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ON THE UNITARITY OF STOCHASTIC EVOLUTIONS DRIVEN BY THE SQUARE OF WHITE NOISE

LUIGI ACCARDI*

Centro Vito Volterra, Università di Roma Tor Vergata, Via di Tor Vergata, 00133 Roma, Italy

ANDREAS BOUKAS

American College of Greece, Aghia Paraskevi, Athens 15342, Greece

HUI-HSUNG KUO

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA

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Using the closed Itô's table for the renormalized square of white noise, recently obtained by Accardi, Hida, and Kuo in Ref. 4, we consider the problem of providing necessary and sufficient conditions for the unitarity of the solutions of a certain type of quantum stochastic differential equations.

1. Introduction

The renormalized square of white noise (or SWN) $\star$-algebra is generated by operators B_f, B_f^+ and N_f satisfying the commutation relations

$$[B_f, B_g^+] = 2c\langle f, g\rangle + 4N_{\bar{f}g}, \qquad [N_f, B_g^+] = 2B_{fg}^+, \qquad [N_f, B_g] = -2B_{\bar{f}g}\,, \quad (1.1)$$

$$[B_f, B_g] = [B_f^+, B_g^+] = [N_f, N_g] = 0, \qquad B_f\Omega = N_f\Omega = 0\,, \quad (1.2)$$

where $f, g \in L^2 \cap L^\infty(\mathbb{R})$, Ω is the vacuum vector, $c > 0$ comes from renormalization, and $\langle f, g\rangle = \int_{\mathbf{R}} \bar{f}(t)g(t)\,dt$. It was shown in Ref. 2 that the quantum stochastic calculus associated with the SWN operators is included in the representation free calculus of Ref. 3 and satisfies the basic semimartingale inequalities. As shown in Ref. 3, this is sufficient to guarantee the existence and uniqueness theorem for stochastic differential equations with bounded coefficients, driven by the SWN. A similar result could also be established by using the representation of the SWN in

*E-mail: accardi@volterra.mat.uniroma2.it

terms of usual quantum white noise as in Refs. 6 and 7. On the other hand, Accardi, Hida and Kuo in Ref. 4 proved that the SWN differentials

$$dB(t) = B_{\chi_{[t,t+dt)}}, \qquad dB^+(t) = B^+_{\chi_{[t,t+dt)}}, \qquad dN(t) = N_{\chi_{[t,t+dt)}} \tag{1.3}$$

satisfy weakly on the SWN exponential vectors the following closed Itô's table:

	$dB^+(t)$	$dN(t)$	$dB(t)$
$dB(t)$	$\alpha_-^+ dt + \beta_-^+ dB^+(t) + \gamma_-^+ dN(t)$	$\alpha_-^0 dt + \beta_-^0 dB^+(t) + \gamma_-^0 dN(t)$	$\alpha_-^- dt + \beta_-^- dB^+(t) + \gamma_-^- dN(t)$
$dN(t)$	$\alpha_0^+ dt + \beta_0^+ dB^+(t) + \gamma_0^+ dN(t)$	$\alpha_0^0 dt + \beta_0^0 dB^+(t) + \gamma_0^0 dN(t)$	$\alpha_0^- dt + \beta_0^- dB^+(t) + \gamma_0^- dN(t)$
$dB^+(t)$	$\alpha_+^+ dt + \beta_+^+ dB^+(t) + \gamma_+^+ dN(t)$	$\alpha_+^0 dt + \beta_+^0 dB^+(t) + \gamma_+^0 dN(t)$	$\alpha_+^- dt + \beta_+^- dB^+(t) + \gamma_+^- dN(t)$

