Bulg. J. Phys. **36** (2009) 1–16

Renormalized Powers of Quantum White Noise*

L. Accardi[1], **A. Boukas**[2]

[1]Centro Vito Volterra, Università di Roma Tor Vergata, via Columbia 2, 00133 Roma, Italy

[2]Department of Mathematics and Natural Sciences, American College of Greece, Aghia Paraskevi, Athens 15342, Greece

Abstract. Giving meaning to the powers of the creation and annihilation densities (quantum white noise) is an old and important problem in quantum field theory. In this paper we present an account of some new ideas that have recently emerged in the attempt to solve this problem. We emphasize the connection between the Lie algebra of the renormalized higher powers of quantum white noise (RHPWN), which can be interpreted as a suitably deformed (due to renormalization) current algebra over the 1–mode full oscillator algebra, and the current algebra over the centerless Virasoro (or Witt)-Zamolodchikov-w_∞ Lie algebras of conformal field theory. Through a suitable definition of the action on the vacuum vector we describe how to obtain a Fock representation of all these algebras. We prove that the restriction of the vacuum to the abelian subalgebra generated by the field operators gives an infinitely divisible process whose marginal distribution is the beta (or continuous binomial).

PACS number: 03.70.+k; 11.10.-z

1 Introduction

The Hida (quantum) white noise functionals $a_t^\dagger$ (creation density) and a_t (annihilation density) satisfy the Boson commutation relations

$$[a_t, a_s^\dagger] = \delta(t-s), \quad [a_t^\dagger, a_s^\dagger] = [a_t, a_s] = 0, \tag{1}$$

where $t, s \in \mathbb{R}$ and δ is the Dirac delta function, as well as the duality relation

$$(a_s)^* = a_s^\dagger \,. \tag{2}$$

*The paper was represented at the VII International Workshop on *Lie Theory and Its Applications in Physics* held in Varna, Bulgaria, 18-24 June 2007.

Here (and in what follows) $[x,y] := xy - yx$ is the usual operator commutator. For all $t, s \in \mathbb{R}$ and integers $n, k, N, K \geq 0$ we have (cf. [6])

$$[a_t^{\dagger^n} a_t^k, a_s^{\dagger^N} a_s^K] = \epsilon_{k,0}\, \epsilon_{N,0} \sum_{L\geq 1} \binom{k}{L} N^{(L)}\, a_t^{\dagger^n}\, a_s^{\dagger^{N-L}}\, a_t^{k-L}\, a_s^K\, \delta^L(t-s)$$
$$- \epsilon_{K,0}\, \epsilon_{n,0} \sum_{L\geq 1} \binom{K}{L} n^{(L)}\, a_s^{\dagger^N}\, a_t^{\dagger^{n-L}}\, a_s^{K-L}\, a_t^k\, \delta^L(t-s), \quad (3)$$

where for $n, k \in \{0, 1, 2, ...\}$ we have used the notation $\epsilon_{n,k} := 1 - \delta_{n,k}$, where $\delta_{n,k}$ is Kronecker's delta and $x^{(y)} = x(x-1)\cdots(x-y+1)$ with $x^{(0)} = 1$. In order to consider the smeared fields defined by the higher powers of a_t and $a_t^\dagger$, for a test function f and $n, k \in \{0, 1, 2, ...\}$, we define the sesquilinear form

$$B_k^n(f) := \int_{\mathbb{R}} f(t)\, a_t^{\dagger^n}\, a_t^k\, dt \quad (4)$$

with involution

$$(B_k^n(f))^* = B_n^k(\bar{f}). \quad (5)$$

For $l = 2, 3, ...$, using the renormalization

$$\delta^l(t) = c^{\,l-1}\, \delta(t), \quad (6)$$

where $c > 0$ is an arbitrary real constant (cf. [6,8]), by multiplying both sides of (3) by test functions $f(t)\, g(s)$ and then formally integrating the resulting identity (*i.e.*, taking $\int\int\ \ldots\ dsdt$ of both sides) we obtain the Lie algebra commutation relations

$$[B_K^N(g), B_k^n(f)] = \sum_{L=1}^{(K\wedge n)\vee(k\wedge N)} \theta_L(N, K; n, k)\, c^{L-1}\, B_{K+k-L}^{N+n-L}(gf), \quad (7)$$

where, using the notation

$$\binom{y, z}{x} := \binom{y}{x} z^{(x)}, \quad (8)$$

we have

$$\theta_L(N, K; n, k) := H(L-1) \left(\epsilon_{K,0}\, \epsilon_{n,0} \binom{K, n}{L} - \epsilon_{k,0}\, \epsilon_{N,0} \binom{k, N}{L} \right), \quad (9)$$

where

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is the Heaviside function. Commutation relations (7) do not admit a common (*i.e.*, valid for all $n, k, N, K \in \{0, 1, 2, ...\}$) Fock space representation. The proof of the following theorem can be found in [2].

Theorem 1 *Let $n \geq 3$ and suppose that an operator $*$–Lie sub–algebra $\mathcal{L}$ of the renormalized higher powers of white noise (or RHPWN) Lie algebra (7) contains (the basic creator operator) B_0^n. Then $\mathcal{L}$ does not admit a Fock space representation.*

Theorem 1 leaves us with the following options (explored separately in Sections 2–4 below): (i) Further renormalize commutation relations (7) (ii) Look for a new renormalization for $\delta^l(t)$ (iii) Study the sub-algebras of the universal enveloping algebra (or 1-mode full oscillator algebra) FOA(1) generated by $\{a, a^\dagger\}$.

2 Further Renormalization of Commutation Relations

Intuitively the constant c appearing in the renormalization prescription (6) and in the commutation relations (7) is equal to $\delta(0)$. So we can think of c as a very large positive number. Moreover, the commutator $[B_K^N(\bar{g}), B_k^n(f)]$ of (7) is a polynomial in c of degree $(K\wedge n)\vee(k\wedge N)-1$. This justified the consideration in [5] of the truncation of (7) keeping a certain number of dominant c–terms. This approach led to Heisenberg–Weyl (CCR) and Renormalized Square of White Noise (RSWN) Lie algebra structures described in Proposition 1 below.

