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QUANTUM STOCHASTIC PROCESSES

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Contents

1.) Quantum stochastic processes.

Let $\mathcal{B}$ be a $*$-algebra with identity (usually it wil be a C*- or a W*-algebra). A quantum stochastic process over $\mathcal{B}$ indexed by $\mathbb{R}$ is defined by a triple $\{\mathcal{A}, (j_t)_{t\in\mathbb{R}}, \varphi\}$ where

- $\mathcal{A}$ is a $*$-algebra with identity.
- $j_t : \mathcal{B} \hookrightarrow \mathcal{A}$ is an embedding ($t\in\mathbb{R}$).
- φ is a state on $\mathcal{A}$.

Example 1.) Classical real valued stochastic processes.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X_t : (\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$ ($t\in\mathbb{R}$) be a real valued stochastic process. By choosing

- $\mathcal{B} = L^{\infty}(\mathbb{R})$ = algebra of all complex valued, Borel-measurable functions on $\mathbb{R}$.
- $\mathcal{A} = L^{\infty}(\Omega, \mathcal{F}, P)$.
- $j_t : f\in\mathcal{B} \hookrightarrow j_t(f) = f\circ X_t = f(X_t)$; $(t\in\mathbb{R})$ (1.1
- $\varphi(a) = \int_{\Omega} a dP$; $a\in\mathcal{A}$.

The triple $\{\mathcal{A}, (j_t)_{t\in\mathbb{R}}, \varphi\}$ is a quantum stochastic process in the sense defined above. Conversely, one easily sees that to a given a quantum stochastic process $\{\mathcal{A}, (j_t), \varphi\}$ such that $\mathcal{A}$ is an abelian C*-algebra, one can associate a classical stochastic process, characterized (up to stochastic equivalence) by the property of having the same finite dimensional correlation functions as the initial one. Thus, since the quantum stochastic processes include the classical ones, in the following we shal only speak of stochastic processes.

Example 2.) (A "small" quantum system interacting with a "larger" one).

Let H_0 and F be two Hilbert spaces. One might regard H_0 as the

quantum state space of a "small system" interacting with an "extended system" with state space F (a typical situation is : $H_0 \cong \mathbb{C}^n$; F - some Fock space); in this case $H_0 \otimes F$ will be the state space of the "composite system". The evolution of the "composite system" is given by a 1-parameter group ($\mathcal{U}_t$) of unitary operators on $H_0 \otimes F$:

$$\mathcal{U}_t \in \mathcal{B}(H_0 \otimes F) \cong \mathcal{B}(H_0) \otimes \mathcal{B}(F) \tag{1.2}$$

and there is a natural embedding $j_0 : \mathcal{B}(H_0) \hookrightarrow \mathcal{B}(H_0) \otimes \mathcal{B}(F)$ of the algebra of the "small system" into the algebra of the composite system, given by :

$$j_0 : b \in \mathcal{B}(H_0) \hookrightarrow j_0(b) = b\otimes 1 \in \mathcal{B}(H_0)\otimes\mathcal{B}(F) \tag{1.3}$$

denoting, for each $t\in\mathbb{R}$ and $a\in\mathcal{B}(H_0)\otimes\mathcal{B}(F)$:

$$u_t(a) = \mathcal{U}_t \cdot a \cdot \mathcal{U}_t^+ \tag{1.4}$$

one can define, for each $t\in\mathbb{R}$, the embedding :

$$j_t : b\in\mathcal{B}(H_0) \hookrightarrow j_t(b) = u_t(j_0(b)) \in \mathcal{B}(H_0\otimes F) \tag{1.5}$$

Usually a state $\overline{\varphi}$ on $\mathcal{B}(H_0\otimes F)$ is given ($\overline{\varphi}\geq 0$ and $\overline{\varphi}(1_{H_0}\otimes 1_F) = 1$) and, if we are interested only in the time evolution of the "small system", then all the interesting physical quantities can be expressed in terms of the correlation functions :

$$\overline{\varphi}(j_{t_1}(b_1)\cdot\ldots\cdot j_{t_n}(b_n)) \tag{1.6}$$

where $b_j \in \mathcal{B}(H_0)$ $(j = 1,\ldots,n)$ and $t_1,\ldots,t_n$ are real numbers which need not to be neither time-ordered nor mutually different. Choosing $\mathcal{B} = \mathcal{B}(H_0)$, $\mathcal{A} = \mathcal{B}(H_0\otimes F)$, and $(j_t)_{t\in\mathbb{R}}$ as in (1.5), one obtains a quantum stochastic process in the sense defined above.

Remark 1.) Both in examples (1.) and (2.) one could have choosen a smaller algebra $\mathcal{A}$ - for example the norm (in $L^{\infty}(\Omega, \mathcal{F}, P)$ or in $\mathcal{B}(H\otimes F)$) closure of the $*$-algebra generated by the family $\{j_t(\mathcal{B}) : t\in\mathbb{R}\}$. In general, if $\mathcal{A}$ is generated, algebraically or topologically, by the family $\{j_t(\mathcal{B}) : t\in\mathbb{R}\}$, we say that the stochastic process $\{\mathcal{A}, (j_t), \overline{\varphi}\}$ is minimal. In the following, unless explicitaly stated, by "stochastic process" we will mean "minimal stochastic process".

Remark 2.) The occurrence of not necessarily time-ordered correlation functions in (1.6) arises naturally, for example in the computation of moments of observables of the form

$$\sum_{k=1}^{n} j_{s_k}(b_k) \ ; \ s_1 < \ldots < s_n \ ; \ b_1,\ldots,b_n \in \mathscr{B}(H_0)$$

Usually some commutation or anti-commutation relations (arising for example from Einstein causality) are available, and one is reduced to time-ordered correlations. Finally, by polarization and eventually choosing some b_j equal to 1, one verifies that the correlations (1.6) are uniquely determined by the so called correlation Kernels :

$$W^{\overline{\varphi}}_{t_1,\ldots,t_n}(b_1,\ldots,b_n) = \overline{\varphi}(|j_{t_1}(b_1)\cdot\ldots\cdot j_{t_n}(b_n)|^2) \tag{1.7}$$

($b_j \in \mathscr{B}$; $t_j \in \mathbb{R}$; $j = 1,\ldots,n$). In [3] an abstract characterization of the correlation Kernels is given, and it is shown that any family of correlation Kernels defines (uniquely up to stochastic equivalence) a stochastic process.

