in § (6) we show that, just as the analysis of projective families of measures leads naturally to the concept of <u>martingale</u>, the analysis of projective systems of conditional expectations -i.e. Dobrushin's theory-leads naturally to the concept of <u>conditional martingale</u>. In particular this concept provides a simple method to construct a new projective family of conditional expectations out of a given one. Finally, in §(7) we show how our approach to de Finetti's theorem extends, just through a change of notations, to the quantum (i.e. non-commutative) case.
In particular, as a Corollary of the general theory formulated in § (5) we obtain a new proof of a result of E.Störmer (Theorem (2.7) in [29]).

We included this last paragraph to show how the problem formulated many years

ago by B. de Finetti continues to be a fruitful and stimulating one not only in the field of classical statistics but also in quantum probability theory. (°)

1. ALGEBRAIC CHARACTERIZATION OF CONDITIONAL EXPECTATIONS

In this § we introduce some notations which will be used throughout the paper. Let (Ω,θ) be a measurable space. We denote $L^\infty(\Omega,\theta)$ the space of bounded, θ-mesurable, complex valued functions on Ω; this is an algebra for pointwise multiplication, and the involution given by complex conjugation will be denoted $f\to f^*$. If μ is a probability measure on (Ω,θ) (by this we mean a countably additive one), the integral associated to μ defines a positive, linear functional on $L^\infty(\Omega,\theta)$. We will always use the notation

$$(1.1)\qquad \mu(f) = \int_\Omega f\, d\mu\ ;\ f\in L^\infty(\Omega,\theta)$$

i.e., we denote with the same symbol a measure and the expectation functional associated to it. The linear functional defined by (1.1) is <u>normalized</u>, in the sense that $\mu(1) = 1$.

Any positive, linear, normalized functional on $L^\infty(\Omega,\theta)$ will be called a <u>state</u>.

States induced by countably additive measures will be called <u>normal</u>; they are characterized by the property:

$$(1.2)\qquad f_n \downarrow 0 \Rightarrow \mu(f_n)\downarrow 0;\quad f_n\in L^\infty(\Omega,\theta)$$

If μ is a measure (countably or finitely additive) on (Ω,θ) we denote $L^\infty(\Omega,\theta,\mu)$ the quotient algebra of $L^\infty(\Omega,\theta)$ by the μ-null functions (those f for which $\mu(|f|^2) = 0$), i.e. the algebra of μ-classes of measurable functions.

<u>Definition (1.1)</u>. Let θ_o be a sub-σ-algebra of θ. A linear map $E^\circ : L^\infty(\Omega,\theta)\to L^\infty(\Omega,\theta_o)$ is called a <u>conditional expectation</u> if

$$(1.3)\qquad f\in L^\infty(\Omega,\theta)\ ;\ f\geq 0 \Rightarrow E^\circ(f)\geq 0$$

$$(1.4)\qquad E^\circ(1) = 1$$

$$(1.5)\qquad E^\circ(f_o f) = f_o E^\circ(f)\ ;\ f_o\in L^\infty(\Omega,\theta_o);\ f\in L^\infty(\Omega,\theta)$$

(°) After the completion of this paper we became acquainted with the preprint [18] of S.L.Lauritzen, in which a program similar to ours is advocated and illustrated with many exemples of concrete statistical interest. Our scheme includes random fields, allows to deal with algebras of "test funtions" different from the continuous ones, and admits a natural extension to the quantum mechanical case.

$E°$ is called normal if

(1.6) $$f_n \downarrow 0 \Rightarrow E°(f_n) \downarrow 0 \ ; \quad f_n \in L^\infty(\Omega,\theta)$$

Remark. If $E° : L^\infty(\Omega,\theta) \to L^\infty(\Omega,\theta_0)$ is a conditional expectation in the above sense then for each θ-measurable function K such that $K \geq 0$ and such that $E°(K) = 1$, the map $E : L^\infty(\Omega,\theta) \to L^\infty(\Omega,\theta_0)$ defined by

(1.7) $$E(f) = E°(K.f); \quad f \in L^\infty(\Omega,\theta)$$

is a conditional expectation; E is normal if $E°$ is.

In particular, (1,5) implies that:

(1.8) $$E°^2(f) = E°(E°(f)) = E°(f)$$

A linear map $E°: L^\infty(\Omega,\theta) \to L^\infty(\Omega,\theta_0)$ satisfying (1.4) and (1.8) is called a norm one projection. A theorem due to J. Tomijama [30] asserts that any norm one projection from $L^\infty(\Omega,\theta)$ to $L^\infty(\Omega,\theta_0)$ automatically satisfies (1.5) and (1.3) - i.e. is a conditional expectation. (Tomijama's theorem holds in the much more general contex context of C^*-algebras).

The theoretical justification of Definition (1.1) is given by the following.

Theorem (1.2) (Moy [23]). Let μ be a probability measure on (Ω,θ) and let $E° : L^\infty(\Omega,\theta) \to L^\infty(\Omega,\theta_0)$ be a normal conditional expectation. Then

(i) $E°$ is a version of the μ-conditional expectation on θ_0 if and only if

(1.9) $$\mu_\circ E° = \mu (\text{i.e. } \mu(f) = \mu(E°(f)); \quad f \in L^\infty(\Omega,\theta))$$

(ii) There exists a $K \in L^1(\Omega,\theta,\mu)$ such that $K \geq 0$, μ-a.e. and

(1.10) $$E°(f) = E^\mu_{\theta_0}(K f); \quad \mu\text{-a.e.} \quad f \in L^\infty(\Omega,\theta)$$

where $E^\mu_{\theta_0}$ denotes the μ-conditional expectation on θ_0.

If $E : L^\infty(\Omega,\theta) \to L^\infty(\Omega,\theta_0)$ is a normal conditional expectation then defining, for each $\omega \in \Omega$

(1.11) $$Q(B,\omega) = E(\chi_B)(\omega); \quad B \in \theta$$

the family $\{Q(., \omega) : \omega \in \Omega\}$ is a family of transition (or markovian) kernels.

Conversely, every such a family defines a normal conditional expectation.

Example (1). Let G be a compact group acting on (Ω,θ) and λ a left invariant measure on G, then the linear map

(1.12) $$E(f)(\omega) = \frac{1}{\lambda(G)} \int_G f(g\omega)\, \lambda(dg) \ ; \quad f \in L^\infty(\Omega,\theta)$$

is a normal conditional expectation whose range is tha algebra of measurable G-invariant functions (i.e. in this case θ_0 is the σ-algebra generated by the "G-symmetric" functions).

Moreover E satisfies:

(1.13) $$g \,.\, E = E \,.\, g = E \qquad \forall g \in G$$

The assignment of a sufficient statistic is equivalent to the assignment of an abstract conditional expectation on some $L^\infty(\Omega,\theta)$ (or on some $L^\infty(\Omega,\theta,\mu)$ - defined by an "universally dominating" measure λ).

In the following we will only consider conditional expectations defined on algebras of the form $L^\infty(\Omega,\theta)$, but in some problems it is convenient to deal with different algebras (say: continuous functions, polinomials in a set of random variables, ...).

Remark. Probably the first to make (implicitely) use of the properties of the abstract (i. e. defined independently on any measure) conditional expectation was D.Hilbert in the proof of his famous Basis Theorem. In the terminology of Example (1.) above, Hilbert considered the case in which $G=SL(m,\mathbb{C})$ acting, through some finite dimensional representation on $\mathbb{C}^n \cong \Omega$. The conditional expectation considered by Hilbert has the form (1.12) and acts on the $\mathbb{C}$-algebra (ring) of polinomials in n variables. The range of this conditional expectation is the ring of invariants (cf. [24]).

2. de FINETTI'S THEOREM

Denote, for $N \in \mathbb{N}$, S_N the permutation group on $\{1,\dots, N\}$. With the usual convention:

(2.1) $$\sigma(K) = K \text{ for } K > N \ ; \quad \sigma \in S_N$$

we consider S_N as acting on $\mathbb{N}$. With this convention

$$S_N \subseteq S_{N+1} \qquad \text{and} \qquad S_\infty = \bigcup_{N \in \mathbb{N}} S_N$$

is a group of transformations of $\mathbb{N}$, called the symmetric group on $\mathbb{N}$.

Let now $(X,\mathcal{B})$ be a measurable space; $(\Omega,\theta) = \Pi_{\mathbb{N}}(X,\mathcal{B})$ the product of countably many copies of $(X,\mathcal{B})$ and $\xi_n : (\Omega,\theta) \to (X,\mathcal{B})$ the canonical projection onto the n-th factor ($n \in \mathbb{N}$). There is a natural action of S_∞ on Ω characterized by the property

$$\xi_n \circ \pi = \xi_{\pi(n)} \; ; \quad \pi \in S_\infty \tag{2.3}$$

and with this action S_∞ is identified to a group of automorphisms (one-to-one bimeasurable transformations) of (Ω,θ).

Definition (2.1). A probability measure μ on (Ω,θ) will be called exchangeable (or symmetric) if:

$$\mu(\pi(A)) = \mu(A) \; ; \; \forall A \in \theta \; ; \qquad \forall \pi \in S_\infty \tag{2.4}$$

Thus the exchangeable probability measures are the fixed points for the natural action of S_∞ on the probability measures on (Ω,θ).

The action of S_∞ on Ω induces an action of S_∞ on $L^\infty(\Omega,\theta)$ - the algebra of bounded, measurable, complex valued functions on Ω-defined by

$$\pi f = f \circ \pi^{-1} \quad ; \quad \pi \in S_\infty \; ; \; f \in L^\infty(\Omega,\theta) \tag{2.5}$$

and with this action S_∞ is identified to a group of $*$-automorphisms of $L^\infty(\Omega,\theta)$, i.e.

$$\pi(f.g) = \pi(f) \, . \, \pi(g) \; ; \quad \pi(f^*) = \pi(f)^*$$

where $*$ denotes complex conjugation, $\pi \in S_\infty$, $f,g \in L^\infty(\Omega,\theta)$. For each $N \in \mathbb{N}$, define

$$E_N(f) = \frac{1}{N!} \sum_{\pi \in S_N} \pi f ; \quad f \in L^\infty(\Omega,\theta) \tag{2.6}$$

Proposition (2.2). The maps E_N have the following properties:

(i) $E_N = E_N \circ \pi = \pi \circ E_N$; $\forall N \in \mathbb{N}$; $\forall \pi \in S_N$ (2.7)

(ii) E_N is a norm- one projection; $\forall N \in \mathbb{N}$

(Iii) For each $N \in \mathbb{N}$ the fixed points of E_N coincide with the fixed points of S_N (i.e. those $f \in L^\infty(\Omega,\theta)$ which are symmetric in the first N variables).