Definition: For $n, k, N, K \in \{0, 1, 2, ...\}$ with $(K \wedge n) \vee (k \wedge N) \geq 1$ and θ as in (9) we define the truncated commutators

$$[B_K^N(g), B_k^n(f)]_1 := \theta_{(K\wedge n)\vee(k\wedge N)}(N, K; n, k)\, c^{(K\wedge n)\vee(k\wedge N)-1}\, B_{K+k-(K\wedge n)\vee(k\wedge N)}^{N+n-(K\wedge n)\vee(k\wedge N)}(gf), \quad (10)$$

i.e., $[B_K^N(g), B_k^n(f)]_1$ is the leading term in the expansion of $[B_K^N(g), B_k^n(f)]$ as a polynomial in c, and

$$[B_K^N(g), B_k^n(f)]_2 := \theta_{(K\wedge n)\vee(k\wedge N)}(N, K; n, k)\, c^{(K\wedge n)\vee(k\wedge N)-1} B_{K+k-(K\wedge n)\vee(k\wedge N)}^{N+n-(K\wedge n)\vee(k\wedge N)}(gf) + \theta_{(K\wedge n)\vee(k\wedge N)-1}(N, K; n, k) c^{(K\wedge n)\vee(k\wedge N)-2} B_{K+k-(K\wedge n)\vee(k\wedge N)+1}^{N+n-(K\wedge n)\vee(k\wedge N)+1}(gf), \quad (11)$$

i.e., $[B_K^N(g), B_k^n(f)]_2$ is the sum of the two leading terms in the expansion of $[B_K^N(g), B_k^n(f)]$ as a polynomial in c.

In general, the truncated commutators $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ do not satisfy the conditions of a Lie algebra commutator. However, the following Proposition was proved in [5].

Proposition 1 *(i) For $n \geq 1$ and $1 \leq k \leq n$, $B_0^n(\cdot)$, $B_n^0(\cdot)$, and $B_k^k(\cdot)$ generate a Heisenberg-Weyl type Lie–algebra with respect to $[\cdot,\cdot]_1$, i.e.,*

$$[B_n^0(g), B_0^n(f)]_1 \,=\, n!\, c^{n-1} \int_{\mathbb{R}} g(t)\, f(t)\, dt \tag{12}$$

$$[B_k^k(f), B_0^n(g)]_1 \,=\, n^{(k)}\, c^{k-1}\, B_0^n(fg) \tag{13}$$

and

$$[B_n^0(g), B_k^k(f)]_1 = n^{(k)}\, c^{k-1}\, B_n^0(gf) \tag{14}$$

(ii) For $n \geq 2$, $B_n^0(\cdot)$, $B_0^n(\cdot)$, and $B_1^1(\cdot)$ generate a RSWN–type Lie algebra (cf. [8]) with respect to $[\cdot,\cdot]_2$, i.e.,

$$[B_n^0(g), B_0^n(f)]_2 \,=\, n! \left(c^{n-1} \int_{\mathbb{R}} g(t)\, f(t)\, dt + n\ c^{n-2}\, B_1^1(gf) \right) \tag{15}$$

$$[B_1^1(g), B_0^n(f)]_2 \,=\, n\, B_0^n(gf) \tag{16}$$

and

$$[B_n^0(f), B_1^1(g)]_2 = n\, B_n^0(fg). \tag{17}$$

3 A New Renormalization for $\delta^l(t)$

To overcome the negative result of Theorem 1 we introduce (see [1] and [2]) the convolution type renormalization

$$\delta^l(t-s) = \delta(s)\, \delta(t-s), \quad l = 2, 3, ... \tag{18}$$

Then (3) yields

$$\begin{aligned} [B_k^n(g), B_K^N(f)] &= (\epsilon_{k,0}\, \epsilon_{N,0}\, k\, N - \epsilon_{K,0}\, \epsilon_{n,0}\, K\, n)\ B_{K+k-1}^{N+n-1}(gf) \\ &+ \sum_{L=2}^{(K\wedge n)\vee(k\wedge N)} \theta_L(n,k;N,K)\, \bar{g}(0)\, f(0)\, {a_0^\dagger}^{N+n-l}\, a_0^{K+k-l}. \end{aligned} \tag{19}$$

We eliminate the singular terms ${a_0^\dagger}^{N+n-l}\, a_0^{K+k-l}$ from (19) by restricting to test functions f, g that satisfy $f(0) = g(0) = 0$. We then arrive to the following Definition of the RHPWN Lie–algebra commutator.

Definition: For right-continuous step functions f, g, such that $f(0) = g(0) = 0$, we define

$$[B_k^n(g), B_K^N(f)]_{RHPWN} := (k\, N - K\, n)\ B_{k+K-1}^{n+N-1}(gf). \tag{20}$$

We can think of (20) as the second-quantization of the commutation relations

$$[B_k^n, B_K^N]_{RHPWN} := (k\,N - K\,n)\; B_{k+K-1}^{n+N-1} \tag{21}$$

with involution

$$(B_k^n)^* = B_n^k, \tag{22}$$

which bear a striking similarity (except for the adjointness condition) with the commutation relations of the Virasoro-Zamolodchikov-w_∞ algebra defined in the following:

Definition: The centerless Virasoro (or Witt)-Zamolodchikov-w_∞ algebra of conformal field theory (quantum gravity, high energy physics, cf. [20]- [22]) is the infinite dimensional non-associative Lie algebra spanned by the generators $\hat{B}_k^n$, where $n, k \in \mathbb{Z}$ with $n \geq 2$, with commutation relations

$$[\hat{B}_k^n, \hat{B}_K^N]_{w_\infty} = (k\,(N-1) - K\,(n-1))\; \hat{B}_{k+K}^{n+N-2} \tag{23}$$

and adjoint condition

$$\left(\hat{B}_k^n\right)^* = \hat{B}_{-k}^n. \tag{24}$$

In particular,

$$[\hat{B}_k^2, \hat{B}_K^2]_{Vir} := (k-K)\,\hat{B}_{k+K}^2 \tag{25}$$

are the centerless Virasoro (or Witt) algebra commutation relations.