2.) The local algebras associated to a stochastic process

Given a stochastic process $\{\mathscr{A}, (j_t)_{t\in\mathbb{R}}, \varphi\}$ over a $*$-algebra with identity $\mathscr{B}$, one can define, for each sub-set $I \subseteq \mathbb{R}$, the algebra

$$\mathscr{A}_I = \bigvee_{t\in I} j_t(\mathscr{B}) \tag{2.1}$$

where the right-hand side of (2.1) denotes the algebra generated by the set $\{j_t(\mathscr{B}) : t \in I\}$ (we leave unspecified the topology under which this algebra is closed : this will be clear, case by case, from the context). We will use the notations :

$$\mathscr{A}_{t]} = \bigvee_{s\leq t} j_s(\mathscr{B}) \tag{2.2}$$

$$\mathscr{A}_{[t} = \bigvee_{s\geq t} j_s(\mathscr{B}) \tag{2.3}$$

$$\mathscr{A}_t = j_t(\mathscr{B}) \tag{2.4}$$

Clearly

$$s \leq t \Longrightarrow \mathscr{A}_{s]} \leq \mathscr{A}_{t]} \tag{2.5}$$

A family $(\mathscr{A}_{s]})_{s\in\mathbb{R}}$ of sub-algebras of $\mathscr{A}$, satisfying (2.5), is called a filtration. Given a family $\mathscr{T}$ of sub-sets of $\mathbb{R}$ a family $(\mathscr{A}_I)$ of

sub-algebras of $\mathscr{A}$ satisfying :

$$I \subseteq J \Longrightarrow \mathscr{A}_I \subseteq \mathscr{A}_J \tag{2.6}$$

is called a family of local sub-algebras of $\mathscr{A}$ or simply a localization on $\mathscr{A}$ based $\mathscr{T}$.

Example. In the case of a classical stochastic process $(X_t)_{t\in\mathbb{R}}$ cf. the Example (1.) in Section (1.), the local algebras $\mathscr{A}_I$ ($I \subseteq \mathbb{R}$) are sub-algebras of $L^\infty(\Omega, \mathscr{F}_I, P)$, where $\mathscr{F}_I$ is the σ-algebra generated by the random variables $(X_t)_{t\in I}$.

Given a family $(\mathscr{A}_I)_{I\subseteq\mathbb{R}}$ of local algebras ($\subseteq \mathscr{A}$) a 1-parameter group of automorphisms (sometimes endomorphisms) of $\mathscr{A}$ is called a shift (with respect to that localization) if :

$$u_t \mathscr{A}_I = \mathscr{A}_{I+t} \ ; \forall t \in \mathbb{R} \ ; \ I \subseteq \mathbb{R} \ ; \ \text{(covariance)} \tag{2.7}$$

for any $I \subseteq \mathbb{R}$ and $t \in \mathbb{R}$. If the localization $(\mathscr{A}_I)$ is defined by a stochastic process through (2.1), then (2.7) is equivalent to :

$$u_t \cdot j_s = j_{s+t} \quad ; \quad \forall \ s,t \in \mathbb{R} \tag{2.8}$$

Example. For a classical stochastic process (X_t), one has

$$j_t : f \in L^\infty(\mathbb{R}) \hookrightarrow j_t(f) = f(X_t) \in L^\infty(\Omega, \mathscr{F}, P) \tag{2.9}$$

$$u_t(f(X_s)) = f(X_{s+t}) \quad ; \ s,t \in \mathbb{R} \tag{2.10}$$

A stochastic process $\{\mathscr{A}, (j_t), \varphi\}$ on $\mathscr{B}$ is called stationary, if there exists a shift (u_t) on $\mathscr{A}$ (i.e. a 1-parameter automorphisms group of $\mathscr{A}$ satisfying (2.8)) such that :

$$\varphi \cdot u_t = \varphi \quad ; \forall \ t \in \mathbb{R} \tag{2.11}$$

Recall that a conditional expectation from $\mathscr{A}$ onto a sub-algebra $\mathscr{C}$ is a linear map $E : \mathscr{A} \longrightarrow \mathscr{C}$ satisfying :

$$E(1) = 1 \ ; \ E(ca) = cE(a) \ ; \ \forall a \in \mathscr{A} \ ; \ \forall c \in \mathscr{C} \tag{2.12}$$

Sometimes (in classical stochastic processes - always for a natural choice of the local algebras $(\mathscr{A}_I)$) for any local algebra $\mathscr{A}_I$ ($I \subseteq \mathbb{R}$) there exists a conditional expectation $E_I : \mathscr{A} \longrightarrow \mathscr{A}_I$ such that

$$\varphi \cdot E_I = \varphi \tag{2.13}$$

i.e. compatible with the state φ. The family (E_I) satisfies

$$I \subseteq J \Longrightarrow E_I \cdot E_J = E_I \text{ (projectivity)} \qquad (2.14$$

and if the state φ is shift-invariant, then :

$$u_t \cdot E_I = E_{I+t} \cdot u_t \quad ; \ \forall\, t, \forall\, I \qquad (2.15$$

Any family (E_I) of surjective conditional expectations $E_I : \mathscr{A} \longrightarrow \mathscr{A}_I$ will be called <u>projective</u> if it satisfies (2.14) and <u>covariant</u> if it satisfies (2.15). In particular, in case of a filtration (a localization based on the "past half-lines" $\{(-\infty,t] : t \in \mathbb{R}\}$) conditions (2.14) and (2.15) become :

$$s \leq t \Longrightarrow E_{s]} \cdot E_{t]} = E_{s]} \qquad (2.16$$

$$u_t \cdot E_{s]} = E_{s+t]} \cdot u_t \qquad (2.17$$

3.) <u>Markov processes and dilations.</u>

A <u>markovian</u> <u>stucture</u> on a $*$-algebra $\mathscr{A}$ is defined by the assignment of :