Proof. (i) is a simple computation. (ii) and (iii) follow from (i) and (2.6).

The exchangeable probability measures can be characterized in terms of the E_N's. In fact:

Proposition (2.3). A probability measure μ on (Ω,θ) is exchangeable if and only if

$$\mu \circ E_N = \mu \; ; \quad N \in \mathbb{N} \tag{2.8}$$

Proof. If μ is exchangeable then for each $N \in \mathbb{N}$:

$$\mu \circ E_N = \frac{1}{N!} \sum_{\pi \in S_N} \mu \circ \pi = \mu$$

Conversely, if (2.8) holds, then for each $N \in \mathbb{N}$ and $\pi \in S_N$ one has, using (2.7):

$$\mu \circ \pi = \mu \circ E_N \circ \pi = \mu \circ E_N = \mu$$

hence μ is exchangeable.

By (ii) of Proposition (2.2), each E_N is a conditional expectation on $L^\infty(\Omega,\theta)$ and its range coincides with the algebra $A(S_N)$ of its fixed points (cfr. [23]) which are also the fixed points of S_N. Since $S_N \subseteq S_{N+1}$ one has also

$$A(S_N) \supseteq A(S_{N+1}) \tag{2.9}$$

Lemma (2.4)(*). The family (E_N) enjoys the following properties

$$E_N \circ E_M = E_N \quad \text{if} \quad M \leqslant N \tag{2.10}$$

$$E_N(f(\xi_k)) = \frac{1}{N} \sum_{j=1}^{N} f(\xi_j) \; ; \quad \text{if} \quad K \leqslant N \tag{2.11}$$

$$\lim_{N \to \infty} \{E_N [f(\xi_k) \, . g(\xi_{k+1}, \ldots, \xi_n)] - E_N(f(\xi_k)) \, . \, E_N(g(\xi_{k+1}, \ldots, \xi_n))\} = 0 \tag{2.12}$$

for $K < n$ $f \in L^\infty(\chi,\mathcal{B})$ and $g \in L^\infty(\prod_1^{n-k-1} (X,\mathcal{B}))$. The limit in (2.12) being meant in the sup-norm on $L^\infty(\Omega,\theta)$.

Proof. (2.10) is an immediate consequence of (2.6), (2.7). Moreover

$$E_N(f(\xi_k)) = \frac{1}{N!} \sum_{\pi \in S_N} \pi f(\xi_k) =$$

$$= \frac{1}{N!} \sum_{j=1}^{N} \sum_{\{\pi \in S_N : \, \pi(k) = j\}} f(\xi_{\pi(k)}) = \frac{1}{N} \sum_{j=1}^{N} f(\xi_j)$$

(*) The authors are grateful to M.Cawling for pointing out an inaccuracy in a previous formulation of this lemma.

and this proves (2.11). Finally using (2.11) one finds that:

$$E_N(f(\xi_k)\cdot g(\xi_{k+1},\ldots,\xi_n)) - E_N(f(\xi_k))\cdot E_N(g(\xi_{k+1},\ldots,\xi_n)) =$$
$$= \frac{1}{N}\sum_{j=1}^{N} f(\xi_j)\cdot\frac{1}{N!}\sum_{\ell=1}^{N}\Big[\sum_{\{\pi\in S_N:\ \pi(k)=j\}} g(\xi_{\pi(k+1)},\ldots,\xi_{\pi(n)}) -$$
$$- \sum_{\{\sigma\in S_N:\sigma(k)=\ell\}} g(\xi_{\sigma(k+1)},\ldots,\xi_{\sigma(n)})\Big]$$

and since, for each couple $j\neq\ell$, the non zero terms in the square bracket are less than $\{N(N-1)\cdot\ldots\cdot(N-[n-k-1])\}^2$, the above written difference is majorized in modulus by $2\|f\|_\infty\cdot\|g\|_\infty\cdot\frac{N(N-1)^2\,(N-2)\cdot\ldots\cdot(N-[n-k-1])}{(N-[n-k])!}$, which tends to zero as $N\to\infty$.

An immediate consequence of (2.12) is that, if $f_1,\ldots,f_n\in L^\infty(X,\mathcal{B})$, then

$$\lim_{N\to\infty}\{E_N(f_1(\xi_1)\cdot\ldots\cdot f_n(\xi_n)) - E_N(f_1(\xi_1))\cdot\ldots\cdot E_N(f_n(\xi_n))\} = 0 \text{ (in sup-norm)} \tag{2.13}$$

Let now μ be an exchangeable probability measure on (Ω,θ). From (2.8) and the uniqueness of the conditional expectation it follows that

$$E_N = E^\mu(\,\cdot\,|\,S_N)\quad \mu\text{-a.e.} \tag{2.14}$$

where S_N denotes the sub-σ-algebra of θ generated by the measurable functions, symmetric in the variables $\xi_1,\ldots,\xi_N$ and $E^\mu(\cdot|S_N)$- the μ-conditional expectation on S_N. Moreover, (2.8) implies that E_N maps μ-null functions into μ-null functions hence the action of E_N on $L^\infty(\Omega,\theta)$ can be carried over to $L^\infty(\Omega,\theta,\mu)$ - the quotient algebra of $L^\infty(\Omega,\theta)$ by the ideal of μ-null functions - and this action will be denoted E^μ_N. By Doob's martingale theorem one has:

$$\lim_{N\to\infty} E^\mu_N = E^\mu_S \tag{2.15}$$

where E^μ_S in the μ-conditional expectation on the σ-algebra

$$S = \bigcap_{N\geq 1} S_N \tag{2.16}$$

and the limit takes place pointwise μ-a.e. (as well as in $L^p(\Omega,\theta,\mu)$ for $1\leq p<+\infty$).

From (2.13) we deduce that

$$E^\mu_S(f_1(\xi_1)\cdot\ldots\cdot f_n(\xi_n)) = E_S(f_1(\xi_1))\cdot\ldots\cdot E_S(f_n(\xi_n)) \tag{2.17}$$

for each $n\in\mathbb{N}$ and $f_1,\ldots,f_n\in L^\infty(X,\mathcal{B})$.

Hence the random variables (ξ_k) are conditionally independent over S. The algebra

$$A_S = L^\infty(\Omega,S,\mu) = \bigcap_{N\geq 1} L^\infty(\Omega,S_N,\mu) \tag{2.18}$$

will be called the μ-<u>symmetric algebra at infinity</u>.

There are two more tail σ-algebras naturally associated to the stochastic process (ξ_k) :

one is defined by

$$\theta_\infty = \bigcap_{N\geq 1}\theta_{[N,\infty[} \tag{2.19}$$

where $\theta_{[N,\infty[}$ is the σ-algebra generated by the ξ_k's with $k\geq N$. And the other is the σ-algebra θ_T generated by the shift-invariant functions i.e.

$$L^\infty(\Omega,\theta_T,\mu) = A^\mu_T = \{f\in L^\infty(\Omega,\theta,\mu) : Tf = f\} \tag{2.20}$$

where $T : L^\infty(\Omega,\theta,\mu)\to L^\infty(\Omega,\theta,\mu)$ is the shift-endomorphism, uniquely defined by the property

$$T f(\xi_k) = f(\xi_{k+1}) \tag{2.21}$$

for each $k\in\mathbb{N}$ and each $f\in L^\infty(X,\mathcal{B})$.

<u>Theorem</u> (2.5). (Hewitt-Savage [14]) If μ is an exchangeable probability measure on (Ω,θ) then $S=\theta_T=\theta_\infty\mu$ or, equivalently:

$$A^\mu_S = A^\mu_T = L^\infty(\Omega,\theta_\infty,\mu) \tag{2.22}$$

<u>Proof</u>. Let $n\in\mathbb{N}$ and $f_1,\ldots,f_n\in L^\infty(X,\mathcal{B})$. From (2.11) and (2.13) we deduce that $\forall\, N\in\mathbb{N},\ n\leq N,$

$$\lim_{N\to\infty} E_N(f_1(\xi_1)\cdot\ldots\cdot f_n(\xi_n)) = \tag{2.23}$$
$$= \lim_{N\to\infty}\Big[\frac{1}{N}\sum_{j=1}^{N} f_1(\xi_j)\Big]\cdot\ldots\cdot\Big[\frac{1}{N}\sum_{j=1}^{N} f_n(\xi_j)\Big]$$

Remark that each exchangeable measure is T-invariant (i.e. $\mu\circ T=\mu$) hence we can take the limit of (2.23) for $N\to\infty$ using the martingale theorem for the left hand side and the ergodic theorem for the right hand side. The result is:

(2.24). $$E^\mu_S(f_1(\xi_1)\cdot\ldots\cdot f_n(\xi_n)) = E^\mu_T(f_1(\xi_1)\cdot\ldots\cdot E^\mu_T(f_n(\xi_n))$$

where $E^\mu_T = E^\mu(\cdot\,|\theta_T)$ denotes the conditional expectation on θ_T. But the right hand side of (2.24) is in A^μ_T; hence, by a density argument

(2.25) $$A^\mu_S = E^\mu_S(L^\infty(\Omega,\theta,\mu)) \subseteq A^\mu_T$$

Moreover, for each $N \in \mathbb{N}$

$$A^\mu_T = T^N A^\mu_T \subseteq T^N L^\infty(\Omega,\theta,\mu) = L^\infty(\Omega,\theta_{[N,\infty[},\mu)$$

hence

(2.26) $$A^\mu_T \subseteq \bigcap_N L^\infty(\Omega,\theta_{[N,\infty[},\mu) = L^\infty(\Omega,\theta_\infty,\mu)$$

Finally for any $M \in \mathbb{N}$, $\pi \in S_M$, $f \in L^\infty(\Omega,\theta,\mu)$ and $f_N \in L^\infty(\Omega,\theta_{[N,\infty[},\mu)$ with $N > M$, one has:

$$\mu(f_N\cdot f) = \mu(f_N\cdot E^\mu_{[N,\infty[}(f))$$

where $E^\mu_{[N,\infty[} = E^\mu(\cdot\,|\theta_{[N,\infty[})$ is the conditional expectation on $\theta_{[N,\infty[}$ and also:

$$\mu(f_N\cdot f) = \mu(\pi(f_N)\cdot\pi(f)) = \mu(f_N\cdot\pi(f)) =$$

$$= \mu(f_N E^\mu_{[N,\infty[}(\pi(f)))$$

This means that $E^\mu_{[N,\infty[}\cdot\pi = E^\mu_{[N,\infty[}$ hence by the martingale theorem

(2.27) $$E^\mu_\infty = E^\mu(\cdot\,|\theta_\infty) = E^\mu_\infty\cdot\pi;\quad \forall\,\pi\in S_M$$

From (2.6) and (2.27) we deduce that for each $M \in \mathbb{N}$, $E^\mu_\infty = E^\mu_\infty \circ E^\mu_M$, hence again by the martingale theorem

$$E^\mu_\infty = E^\mu_\infty \circ E^\mu_S$$

But (2.25) and (2.26) also imply

$$E^\mu_\infty \circ E^\mu_S = E^\mu_S$$

Thus

$$A^\mu_S \subseteq A^\mu_T \subseteq L^\infty(\Omega,\theta_\infty,\mu) = A^\mu_S$$

and this ends proof.