We remark that the centerless Virasoro (or Witt)-Zamolodchikov-w_∞ algebra is a special case of the atavistic Lie algebra of Fairlie and Zachos (cf. [14]).

Definition: For scalar-valued differentiable functions $f(x,y)$ and $g(x,y)$, the Poisson bracket $\{f,g\}$ is defined by

$$\{f,g\} := \frac{\partial f}{\partial x}\,\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\,\frac{\partial g}{\partial x}\,.$$

A Poisson–brackets representation of (21), (22) and (23), (24) is provided in the following Proposition whose proof can be found in [2].

Proposition 2

(a) For $n, k \in \mathbb{Z}$ with $n \geq 2$, let $f_{n,k} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be defined by

$$f_{n,k}(x,y) = e^{ikx}\, y^{n-1}. \tag{26}$$

Then

$$\{f_{n,k}(x,y), f_{N,K}(x,y)\} = i(k(N-1) - K(n-1))f_{n+N-2,k+K}(x,y) \tag{27}$$

and

$$\overline{f}_{n,k}(x,y) = f_{n,-k}(x,y) \tag{28}$$

(b) For $n,k \geq 0$, let $g_{n,k} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be defined by

$$g_{n,k}(x,y) = \left(\frac{x+iy}{\sqrt{2}} \right)^n \left(\frac{x-iy}{\sqrt{2}} \right)^k . \tag{29}$$

Then

$$\{g_{n,k}(x,y), g_{N,K}(x,y)\} = i\ (kN - nK)\ g_{n+N-1,k+K-1}(x,y) \tag{30}$$

and

$$\overline{g}_{n,k}(x,y) = g_{k,n}(x,y). \tag{31}$$

The following Definition provides the white noise form of the w_∞ generators.

Definition: For right-continuous step functions f, g such that $f(0) = g(0) = 0$ and for $n, k \in \mathbb{Z}$ with $n \geq 2$, we define

$$\hat{B}^n_k(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(a_t - a_t^\dagger)} \left(\frac{a_t + a_t^\dagger}{2} \right)^{n-1} e^{\frac{k}{2}(a_t - a_t^\dagger)}\, dt \tag{32}$$

with involution

$$\left(\hat{B}^n_k(f) \right)^* = \hat{B}^n_{-k}(\bar{f}). \tag{33}$$

In particular,

$$\hat{B}^2_k(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(a_t - a_t^\dagger)} \left(\frac{a_t + a_t^\dagger}{2} \right) e^{\frac{k}{2}(a_t - a_t^\dagger)}\, dt \tag{34}$$

is the RHPWN form of the Virasoro operators. We call $\hat{B}^n_k(f)$ the second quantized version of $\hat{B}^n_k$.

The integral on the right hand side of (32) is meant in the sense that one expands the exponential series, applies the commutation relations (20) to bring the resulting expression to normal order, introduces the renormalization prescription (18), integrates the resulting expressions after multiplication by a test function, and interprets the result as a quadratic form on the exponential vectors. Moreover, we may analytically continue the parameter k in the definition of $\hat{B}^n_k(f)$ to an arbitrary complex number $k \in \mathbb{C}$ and to $n \geq 1$. A detailed proof of Theorems 2 and 3 below can be found in [1], [2], and [3]. Theorem 3 provides the expression of the generators of the analytic continuation of the second quantized w_∞ and of the RHPWN Lie algebras in terms of each other.

Theorem 2 *If f, g are right-continuous step functions such that $f(0)=g(0)=0$ and the powers of the delta function are renormalized by the prescription (18) then*

$$[\hat{B}^n_k(g), \hat{B}^N_K(f)] = (k\,(N-1) - K\,(n-1))\; \hat{B}^{n+N-2}_{k+K}(g\,f), \qquad (35)$$

i.e., the operators $\hat{B}^n_k$ of Definition 3 satisfy the commutation relations of the w_∞ algebra. In particular,

$$[\hat{B}^2_k(g), \hat{B}^2_K(f)] = (k - K)\; \hat{B}^2_{k+K}(g\,f), \qquad (36)$$

i.e., the operators $\hat{B}^2_k$ of Definition 3 satisfy the commutation relations of the Virasoro algebra.

Theorem 3 *(i) Let $n \geq 2$ and $k \in \mathbb{Z}$. Then for all right-continuous step functions f such that $f(0) = 0$*

$$\hat{B}^n_k(f) = \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^p \, \frac{k^{p+q}}{p!\,q!} \, B^{m+p}_{n-1-m+q}(f) \qquad (37)$$

(ii) If $n, k \in \{0, 1, 2, ...\}$ then

$$B^n_k(f) = \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} \binom{k}{\rho} \binom{n}{\sigma} \frac{(-1)^\rho}{2^{\rho+\sigma}} \, \frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}|_{z=0} \, \hat{B}^{k+n+1-(\rho+\sigma)}_z(f). \qquad (38)$$

4 Lie $*$–Sub–Algebras of the 1-mode Full Oscillator Algebra FOA(1)

The program of studying sub-algebras of the (normally ordered) universal enveloping algebra (or 1-mode full oscillator algebra) FOA(1) generated by $\{a, a^\dagger\}$, where $[a, a^\dagger] = 1$, was initiated in [5] where the following Proposition was proved.