- a "past-filtration" $(\mathscr{A}_{t]})_{t \in \mathbb{R}}$ on $\mathscr{A}$.
- a "future-filtration" $(\mathscr{A}_{[s})_{s \in \mathbb{R}}$ on $\mathscr{A}$.
- for each $t \in \mathbb{R}$ – an "algebra at time t", $\mathscr{A}_t$ such that :

$$\mathscr{A}_t \subseteq \mathscr{A}_{t]} \cap \mathscr{A}_{[t} \qquad (3.1$$

- A projective system of conditional expectations $E_{t]} : \mathscr{A} \longrightarrow \mathscr{A}_{t]}$ i.e. :

$$s \leq t \Longrightarrow E_{s]} \cdot E_{t]} = E_{s]} \qquad (3.2$$

enjoying the Markov property :

$$E_{t]}(\mathscr{A}_{[t}) \subseteq \mathscr{A}_t \quad ; \ \forall\, t \in \mathbb{R} \qquad (3.3$$

If the localization $\{(\mathscr{A}_{t]}), (\mathscr{A}_{[t}), (\mathscr{A}_t)\}$ admits a shift (u_t), i.e.

$$u_t \cdot \mathscr{A}_{s]} = \mathscr{A}_{s+t]} \ ; \ u_t \cdot \mathscr{A}_{[s} = \mathscr{A}_{[s+t} \ ; \ u_t \cdot \mathscr{A}_s = \mathscr{A}_{s+t} \qquad (3.4$$

and if the family $(E_{t]})$ of conditional expectations is covariant, i.e.

$$u_t \cdot E_{s]} = E_{s+t]} \cdot u_t \qquad (3.5$$

then we speak of a <u>covariant</u> <u>markovian</u> <u>structure</u> .

<u>Example</u> Let $X_t : (\Omega, \mathscr{F}, P) \longrightarrow \mathbb{R}$ $(t \in \mathbb{R})$ be a classical Markov process;

let $\mathscr{F}_{t]}$, $\mathscr{F}_t$, $\mathscr{F}_{[t}$ be respectively the past, present and the future σ-algebras; denote

$\mathscr{A} = L^\infty(\Omega, \mathscr{F}, P)$; $\mathscr{A}_{t]} = L^\infty(\Omega, \mathscr{F}_{t]}, P)$; ...

and let $E_{t]} = E^P(\cdot \mid \mathscr{F}_{t]})$ be the P-conditional expectation on $\mathscr{F}_{t]}$. Clearly these objects define a markovian structure on $\mathscr{A}$. – a covariant markovian structure if the process (X_t) is stationary.

The connection between markovian structures and semi-groups is made precise by the following

<u>Theorem (3.1)</u> Let $(E_{s]})$ be a family of maps (not necessarily conditional expectations) satisfying conditions (3.2) – projectivity – and (3.5) – covariance. Denote $\tilde{\mathscr{A}}_o$ the vector space generated by the family:

$$\{E_{o]} \cdot u_t \cdot E_{o]}(\mathscr{A}_o) : t \geq 0\} \qquad (3.6$$

($\tilde{\mathscr{A}}_o = \mathscr{A}_o$ in the markovian case), and define

$$P^t = E_{o]} \cdot u_t \big|_{\tilde{\mathscr{A}}_o} \quad ; \ t \in \mathbb{R} \qquad (3.7$$

then the family (P^t) is a semi-group of $\tilde{\mathscr{A}}_o$ into itself.

<u>Proof.</u> For $a_o \in \tilde{\mathscr{A}}_o$ and $s,t \in \mathbb{R}$ one has :

$$P^s P^t(a_o) = E_{o]} \cdot u_s \cdot E_{o]} \cdot u_t(a_o) = E_{o]} E_{s]} u_{s+t}(a_o) =$$

$$= E_{o]} u_{s+t}(a_o) = P^{s+t}(a_o)$$

If the maps $(E_{t]})$ are completely positive, identity preserving, (e.g. conditional expectations) then the semi-group (P^t) is completely positive identity preserving. Any such a semi-group will be called a <u>markovian</u> <u>semi-group</u> on $\tilde{\mathscr{A}}_o$. If $\tilde{\mathscr{A}}_o$ is a non-commutative algebra, one also speaks of a quantum dynamical semi-group.

In the following we shall only consider the markovian case, i.e. $\tilde{\mathscr{A}}_o = \mathscr{A}_o$. Thus, denoting $\mathscr{B} = \mathscr{A}_o$ and j_0 = the identity embedding $\mathscr{A}_o \hookrightarrow \mathscr{A}$, one obtains the commutative diagramme :

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{\ t\ } & \mathscr{A} & \xrightarrow{\ E_{o]}\ } & \mathscr{A} \\ j_0 \uparrow & & & & \downarrow j_0^{-1} \\ \mathscr{B} & & \xrightarrow[\ P^t\ ]{} & & \mathscr{B} \end{array} \qquad (3.8$$

where j_o^{-1} denotes the left inverse of j_o.

Definition (3.2) Let $\mathcal{B}$ be a $\mathcal{C}$*-algebra (with identity) and (P^t) - a markovian semi-group on $\mathcal{B}$. A $\mathcal{C}$*-dilation of the pair $\{\mathcal{B}, (P^t)\}$ is a quadruple $\{\mathcal{A}, j_o, E_{o]}, (u_t)\}$ making commutative the diagramme (3.8) and such that $j_o : \mathcal{B} \hookrightarrow \mathcal{A}$ is a $\mathcal{C}$*-embedding; (u_t) is a 1-parameter automorphisms group of $\mathcal{A}$; $E_{o]} : \mathcal{A} \longrightarrow \mathcal{A}$ is a norm-one projection satisfying :

$$E_{o]} \cdot u_t(j_o(\mathcal{B})) \subseteq j_o(\mathcal{B}) \ ; \forall \ t \geq 0 \qquad (3.9$$

If, moreover, denoting $\mathcal{A}_{[t}$ - the algebra generated by $\{u_s \cdot j_o(\mathcal{B}) : s \geq t\}$ one has :

$$E_{o]} \cdot u_t \cdot E_{o]} \cdot u_t^{-1} | \mathcal{A}_{[t} = E_{o]} | \mathcal{A}_{[t} \ ; \ t \geq 0 \qquad (3.10$$

then we speak of a (covariant) markovian dilation of $\{\mathcal{B}, (P^t)\}$.
Finally if there exists a state (weight) φ on $\mathcal{A}$ satisfying :

$$\varphi \cdot E_{t]} = \varphi \quad ; \quad \varphi \cdot u_t = \varphi \quad ; \ t \geq 0 \qquad (3.11$$

then we speak of a stationary markovian dilation of $\{\mathcal{B}, (P^t)\}$.