Corollary (2.6). For an exchangeable μ on (Ω,θ) the following assertions are equivalent:

(i) there exists a probability measure m on $(X,\mathcal{B})$ such that

$$\mu = \otimes_{\mathbb{N}}\, m$$

(ii) The algebra A^μ_S (or equivalently A^μ_T or $L^\infty(\Omega,\theta_\infty,\mu)$) is trivial.

Proof. (i) $\Longrightarrow$ (ii) A homogeneous product measure is ergodic, hence A^μ_T is trivial. (ii) $\Longrightarrow$ (i) From (2.22) follows that, if A^μ_T is trivial, $E^\mu_T(f(\xi_k))$ is a scalar, hence it must be equal to $\mu(f(\xi_k)) = m(f)$ where m is a probability measure on $(X,\mathcal{B})$, independent on k by stationarity. The thesis then follows from (2.24).

3. SUFFICIENT STATISTICS

Let (Ω,θ) be a measurable space and $\mathcal{P}$-a family of probability measures on (Ω,θ). The triple $(\Omega,\theta,\mathcal{P})$ is called a statistical model and a sub-σ-algebra $\theta_0 \subseteq \theta$ is called sufficient with respect to this statistical structure if for each $P \in \mathcal{P}$ and for each $f \in L^\infty(\Omega,\theta)$ there exists a version of

$E^P_{\theta_0}(f)$ = P-conditional expectation of f on θ_0 which is independent on P. More precisely, we can say that $\theta_0 \subset \theta$ is sufficient with respect to $(\Omega,\theta,\mathcal{P})$ if there is a conditional expectation (in the sense of §(1)).

(3.1) $$E_{\theta_0} : L^\infty(\Omega,\theta) \to L^\infty(\Omega,\theta_0)$$

such that for each $f \in L^\infty(\Omega,\theta)$ and for each $P \in \mathcal{P}$

(3.2) $$E_{\theta_0}(f) = E^P_{\theta_0}(f) \qquad P\text{-a.e.}$$

As usual there is some arbitrariness in the choice of the algebra on which E_{θ_0} is defined but for concreteness we shall only consider the L^∞ - case.

The sufficiency problem in statistics consists in deciding whether a given σ-algebra θ_0, or a family thereof is sufficient for a given family $\mathcal{P}$ of probability measures. The inverse problem of sufficiency is also frequently met in the applications, expecially in a bayesian context. Namely: given a family of σ-algebras $(\theta_{\alpha'})_{\alpha' \in F'}$ describe the set P of probability measures such that each σ-algebra $\theta_{\alpha'}$, is a sufficient σ-algebra for (Ω,θ,P). In this problem one assumes that the projective family $(E_{\alpha'})$ of conditional expectations $E_{\alpha'} : L^\infty(\Omega,\theta)\to L^\infty(\Omega,\theta_{\alpha'})$ is given a priori. Therefore the inverse problem of sufficiency consists in the description of the family P of probability measures on (Ω,θ) - or, more generally, of states on $L^\infty(\Omega,\theta)$ - such that

$$\mu \circ E_{\alpha'} = \mu; \quad \forall\, \mu \in P; \quad \forall \alpha' \in F' \tag{3.3}$$

It is not an essential restriction to assume that the family $(\theta_{\alpha'})_{\alpha' \in F'}$ is closed under finite intersections, in fact if the σ-algebras $\theta_{\alpha'_1},\dots,\theta_{\alpha'_r}$ are sufficient for P then also $\theta_{\alpha'_1}\cap\dots\cap\theta_{\alpha'_n}$ is sufficient for P since for each $\mu\in P$ one has

$$E^\mu(\cdot \mid \bigcap_{j=1}^{n} \theta_{\alpha'_j}) = \lim_{N\to\infty} (E^\mu_{\alpha'_1} \circ \dots \circ E^\mu_{\alpha'_n})^N \tag{3.4}$$

$$= \lim_{N\to\infty} (E_{\alpha'_1} \circ \dots \circ E_{\alpha'_n})^N$$

the limit being meant in the sense of the strong convergence in $L^2(\Omega,\theta,\mu)$ (cfr.[2]).

Since the σ-algebras $\theta_{\alpha'}$ are completely determined by the $E_{\alpha'}$'s, the inverse problem of sufficiency can be reformulated as follows; let be given a filtering decreasing (for an order $>$) family of indices F' and a projective family $(E_{\alpha'})_{\alpha' \in F'}$ of norm one projections (i.e.-conditional expectations -cf. §(1)) defined on $L^\infty(\Omega,\theta)$; describe the set I of states on $L^\infty(\Omega,\theta)$ such that

$$\mu \circ E_{\alpha'} = \mu; \quad \forall \alpha' \in F'; \quad \forall \mu \in I \tag{3.5}$$

Remark. The inverse problem of sufficiency can be formulated on algebras different from $L^\infty(\Omega,\theta)$. For example if Ω is a topological space and θ is the Borel σ-algebra, one might consider the algebra $C_b(\Omega)$ of the bounded, continuous complex valued functions on Ω. In this case however it is necessary to assume that the conditional expectations $E_{\alpha'}$ map continuous functions into continuous functions. A projective family $(E_{\alpha'})$ of conditional expectations with this property will be called a Feller family. The general scheme described by Lauritzen in [18] corresponds to Dobruscin's theory for Feller families.

4. MARKOVIANITY AND PREDICTIVE SUFFICIENCY

Historically the Markov property was introduced to describe a situation in which the knowledge of the present is sufficient; with respect to the knowledge of all the past, for the statistical prediction of the future. This property has been subsequently generalized in various contexts (cfr. for example [19],[21],[25], [2], [13])

In all these generalizations the main original idea has been preserved, namely: one is given three σ-algebras θ_1, θ_2 and $\theta_3 \subseteq \theta_1$, and requires that the conditioning of θ_2 on θ_1 is the same as the conditioning of θ_1 on θ_3.

For the reasons mentioned in the introduction it is convenient to give a more algebraic formulation of this property. To fix the ideas we will discuss the generalized Markov property only in the framework of algebras of the form $L^\infty_{\mathbb{R}}(\Omega,\theta)$, the extension to more general algebras being straightforward. If $\theta_\circ$ is a sub-σ-algebra of θ, we will denote $E_{\theta_\circ}: L^\infty(\Omega,\theta)\to L^\infty(\Omega,\theta_\circ)$ a conditional expectation onto $L^\infty(\Omega,\theta_\circ)$, in the sense specified in §(1). When dealing simultaneously with many sub-σ-algebras we will always assume that the corresponding conditional expectations are compatible (i.e. if $\theta_\circ \subseteq \theta_1$ then $E_{\theta_\circ} \circ E_{\theta_1} = E_{\theta_\circ}$). The following results are essentially known. In our proofs we underline their purely algebraic character, (i.e. without reference to any a priori given probability measure).

Proposition (4.1). Let (Ω,θ) be a measurable space and let θ_1, θ_2, θ_3 be sub-σ-algebras of θ such that $\theta_3 \subseteq \theta_1$. Consider the following identities:

$$E_{\theta_1} \mid L^\infty(\Omega, \theta_2) = E_{\theta_3} \mid L^\infty(\Omega,\theta_2) \tag{4.1}$$

$$E_{\theta_1}(L^\infty(\Omega,\theta_2)) \subseteq L^\infty(\Omega, \theta_3) \tag{4.2}$$

$$E_{\theta_3}(f_1 f_2) = E_{\theta_3}(f_1)E_{\theta_3}(f_2); \ \forall f_1 \in L^\infty(\Omega, \theta_1), \ \forall f_2 \in L^\infty(\Omega, \theta_2) \tag{4.3}$$

$$E_{\theta_2 \vee \theta_3} \mid L^\infty(\Omega,\theta_1) = E_{\theta_3} \mid L^\infty(\Omega,\theta_1) \tag{4.4}$$

then

(i) (4.1) $\Longleftrightarrow$ (4.2) $\Longrightarrow$ (4.3) $\Longleftarrow$ (4.4)

(ii) If (4.3) holds then

(4.5) $$E_{\theta_3}(|E_{\theta_3}(f_2) - E_{\theta_1}(f_2)|^2) = 0 \;;\; \forall f_2 \in L^\infty(\Omega,\theta_2)$$

and, if E_{θ_3} is continuous for the pointwise convergence:

(4.6) $$E_{\theta_3}(|E_{\theta_3}(f_1) - E_{\theta_2 \vee \theta_3}(f_1)|^2) = 0 \;;\; \forall f_1 \in L^\infty(\Omega,\theta_1)$$

Proof. (4.1) ==> (4.2). If (4;1) holds, then

$$E_{\theta_1}(L^\infty(\Omega,\theta_2)) = E_{\theta_3}(L^\infty(\Omega,\theta_2)) \subseteq L^\infty(\Omega,\theta_3)$$

which is (4.2). Conversely, if (4.2) holds, then for any $f_2 \in L^\infty(\Omega,\theta_2)$:

$$E_{\theta_3}(f_2) = E_{\theta_3}(E_{\theta_1}(f_2)) \quad (\text{since } \theta_3 \subseteq \theta_1)$$

$$= E_{\theta_1}(f_2) \quad (\text{because of } (4.2))$$

and this is (4.1).