where

$$\begin{aligned}
\alpha_+^+ &= 8ck(\partial_t^*)^2, & \beta_+^+ &= 16k(\partial_t^*)^2\partial_t, & \gamma_+^+ &= 8k(\partial_t^*)^2,\\
\alpha_0^+ &= 64ck(\partial_t^*)^3\partial_t^2, & \beta_0^+ &= 2 + 8k\partial_t\partial_t^*, & \gamma_0^+ &= 64k(\partial_t^*)^3\partial_t^2,\\
\alpha_-^+ &= (2c - 32c(\partial_t^*\partial_t)^2 & \beta_-^+ &= 16\partial_t^*\partial_t^2 k, & \gamma_-^+ &= (4 - 64(\partial_t^*\partial_t)^2\\
&\quad +128c(\partial_t^*\partial_t)^3)k, & & & &\quad +128(\partial_t^*\partial_t)^3)k,\\
\alpha_+^0 &= 64c(\partial_t^*)^3\partial_t^2 k, & \beta_+^0 &= 8\partial_t^*\partial_t k, & \gamma_+^0 &= 64(\partial_t^*)^3\partial_t^2 k,\\
\alpha_0^0 &= 32c(\partial_t^*\partial_t)^2 k, & \beta_0^0 &= 4\partial_t k, & \gamma_0^0 &= 32(\partial_t^*\partial_t)^2 k,\\
\alpha_-^0 &= 64c(\partial_t^*)^2\partial_t^3 k, & \beta_-^0 &= 2\frac{\partial_t}{\partial_t^*} + 8\partial_t^2 k, & \gamma_-^0 &= 64(\partial_t^*)^2\partial_t^3 k,\\
\alpha_+^- &= 128c(\partial_t^*\partial_t)^3 k, & \beta_+^- &= 16\partial_t^*\partial_t^2 k, & \gamma_+^- &= 128(\partial_t^*\partial_t)^3 k,\\
\alpha_0^- &= 64c(\partial_t^*)^2\partial_t^3 k, & \beta_0^- &= 8\partial_t^2 k, & \gamma_0^- &= 64(\partial_t^*)^2\partial_t^3 k,\\
\alpha_-^- &= 8c\partial_t^2 k, & \beta_-^- &= 16\partial_t^3 k, & \gamma_-^- &= 8\partial_t^2 k,
\end{aligned}$$

where $k = \frac{1}{1-16(\partial_t^*\partial_t)^2}$, ∂_t and ∂_t^* are the Hida derivative and its adjoint and, for an analytic function $F(x,y) = \sum_{m,n\geq 0} a_{m,n}x^m y^n$ and any operator M in the algebra generated by B_f, B_f^+ and N_f, the sesquilinear form $MF(\partial_t^*, \partial_t)$ is defined by

$$\begin{aligned}
MF(\partial_t^*, \partial_t)(\psi(f), \psi(g)) &= \sum_{m,n\geq 0} a_{m,n}\bar{f}(t)^m g(t)^n \langle\psi(f), M\psi(g)\rangle\\
&= F(\bar{f}(t), g(t))\langle\psi(f), M\psi(g)\rangle\,.
\end{aligned} \tag{1.4}$$

Similarly

$$\begin{aligned}
MF(\partial_t, \partial_t^*)(\psi(f), \psi(g)) &= \sum_{m,n\geq 0} a_{m,n} g(t)^m \bar{f}(t)^n \langle\psi(f), M\psi(g)\rangle\\
&= F(g(t), \bar{f}(t))\langle\psi(f), M\psi(g)\rangle\,,
\end{aligned} \tag{1.5}$$

where $\psi(f)$, $\psi(g)$ are SWN exponential vectors. Notice that, by construction, $MF(\partial_t^*, \partial_t) = F(\partial_t^*, \partial_t)\, M$ in the sense of equality of sesqilinear forms. Moreover, $[\partial_t, \partial_t^*] = 0$. We denote by $\bar{F}$ the adjoint form of F. By avoiding test functions f, g for which the denominator vanishes we can extend definition (1.4) to more general rational functions $F(\partial_t^*, \partial_t)$, for example of the form $F(\partial_t^*, \partial_t) = 1/\partial_t^*\partial_t$. This possibility will be used freely in the following, in particular in the example of Sec. 4.

It is therefore natural to combine the above-mentioned results and to try to obtain the unitarity conditions for stochastic differential equations driven by the square of white noise. Since the SWN Itô table involves "operators" of the form (1.4), (1.5), it is also natural to expect that the coefficients of an equation, admitting a unitary solution, will depend on such "operators". This means that, as already discussed in Ref. 1, such equations must be interpreted as ordinary differential equations on sesquilinear forms and only *a posteriori* one has to prove that these quadratic forms are induced by unitary operators.