Remark One easily sees that there is a one-to-one correspondence between covariant markovian dilations of $\{\mathcal{B}, (P^t)\}$ and covariant markovian structures (as defined at the beginning of this section) with $\mathcal{A}_o \cong \mathcal{B}$ and $E_{o]} \cdot u_t \cdot j_o = P^t$.

A beautiful classification theory of dilations of completely positive semi-groups has been developed by B. Kummerer and W. Schroder. In the classical case, i.e. when $\mathcal{B}$ is abelian we know that :

i) any markovian semi-group (P^t) on $\mathcal{B}$ has a covariant markovian dilation (obtained through the well known Daniell-Kolmogorov construction).

ii) (P^t) has a stationary markovian dilation if and only if there exists a state (weight) φ_o on $\mathcal{B}$ such that

$$\varphi_o = \varphi_o \cdot P^t \qquad (3.12$$

In the quantum case the situation is more complicated and only recently R. Hudson and K.R. Parthasarathy [6] have shown that the statement (i) holds; while A. Frigerio [5] (cf. also the paper by A. Frigerio and V. Gorini [4]) has found the correct quantum analogue of the statement (ii).

In the following sections we will describe the main technical tools through which the solution of the above mentioned problems has been achieved.

4.) Perturbations of semi-group : the Feynman-Kac formula.

Let $\{\mathcal{A}, (\mathcal{A}_{t]}), (\mathcal{A}_t), (\mathcal{A}_{[t}), (u_t^o), (E_{t]})\}$ be a given covariant markovian structure, and let be given a covariant family of local algebras $(\mathcal{A}_{[s,t]})$ $(s \leq t \ ; \ s,t \in \mathbb{R}_+)$ such that

$$\mathcal{A}_{[s,t]} \subseteq \mathcal{A}_{[s} \cap \mathcal{A}_{t]} \qquad (4.1$$

A markovian cocycle (with respect to the structure defined above) is a 1-parameter family $(M_{o,s})_{s \geq o}$ of elements of $\mathcal{A}$ such that :

$$M_{o,t} \in \mathcal{A}_{[o,t]} \quad ; \forall \ t \geq 0 \ ; \ \text{(markovianity)} \qquad (4.2$$

$$M_{o,t+s} = M_{o,s} \cdot u_s^o(M_{o,t}) \ ; \ \text{(cocycle property)} \qquad (4.3$$

Denoting, for $s < t$, $M_{s,t} = u_s(M_{o,t-s})$, then the two parameter family $(M_{s,t})s<t$ is such that :

$$M_{s,t} \in \mathcal{A}_{[s,t]} \quad ; \forall \ s \leq t \qquad (4.4$$

$$M_{r,s} \cdot M_{s,t} = M_{r,t} \quad ; \ r<s<t \qquad (4.5$$

$$u_t(M_{r,s}) = M_{r+t,s+t} \qquad (4.6$$

and the three conditions above are those which, in classical probability theory, define the so-called multiplicative functionals associated to a given family $\{\mathcal{F}_{[s,t]}\}$ of σ-algebras. Typical examples are given by : $\mathcal{A} = L^\infty(\Omega, \mathcal{F}, P)$; $(\Omega, \mathcal{F}, P)$ - a Wiener probability space; $(W_t)_{t \geq 0}$ - a real valued Wiener process;

$$M_{s,t} = \exp \tfrac{1}{2} \{-\int_s^t V(W_r)dr + \int_s^t a(W_r)dW_r\} \qquad (4.7$$

with $V,a : \mathbb{R} \longrightarrow \mathbb{R}$ - sufficiently regular functions.

Theorem (4.1) Let $(M_{o,s})_{s>o}$ be a markovian cocycle and define, for $t>0$

$$P^t(a_o) = E_{o]}(M_{o,t} \cdot u_t^o(a_o) \cdot M_{o,t}^+) \ ; \ a_o \in \mathcal{A}_o \qquad (4.8$$

It follows that (P^t) is a semi-group $\mathcal{A}_o \longrightarrow \mathcal{A}_o$.

Proof. For $a_o \in \mathcal{A}_o$ and $s,t \in \mathbb{R}_+$, one has :

$$P^t P^s(a_o) = E_{o]}(M_{o,t}[u_t^o \cdot \{E_{o]}(M_{o,s} \cdot u_s^o(a_o) \cdot M_{o,s}^+)\}] \cdot M_{o,t}^+) =$$

$$= E_{o]}(M_{o,t} \cdot E_{t]}[u_t^o(M_{o,s}) \cdot u_{s+t}^o(a_o) \cdot u_t^o(M_{o,s})^+] \cdot M_{o,t}^+) =$$

$$= E_{o]} \cdot E_{t]}(M_{o,t} \cdot u_t^o(M_{o,s}) \cdot u_{s+t}^o(a_o) \cdot u_t^o(M_{o,s})^+ \cdot M_{o,t}^+) =$$

$$= E_{o]}(M_{o,t+s} \cdot u_{t+s}(a_o) \cdot M_{o,t+s}^+) = P^{t+s}(a_o)$$

Any semi-group (P^t) defined as above, will be called a Feynman-Kac perturbation of the semi-group $P_o^t = E_{o]} \cdot u_t^o$, $(t > 0)$.

Formula (4.8) will be referred to as the Feynman-Kac formula. This formula generalizes several known constructions :

1.) The classical Feynman-Kac formula. This is obtained by choosing, in the notations of formula (4.7) :

$$M_{o,t} = \exp - {}^1\!/_2 \int_o^t V(W_s) ds \tag{4.9}$$

where V is a suitably regular function (e.g. measurable bounded below).