(4.1) ==> (4.3) For any $f_1 \in L^\infty(\Omega, \theta_1)$, $f_2 \in L^\infty(\Omega,\theta_2)$ one has, if (4.1) holds:

(4.7) $$E_{\theta_3}(f_1 f_2) = E_{\theta_3}(f_1 E_{\theta_1}(f_2)) = E_{\theta_3}(f_1)\, E_{\theta_3}(f_2)$$

which is (4.3).

(4.4) ==> (4.3) . Substitute $\theta_2 \vee \theta_3$ for θ_1 in (4.7) and exchange the roles of f_1 and f_2.

(4.3) ==> (4.5). If (4.3) holds then for f_1, f_2 as above

$$E_{\theta_3}(f_1 E_{\theta_3}(f_2)) = E_{\theta_3}(f_1 f_2) = E_{\theta_3}(f_1 E_{\theta_1}(f_2))$$

thus $E_{\theta_3}(f_1[E_{\theta_3}(f_2) - E_{\theta_1}(f_2)]) = 0$ and, since $f_1 \in L^\infty(\Omega,\theta_1)$ is arbitrary and $\theta_3 \subset \theta_1$, we can choose $f_1 = E_{\theta_3}(f_2) - E_{\theta_1}(f_2)$ obtaining (4.5).

(4.3) ==> (4.6). Let $f_j \in L^\infty(\Omega,\theta_j)$, with j=1,2,3. Then:

$$E_{\theta_3}(E_{\theta_3}(f_1) f_2 f_3) = E_{\theta_3}(f_1 f_2 f_3) = E_{\theta_3}(E_{\theta_2 \vee \theta_3}(f_1)\, f_2\, f_3)$$

Since linear combinations of products of the type $f_2 \cdot f_3$ are pointwise dense in $L^\infty(\Omega,\theta_2 \vee \theta_3)$ the above equalities imply:

$$E_{\theta_3}([E_{\theta_3}(f_1) - E_{\theta_2 \vee \theta_3}(f_1)]\, F_{2,3}) = 0$$

for each $F_{2,3} \in L^\infty(\Omega, \theta_2 \vee \theta_3)$, and the thesis follows by choosing $F_{2,3} = E_{\theta_3}(f_1) - E_{\theta_2 \vee \theta_3}(f_1)$.

Remark (1). In (4.6) the condition of continuity of E_{θ_3} for monotone converge is sufficient.

Remark (2). In particular, if E_{θ_3} is monotone continuous and faithful (i.e. $f \geq 0$ and $E_{\theta_3}(f) = 0 \Rightarrow f = 0$) then the four conditions (4.1),...,(4.4) are equivalent. In the following we shall always assume that the conditonal ecpectations which we consider enjoy the above mentioned properties, hence the four properties in Proposition (4.1) will be considered equivalent.

Definition (4.2) . Let (Ω,θ) be a measurable space, and let $\theta_1, \theta_2, \theta_3$ be sub-σ-algebras of θ such that $\theta_3 \subseteq \theta_1$. Two compatible conditional expectations $E_{\theta_j}: L^\infty(\Omega,\theta) \to L^\infty(\Omega,\theta_j)$ (j=1 or 3) are said to be markovian with respect to the triple $\theta_1, \theta_2, \theta_3 \subseteq \theta_1$ if

(4.8) $$E_{\theta_1}(f_2) = E_{\theta_3}(f_2) \quad \forall f_2 \in L^\infty(\Omega,\theta_2)$$

Remark (1.) The above notion of markovianity extends the notion of predictive sufficiency considered by M. Campanino and F. Spizzichino in [4] and by D. Cifarelli and E. Regazzini [5], [6]. The latter notion is recovered when a probability measure μ is given on (Ω,θ), E_{θ_1} (resp. E_{θ_3}) is the μ-conditional expectation on θ_1

(resp. θ_3) and the equality in (4.8) holds μ-almost everywhere; in this case we will say that μ has the Markov property with respect to $\theta_1, \theta_2, \theta_3 \subseteq \theta_1$. In these notations the main result of Regazzini and Cifarelli [6] can be formulated as follows:

<u>Theorem</u> (4.3). Let (Ω,θ,μ) be a probability space, and let for each $J \subseteq \mathbb{N}$ be given a sub-σ-algebra $\theta_J \subseteq \theta$ so that:

(4.9) $$I \subseteq J \Longrightarrow \theta_I \subset \theta_J$$

(for example, the σ-algebras generated by a stochastic process $(X_K)_{K\in\mathbb{N}}$). Assume that for each non empty interval $[m, m+n-1]$ it is given a σ-algebra $\mathcal{T}_{m,n} \subseteq \theta_{[m,m+n-1]}$ such that μ has the Markov property with respect to the triple $\theta_{[m,m+n-1]}$, $\theta_{[1,m-1]}$, $\mathcal{T}_{[m,n]} \subseteq \theta_{[m,m+n-1]}$.

Then, denoting

$$\mathcal{T}^\infty_m = \bigcap_{j\geq 1} \Big(\bigcup_{k\geq j} \mathcal{T}_{m,k}\Big) \subseteq \theta_{[m,\infty[}$$

$$\mathcal{T}_\infty = \bigcap_{m\geq 1} \Big(\bigcup_{k\geq m} \mathcal{T}^\infty_k\Big) \subseteq \theta_\infty = \bigcap_{m\geq 1} \theta_{[m,\infty[}$$

μ has the Markov property with respect to the triples:

$$\theta_{[m,\infty[}\,,\ \theta_{[1,m-1]}\,,\ \mathcal{T}^\infty_m \subseteq \theta_{[m,\infty[}$$

$$\theta_\infty\,,\ \theta_{[1,m-1]}\,,\ \mathcal{T}_\infty \subseteq \theta_\infty$$

The following extends a result of Lauritzen [17]:

<u>Proposition</u> (4.4). Let (Ω,θ,μ) be a probability space and let

$$\theta_1, \theta_2, \theta_3 \subseteq \theta_1 \ ; \qquad \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1$$

be two triples of sub-σ-algebras of θ. Assume that μ has the Markov property with respect to both triples of σ-algebras. Then, if

(4.10) $$\theta_1 \subseteq \mathcal{F}_2; \qquad \mathcal{F}_1 \subseteq \theta_2$$

the measure μ has the Markov property with respect to any triple $\theta_1 \vee \mathcal{F}_1$, $\mathcal{A}_2$, $\theta_3 \vee \mathcal{F}_3 \subseteq \theta_1 \vee \mathcal{F}_1$ where $\mathcal{A}_2$ is any sub-σ-algebra of $\theta_2 \cap \mathcal{F}_2$.

<u>Example</u>: Let θ be the σ-algebra generated by a stochastic process $(X_t)_{t\in T}$; denote, for $I \subseteq T$, $\theta_I = \sigma\{X_t : t \in I\}$ and choose $T_o, T_1 \subseteq T$ with $T_o \cap T_1 = \phi$. Then choosing $\theta_1 = \theta_{T_o}$; $\mathcal{F}_1 = \theta_{T_1}$; $\theta_2 = \theta_{T-T_o}$; $\mathcal{F}_2 = \theta_{T-T_1}$; condition (4.10) is satisfied.

<u>Proof</u>. Fix $\mathcal{A}_2 \subseteq \theta_2 \cap \mathcal{F}_2$ and choose $f_o \in L^\infty(\Omega,\theta_1)$, $f_1 \in L^\infty(\Omega,\mathcal{F}_1)$, $f_2 \in L^\infty(\Omega,\mathcal{A}_2)$.

Then, since $\mathcal{F}_3 \subseteq \mathcal{F}_1 \subseteq \theta_2$, one has:

$$E_{\theta_2 \vee \mathcal{F}_3}(f_o f_1 f_2) = E_{\theta_3 \vee \mathcal{F}_3}(E_{\theta_3 \vee \theta_2}(f_1 f_2 f_3)) =$$

$$= E_{\theta_3 \vee \mathcal{F}_3}(E_{\theta_3 \vee \theta_2}(f_o) f_1 f_2) = \qquad \text{(since } f_1 \cdot f_2 \hat{\in} \theta_2)$$

$$= E_{\theta_3}(f_o)\, E_{\theta_3 \vee \mathcal{F}_3}(f_1 f_2) = \qquad \text{(using (4.4))}$$

$$= E_{\theta_3}(f_o)\, E_{\theta_3 \vee \mathcal{F}_3}(E_{\mathcal{F}_2 \vee \mathcal{F}_3}(f_1) f_2$$

$$= E_{\theta_3}(f_o)\, E_{\mathcal{F}_3}(f_1)\, E_{\theta_3 \vee \mathcal{F}_3}(f_2)$$

in particular, letting $f_2 = 1$, we obtain

(4.11) $$E_{\theta_3 \vee \mathcal{F}_3}(f_o f_1) = E_{\theta_3}(f_o)\, E_{\mathcal{F}_3}(f_1)$$

therefore

(4.12) $$E_{\theta_3 \vee \mathcal{F}_3}(f_o f_1 f_2) = E_{\theta_3 \vee \mathcal{F}_3}(f_o f_1)\, E_{\theta_3 \vee \mathcal{F}_3}(f_2)$$

Since sums of products of the type $f_o \cdot f_1$ are dense in $L^\infty(\Omega, \theta_1 \vee \mathcal{F}_1, \mu)$, condition (4.11) implies the markovianity of μ with respect to the triple

$$\theta_1 \vee \mathcal{F}_1,\ \mathcal{A}_2,\ \theta_3 \vee \mathcal{F}_3 \subseteq \theta_1 \vee \mathcal{F}_1.$$

<u>Remark</u>. For simplicity we have considered only the case in which a probability measure μ is given a priori. However the proof of (4.11) and (4.12) is purely algebraic and from (4.12) the statement of Proposition (4.4) follows just using the monotone continuity of E.