Proposition 3 *For integers $n, k \in \{0, 1, 2, ...\}$ define*

$$A_k(n) := N^n\, a^k \qquad , \qquad A_k(n)^\dagger := (a^\dagger)^k\, N^n, \qquad (39)$$

where $N := a^\dagger\, a$. Then for all $\gamma, \gamma', n, k \in \{0, 1, 2, ...\}$

$$[A_k(\gamma)^\dagger, A_k(\gamma')] = \sum_{\alpha=0}^{\gamma} \sum_{\beta=0}^{\gamma'} S_{\gamma,\alpha}\, S_{\gamma',\beta}\, \{\epsilon_{\alpha,0}\, \epsilon_{\beta,0} \sum_{l \geq 1} \binom{\alpha}{l} \beta^{(l)} \sum_{m=0}^{\alpha+\beta+k-l} s_{\alpha+\beta+k-l,m}\, N^m$$
$$- \sum_{L \geq 1} \binom{\beta+k}{L} (\alpha+k)^{(L)} \sum_{m'=0}^{\alpha+\beta+k-L} s_{\alpha+\beta+k-L,m'}\, N^{m'}\} \qquad (40)$$

$$[A_k(\gamma), N^n] = \sum_{m=0}^{n} S_{n,m}\, \epsilon_{m,0} \sum_{l\geq 1} \binom{k}{l} m^{(l)} \sum_{w=0}^{m-l} s_{m-l,w}\, A_k(w+\gamma) \tag{41}$$

$$[N^n, A_k(\gamma)^\dagger] = \sum_{m=0}^{n} S_{n,m}\, \epsilon_{m,0} \sum_{l\geq 1} \binom{k}{l} m^{(l)} \sum_{w=0}^{m-l} s_{m-l,w}\, A_k(w+\gamma)^\dagger \tag{42}$$

$$[A_k(\gamma), A_k(\gamma')] = \sum_{m'=0}^{\gamma'} S_{\gamma',m'}\, \epsilon_{m',0} \sum_{l\geq 1} \binom{k}{l} m'^{(l)} \sum_{\lambda'=0}^{m'-l} s_{m'-l,\lambda'}\, A_{2k}(\gamma+\lambda') - \sum_{m=0}^{\gamma} S_{\gamma,m}\, \epsilon_{m,0} \sum_{L\geq 1} \binom{k}{L} m^{(L)} \sum_{\lambda=0}^{m-L} s_{m-L,\lambda}\, A_{2k}(\gamma'+\lambda) \tag{43}$$

$$[A_k(\gamma')^\dagger, A_k(\gamma)^\dagger] = \sum_{m'=0}^{\gamma'} S_{\gamma',m'}\, \epsilon_{m',0} \sum_{l\geq 1} \binom{k}{l} m'^{(l)} \sum_{\lambda'=0}^{m'-l} s_{m'-l,\lambda'}\, A_{2k}(\gamma+\lambda')^\dagger - \sum_{m=0}^{\gamma} S_{\gamma,m}\, \epsilon_{m,0} \sum_{L\geq 1} \binom{k}{L} m^{(L)} \sum_{\lambda=0}^{m-L} s_{m-L,\lambda}\, A_{2k}(\gamma'+\lambda)^\dagger \tag{44}$$

and

$$[N^\gamma, N^n] = 0, \tag{45}$$

where $s_{n,k}$ and $S_{n,k}$ are the Stirling numbers of the first and second kind respectively, with $s_{0,0} = S_{0,0} = 1$ and $s_{0,k} = s_{n,0} = S_{0,k} = S_{n,0} = 0$ for all $n, k \geq 1$. Moreover, for fixed $k \in \mathbb{N}$, the $*$–linear subspace $\Lambda(k)$ of the FOA (1) generated by the set $\{N^m, A_{2^\alpha k}(n), A^\dagger_{2^{\alpha_k}}(n) : m, n, \alpha \in \mathbb{N}\}$ is a $*$–Lie algebra with structure constants given by (40),..., (45).

To study the current $*$–Lie algebra corresponding to the $*$–Lie algebra $\mathcal{L}(k)$ of Proposition 3 and the renormalization prescription (18) we proceed as follows.

Proposition 4 *For $t, s \in \mathbb{R}$ and $n \in \{0, 1, 2, ...\}$*

$$(a_t^\dagger)^n\, (a_s)^n = \sum_{k=0}^{n} s_{n,k}\, (a_t^\dagger\, a_s)^k\, \delta^{n-k}(t-s) \tag{46}$$

and

$$(a_t^\dagger\, a_s)^n = \sum_{k=0}^{n} S_{n,k}\, (a_t^\dagger)^k\, (\, a_s)^k\, \delta^{n-k}(t-s), \tag{47}$$

where $s_{n,k}$ and $S_{n,k}$ are as in Proposition 3 and $\delta^0(t-s) := 1$.

Proof: It is well known in the literature (see *e.g.* [19]) that if $[b, b^\dagger] = 1$ then for $n \geq 0$

$$(b^\dagger\, b)^n = \sum_{k=0}^{n} S_{n,k}\, (b^\dagger)^k\, (b)^k \tag{48}$$

and

$$(b^\dagger)^n\, (b)^n = \sum_{k=0}^{n} s_{n,k}\, (b^\dagger\, b)^k. \tag{49}$$

For fixed $t, s \in \mathbb{R}$ formally define

$$b^\dagger := \frac{a_t^\dagger}{\delta(t-s)^{1/2}}, \quad b := \frac{a_s}{\delta(t-s)^{1/2}}. \tag{50}$$

Then $[b, b^\dagger] = 1$ and so (48) implies

$$(a_t^\dagger\, a_s)^n\, \delta(t-s)^{-n} = \sum_{k=0}^{n} S_{n,k}\, (a_t^\dagger)^k\, (a_s)^k\, \delta(t-s)^{-k} \tag{51}$$

from which we obtain (47). The proof of (46) is similar.