2.) The interaction representation . This is obtained by choosing the markovian structure to be trivial (i.e. all the local algebras are equal to $\mathcal{A}$ and $E_{o]}$ is the identity map on $\mathcal{A}$), and the cocycle $M_{o,t} = U_{o,t}$ to be unitary. In this case, writing u_t instead of P^t the Feynman-Kac formula becomes :

$$u_t(a) = U_{o,t} \cdot u_t^o(a) \cdot U_{o,t}^+ \quad ; \quad a \in \mathcal{A} \tag{4.10}$$

The cocycle property then assures that (u_t) is a 1-parameter automorphisms group of $\mathcal{A}$ (cf. the proof of Theorem (4.1), with all the conditional expectations equal to the identity).

The pair $\{(u_t^o), (U_{o,t})\}$ where (u_t^o) is a 1-parameter automorphisms group and $(U_{o,t})$ is a unitary (markovian) (u_t^o)-cocycle is called an interaction representation for the 1-parameter automorphisms group (u_t) defined by (4.10). The connection with the notion of interaction representation usually met in physics is given by the following formal considerations : let (u_t^o) be of the form :

$$u_t^o(a) = \mathcal{U}_t^o \cdot a \cdot \mathcal{U}_t^{o\,+} \quad ; \quad a \in \mathcal{A} \tag{4.11}$$

with $\mathcal{U}_t^o = \exp itH_0$ - a unitary in $\mathcal{A}$, and let $H_I \in \mathcal{A}$ be a self-adjoint operator. Define

$$H_I(t) = u_t^o(H_I) = \mathcal{U}_t^o \cdot H_I \cdot \mathcal{U}_t^{o+} \quad ; \quad t \in \mathbb{R} \tag{4.12}$$

and let $(U_{o,t})$ be defined by :

$$\frac{d}{dt} U_{o,t} = iU_{o,t} \cdot H_I(t) \quad ; \quad U_{o,o} = 1 \tag{4.13}$$

then $(U_{o,t})$ is a unitary (u_t^o)-cocycle (markovian in an appropriate localization) and

$$\mathcal{U}_t = U_{o,t} \cdot \mathcal{U}_t^o \quad ;$$

is a 1-parameter unitary group in $\mathcal{A}$ satisfying the formal equation

$$\frac{d}{dt} \mathcal{U}_t = i\,\mathcal{U}_t \cdot [H_0 + H_I] \tag{4.14}$$

In many concrete examples either $H_0 + H_I$ or $H_I(t)$ are not well defined as operators so that equation (4.13) or (4.14) has no rigorous meaning. But we will see that in many cases it is still possible to define, using quantum stochastic calculus, a markovian cocycle $(U_{o,t})$ and a 1-parameter unitary group $(\mathcal{U}_t)$ having all the properties of the formal solutions of the equations (4.13) and (4.14) (cf. Section (6.) in the following).

3.) Perturbations of the identity semi-group. Consider a markovian structure as in the beginning of this section, and let $\mathcal{A}$ be of the form :

$$\mathcal{A} \cong \mathcal{B}(H_0) \otimes \mathcal{B}(F) \cong \mathcal{B}(H_0 \otimes F) \tag{4.15}$$

where H_0 and F are complex separable Hilbert spaces. Assume that the shift (u_t^o) as the form :

$$u_t^o = \ell_0 \times V_t^o \tag{4.16}$$

where ℓ_0 is the identity map on $\mathcal{B}(H_0)$ and (V_t^o) is a 1-parameter automorphisms group of $\mathcal{B}(F)$. In this case the semi-group $P_o^t = E_{o]} \cdot u_t^o$ is the identity semi-group on $\mathcal{A}_o \cong \mathcal{B}(H_0) \otimes 1$, and its Feynman-Kac perturbation with respect to a unitary markovian cocycle $(U_{o,t})$ has the form:

$$P^t(a_o) = E_{o]}(U_{o,t} \cdot a_o \cdot U_{o,t}^+) \tag{4.17}$$

A semi-group of this form will be called a Feynman-Kac perturbation of the identity semi-group.

Theorem (4.2) (cf. R. Hudson - K.R. Parthasarathy [6] , A. Frigerio, V.Gorini [4] ,A. Frigerio [5]). Let H_0 be a complex separable

Hilbert space. Any markovian semi-group on $\mathcal{B}(H_0)$ admitting a Lindblad generator has a covariant markovian dilation which is a Feynman-Kac perturbation of the identity semi-group.

5.) Perturbation of stochastic process .

In the preceeding section we have shown that any markovian cocycle gives rise to a perturbation of a markovian semi-group. In this section we show that any unitary markovian cocycle gives rise to a perturbation of a covariant markovian structure which is still a covariant markovian structure. This is a purely quantum-probabilistic phenomenon, since in the abelian case unitary markovian cocycles give rise only to trivial (i.e. identity) perturbations.

Let $\mathcal{A}, (\mathcal{A}_{t]}), (\mathcal{A}_t), (\mathcal{A}_{[t}), (\mathcal{A}_{[s,t}), (u^o_t), (E_{t]})$ be as in Section (5.); let $(U_{o,t})$ be a unitary markovian cocycle, and define

$$u_t(a) = U_{o,t}\cdot u^o_t(a)\cdot U^+_{o,t} \quad ; \quad a\in\mathcal{A} \tag{5.1}$$

Then (u_t) is a 1-parameter antomorphisms group of $\mathcal{A}$ and defining :

$$\mathcal{B}_0 = \mathcal{A}_0 \quad ; \quad \mathcal{B}_t = u_t(\mathcal{A}_0) \subseteq \mathcal{A}_{[o,t]} \tag{5.2}$$

one easily verifies that for each $a\in\mathcal{A}$:

$$u_t\cdot E_{s]}(a) = E_{s+t]}\cdot u_t(a) \quad ; \tag{5.3}$$

thus the family $(E_{t]})$ is also covariant for the evolution (u_t) defined by (5.1).