Now, if μ has the Markov property with respect to the triples $\theta_1, \theta_2, \mathcal{F}_j \subseteq \theta_1$

(j=1,...,k), then for each $f_2 \in L^\infty(\Omega,\theta_2)$ one has:

$$E_{F_1}\circ\cdots\circ E_{F_k}(f_2) = E_{\theta_1}(f_2).$$

Hence, by the same argument as in §(3), μ has the Markov property with respect to the triple $\theta_1, \theta_2, \bigcap_{j=1}^{k} F_j \subseteq \theta_1$. A martingale argument extends this result to arbitrary intersections and, since the Markov property trivially takes place for any triple of the form $\theta_1, \theta_2, \theta_1$, it follows that for any given $\theta_1, \theta_2 \subseteq \theta$ there exists an $\theta_3 \subseteq \theta_1$ such that μ has the Markov property with respect to the triple $\theta_1, \theta_2, \theta_3 \subseteq \theta_1$ and if F_3 is another sub-σ-algebra of θ_1 with this property then $F_3 \supseteq \theta_3$. Following a terminology used in filtering theory, we call an θ_3 with the above property -the splitting σ-algebra between θ_1 and θ_2 (cfr. [21]).

Let us now briefly illustrate how the concept of total sufficiency introduced by Lauritzen [17] is related to the inverse problem of sufficiency formulated in §(3).

Let $(X_t)_{t\in T}$ (T- a set) be a family of measurable functions defined on (Ω,θ).

Denote for any sub-set $T_0 \subseteq T$,

$$\theta_{T_0} = \sigma\{X_t : t\in T_0\} = \sigma\text{-algebra generated by } \{X_t : t\in T_0\}$$

and let T_0' be the set-theoretical complement of T_0 in T.

Definition (4.5) (cf. [17]). Let $T_0 \subseteq T$, F_{T_0} - a sub-σ- algebra of θ_{T_0}, $E_{F_{T_0}} : L^\infty(\Omega,\theta_{T_0}) \to L^\infty(\Omega, F_{T_0})$ be a normal (i.e. monotone continuous) conditional expectation and let P-be a family of probability measures on (Ω,θ). The couple $\{F_{T_0}, E_{F_{T_0}}\}$ is called totally sufficient for P relatively to T_0 if:

(i) For every $\mu\in P$, $E_{F_{T_0}}$ is a version of the restriction of

$$E^\mu(\cdot\,|F_{T_0}) \text{ to } L^\infty(\Omega,\theta_{T_0}).$$

(ii) Every $\mu\in P$ has the Markov property with respect to the triple

$$\theta_{T_0},\ \theta_{T_0'},\ F_{T_0} \subseteq \theta_{T_0}$$

From conditions (i) and (ii) one can construct a conditional expectation $E_{F_{T_0}\vee\theta_{T_0'}} : L^\infty(\Omega,\theta) \to L^\infty(\Omega,F_{T_0}\vee\theta_{T_0'})$ characterized by the property:

(4.13) $$E_{F_{T_0}\vee\theta_{T_0'}}(f_{T_0} f_{T_0'}) = E_{F_{T_0}}(f_{T_0})\, f_{T_0'}$$

for every $f_{T_0}\in L^\infty(\Omega,\theta_{T_0})$, $f_{T_0'} \in L^\infty(\Omega,\theta_{T_0'})$.

It is easy to verify that, for all $\mu\in P$, $E_{F_{T_0}\vee\theta_{T_0'}}$ is a version of $E^\mu(\cdot\,|F_{T_0}\vee\theta_{T_0'})$ and that

(4.14) $$E_{F_{T_0}\vee\theta_{T_0'}}\big|\, L^\infty(\Omega,\theta_{T_0}) = E_{F_{T_0}}$$

Conversely if $E_{F_{T_0}\vee\theta_{T_0'}} : L^\infty(\Omega,\theta) \to L^\infty(\Omega,F_{T_0}\vee\theta_{T_0'})$ is any conditional expectation satisfying (4.14), then the couple F_{T_0}, $E_{F_{T_0}}$ is totally sufficient for the family of all the probability measures compatible with $E_{F_{T_0}\vee\theta_{T_0'}}$.

Let now A be a net (increasing by inclusion) of subsets $T_0 \subseteq T$ and assume that for each $T_0 \in A$ it is given a σ-algebra $F_{T_0} \subseteq \theta_{T_0}$ and a conditional expectation $E_{F_{T_0}\vee\theta_{T_0'}} : L^\infty(\Omega,\theta) \to L^\infty(\Omega,\theta_{F_{T_0}}\vee\theta_{T_0'})$ satisfying (4.14) (Markov property). Then one can state the inverse sufficiency problem for the family $\{E_{F_{T_0}\vee\theta_{T_0'}} : T_0\in A\}$ even if no compatibility condition is a priori required. However it can easily be shown that, if $\forall\, T_0\in A$, F_{T_0} is the splitting σ-algebra between θ_{T_0} and $\theta_{T_0'}$ (i.e. F_{T_0} is minimal among the σ-algebras satisfying (4.13)) then, by Proposition (4.4), for every $T_\alpha \supseteq T_0$ one has

(4.15) $$F_{T_\alpha} \subseteq \theta_{T_0'} \cap \theta_{T_\alpha}$$

In particular if the family A is an increasing sequence (T_n) and the above mentioned minimality conditon is satisfied, then one easily verifies that the sequence of σ-algebras (F_{T_n}) is a Markov chain with respect to each probability measure compatible with the family $(E_{F_{T_n}\vee\theta_{T_n'}})_n$.

Conversely if μ is e probability measure on (Ω,θ) such that the σ-algebras (F_{T_n}) are a μ-Markov chain, then μ is compatible with the family $(E_{F_{T_n}\vee\theta_{T_n'}})_n$.

The above remark, due to Lauritzen [18], allows to reduce the inverse problem of sufficiency for the family $(E_{F_{T_n}\vee\theta_{T_n'}})$ to the same problem for a Markov chain.

5. DOBRUSHIN'S THEORY

The main problem dealt with by Dobrushin's theory can be heuristically formulated as follows [11], [2]: given, on a measurable space, a family of conditional probability measures with respect to a decreasing net of σ-algebras, describe the structure of the set of all probability measures whose family of conditional probabilities coincides with the given one.

In order to give a precise formulation of the problem it is convenient to deal with conditional expectations, rather then with conditional probabilities. In the remaining of this § we will follow the approach suggested in [2] (where the general non-commutative case in dealt with).

Let (Ω,θ) be a measurable space and let $F' = \{\alpha'\}$ be a decreasing net, i.e. on F' there is a partial order denoted $>$ and for any $\alpha', \beta' \in F'$ there is $\gamma' \in F'$ such that $\alpha' > \gamma'$ and $\beta' > \gamma'$ (often F' is a family of sub-sets of a given set T -typically $T = \mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$ ... each α' is the set-theoretical complement of a set $\alpha \in T$, and $\alpha' > \beta'$ means that β' is contained in α' or that $\alpha \subseteq \beta$). For each α' a σ-algebra $\theta_{\alpha'} \subseteq \theta$ is given such that

(5.1) $$\alpha' > \beta' \implies \theta_{\alpha'} \supseteq \theta_{\beta'}$$

and for each α' a conditional expectation $E_{\alpha'} : L^\infty(\Omega,\theta) \to L^\infty(\Omega,\theta_{\alpha'})$ is given such that the projectivity condition:

(5.2) $$\alpha' > \beta' \implies E_{\beta'} \cdot E_{\alpha'} = E_{\beta'}$$

is satisfied.

<u>Remark</u>. In many cases it is convenient to deal with algebras which are not of L^∞ -type (say, for example, algebras of continuous functions or algebras of polynomials in certain random variables). In the following we restrict ourselves to the formulation of the problem given above, and refer to [2] for the general case.

A <u>state</u> on $L^\infty(\Omega,\theta)$ is a linear functional $\mu : L^\infty(\Omega,\theta) \to \mathbb{C}$ such that

(5.3) $$f \in L^\infty(\Omega,\theta) \ , \quad f \geq 0 \implies \mu(f) \geq 0$$

(5.4) $$\mu(1) = 1$$

A state μ on $L^\infty(\Omega,\theta)$ is called <u>compatible</u> with the family $(E_{\alpha'})$, or simply $(E_{\alpha'})$ -invariant if

(5.5) $$\mu \circ E_{\alpha'} = \mu \quad ; \qquad \forall \alpha' \in F'$$

If μ is defined by a countably additive measure on (Ω,θ) -still denoted μ-then condition (5.5) is equivalent to:

(5.6) $$E_{\alpha'} = E^\mu(\cdot \mid \theta_{\alpha'}) \ ; \ \mu\text{- a.e};$$

where $E^\mu_\alpha(\cdot \mid \theta_{\alpha'})$ denotes the μ-conditional expectation on $\theta_{\alpha'}$.

<u>Theorem</u> (5.1) (cf. [2], Theorem (1.1) and Theorem (3.2)). The set of $(E_{\alpha'})$ -invariant states is non empty. It is a Choquet simplex.

The main problems dealt with by Dobrushin's theory are:

<u>Problem</u> 1). In which cases is the set of $(E_{\alpha'})$ -invariant state, induced by a countably additive measure, non empty?

<u>Problem</u> 2). Under which conditions there exists exactly one $(E_{\alpha'})$ -invariant state?

<u>Problem</u> 3). Describe the structure of the extremal $(E_{\alpha'})$ -invariant states.

There are examples in which $(\Omega,\theta) = \Pi_{\mathbb{Z}} (X, \mathcal{B})$ where X is a countable set, $\mathcal{B}$ is the family of parts of X, $\alpha =]m,n[$ with m n, $\alpha' = C\,]m,n[= \{k \in \mathbb{Z} : k \notin]m,n[\}$, $\theta_{\alpha'}$, is the σ-algebra generated by the canonical projections $\xi_k : \Omega \to X$ with $k \notin \alpha$, and the conditional expectations $E_{\alpha'}$ satisfy the Markov property, but in which the set of $(E_{\alpha'})$-invariant countably additive measures in empty (cf. [27]). In general, the existence of countably additive $(E_{\alpha'})$- invariant measures is connected with equicontinuity properties of the action of the family $(E_{\alpha'})$ on the space of countably additive measures.

Concerning the problem of the extremal invariant state one knows (cf. [2], theorem (3.2)).

<u>Theorem</u> (5.2). Let μ be an $(E_{\alpha'})$ -invariant state. The following assertions are equivalent:

(i) is an extremal $(E_{\alpha'})$ -invariant state.

(ii) The tail algebra is trivial, i.e.