In this note we derive these unitarity conditions. However we prove, by providing a counterexample, that the SWN differentials (1.3) are not linearly independent on the algebra generated by the sesquilinear forms (1.4), (1.5). This implies that, without additional information, one cannot conclude that the sufficient conditions for unitarity, deduced from the SWN Itô table in Sec. 2 below, are also necessary.

In fact we are able, by explicit calculations, to determine the form-coefficients of the stochastic equations satisfied by the SWN analogue of the Poisson process (which includes the SWN Weyl operators). These processes are unitary by construction, but we prove that their coefficients do not satisfy the sufficient conditions of Sec. 2. Finally we construct an example of an equation which satisfies the above-mentioned sufficient conditions.

2. Unitarity Conditions for Evolutions Driven by the SWN

Let H_0 be a complex separable Hilbert space and let $S = \text{span}\{\psi(f)\}$ and $\mathcal{E} = H_0 \otimes S$. We consider stochastic evolutions of the form

$$dU(t) = (\mathcal{A}(t)dt + \mathcal{B}(t)dB(t) + \mathcal{C}(t)dB^+(t) + \mathcal{D}(t)dN(t))U(t) \tag{2.1}$$

and its adjoint

$$dU^*(t) = U(t)^*(\mathcal{A}^*(t)dt + \mathcal{B}^*(t)dB^+(t) + \mathcal{C}^*(t)dB(t) + \mathcal{D}^*(t)dN(t)) \tag{2.2}$$

with initial conditions

$$U(0) = U^*(0) = I, \qquad 0 \leq t \leq t_0 < +\infty\,,$$

where the solution $U = \{U(t), 0 \leq t \leq t_0 < +\infty\}$ is defined as a sesquilinear form on $\mathcal{E} \times \mathcal{E}$, for each t the coefficients $\mathcal{A}(t), \mathcal{B}(t), \mathcal{C}(t)$ and $\mathcal{D}(t)$ are, in general, finite linear combinations of elements of the form $R(t) \otimes F(\partial_t^*, \partial_t)$ where $R(t)$ is a bounded linear operator on H_0 and $F(\partial_t^*, \partial_t)$ is as above, and for $X \in \{B, B^+, N\}$ we define $dX \equiv I \otimes dX$ where I is the identity on H_0 and the dX on the right is defined on $S = \text{span}\{\psi(f)\}$ in the standard way.

The above form of the coefficients is suggested by the SWN Itô table. For such coefficients the stochastic differentials (1.3) are not linearly independent in the sense that the equation

$$A_1(t)\otimes\alpha_1(\partial_t^*,\partial_t)dt + A_2(t)\otimes\alpha_2(\partial_t^*,\partial_t)dB(t) + A_3(t)\otimes\alpha_3(\partial_t^*,\partial_t)dB^+(t)$$
$$+ A_4(t)\otimes\alpha_4(\partial_t^*,\partial_t)dN(t) = 0 \tag{2.3}$$

meant in the sense of sesquilinear forms, does not imply

$$A_i(t)\otimes\alpha_i(\partial_t^*,\partial_t) = 0 \tag{2.4}$$

for all t and $i = 1,2,3,4$. To see this let $A_1(t) = A_2(t) = A_3(t) = A_4(t) = I$ for all t and, assuming that, for each $i = 1,2,3,4$,

$$\alpha_i(\partial_t^*,\partial_t) = \sum_{n,k=0}^{\infty} \alpha_i^{n,k}\partial_t^{*n}\partial_t^{k},$$

choose $\alpha_0^{n,k} = \alpha_3^{n,0} = \alpha_2^{0,k} = 0$ for all $n,k = 0,1,\ldots$, and for all $n,k = 1,2,\ldots$ choose $\alpha_2^{n,k-1} + \alpha_3^{n-1,k} + 2\alpha_4^{n-1,k-1} = 0$, for example

$$\alpha_2^{n,k-1} = \frac{1}{n!(k-1)!},$$

$$\alpha_3^{n-1,k} = \frac{1}{(n-1)!k!},$$

$$\alpha_4^{n-1,k-1} = -\frac{1}{2}\left(\frac{1}{n!(k-1)!} + \frac{1}{(n-1)!k!}\right).$$

Then, using (1.4), (1.5) and Proposition 2.1 of Ref. 2 to compute the matrix elements, we see that (2.3) is satisfied but (2.4) is not.