Define now, for $t \geq 0$:

$$\mathcal{B}_{[t} = \bigvee_{s\geq t} u_s(\mathcal{B}_0) = \bigvee_{s\geq t} u_s(\mathcal{A}_0) \tag{5.4}$$

and similarly for $\mathcal{B}_{t]}$. Remark that :

$$\mathcal{B}_{[t} \subseteq U_{o,t}\cdot\mathcal{A}_{[t}\cdot U^+_{o,t} \tag{5.5}$$

whence, due to the Markov property of (E_t) :

$$E_{t]}(\mathcal{B}_{[t})\subseteq U_{o,t}\cdot E_{t]}(\mathcal{A}_{[t})\cdot U^+_{o,t}\subseteq U_{o,t}\cdot\mathcal{A}_t\cdot U^+_{o,t} = u_t(\mathcal{A}_0) = \mathcal{B}_t \tag{5.6}$$

Thus : $(E_{t]})$ is markovian also with respect to the localization $(\mathcal{B}_{t]})$, $(\mathcal{B}_t), (\mathcal{B}_{[t})$ or equivalently, defining :

$$\mathcal{B} = \bigvee_{t\in\mathbb{R}_+} u_t(\mathcal{A}_0) \tag{5.7}$$

the family $\{\mathcal{B}, (\mathcal{B}_{t]}), (\mathcal{B}_t), (\mathcal{B}_{[t}), (u_t), (E_{t]})\}$ is still a covariant markovian structure. In particular, for any state φ_o on $\mathcal{B}_o = \mathcal{A}_o$, defining $\varphi = \varphi_o\cdot E_{o]}$ (state on $\mathcal{B}$) ; j_0 = the identity embedding $\mathcal{B}_o \hookrightarrow \mathcal{B}$; $j_t = u_t\cdot j_0$ $(t \geq 0)$, the triple $\{\mathcal{A}, (j_t)_{t\geq o}, \varphi\}$ is a (markovian) stochastic process over $\mathcal{A}_o$, in the sense defined at the beginning of Section (1.). As shown by A. Frigerio and V. Gorini [4], [5], (in the case of boson dilations) the process will be stationary if and only if the associated semi-group satisfies a detailed balance conditions. More generally, in the framework of local algebras, it can be shown that the stationarity of the process is related to the behaviour of the semi-group under appropriate "time-reflections" (cf. [1], [2]).

6.) The Wigner-Weisskopf atom .

In this section I will outline some results obtained in collaboration with D. Applebaum and which will be published elsewhere. For the description of the Wigner-Weisskopf model we follow the exposition given by W. von Waldenfels in [9] and we also refer to this paper for a more complete discussion of the physical limits of this approximation. In its simplest version the model describes a 2-levels atom in interaction with an electro-magnetic field. In the "rotating wave approximation" the system is described on the Hilbert space

$$\mathcal{H} = \mathbb{C}^2\otimes\mathcal{F}(\mathbb{C}^{|\Lambda|}) \cong \mathbb{C}^2\otimes[\otimes_{\lambda\in\Lambda}\mathcal{F}_\lambda] \tag{6.1}$$

where Λ is a finite set (indexing the frequencies of the EM field), $|\lambda|$ denotes the cardinality of Λ and, for each $\lambda\in\Lambda$, $\mathcal{F}_\lambda \cong \Gamma(\mathbb{C})$ is the Fock space over the Hilbert space $\mathbb{C}$ (with scalar product $\langle u,v\rangle = \bar{u}v$; $u,v\in\mathbb{C}$). On each space $\mathcal{F}_\lambda$ the creation and annihilation operators B^+_λ, B_λ are defined in the usual way and they satisfy the commutation relations :

$$[B_\lambda, B^+_{\lambda'}] = \delta_{\lambda\lambda'} \; ; \; [B_\lambda, B_{\lambda'}] = [B^+_\lambda, B^+_{\lambda'}] = 0 \tag{6.2}$$

Introducing the spin matrices :

$$\sigma_+ = \begin{pmatrix}0&0\\1&0\end{pmatrix} \; ; \; \sigma_- = \begin{pmatrix}0&1\\0&0\end{pmatrix} \; ; \; \sigma_3 = \tfrac{1}{2}\begin{pmatrix}-1&0\\0&1\end{pmatrix} \tag{6.3}$$

The hamiltonian of the system in the rotating wave approximation is :

$$H_{tot.} = H_{at.} + H_{EM} + H_I =$$
$$= (\omega_o\sigma_3 \otimes 1) + [\sum_{\lambda\in\Lambda}(\omega_o + \omega_\lambda) 1\otimes B^+_\lambda B_\lambda] +$$
$$+ [\sum_{\lambda\in\Lambda}(g_\lambda\sigma_+\otimes B_\lambda + \bar{g}_\lambda\sigma_-\otimes B^+_\lambda)]$$

where $\omega_o + \omega_\lambda$ is the frequency of the λ-th oscillator and g_λ is the coupling constant of the atom with the λ-th oscillator. Rewriting the hamiltonian as

$$H_{tot.} = H_o + H_1 = \omega_o[\sigma_3\otimes 1 + \sum_{\lambda\in\Lambda} 1\otimes B^+_\lambda B_\lambda] +$$
$$+ [\sum_{\lambda\in\Lambda}(\omega_\lambda 1\otimes B^+_\lambda B_\lambda + g_\lambda\sigma_+\otimes B_\lambda + \bar{g}_\lambda\sigma_-\otimes B^+_\lambda)]$$

and remarking that H_o and H_1 commute, we we reduce ourselves to the consideration of the single term

$$H_1 = H_{1o} + H = [\sum_{\lambda\in\Lambda}\omega_\lambda 1\otimes B^+_\lambda B_\lambda] + [\sum_{\lambda\in\Lambda}(g_\lambda\sigma_+\otimes B_\lambda + \bar{g}_\lambda\sigma_-\otimes B^+_\lambda)]$$

and H_1 is described in interaction representation using H_{1o} as "free part" and H as "interaction part". This leads to the unitary cocycle $U_t = U_{o,t}$ defined by the equation

$$\frac{d}{dt} U_t = -iU_t H_\Lambda(t) \quad ; \quad U_o = 1$$
$$H_\Lambda(t) = \sum_{\lambda\in\Lambda}(g_\lambda\sigma_+\otimes B_\lambda + \bar{g}_\lambda\sigma_-\otimes B^+_\lambda) = \sigma_+\otimes B_\Lambda(t) + \sigma_-\otimes B^+_\Lambda(t)$$

where

$$B_\Lambda(t) = \sum_{\lambda\in\Lambda} g_\lambda\sigma_+\otimes B_\lambda\cdot e^{-i\omega_\lambda t}$$

The commutator between $B_\Lambda(t)$ and $B^+_\Lambda(s)$ is :

$$[B_\Lambda(t) , B^+_\Lambda(s)] = \sum |g_\lambda|^2 e^{-i\omega_\lambda(t-s)} = K_\Lambda(t-s)$$

while all the other commutators vanish.