Definition: For a test function f as in Definition 3 and for integers $n, k \geq 0$ we define

$$A_k^n(f) := \int_{\mathbb{R}} f(t)\, N_t^n\, a_t^k\, dt \tag{52}$$

$$A_k^n(f)^\dagger := \int_{\mathbb{R}} f(t)\, {a_t^\dagger}^k\, N_t^n\, dt \tag{53}$$

$$N^n(f) := \int_{\mathbb{R}} f(t)\, N_t^n\, dt\,, \tag{54}$$

where $N_t := a_t^\dagger\, a_t$. Notice that by (39) and (2)

$$(A_k^n(f))^* = A_k^n(\bar{f})^\dagger \;\; ; \;\; (N^n(f))^* = N^n(\bar{f}) \tag{55}$$

while for $k = 0$ we have

$$A_0^n(f) = A_0^n(f)^\dagger = N^n(f). \tag{56}$$

Proposition 5 *With the renormalization prescription (18), for all test functions f as in Definition 3 and for all $n, k \geq 0$*

$$B_k^n(f) = \begin{cases} A_{n-k}^k(f)^\dagger & \text{if } n \geq k \\ A_{k-n}^n(f) & \text{if } n < k \end{cases} \;. \tag{57}$$

In particular, for $k = n$ using (56) we obtain

$$B_n^n(f) = N^n(f) \tag{58}$$

while for $k = 0$ and $n \geq 0$ we have

$$B_0^n(f) = A_n^0(f)^{\dagger} \tag{59}$$

whose adjoint is

$$B_n^0(f) = A_n^0(f). \tag{60}$$

Proof: Let f be a test function and let $n \geq k$. By Proposition 4

$$f(t)\,(a_t^{\dagger})^n\,(a_s)^k\,\delta(t-s) = f(t)\,(a_t^{\dagger})^{n-k}\,(a_t^{\dagger})^k\,(a_s)^k\,\delta(t-s)$$
$$= \sum_{m=0}^{k} s_{k,m}\,f(t)\,(a_t^{\dagger})^{n-k}\,(a_t^{\dagger}\,a_s)^m\,\delta^{k-m+1}(t-s). \tag{61}$$

Taking $\int_{\mathbb{R}}\int_{\mathbb{R}} \ldots\, dsdt$ of both sides of (61) and using the fact that by the renormalization prescription (18) and the condition that f vanishes at zero only the $m = k$ term on the right hand side of (61) survives, we obtain (57) for $n \geq k$. If $n < k$, then

$$B_k^n(f) = (B_n^k(\overline{f}))^* = (A_{k-n}^n(\bar{f})^{\dagger})^* = A_{k-n}^n(f). \tag{62}$$

Proposition 6 *With the renormalization prescription (18), for all test functions f as in Definition 3 and for all integers $n, k \geq 0$*

$$A_k^n(f) = B_{n+k}^n(f) \tag{63}$$

and

$$A_k^n(f)^{\dagger} = B_n^{n+k}(f). \tag{64}$$

Proof: By Proposition 5

$$A_k^n(f) = A_{(k+n)-n}^n(f) = B_{n+k}^n(f) \tag{65}$$

which is (63). Equation (64) is the adjoint of (65).

The current $*$–Lie algebra corresponding to the $*$–Lie algebra $\mathcal{L}(k)$ of Proposition 3 and the renormalization prescription (18) is described in the following.

Proposition 7 *With the renormalization prescription (18), for all test functions f and g as in Definition 3 and for all integers $k, k', \gamma, \gamma' \geq 0$*

$$[A_{k'}^{\gamma'}(g), A_k^{\gamma}(f)^{\dagger}] = \begin{cases} (k'\gamma + k\gamma' + k\,k')A_{k-k'}^{\gamma+\gamma'+k'-1}(fg)^{\dagger} & \text{if } k \geq k' \\ (k'\gamma + k\gamma' + k\,k')A_{k'-k}^{\gamma+\gamma'+k-1}(fg) & \text{if } k \leq k' \end{cases} \tag{66}$$

and

$$[A_{k'}^{\gamma'}(g), N^{\gamma}(f)] = k'\,\gamma\,A_{k'}^{\gamma+\gamma'-1}(f\,g) \tag{67}$$

$$[N^{\gamma'}(g), A_k^{\gamma}(f)^{\dagger}] = k\,\gamma'\,A_k^{\gamma+\gamma'-1}(f\,g)^{\dagger} \tag{68}$$

$$[A_k^{\gamma}(f), A_{k'}^{\gamma'}(g)] = (k\,\gamma' - \gamma\,k')\,A_{k+k'}^{\gamma+\gamma'-1}(f\,g) \tag{69}$$

$$[A_{k'}^{\gamma'}(f)^{\dagger}, A_k^{\gamma}(g)^{\dagger}] = (k\,\gamma' - \gamma\,k')\,A_{k+k'}^{\gamma+\gamma'-1}(f\,g)^{\dagger} \tag{70}$$

$$[N^{\gamma}(f), N^{\gamma'}(g)] = 0\,. \tag{71}$$

In particular, for $k = k'$ (66), (69) and (70) reduce to

$$[A_k^{\gamma'}(g), A_k^{\gamma}(f)^{\dagger}] = k\,(\gamma + \gamma' + k)\,N^{\gamma+\gamma'+k-1}(f\,g) \tag{72}$$

$$[A_k^{\gamma}(f), A_k^{\gamma'}(g)] = k\,(\gamma' - \gamma)\,A_{2k}^{\gamma+\gamma'-1}(f\,g) \tag{73}$$

$$[A_k^{\gamma'}(f)^{\dagger}, A_k^{\gamma}(g)^{\dagger}] = k\,(\gamma' - \gamma)\,A_{2k}^{\gamma+\gamma'-1}(f\,g)^{\dagger}\,. \tag{74}$$