To obtain unitarity conditions for U we start with

$$U(t)U^*(t) = U^*(t)U(t) = I, \qquad U(0) = U^*(0) = I \tag{2.5}$$

which are equivalent to

$$d(U(t)U^*(t)) = dU(t)U^*(t) + U(t)dU^*(t) + dU(t)dU^*(t) = 0 \tag{2.6}$$

and

$$d(U^*(t)U(t)) = dU^*(t)U(t) + U^*(t)dU(t) + dU^*(t)dU(t) = 0 \tag{2.7}$$

and then using the Itô table and equating coefficients of the time and noise differentials to zero we obtain:

Theorem 2.1. *If for each* t

$$\mathcal{A} + \mathcal{A}^* + \mathcal{B}\mathcal{B}^*\alpha_-^+ + \mathcal{B}\mathcal{C}^*\alpha_-^- + \mathcal{B}\mathcal{D}^*\alpha_-^0 + \mathcal{C}\mathcal{B}^*\alpha_+^+ + \mathcal{C}\mathcal{C}^*\alpha_+^- + \mathcal{C}\mathcal{D}^*\alpha_+^0$$
$$+ \mathcal{D}\mathcal{B}^*\alpha_0^+ + \mathcal{D}\mathcal{C}^*\alpha_0^- + \mathcal{D}\mathcal{D}^*\alpha_0^0 = 0, \tag{2.8}$$

$$\mathcal{B} + \mathcal{C}^* = 0, \tag{2.9}$$

$$\mathcal{C} + \mathcal{B}^* + \mathcal{B}\mathcal{B}^*\beta_-^+ + \mathcal{B}\mathcal{C}^*\beta_-^- + \mathcal{B}\mathcal{D}^*\beta_-^0 + \mathcal{C}\mathcal{B}^*\beta_+^+ + \mathcal{C}\mathcal{C}^*\beta_+^- + \mathcal{C}\mathcal{D}^*\beta_+^0$$
$$+ \mathcal{D}\mathcal{B}^*\beta_0^+ + \mathcal{D}\mathcal{C}^*\beta_0^- + \mathcal{D}\mathcal{D}^*\beta_0^0 = 0, \tag{2.10}$$

$$\mathcal{D} + \mathcal{D}^* + \mathcal{B}\mathcal{B}^*\gamma_-^+ + \mathcal{B}\mathcal{C}^*\gamma_-^- + \mathcal{B}\mathcal{D}^*\gamma_-^0 + \mathcal{C}\mathcal{B}^*\gamma_+^+ + \mathcal{C}\mathcal{C}^*\gamma_+^- + \mathcal{C}\mathcal{D}^*\gamma_+^0$$
$$+ \mathcal{D}\mathcal{B}^*\gamma_0^+ + \mathcal{D}\mathcal{C}^*\gamma_0^- + \mathcal{D}\mathcal{D}^*\gamma_0^0 = 0, \tag{2.11}$$

$$\mathcal{A}^* + \mathcal{A} + \mathcal{B}^*\mathcal{B}\alpha_+^- + \mathcal{B}^*\mathcal{C}\alpha_+^+ + \mathcal{B}^*\mathcal{D}\alpha_+^0 + \mathcal{C}^*\mathcal{B}\alpha_-^- + \mathcal{C}^*\mathcal{C}\alpha_-^+ + \mathcal{C}^*\mathcal{D}\alpha_-^0$$
$$+ \mathcal{D}^*\mathcal{B}\alpha_0^- + \mathcal{D}^*\mathcal{C}\alpha_0^+ + \mathcal{D}^*\mathcal{D}\alpha_0^0 = 0, \tag{2.12}$$