Introducing on $\mathcal{B}(\mathcal{F}(\mathbb{C}^{|\Lambda|}))$ the quasi-free state $\mathcal{E}$ characterized by

$$\mathcal{E}(B_\lambda B_\mu) = \mathcal{E}(B^+_\lambda B^+_\mu) = 0$$
$$\mathcal{E}(B^+_\lambda B_\mu) = \delta_{\lambda\mu}\theta_\lambda$$

(θ_λ a physical constant), one finds

$$\mathcal{E}(B_\Lambda(t)\cdot B_\Lambda(s)) = \mathcal{E}(B^+_\Lambda(t)B^+_\Lambda(s)) = 0$$
$$\mathcal{E}(B_\Lambda(t)\cdot B^+_\Lambda(s)) = \sum |g_\lambda|^2(1+\theta_\lambda)e^{-i\omega_\lambda(t-s)}$$

The Wigner-Weisskopf approximation is obtained, from the rotating wave approximation, by replacing :

$$K_\Lambda(t-s) = x\cdot\delta(t-s) \quad (x\in C)$$
$$\theta_\lambda = \theta = \frac{\exp(-h\omega_o/KT)}{1-\exp(-h\omega_o/KT)}$$

This means that one substitutes for $B_\Lambda(t)$ and $B^+_\Lambda(t)$ two operators $F(t)$, $F^+(t)$ satisfying :

$$[F(t) , F(s)] = [F^+(t) , F^+(s)] = 0 \qquad (6.4$$
$$[F(t) , F^+(s)] = x\delta(t-s) \qquad (6.5$$

and on the algebra generated by the family $\{F(t), F^+(t)\}$ one introduces the quasi-free state characterized by

$$\mathcal{E}(F(t)\cdot F(s)) = \mathcal{E}(F^+(t)\cdot F^+(s)) = 0 \qquad (6.6$$
$$\mathcal{E}(F(t)\cdot F^+(s)) = (1+\theta)\cdot\delta(t-s) \qquad (6.7$$

With these approximations the equation for the unitary cocycle becomes

$$\frac{d}{dt} U_t = -iU_t\cdot H(t) \quad ; \quad U_o = 1 \qquad (6.8$$
$$H(t) = \sigma_+\otimes F(t) + \sigma_-\otimes F^+(t) \qquad (6.9$$

Equation (6.8) is purely formal because, due to (6.5), (6.7) and (6.9), $H(t)$ is not a well defined operator but an operator valued distribution. In analogy with the classical procedure von Waldenfels [9] introduced to methods for the solution of equation (6.8) :

I.) The "Stratonovich method", corresponding to the "singular coupling limit method" in the physical literature, consisting in three steps :

i) regularize the covariance with the substitution, in (6.5) and (6.7) $\delta(t-s) \longrightarrow K_\varepsilon(t-s)$ for some smooth function $K_\varepsilon(\cdot)$.

ii) solve the corresponding ordinary differential equation, finding a cocycle $U_\varepsilon(0,t)$.

iii) determine the limit of $U_\varepsilon(0,t)$ - and of the associated process (Section (5)) as $\varepsilon \longrightarrow 0$ and $K_\varepsilon(t-s) \longrightarrow \delta(t-s)$.

II.) The "multiplicative Ito integral method", (corresponding to the approximation methods in classical probability) in which - instead of the covariance - you regularize the fields. This can be done in several ways. In [9] one considers for each fixed $T\in\mathbb{R}_+$ a partition $z = \{0=t_o < t_1 < \ldots < t_n = T\}$ of the interval $[0,T]$ and introduces the piecewise constant fields :

$$F_z(t) = \frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}} F(\tau)d\tau = F(\chi_{(t_k,t_{k+1}]}) \quad : \quad t_k < t \le t_{k+1}$$

One then solves the ordinary differential equation :

$$\frac{d}{dt} U_z(t) = -iU_z(t)\cdot H_z(t)$$

and studies the limit of $U_z(t)$ (and of the corresponding process) as $|z| = \max_k (t_{k+1}-t_k) \longrightarrow 0$.

For the Wigner-Weisskopf model the existence of the limiting cocycle (and of the corresponding process) was established by von Waldenfels [9] in both cases (I.) and (II.). A third possibility, is to interpret (6.8) as a quantum stochastic differential equation and use the results of R.Hudson and K.R. Parthasarathy [6] to establish the existence, uniqueness and unitarity of the cocycle U(t). Namely, one considers the Hilbert space

$$\Gamma(L^2(\mathbb{R}^+,dt))\otimes\Gamma(L^2(\mathbb{R}^+,dt)^-) = \mathscr{H}$$

where $\Gamma(H)$ denotes the (boson) Fock space of H and H^- denotes the conjugate Hilbert space of H. On this Hilbert space one considers the representation of the CCR with creation and annihilation operators given by :

$$F(t) = \sqrt{\gamma}\cosh\Phi\cdot a(\chi_{[o,t]})\otimes 1 + \sqrt{\gamma}\sinh\Phi\cdot 1\otimes a^+(\bar{\chi}_{[o,t]})$$

$$F^+(t) = \sqrt{\gamma}\cosh\Phi\cdot a^+(\chi_{[o,t]})\otimes 1 + \sqrt{\gamma}\sinh\Phi\cdot 1\otimes a(\bar{\chi}_{[o,t]})$$

where $a(\cdot)$ and $a^+(\cdot)$ are the annihilation and creation operators over $\Gamma(L^2(\mathbb{R}^+))$, and by definition, $\gamma = 2$ Rex, and :

$$\cosh^2\Phi = \frac{1}{1-\exp(-\omega_o/KT)} \quad ; \quad \sinh^2\Phi = \frac{\exp(-\omega_0/KT)}{1-\exp(-\omega_o/KT)} = \theta$$