(5.7) $$A^\mu_\infty = \bigcap_{\alpha' \in F'} L^\infty(\Omega, \theta_{\alpha'}, \mu) = \mathbb{C} \cdot 1$$

(iii) for any $\varepsilon > 0$ and $f \in L^\infty(\Omega,\theta)$ there exists a $\beta'_0 = \beta'_0(\varepsilon, f)$ such that for any β' such that $\beta'_0 > \beta'$ and $\forall\, g_\beta \in L^\infty(\Omega,\theta_{\beta'})$

(5.8) $$| \mu(f \cdot g_{\beta'}) - \mu(f)\, \mu(g_{\beta'}) | \leq \varepsilon \, \| g_{\beta'} \|_\infty$$

where $\| \cdot \|_\infty$ denotes the μ -ess-sup-norm.

The fine structure of the extremal states is known only in partiuclar cases, for example, for the 2-dimensional Ising model (cf. [22]) or for the problem of exchangeable measures on $\Pi_{\mathbb{N}}(X,\mathcal{B})$ de Finetti's theorem).

Concerning the uniqueness problem, it is known (cf. [2], Corollary (6.3)) that:

Theorem (5.3). Let μ be an $(E_{\alpha'})$ -invariant state on $L^{\infty}(\Omega,\theta)$. A necessary condition for μ to be the only $(E_{\alpha'})$ -invariant state is that $\forall\, \varepsilon > 0$, $\forall\, a \in A$, there exists a $\beta_0 = \beta_0(\varepsilon, a)$ such that if $\beta'_0 > \beta'$

$$| \mu(a\cdot b_{\beta'}) - \mu(a)\, \mu(b_{\beta'}) | \leqslant \varepsilon\, \mu(b_{\beta'}) = \varepsilon\, \|b_{\beta'}\|_{L^1(\mu)} \tag{5.9}$$

uniformly in $b_{\beta'} \in L^{\infty}(\Omega,\theta_{\beta'},\mu)$ $b_{\beta'} \geqslant 0$.

If the family $(E_{\alpha'})$ is ergodic (cf. Definition (5.4) below) then (5.9) is also a sufficient condition for the uniqueness of the $(E_{\alpha'})$ invariant state.

Definition (5.4). A family $(E_{\alpha'})$ of conditional expectations satisfying (5.2) is called ergodic if $\forall\, a \in L^{\infty}(\Omega,\theta)$ and for each state ψ on $L^{\infty}(\Omega,\theta)$, there exists an $\alpha' \in F'$ such that $\psi(E_{\alpha'}(a)) > 0$.

Remark 1). In case of discrete systems, the ergodicity condition is just the statement that all the conditional probabilities defined by the conditional expectations E are strictly positive.

Remark 2). A comparison between (5.8) and (5.9) and the remark that $\mu(b_{\beta'})$ is just the $L^1(\Omega,\theta,\mu)$ -norm of $b_{\beta'}$ shows that both extremality and uniqueness are properties of strong mixing type; the difference being only in the rate at which the asympotic factorization takes place.

Even if μ is not induced by a countably additive measure, there exists a conditional expectation $E^{\mu}_{\infty} : L^{\infty}(\Omega,\theta,\mu) \to A^{\mu}_{\infty}$ compatible with μ (cf. [2], Lemma (3.1)).

By the Gelfand isomorphism theorem A^{μ}_{∞} can be realized as the algebra $C(K)$ of all the continuous complex functions on a compact Hausdorf space K. With this identification, and with the notation $\mu_{\infty} = \mu \mid A^{\mu}_{\infty} \cong C(\mathbf{K})$, we have

$$\mu(a) = \mu_{\infty}(E^{\mu}_{\infty}(a)) = \int_K E^{\mu}_{\infty}(a)(\omega)\mu_{\infty}(d\omega) \tag{5.10}$$

for each $a \in L^{\infty}(\Omega,\theta)$. For each $\omega \in K$, the map $a \in L^{\infty}(\Omega,\theta) \to E^{\mu}_{\infty}(a)(\omega)$ defines an $(E_{\alpha'})$ -invariant state and we will now show that it is an extremal $(E_{\alpha'})$ -invariant state. Therefore (5.10) defines an integral decomposition of μ trough extremal $(E_{\alpha'})$ -invariant states.

Lemma (5.5). In the notations above, if the net F' is countably generated, then for each $\omega \in K$, the state: $a \in L^{\infty}(\Omega,\theta) \to E^{\mu}_{\infty}(a)(\omega)$ is extremal $(E_{\alpha'})$-invariant.

Proof. That $a \to E^{\mu}_{\infty}(a)(\omega)$ is $(E_{\alpha'})$ -invariant follows from the definition of E^{μ}_{∞}. To prove extremality let us denote for any $a \in L^{\infty}(\Omega,\theta)$ $\omega \in k$, and $\beta' \in F'$:

$$f^a_{\beta'}(\omega) = \sup_{b_{\beta'} \in A_{\beta'},\ \|b_{\beta'}\|_{\infty} = 1} | E^{\mu}_{\infty}(ab_{\beta'})(\omega) - E^{\mu}_{\infty}(a)(\omega)E^{\mu}_{\infty}(b_{\beta'})(\omega)|$$

(here and in the remaining of the proof we use the notation: $A_{\beta'} = L^{\infty}(\Omega,\theta_{\beta'})$. Since, for $\beta' > \gamma'$, $A_{\beta'} \supseteq A_{\gamma'}$ we have $\beta' > \gamma' \Longrightarrow f^a_{\beta'}(\omega) \geqslant f^a_{\gamma'}(\omega)$; $\forall\, \omega$, i.e. $(f^a_{\beta'})$, is a monotone decreasing net of continuous functions on K. Therefore thet limit $g^a(\omega) = \lim_{\beta'} f^a_{\beta'}(\omega)$ exists for each ω. Since F' is countably generated, g^a is also the limit of a decreasing sequence of continuous functions on K hence, by Dini's theorem, g^a is continuous.

Our thesis is equivalent to the statement that $g^a \equiv 0$. Assume the contrary. Then, since $g^a \geqslant 0$, there exists an open set $U \subseteq K$ and a $\delta > 0$ such that $g^a(\omega) \geqslant \delta$ $\forall\, \omega \in U$ and therefore, by monotonicity:

$$\sup_{\|b_{\beta'}\| = 1} | E^{\mu}_{\infty}(ab_{\beta'})(\omega) - E^{\mu}_{\infty}(a)(\omega)\, E^{\mu}_{\infty}(b_{\beta'})(\omega)| > \delta,\ \forall\, \omega \in U$$

Passing eventually to a sub-net (or sub-sequence) one can always assume that for each $\beta' \in F'$ there exists a $b_{\beta'} \in A_{\beta'}$ with $b_{\beta'} \geqslant 0$, $\|b_{\beta'}\| = 1$, such that the difference

$$E^{\mu}_{\infty}(ab_{\beta'})(\omega) - E^{\mu}_{\infty}(a)(\omega)\, E^{\mu}_{\infty}(b_{\beta'})(\omega)$$

has constant sign for each $\omega \in U$ and β'.

Assuming that this sign is positive (otherwise we substitute $- b_{\beta'}$ for $b_{\beta'}$) we have that $\forall\, \omega \in U$ and $\forall\, \beta'$:

$$E^{\mu}_{\infty}(ab_{\beta'})(\omega) - E^{\mu}_{\infty}(a)(\omega)\, E^{\mu}_{\infty}(b_{\beta'})(\omega) \geqslant \delta$$

from which we deduce; denoting by X_U the characteristic function of the set U,

$$| \mu(ab_{\beta'}) - \mu(E^{\mu}_{\infty}(a)\, E^{\mu}_{\infty}(b_{\beta'})) | \geqslant$$

$$\geqslant \Big| \, | \mu(\{E^{\mu}_{\infty}(ab_{\beta'}) - E^{\mu}_{\infty}(a)\, E^{\mu}_{\infty}(b_{\beta'})\} \cdot X_U)\Big| -$$

$$- \left|\mu(\{E^{\mu}_{\infty}(ab_{\beta'}) - E^{\mu}_{\infty}(a)\, E^{\mu}_{\infty}(b_{\beta'})\}\, \chi_{K-U})\right| \Big| \geqslant$$

$$\geqslant \Big| \delta - \left| \mu\, \{E^{\mu}_{\infty}(ab_{\beta'}) - E^{\mu}_{\infty}(a)\, E^{\mu}_{\infty}(b_{\beta'})\}\, \chi_{K-U} \right| \Big|$$

$$\Big| \delta - \left| \mu(ab_{\beta'}\, \chi_{K-U}) - \mu(E^{\mu}_{\infty}(a)\, b_{\beta'}\, \chi_{K-U}) \right| \Big|$$

But since $\mu(U) > 0$ (μ is faithful on $L^{\infty}(\Omega,\theta,\mu)$), the above inequality is impossible, as the following Lemma shows.

Lemma (5.6). Let $a \in L^{\infty}(\Omega,\theta)$, $b_{\beta'} \in A_{\beta'}, \chi \in A^{\mu}_{\infty}$. Then for each $\varepsilon > 0$ there exists $\beta'_0 = \beta'_0\,(a,\varepsilon)$ such that if $\beta'_0 > \beta'$

$$\left| \mu(ab_{\beta'}\chi) - \mu(E^{\mu}_{\infty}(a)\, b_{\beta'}\chi)\right| \leqslant \varepsilon \|b_{\beta'}\|_{\infty}$$

in particular, if $\|b_{\beta'}\|_{\infty} = 1$, $\forall_{\beta'}$, then

$$\lim_{\beta'} \left|\mu(ab_{\beta'}\chi) - \mu(E^{\mu}_{\infty}(a)\, b_{\beta'}\, \chi)\right| = 0$$

Proof. One has, in the Hilbert space notations of [2], § 3:

$$\left| \mu(ab_{\beta'}\chi) - \mu(E^{\mu}_{\infty}(a)\, b_{\beta'}\chi)\right| =$$

$$\left|\langle a^{*} 1_{\mu},\, b_{\beta'}\chi 1_{\mu}\rangle - \langle e^{\mu}\, a^{*} 1_{\mu},\, b_{\beta'}\, \chi 1_{\mu} \rangle\right| =$$

$$= \left|\langle [e^{\mu}_{\beta'}\, a^{*} 1_{\mu} - e^{\mu}\, a^{*}\, 1_{\mu}],\, b_{\beta'}\, \chi 1_{\mu} \rangle\right| \leqslant$$

$$\leqslant \|b_{\beta'}\|_{\infty}\, \|\chi\|_{\infty} \|e^{\mu}_{\beta'}\, a^{*}\, 1_{\mu} - e^{\mu}\, a^{*}\, 1_{\mu}\|_{2}$$

and the assertion follows since strong- $\lim_{\beta'} e^{\mu}_{\beta'} = \mu e$ in $L^{2}(\Omega,\theta,\mu)$.