$$\mathcal{B}^* + \mathcal{C} + \mathcal{B}^*\mathcal{B}\beta_+^- + \mathcal{B}^*\mathcal{C}\beta_+^+ + \mathcal{B}^*\mathcal{D}\beta_+^0 + \mathcal{C}^*\mathcal{B}\beta_-^- + \mathcal{C}^*\mathcal{C}\beta_-^+ + \mathcal{C}^*\mathcal{D}\beta_-^0$$
$$+ \mathcal{D}^*\mathcal{B}\beta_0^- + \mathcal{D}^*\mathcal{C}\beta_0^+ + \mathcal{D}^*\mathcal{D}\beta_0^0 = 0, \tag{2.13}$$

$$\mathcal{D}^* + \mathcal{D} + \mathcal{B}^*\mathcal{B}\gamma_+^- + \mathcal{B}^*\mathcal{C}\gamma_+^+ + \mathcal{B}^*\mathcal{D}\gamma_+^0 + \mathcal{C}^*\mathcal{B}\gamma_-^- + \mathcal{C}^*\mathcal{C}\gamma_-^+ + \mathcal{C}^*\mathcal{D}\gamma_-^0$$
$$+ \mathcal{D}^*\mathcal{B}\gamma_0^- + \mathcal{D}^*\mathcal{C}\gamma_0^+ + \mathcal{D}^*D\gamma_0^0 = 0, \tag{2.14}$$

then the solution $U = \{U(t)\colon 0 \le t \le t_0 < +\infty\}$ *of the initial value problem* (2.1) *is unitary.*

It should be pointed out that in conditions (2.8)–(2.14), $\alpha_\varepsilon^{\varepsilon'}$, $\beta_\varepsilon^{\varepsilon'}$, $\gamma_\varepsilon^{\varepsilon'}$, $\varepsilon,\varepsilon' \in \{+,-,0\}$ stand for $I\otimes\alpha_\varepsilon^{\varepsilon'}, I\otimes\beta_\varepsilon^{\varepsilon'}, I\otimes\gamma_\varepsilon^{\varepsilon'}$ respectively, where I is the identity on H_0.

For a detailed exposition of how existence, uniqueness and unitarity of solutions of quantum stochastic differential equations driven by nonlinear noise can be formulated in the language of sesquilinear forms, we refer to Ref. 1.

3. Examples of Unitary SWN Stochastic Evolutions

In this section, we obtain the quantum stochastic differential equation (QSDE) satisfied by the SWN Weyl operator. It will be seen that it is a QSDE of the type considered in Sec. 2 above. For $t \ge 0$, $\lambda, k \in \mathbb{R}$ and $z \in \mathbb{C}$ with $z + \bar{z} + k \ne 0$ let $A(t) = \lambda t + zB(t) + \bar{z}B^\dagger(t) + kN(t)$ and consider $U(t) = e^{iA(t)}$. Notice that $A(t)$ can either be viewed as acting on the noise space only or, by looking at e.g. $zB(t)$ as $zI\otimes B(t)$, on the tensor product of an initial Hilbert space and the noise space. Computing the differential of $U(t)$ we find

$$\begin{aligned} dU(t) &= d(e^{iA(t)}) \\ &= e^{iA(t+dt)} - e^{iA(t)} \\ &= e^{i(A(dt)+A(t))} - e^{iA(t)} \\ &= e^{iA(dt)}e^{iA(t)} - e^{iA(t)} \quad \text{(by the commutativity of } A(dt) \text{ and } A(t)) \end{aligned}$$

$$= e^{iA(t)}[e^{idA(t)} - I]$$

$$= U(t) \sum_{n=1}^{\infty} \frac{(idA(t))^n}{n!} . \tag{3.1}$$

With $\alpha_i^j, \beta_i^j, \gamma_i^j$, $i, j \in \{+, -, 0\}$ as in Sec. 1, let