Proof: We will use Propositions 5 and 6. For $k \geq k'$ we have

$$\begin{aligned}[A_{k'}^{\gamma'}(g), A_k^{\gamma}(f)^{\dagger}] &= [B_{\gamma'+k'}^{\gamma'}(g), B_{\gamma}^{\gamma+k}(f)]_{RHPWN} \\ &= ((\gamma' + k')\,(\gamma + k) - \gamma\,\gamma')\,B_{\gamma+\gamma'+k'-1}^{\gamma+\gamma'+k-1}(f\,g) \\ &= (k'\,\gamma + k\,\gamma' + k\,k')\,A_{k-k'}^{\gamma+\gamma'+k'-1}(f\,g)^{\dagger}\end{aligned} \tag{75}$$

which is the first half of (66). The second half of (66) is proved similarly. To prove (67) we have

$$\begin{aligned}[A_{k'}^{\gamma'}(g), N^{\gamma}(f)] &= [B_{\gamma'+k'}^{\gamma'}(g), B_{\gamma}^{\gamma}(f)]_{RHPWN} \\ &= ((\gamma' + k')\,\gamma - \gamma\,\gamma')\,B_{\gamma+\gamma'+k'-1}^{\gamma+\gamma'-1}(f\,g) \\ &= k'\,\gamma\,A_{k'}^{\gamma+\gamma'-1}(f\,g)\,.\end{aligned} \tag{76}$$

Equation (68) is the adjoint of (67). For (69) we have

$$\begin{aligned}[A_k^{\gamma}(g), A_{k'}^{\gamma'}(f)] &= [B_{\gamma+k}^{\gamma}(g), B_{\gamma'+k'}^{\gamma'}(f)]_{RHPWN} \\ &= (\gamma'(\gamma + k) - (\gamma' + k')\,\gamma)\,B_{\gamma+\gamma'+k+k'-1}^{\gamma+\gamma'-1}(f\,g) \\ &= (k\,\gamma' - \gamma\,k')\,A_{k+k'}^{\gamma+\gamma'-1}(f\,g)\,.\end{aligned} \tag{77}$$

Equation (70) is the adjoint of (69) and equation (71) follows from (66) for $k = k' = 0$.

In view of Propositions 5, 6 and 7, the study of the Fock representation of the RHPWN Lie algebra of Definition 3 described in the next section, applies equally well to the Lie algebras generated by the operators of Definition 4.

Note: For anti–normally ordered generators, in the 1-mode case $[a, a^\dagger] = 1$ we have

$$a^n (a^\dagger)^m = \begin{cases} a^n (a^\dagger)^n a^{\dagger (m-n)} \,, & \text{if} \quad m \geq n \\ a^{n-m} a^m (a^\dagger)^m \,, & \text{if} \quad n \geq m. \end{cases} \tag{78}$$

This leads to the class of generators

$$B_k(n) := a^k N_a^n;\; B_k^\dagger(n) = N_a^n (a^\dagger)^k;\; N_a = a a^\dagger;\; k, n \in \{0, 1, 2, ...\}. \tag{79}$$

We have

$$B_k(n) = \sum_{h=0}^{n} \binom{n}{h} a^k N^h = \sum_{h=0}^{n} \binom{n}{h} A_k(h). \tag{80}$$

A similar expression can be also obtained for the $A_k(h)$. The following proposition (which gives the well known oscillator representation of the Virasoro algebra and whose proof can be found in [13]), shows that inside $\Lambda(1)$ (see Definition 3) there are strictly smaller Lie sub–algebras. It is not clear if there are strictly smaller (*i.e.*, not obtained by restricting the powers of the number operator to be larger than a fixed number) $*$–Lie sub–algebras.

Proposition 8 *For $m \in \mathbb{N}$, let*

$$L_m := \frac{1}{\sqrt{2}} a^{2m+1} a^\dagger = \frac{1}{\sqrt{2}} a^{2m} N_a. \tag{81}$$

Then, the linear space generated by the set

$$\{L_m : m \in \mathbb{N}\}$$

is a Lie subalgebra of the FOA (1), isomorphic to the Virasoro algebra:

$$[L_n, L_m] = (n - m) L_{n+m}.$$

5 Truncated Fock Representation of the n-th Order RHPWN $*$–Lie Algebras $\mathcal{L}_n$

Definition: For $n \geq 1$ we define the n-th order RHPWN $*$–Lie algebras $\mathcal{L}_n$ as follows: (i) $\mathcal{L}_1$ is the $*$–Lie algebra generated by B_0^1 and B_1^0, *i.e.*, $\mathcal{L}_1$ is the linear span of $\{B_0^1, B_1^0, B_0^0\}$. (ii) $\mathcal{L}_2$ is the $*$–Lie algebra generated by B_0^2 and B_2^0, *i.e.*, $\mathcal{L}_2$ is the linear span of $\{B_0^2, B_2^0, B_1^1\}$. (iii) For $n \in \{3, 4, ...\}$, $\mathcal{L}_n$ is the $*$–Lie algebra generated by B_0^n and B_n^0 through repeated commutations and linear combinations. It consists of linear combinations of creation/annihilation operators of the form B_y^x, where $x - y = k\, n$, $k \in \mathbb{Z} - \{0\}$, and of number operators B_x^x with $x \geq n - 1$.

Through white noise and norm compatibility considerations, the action of the RHPWN operators on Φ was defined in [4] as follows:

Definition: For $n, k \in \mathbb{Z}$ and test functions f

$$B_k^n(f)\,\Phi := \begin{cases} 0 & \text{if } n < k \text{ or } n \cdot k < 0 \\ B_0^{n-k}(f)\,\Phi & \text{if } n > k \geq 0 \\ \frac{1}{n+1}\int_{\mathbb{R}} f(t)\,dt\,\Phi & \text{if } n = k\,. \end{cases} \tag{82}$$

Remark: In what follows, for all integers n, k we will use the notation $B_k^n := B_k^n(\chi_I)$, where I is some fixed subset of $\mathbb{R}$ of finite measure $\mu := \mu(I) > 0$.