Remark. An easy consequence of the monotone convergence theorem is that, if the measure μ is countably additive, then in the integral decomposition (5.10) the support of the measure μ_{∞} is contained in the set of those $\omega \in K$ such $E^{\mu}_{\infty}(\cdot)\,(\omega)$ is a countably additive measure. Therefore, if there exists a countably additive $(E_{\alpha'})$-invariant measure, there exists also a countably additive extremal $(E_{\alpha'})$-invariant measure (cf. for example [13]).

6. CONDITIONAL MARTINGALES

From the considerations in § 5 it is clear that Dobrushin's theory is the study of projective families of conditional expectations.

It is then natural to ask: how to build concrete and non trivial models of projective families of conditional expectations? A simple example of such a construction is provided by a generalization of our formulation of de Finetti's

theorem (cfr. § (2)) : if $(G_n)_n$ is a sequence (or a net) of compact groups such that $G_n \subseteq G_{n+1}$ and each G_n acts on (Ω,θ) then one easily verifies that the sequence (R_n) defined by:

$$E_n\,(f) = \frac{1}{\lambda_n(G_n)} \int_{G_n} gf\ \lambda_n(dg) \tag{6.1}$$

(λ_n-left invariant measure on G_n) is a projective family of conditional expectations whose range are the G_n -invariant functions.

Another tecnique to produce nontrivial examples of projective families of conditional expectations, widely used in statistical mechanics and quantum field theory, is to "perturb" a given projective family $(E^{0}_{\alpha'})_{\alpha \in F'}$ with some positive functions (K_{α}), such that $E^{0}_{\alpha'}(K_{\alpha}) = 1$; i.e. to define, for each α' a new conditional expectation $E_{\alpha'}$:

$$E_{\alpha'}(f) = E^{0}_{\alpha'}(K_{\alpha}f) \tag{6.2}$$

This procedure is the analogue of the one which allows to build a new projective family of measures by "perturbation" of a given projective family (μ^{0}_{α}) i.e.

$$\mu_{\alpha}\,(f) = \mu^{0}_{\alpha}\,(\Lambda_{\alpha}\ \ f) \tag{6.3}$$

It is well known that the family (μ_{α}) -defined by (6.3) -will be projective if and only if (Λ_{α}) is a μ^{0}-martingale.

(In the following, to fix the ideas, we will always use the symbols $\alpha,\beta,\ldots$ to denote bounded open sets of $\mathbb{R}^{n}$ (or finite sets of $\mathbb{Z}^{n}$) and α', β', ... to denote the complements of α, β, ...).

The analogy with classical martingale theory suggests to call a family $(K_{\alpha})_{\alpha}$ an $(E^{0}_{\alpha'})$-conditional martingale if and only if the family of conditional expectations $(E_{\alpha'})$ defined by (6.1) is projective.

One can easily show (cf. [2]) that (K_{α}) is an $(E^{0}_{\alpha'})$-conditional martingale if and only if for each $\alpha \subseteq \beta$

$$K_{\beta} = K_{\alpha}\ \cdot E^{0}_{\alpha'}(K_{\beta}) \tag{6.4}$$

Example (1). A simple and very well known example of conditional martingale is obtained as follows (cf. [2]): let $\Omega = \prod_{\mathbb{Z}^n} \{0,1\}$ where n is an integer with the natural σ-algebra and let, for any $x \in \mathbb{Z}^n$, $\xi_x : \Omega \to \{0,1\}$ -the natural projection. For any finite part $\alpha \subseteq \mathbb{Z}^n$, denote α' -the complement of α and define the conditional expectation $E^o_{\alpha'} : L(\Omega,\theta) \to L^\infty(\Omega,\theta_{\alpha'})$ (as usual $\theta_I = \sigma\{\xi_x : x \in I\}$; $I \in \mathbb{Z}^n$) by "partial integration" with respect to the measure $\mu^o = \otimes_{\mathbb{Z}^n} \{\frac{1}{2},\frac{1}{2}\}$, i.e.:

$$E^o_{\alpha'}(f)(\xi_{\alpha'}) = \frac{1}{2^{|\alpha|}} \sum_{\substack{\varepsilon_x \in \{0,1\} \\ x \in \alpha}} f(\{\varepsilon_x\}_{x \in \alpha}, \xi_{\alpha'})$$

where $|\alpha|$ is the cardinality of the set α and we have used the notation $\xi_{\alpha'} = \{\xi_x\}_{x \in \alpha'}$.

If $V : \mathbb{Z}^n \to \mathbb{R}$ is any function such that :

$$\sum_{x \in \mathbb{Z}^n} |V(x)| < +\infty; \quad V(x) = V(-x) \tag{6.5}$$

and $\mu \in \mathbb{R}$, $\beta > 0$ are real numbers, define the functional

$$U(\xi_\alpha,\xi_{\alpha'}) = -\mu \sum_{x \in \alpha} \xi_x + \frac{1}{2} \sum_{\substack{x,j \in \alpha \\ x \neq j}} \xi_x \xi_j V(x-j) + \sum_{x \in \alpha} \sum_{j \in \alpha'} \xi_x \xi_j V(x-j) \tag{6.6}$$

then it is easy to verify that the family (K_α) defined by

$$K_\alpha(\xi_\alpha,\xi_{\alpha'}) = \frac{e^{-\beta U(\xi_\alpha,\xi_{\alpha'})}}{E^o_{\alpha'}(e^{-\beta U(\xi_\alpha,\xi_{\alpha'})})} \tag{6.7}$$

is a conditional martingale.

Example (2). Consider a Markov random field on $\mathbb{R}^n$. Let (Ω,θ,μ^o) the associated probability space and for $\alpha \in \mathbb{R}^n$-bounded open- denote θ_α (resp. $\theta_{\alpha'}$) the σ-algebra of the functionals of the field localized on α (resp. α'), and $E^o_{\alpha'}$ the μ^o-conditional expectation on $\theta_{\alpha'}$. If $\alpha \to U_\alpha$ is an additive functional localized on α, i.e.

$$\alpha \cap \beta = \emptyset \Longrightarrow U_{\alpha \cup \beta} = U_\alpha + U_\beta \tag{6.8}$$

and U_α is θ_α -measurable, then the family (K_α) defined by:

$$K_\alpha = e^{-U_\alpha} / E^o_{\alpha'}(e^{-U_\alpha}) \tag{6.9}$$

is a conditional martingale and, moreover, the conditional expectation $E_{\alpha'}$ -defined by (6.1) enjoys the Markov property with respect to the triple $\theta_{\alpha'}, \theta_\alpha, \theta_{\partial\alpha} \subseteq \theta_{\alpha'}$.

One can prove (cfr. [2]) that, conversely, every markovian conditional martingale arises in this way.

7. QUANTUM de FINETTI'S THEOREM

The quantum mechanical generalization of de Finetti's theorem has been considered by several authors [15], [29], [16].

In its most general formulation it can be stated as follows (cf.[29]) -let $\mathcal{B}$ be a C*-algebra (with unit, for simplicity); let $A = \otimes_{\mathbb{N}} \mathcal{B}$ the C*-infinite tensor product of countably many copies of $\mathcal{B}$; denote, for each $n \in \mathbb{N}$, $J_n : \mathcal{B} \to A$ -the natural embedding of $\mathcal{B}$ into A, which maps $\mathcal{B}$ onto the n-th factor of the product. In the notations of § (2.) -the symmetric group S_∞ has a natural action on A by $\star$-automorphisms which is characterized by the property:

$$\pi \circ J_n = J_{\pi(n)} \qquad \forall n \in \mathbb{N}, \forall \pi \in S_\infty \tag{7.1}$$

The following result is due to E. Stromer [29]:

Theorem (7.1). (Quantum de Finetti's theorem).

The S_∞-invariant states on $A \cong \otimes_{\mathbb{N}} \mathcal{B}$ are a Choquet simplex (in the w*-topology) whose extremal points are the symmetric product states, i.e. those states μ on A of the form $\mu = \otimes_{\mathbb{N}} m$ where m is a state on $\mathcal{B}$.

Proof. For each $N \in \mathbb{N}$, the map E_N defined by:

$$E_N(a) = \frac{1}{N!} \sum_{\pi \in S_N} \pi a ; \qquad a \in A \tag{7.2}$$

is a conditional expectation whose range are the fided points of S_N. The family (E_N) is projective and, just as in Proposition (2.3), one sees that the S_∞-invariant states coincide with the (E_N)-invariant states. It is a simple exercise to verify that the family (E_N) is asymptotically abelian (in the sense that for each a, b ∈ A $\|a E_N(b) - E_N(b)\cdot a\|$ tends to zero as $N \to \infty$)

and that the general theory developed in [2], which is the quantum extension of Dobrushin's theory, can be applied. In particular using Theorem (3.2) of [2] we conclude that the (E_N)-invariant states are a Choquet simplex and the extremal points are characterized by the triviality of the tail algebra. But the proof of Lemma (2.4) applies, just with a change of notations, also to this case. Therefore the (E_N)-invariant states are conditionally independent on the tail algebra, and this immediately implies the thesis.

In particular, cfr. [2], § (4), any S_∞-invariant state is the barycenter of a unique probability measure on the symmetric product states and a theorem of R.L.Hudson and G.R.Moody [15] states that if $\mathcal{B} = \mathcal{B}(H)$ = algebra of all the operators on a complex separable Hilbert space then a locally normal S_∞-invariant state is the barycenter of a unique probability measure on the symmetric product states of the form $\otimes_{\mathbb{N}} m$ where m is a normal state on $\mathcal{B}$. The relevance of the quantum de Finetti's theorem for the interpretative problems of quantum mechanics has been discussed by R.L.Hudson in [16].