With these notations the unitary (markovian) cocycle U_t is defined as the solution of the quantum stochastic differential equation

$$dU_t = U_t\{(-i\sigma_+\otimes dF(t) - i\sigma_-\otimes dF^+(t)) - \\ - \gamma/_2(\cosh^2\Phi\sigma_+\sigma_-\otimes 1 + \sinh^2\Phi\sigma_-\sigma_+\otimes 1)dt \qquad (6.10)$$

Denoting $E_{o]}$ the conditional expectation characterized by :

$$E_{o]} : x\otimes(Y\otimes Z) \in \mathscr{B}(\mathbb{C}^2)\otimes\mathscr{B}(\Gamma(L^2(\mathbb{R}_+))\otimes\Gamma(L^2(\mathbb{R}_+)^-)) \longrightarrow \\ \longrightarrow (x\otimes 1\otimes 1)\langle\Omega,Y\Omega\rangle\langle\bar{\Omega},Z\bar{\Omega}\rangle$$

where Ω (resp. $\bar{\Omega}$) denotes the Fock vacuum in $\Gamma(L^2(\mathbb{R}_+))$ (resp. $\Gamma(L^2(\mathbb{R}_+)^-)$) and applying the theory outlined in Section (4), one obtains a semi-group on $\mathscr{B}(\mathbb{C}^2) = \{2\times 2 \text{ matrices}\}$ via the prescription:

$$x\in\mathscr{B}(C^2) \longrightarrow E_{o]}(U_t\cdot(x\otimes 1\otimes 1)U_t^+) \in \mathscr{B}(C^2)\otimes 1\otimes 1 \cong \mathscr{B}(C^2)$$

whose generator is :

$$L(x) = -\tfrac{1}{2}\cosh^2\Phi\cdot\gamma\{\sigma_+\sigma_-,x\} + \cosh^2\Phi\cdot\gamma\cdot\sigma_+\cdot x\cdot\sigma_- + \\ -\tfrac{1}{2}\sinh^2\Phi\cdot\gamma\{\sigma_-\sigma_+,x\} + \sinh^2\Phi\cdot\gamma\cdot\sigma_-\cdot x\cdot\sigma_+$$

($x\in\mathscr{B}(\mathbb{C}^2)$). Referring the algebra of 2x2 complex matrices; $\mathscr{B}(\mathbb{C}^2)$ to the standard basis, we find for L the matrix :

$$\begin{pmatrix} -\gamma\theta & \gamma(\theta+1) & 0 & 0 \\ \gamma\theta & -\gamma(\theta+1) & 0 & 0 \\ 0 & 0 & -\tfrac{1}{2}(2\theta+1)\gamma & 0 \\ 0 & 0 & 0 & \tfrac{1}{2}(2\theta+1)\gamma \end{pmatrix} \qquad (6.11$$

which is exactly the formula found by von Waldenfels via the "multiplicative Ito method" [9] (in his notations $\gamma = 2$ Rex). To obtain the formula found by von Waldenfels via the "Stratonovich method" instead of (6.10) one has to look for the solution of the quantum stochastic differential equation :

$$dU_t = U_t\cdot\{-i\sigma_+\otimes dF(t) - i\sigma_-\otimes dF^+(t) - [\gamma/_2(\cosh^2\Phi\sigma_+\sigma_-\otimes 1 + \\ - \sinh^2\Phi\sigma_-\sigma_+\otimes 1)]dt - i\beta/_2(2\theta+1)[\cosh^2\Phi\sigma_+\sigma_-\otimes 1 + \sinh^2\Phi\sigma_-\sigma_+\otimes 1]dt\}$$

where, in von Waldenfels notations: $\gamma = 2$Rex, $\beta = 2$Imx. The connection between the multiplicative Ito (i.e. singular coupling) method and quantum stochastic differential equations was suggested by Frigerio and Gorini [4] and the explicit form of the semi-group obtained in the Wigner-Weisskopf model in the "multiplicative Ito" case (i.e. corresponding to equation (10)) has been independently obtained by H. Maassen [8].

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ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR SOME MAPS OF THE CIRCLE

P. M. Blecher and M. V. Jakobson

1. Statement of results. We consider the two-parameter family of maps on the circle

$$f_{q,\omega} : x \mapsto x+\omega+(q/2\pi)\cdot \sin 2\pi x \ , \quad x \in S^1 = \mathbb{R}/\mathbb{Z}$$

and we find a set $M = \{(q,\omega)\}$ of positive Lebesgue measure such that $(q,\omega) \in M$ implies the stochastic behaviour of $f_{q,\omega}$. We present analytical and numerical results which describe the structure of M as follows.

There exists a sequence of points $A_k = (q_k,\omega_k)$, $k\in\mathbb{N}$ converging to the limit $A_\infty = (q_\infty,\omega_\infty)$, where $q_\infty = 1{,}169701\ldots$, $\omega_\infty = q_\infty/2\pi$, satisfying

Theorem 1. For any k there exists a set $M_k \subset \mathbb{R}^2$ of positive Lebesgue measure, such that A_k is the density point of M_k , and if $(q,\omega) \in M_k$ then the map $f_{q,\omega}: S^1 \to S^1$ has an absolutely continuous invariant probability measure $\mu_{q,\omega}$. The map $f_{q,\omega}$ cyclically permutes k adjacent intervals $\ell^{(i)}_{q,\omega}$, $i \in [0,k-1]$, $\bigcup_{i=0}^{k-1} \ell^{(i)}_{q,\omega} = S^1$. The support of $\mu_{q,\omega}$ consists of k intervals $s^{(i)}_{q,\omega} \subset \ell^{(i)}_{q,\omega}$ of equal measure. For any i the map $f^k_{q,\omega}$ is an exact endomorphism on the measure space $(s^{(i)}_{q,\omega}, \mu_{q,\omega})$, and its natural extension is a Bernoulli automorphism.

In order to prove Theorem 1 for a given k it suffices to verify some conditions of non-degeneracy, see Sect. 3. For $k=1$ these conditions are verified analytically. For $2\le k\le 7$ they were verified with the help of a computer.