Remark: For all $t \in [0, +\infty)$ and for all integers n, k we will use the notation $B_k^n(t) := B_k^n(\chi_{[0,t]})$.

It was shown in [4] that if the RHPWN action on Φ is that of Definition 5 then the Fock representation no-go theorems of [6] and [2] extend to the RHPWN $*$–Lie algebras $\mathcal{L}_n$, where $n \geq 3$. The generic element of the $*$-Lie algebras $\mathcal{L}_n$ of Definition 5 is B_0^n. All other elements of $\mathcal{L}_n$ are obtained by taking adjoints, commutators, and linear combinations. It thus makes sense to consider $(B_0^n(f))^k\,\Phi$ as basis vectors for the n-th particle space of the Fock space $\mathcal{F}_n$ associated with $\mathcal{L}_n$. A calculation of the Fock kernel $\langle (B_0^n)^k\,\Phi, (B_0^n)^k\,\Phi\rangle$ reveals that it is the terms containing $B_0^{2\,n}\,\Phi$ that prevent the kernel from being positive definite. The $B_0^{2\,n}\,\Phi$ terms appear either directly or by applying Definition 5 to terms of the form $B_y^x\,\Phi$ where $x - y = 2\,n$. In [4] the Fock kernels were computed by applying Definition 5 and by truncating singular terms through the following definition of the action of the principal $\mathcal{L}_n$ number operator B_{n-1}^{n-1} on the basis vectors $(B_0^n(f))^k\,\Phi$. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ do not respectively contain B_0^2 and B_0^4 , singular terms appear only for $n \geq 3$. The Fock spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ are therefore not truncated.

Definition: In the notation of Remark 5, for integers $n \geq 1$ and $k \geq 0$,

$$B_{n-1}^{n-1}\,(B_0^n)^k\,\Phi := \left(\frac{\mu}{n} + k\,n\,(n-1)\right)\,(B_0^n)^k\,\Phi\,, \tag{83}$$

i.e., the number vectors $(B_0^n)^k\,\Phi$ are eigenvectors of B_{n-1}^{n-1} (the principal number operator of $\mathcal{L}_n$) with eigenvalues $\left(\frac{\mu}{n} + k\,n\,(n-1)\right)$.

The proof of all Propositions presented in this Section can be found in [4].

Proposition 9 *For all* $k, n \geq 1$

$$\langle (B_0^n)^k\,\Phi, (B_0^n)^k\,\Phi\rangle = k!\,n^k\,\prod_{i=0}^{k-1}\left(\mu + \frac{n^2\,(n-1)}{2}\,i\right) \tag{84}$$

The $\mathcal{F}_n$ inner product $\langle \psi_n(f), \psi_n(g) \rangle_n$ of the exponential vectors

$$\psi_n(\phi) := \prod_i e^{a_i \, B_0^n(\chi_{I_i})} \, \Phi \, , \tag{85}$$

where $\phi := \sum_i a_i \, \chi_{I_i}$ is a test function, is therefore given by

$$\langle \psi_1(f), \psi_1(g) \rangle_1 := e^{\int_{\mathbb{R}} \bar{f}(t) \, g(t) \, dt} \tag{86}$$

and for $n \geq 2$

$$\langle \psi_n(f), \psi_n(g) \rangle_n := e^{-\frac{2}{n^2 \, (n-1)} \int_{\mathbb{R}} \ln\left(1 - \frac{n^3 \, (n-1)}{2} \, \bar{f}(t) \, g(t)\right) dt} \, , \tag{87}$$

where $|f(t)| < \frac{1}{n} \sqrt{\frac{2}{n \, (n-1)}}$ and $|g(t)| < \frac{1}{n} \sqrt{\frac{2}{n \, (n-1)}}$.

The Fock space inner product (86) is associated with the Heisenberg-Weyl algebra and the quantum stochastic calculus of [18]. For $n = 2$ the Fock space inner product (87) has appeared in the study of the Finite-Difference algebra and the Square of White Noise algebra in [9, 10, 12], and [15].

Definition: The n-th order truncated RHPWN (or TRHPWN) Fock space $\mathcal{F}_n$ is the Hilbert space completion of the linear span of the exponential vectors $\psi_n(f)$ under the inner product $\langle \cdot, \cdot \rangle_n$ of Proposition 9. The full TRHPWN Fock space $\mathcal{F}$ is the direct sum of the $\mathcal{F}_n$'s.

A Fock representation of the TRHPWN operators is given in the following:

Proposition 10 *For all test functions $f := \sum_i a_i \, \chi_{I_i}$ and $g := \sum_i b_i \, \chi_{I_i}$ with $I_i \cap I_j = \emptyset$ for $i \neq j$, and for all $n \geq 1$*

$$B_n^0(f)\psi_n(g) = n \int_{\mathbb{R}} f(t) g(t) dt \, \psi_n(g) + \frac{n^3(n-1)}{2} \frac{\partial}{\partial \epsilon}|_{\epsilon=0} \psi_n(g + \epsilon f g^2) \tag{88}$$

$$B_0^n(f)\psi_n(g) = \frac{\partial}{\partial \, \epsilon}|_{\epsilon=0} \, \psi_n(g + \epsilon \, f) \tag{89}$$

Moreover, for all $n \geq 1$ and test functions f, g, h

$$B_{n-1}^{n-1}(f \, g) \, \psi_n(h) = \frac{1}{n} \int_{\mathbb{R}} f(t) \, g(t) \; \psi_n(h) + \frac{n(n-1)}{2} \frac{\partial^2}{\partial \epsilon \partial \rho}|_{\epsilon=\rho=0} \left(\psi_n(h + \epsilon g + \rho f (h + \epsilon g)^2) - \psi_n(h + \epsilon f h^2 + \rho g) \right) \tag{90}$$

Using the prescription

$$B_{k+K-1}^{n+N-1}(gf) := \frac{1}{k\,N - K\,n}\,\left(B_k^n(g)\,B_K^N(f) - B_K^N(f)\,B_k^n(g)\right) \tag{91}$$

and suitable linear combinations, we obtain the representation of the B_y^x (and therefore of the RHPWN and centerless Virasoro (or Witt)–Zamolodchikov–w_∞ commutation relations) on the appropriate Fock space $\mathcal{F}_n$.