Finally there is a very interesting connection between the classical and the quantum de Finetti's theorem, first remarked by D. Shale and W.F.Stinespring [26]: let $A = \otimes_{\mathbb{N}} \mathcal{B}$ with $\mathcal{B}$ = the algebra of 2x2 complex matrices, and let

(7.3) $$\sigma^1 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix};\quad \sigma^2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};\quad \sigma^3 = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$$

be the Pauli matrices. If $J_m : \mathcal{B} \to A$ is the natural embedding defined at the beginning of this §, denote:

(7.4) $$\sigma^j_n = J_n(\sigma^j);\quad j = 1, 2, 3;\quad n \in \mathbb{N}$$

(7.5) $$\omega_n = \sigma^1_1 \cdot \ldots \cdot \sigma^1_{n-1} \cdot \sigma^2_n$$

(7.6) $$\omega'_n = \sigma^1_1 \cdot \ldots \cdot \sigma^1_{n-1} \cdot \sigma^3_n$$

then one easily verifies that the operators

(7.7) $$b_j = \frac{1}{2}(\omega_j + \omega'_j)$$

satisfy the canonical anti-comutation relations:

(7.8) $$b_i b_j + b_j b_i = 0 \qquad \forall\, i,j$$

(7.9) $$b_i b^*_j + b^*_j b_i = \delta_{ij} 1 \qquad \forall\, i,j$$

Sums of products of the type $\omega_m \cdot \omega'_n$ are dense in A, and one has:

(7.10) $$\omega_j \omega_k + \omega_k \omega_j = \omega'_j \omega'_k + \omega'_k \omega'_j = 2\,\delta_{kj} \cdot 1$$

Let H_ω (resp. $H_{\omega'}$) be the vector space generated by the ω_j's (resp. ω'_j's) and let $U : H_\omega \to H_\omega$ be an operator such that

(7.11) $$(U\omega_j)(U\omega_k) + (U\omega_k)(U\omega_j) = 2\,\delta_{kj}$$

then, identifying H_ω with $H_{\omega'}$, we can let U act on $H_{\omega'}$, and the map

$$\omega_m \cdot \omega'_n \to (U\omega_n)\cdot(U\omega'_m)$$

has a unique extension to an automorphism of A-denoted α_U.

The family of these α_U form a group, denoted G. A state on A invariant under G is called an universally invariant state.

Let now $\Omega = \Pi_{\mathbb{N}}\{0,1\}$ with the product σ-algebra θ. Then $\mathcal{D} = L^\infty(\Omega,\theta) \cong \otimes_{\mathbb{N}} L^\infty(\{0,1\}) \cong \otimes_{\mathbb{N}}$ {diagonal 2 x 2 matrices} can be identified to a sub-algebra of A and there is a natural conditional expectation $E : A \to \mathcal{D}$ (which maps any matrix into its diagonal part). The result of Shale and Stinespring [26] can be formulated as follows (cf. [29]).

Theorem (7.2). A state ρ on A is universally invariant if and only if it has the form

(7.12) $$\rho = \rho_0 \cdot E$$

where ρ_0 is an S_∞-invariant state on $\mathcal{D} \cong L^\infty(\Omega,\theta)$. In particular the extremal universal invariant states are exactly those for which ρ_0 is an exchangeable measure on $\Pi_{\mathbb{N}}\{0,1\}$.

Let us conclude with the remark that recently also de Finetti's principle of exchangeability, which formed the philosophical background and the conceptual motivation of de Finetti's theorem, has been introduced in the debate concerning the foundations of quantum theory (more specifically: the problem of hidden variables) in the paper [32] of P. Suppes and M. Zanotti.

REFERENCES

[1] Accardi, L., On the significance of Dobrushin's theory for sufficient statistics, Communication to the meeting Ottimizzazione stocastica, Milano (1974)

[2] Accardi, L., Local perturbation of Conditional Expectations, J. of Math. Analysis and Applic. (1979) 34-69.

[3] Barra, J.-R., Notions fondamentales de statistique mathematique (Dunod, Paris, 1971).

[4] Campanino, M. and Spizzichino, F., Prediction Sufficiency and representation of infinite sequences of exchangeable random variables, Istituto Matematico "G. Castelnuovo", U. Roma (March 1981).

[5] Cifarelli, D.M. and Regazzini, E., Sul ruolo dei riassunti esaustivi ai fini della previsione in contesto bayesiano (1^ parte), Riv. di Mat.p.Sc.Ec. e Soc. 3 (1980) 109-125.

[6] Cifarelli, D. and Regazzini, E., Sul ruolo dei riassunti esaustivi ai fini della previsione in contesto bayesiano (2^ parte) Riv. MASES.

[7] Dawid A.P., Invariant distributions and Analysis of Variance Models, Biometrica 64 (1977) 291-297.

[8] Dawid A.P., Extendibility of Spherical Matrix Distributions, J. Mult. Anal. 8 (1978) 559-566.

[9] Dawid A.P., Conditional Indipendence in Statistical Theory, J.R. Stat. Soc. 41(1979) 1-31.

[10] Dawid A.P., Conditional Indipendence for Statistical Operations, Ann. of Stat. (1980) 598-617.

[11] Dobrushin, R.L., Description of a random field by means of conditional probabilities and the condition of its regularity, Theory Probability and Applications 13 (1968) 201-229.

[12] Dobrushin, R.L., Funktsionalni. Analizi Ego Prilozheniya 2(1968) 31-43 Funct. Anal. Appl. 2 (1968) 44-57.

[13] Dynkin, E.B., Sufficient statistics and extreme points, Ann. Prob. 6 (1978) 705-730.

[14] Hewith, E. and Savage, L.J., Symmetric measures on cartesian products, Trans. Amer. Math. Soc. 80 (1955), 470-401.

[15] Hudson, R.L. and Moody, G.R., Locally Normal Symmetric States and an Analogue of de Finetti's Theorem, Zeitschr. fur Wahrscheinlich 33 (1976) 343-351.

[16] Hudson, R.L., Analogs of de Finetti's Theorem and Interpretative Problems in Quantum Mechanics, in : Foundations of Physics, October 1981.

[17] Lauritzen S.L., Sufficiency, Prediction and extreme models, Scand. J.Statist. 1 (1974) 128-134.

[18] Lauritzen, S.L., Examples of Extreme point Models in Statistics, paper presented to the 13th European Meeting of Statisticians, Institute of Math.Stat. U. Copenhagen (November 1980).

[19] Levy, P., A Special Problem of Brownian Motion and a General Theory of Gaussian Random Functions, in: Proc. 3rd Berkeley Sympos. Math. Stat. and Probability 1954-'55, vol. II (Univ. California Press, Berkeley, Calif. 1956) 133-175.

[20] Loève, M., Probability Theory, 3rd ed. (Van Nostrand, Princeton, 1963).

[21] Mc Kean, H.P. Jr., Brownian Motion with Several-Dimensional Time, Teor. Verojatnost. i Primenen 8 (1963) 357-378 (in russian), Theor. Probability Appl. 8 (1963).

[22] Massager, A., Miracle-Sole, S., Equilibrium states of the two-dimensional Ising model in the two-phase region, Commun. Math.Phys. 40 (1975), 187-196.

[23] Moy, S.T.C., Characterization of Conditional Expectation as a Transformation on Function Spaces, Pacif J. Math. 4 (1954) 47-64.

[24] Mumford, D., Hilbert's fourteenth problem- the finite generation of sub-rings such as rings of invariants, in: Browder, F.E. (ed.) Mathematical Developments arising from Hilbert Problems (AMS, Providence, 1976).

[25] Nelson, E., The free Markov field, J. Functional Analysis 12 (1973) 211-227.

[26] D. Shale, D. and Stinespring, W.F., States of the Clifford Algebra, Annals of Math. 80 (1964) 365-381.

[27] Spitzer, F., Phase transitions in One-dimensional nearest-neighbor systems, J. of Funct. Anal.20 (1975) 240-255.

[28] Spizzichino, F., Statistiche sufficienti per numeri aleatori scambiabili, in: Atti del Convegno "Induzione, Probabilità, Statistica", Venezia 1978, 335-355.

[29] Stromer, E., Symmetric states on infinite tensor products of C*-algebras, J. Funct. Anal.3 (1969) 48-68.

[30] Tomijama, J., On the projection of norm one in w*-algebras, Proc. Japan Acad. 33 (1957) 608-612.

[31] Urbanik, K., Extreme point methods in probability theory, in: Winter School on Probability, Karpacy (1975).

[32] P. Suppes, M. Zanotti, A new proof of the impossibility of hidden variables using the principle of exchangeability and identity of conditional distributions. In: Studies in the Foundations of Quantum Mechanics. P. Suppes (ed.), Philosophy of Science Association (1980).

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EXCHANGEABILITY IN PROBABILITY AND STATISTICS
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ON THE STRUCTURE AND APPLICATIONS OF RESTRICTED EXCHANGEABILITY

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The structure of complete and restricted exchangeability with some applications is discussed in a general setting. By using extended versions of the probability integral transforms under various structures, the construction of hypotheses-generating groups are investigated.
Based on the previous development, the structure of distribution-free hypotheses testing is also given. Finally, some related points are discussed. The note in essence gives an overview of restricted exchangeability and its usefulness.

1. INTRODUCTION AND PRELIMINARIES

At the occasion of the International Congress of Mathematicians, Toronto, Jules Hagg [15] discussed exchangeable (or symmetric) sequences of random variables. He dealt with only two-valued random variables. For this case he hints at, but does not rigorously state or prove, the representation theorem. Later on De Finetti [9, 10] in his celebrated papers independently extends and proves results in great generality for exchangeable random variables. One of De Finetti's well-known theorems states that a process is a mixture of sequences of independent, identically distributed random variables, if and only if, it is summarised by the order statistics; in other words, if and only if, two finite sequences with the same order statistics are assigned the same probability.

De Finetti's work gave a tremendous impetus to other researchers and consequently a large amount of work has been done in this area. In particular, basic structural results, limit theorems and path properties of such processes were investigated by Hewitt-Savage [16], Freedman [14] Kendall [18], Blum et al. [6], Chatterji [7], Dacunha-Castelle [8], Aldous [5], Eagleson-Weber [13] and Kallenberg [17] among others. There are many applications of exchangeability and its extended versions, for example besides the above mentioned literature see Takács [21], Ahmad [3] and Kingman [19]. The implication of exchangeability in Bayesian statistics and its basic role in parametric and nonparametric statistical inference is given by De Finetti [11,12], Ahmad [1,2,3] and Ahmad and Peterson [4] among many others.

The distribution F on R^p is defined as totally symmetric if almost surely

$$F(x_1,\ldots,x_p) = F(\pi(x)) = F(x_{\pi(1)},\ldots,x_{\pi(p)}) \qquad (1)$$