6 Classical Stochastic Processes on $\mathcal{F}_n$

Definition: A quantum stochastic process $x = \{x(t)\,/\,t \in [0,\infty)\}$ is a family of Hilbert space operators. Such a process is said to be classical if for all $t, s \in [0,\infty)$, $x(t) = x(t)^*$ and $[x(t), x(s)] := x(t)\,x(s) - x(s)\,x(t) = 0$.

The proof of Lemma 4 and Proposition 11 below can be found in [4].

Lemma 4 *For each $t \in [0,\infty)$ let $X(t)$ be a random variable with distribution given by the density*

$$p_t(x) := \frac{2^{t-1}}{2\,\pi}\,B(\frac{t+i\,x}{2}, \frac{t-i\,x}{2})\,, \tag{92}$$

where

$$B(a,c) := \frac{\Gamma(a)\,\Gamma(c)}{\Gamma(a+c)} = \int_0^1 x^{a-1}\,(1-x)^{c-1}\,dx \;\; ; \;\; \Re a > 0\,,\, \Re c > 0 \tag{93}$$

is the Beta function, and for $n \geq 2$ let

$$Y_n(t) := \sqrt{\frac{n^3\,(n-1)}{2}}\,X(t)\,. \tag{94}$$

Then the moment generating function of $Y_n(t)$ with respect to the density

$$q_t := p_{\frac{2\,n}{n^3\,(n-1)}\,t}\,, \tag{95}$$

where $n \in \{1, 2, ...\}$, is

$$\langle e^{s\,Y_n(t)} \rangle = \left(\sec\left(\sqrt{\frac{n^3\,(n-1)}{2}}\,s\right)\right)^{\frac{2\,n\,t}{n^3\,(n-1)}}\,. \tag{96}$$

Proposition 11 (Vacuum moment generating functions) *In the notation of remark 5, for all $s \in [0,\infty)$*

$$\langle e^{s\,(B_0^1(t)+B_1^0(t))}\,\Phi, \Phi\rangle_1 = e^{\frac{s^2}{2}\,t}\,, \tag{97}$$

i.e., $x_1(t) := B_0^1(t) + B_1^0(t)$ *is Brownian motion (cf. [16], [18]) while for* $n \geq 2$

$$\langle e^{s\,(B_0^n(t)+B_n^0(t))}\,\Phi,\Phi\rangle_n = \left(\sec\left(\sqrt{\frac{n^3\,(n-1)}{2}}\,s\right)\right)^{\frac{2\,n\,t}{n^3\,(n-1)}} , \qquad (98)$$

i.e., $x_n(t) := B_0^n(t) + B_n^0(t)$ *is for each* $n \geq 2$ *identified with the continuous binomial/Beta process* $Y_n(t)$ *of Lemma 4.*

References

[1] L. Accardi, A. Boukas (2006) *Inf. Dim. Anal. Quantum Probab. Rel. Topics* **9** 353-360.

[2] L. Accardi, A. Boukas (2006) *Int. J. Math. Comp. Sci.* **1** 315-342.

[3] L. Accardi, A. Boukas (2008) *Rep. Math. Phys.* **61** 1; hep-th/0610302.

[4] L. Accardi, A. Boukas (2008) *J. Phys. A: Math. Theor.* **41** 304001; arXiv:0706.3397v2 [math-ph].

[5] L. Accardi, A. Boukas (2007) *Commun. Stoch. Anal.* **1** 57-69.

[6] L. Accardi, A. Boukas and U. Franz (2006) *Inf. Dim. Anal. Quantum Probab. Rel. Topics* **9** 129-147.

[7] L. Accardi, U. Franz and M. Skeide (2002) *Commun. Math. Phys.* **228** 123-150.

[8] L. Accardi, Y.G. Lu and I.V. Volovich (1999) Lecture Notes of the Volterra International School on *White noise approach to classical and quantum stochastic calculi*, Trento, Italy, Volterra Center preprint 375.

[9] L. Accardi, M. Skeide (2000) *Math. Notes* **68** 683-694.

[10] L. Accardi, M. Skeide (2000) *Inf. Dim. Anal. Quantum Probab. Rel. Topics* **3** 185-189.

[11] I. Bakas, E.B. Kiritsis (1991) *Progr. Theor. Phys. Suppl.* **102** 15.

[12] A. Boukas (1991) *Mh. Math.* **112** 209-215.

[13] E.L. da Graca, H.L. Carrion and R. de Lima Rodrigues (2003) *Braz. J. Phys.* **33** 333-335.

[14] D.B. Fairlie, C.K. Zachos (2006) *Phys. Lett.* **B637** 123-127.

[15] P.J. Feinsilver (1987) *Mh. Math.* **104** 89-108.

[16] P.J. Feinsilver, R. Schott (1971) *Algebraic structures and operator calculus. Volumes I and III*, Wiley.

[17] W. Feller (1993) *Introduction to probability theory and its applications. Volumes I and II*, Kluwer.

[18] R. L. Hudson, K.R. Parthasarathy (1984) *Commun. Math. Phys.* **93** 301-323.

[19] P. Blasiak, K.A. Penson, A.I. Solomon (2003) *Phys. Lett.* **A309** 198-205.

[20] C.N. Pope, Lectures given at the Trieste Summer School in High-Energy Physics, August 1991.

[21] S.V. Ketov (1995) *Conformal Field Theory*, World Scientific.

[22] A.B. Zamolodchikov (1985) *Teor. Mat. Fiz.* **65** 347-359.