$$\alpha = z^2\alpha_-^- + z\bar{z}\alpha_-^+ + zk\alpha_-^0 + \bar{z}z\alpha_+^- + (\bar{z})^2\alpha_+^+ + \bar{z}k\alpha_+^0 + kz\alpha_0^- + k\bar{z}\alpha_0^+ + k^2\alpha_0^0 ,$$
$$\beta = z^2\beta_-^- + z\bar{z}\beta_-^+ + zk\beta_-^0 + \bar{z}z\beta_+^- + (\bar{z})^2\beta_+^+ + \bar{z}k\beta_+^0 + kz\beta_0^- + k\bar{z}\beta_0^+ + k^2\beta_0^0 ,$$
$$\gamma = z^2\gamma_-^- + z\bar{z}\gamma_-^+ + zk\gamma_-^0 + \bar{z}z\gamma_+^- + (\bar{z})^2\gamma_+^+ + \bar{z}k\gamma_+^0 + kz\gamma_0^- + k\bar{z}\gamma_0^+ + k^2\gamma_0^0 ,$$
$$\alpha_1 = \alpha_-^+ + \alpha_+^+ + \alpha_0^+ , \qquad \beta_1 = \beta_-^+ + \beta_+^+ + \beta_0^+ , \qquad \gamma_1 = \gamma_-^+ + \gamma_+^+ + \gamma_0^+ ,$$
$$\alpha_2 = \alpha_-^0 + \alpha_+^0 + \alpha_0^0 , \qquad \beta_2 = \beta_-^0 + \beta_+^0 + \beta_0^0 , \qquad \gamma_2 = \gamma_-^0 + \gamma_+^0 + \gamma_0^0 .$$

Using Itô's table we obtain

$$(\lambda dt + zdB(t) + \bar{z}dB^\dagger(t) + kdN(t))(xdt + ydB^\dagger(t) + wdN(t))$$
$$= (z + \bar{z} + k)((y\alpha_1 + w\alpha_2)dt + (y\beta_1 + w\beta_2)dB^\dagger(t) + (y\gamma_1 + w\gamma_2)dN(t)) . \tag{3.2}$$

Thus

$$dA(t) = \lambda dt + zdB(t) + \bar{z}dB^\dagger(t) + kdN(t) , \tag{3.3}$$

$$(dA(t))^2 = \alpha dt + \beta dB^\dagger(t) + \gamma dN(t) \tag{3.4}$$

and by repeated use of (3.2) we find that for $n \geq 3$, in standard matrix notation with (1×1) matrices identified with their entries,

$$(dA(t))^n = (z + \bar{z} + k)^{n-2}(\beta \quad \gamma) \begin{pmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{pmatrix}^{n-3}$$
$$\times \left[\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} dt + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} dB^\dagger(t) + \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} dN(t) \right] . \tag{3.5}$$

By (3.3)–(3.5), (3.1) implies that

$$dU(t) = U(t)\left(\left(i\lambda - \frac{\alpha}{2} + M_\alpha \right) dt + izdB(t) + \left(i\bar{z} - \frac{\beta}{2} + M_\beta \right) dB^\dagger(t) \right.$$
$$\left. + \left(ik - \frac{\gamma}{2} + M_\gamma \right) dN(t) \right), \tag{3.6}$$

$$U(0) = I ,$$

where the (1×1) matrices $M_\alpha, M_\beta, M_\gamma$ are defined by

$$M_\alpha = M\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \qquad M_\beta = M\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \qquad M_\gamma = M\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix},$$

and the (1×2) matrix M is defined by

$$M = (z + \bar{z} + k)^{-2}(\beta \quad \gamma) \left\{ \exp\left[i(z + \bar{z} + k) \begin{pmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{pmatrix} \right] \right.$$
$$\left. - I - i(z + \bar{z} + k) \begin{pmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{pmatrix} + \frac{(z + \bar{z} + k)^2}{2} \begin{pmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{pmatrix}^2 \right\} \begin{pmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{pmatrix}^{-3}$$

with the exponential and the inverse defined weakly on the exponential vectors. We note that the coefficients of (3.6) do not satisfy the unitarity conditions of Theorem 2.1 which therefore are not necessary. This is due to the linear dependence of the SWN differentials. By suppressing the tensor product notation, the above work transfers word by word to show that if $E(t) = \lambda \otimes tI + z \otimes B(t) + \bar{z} \otimes B^\dagger(t) + k \otimes N(t)$, where λ, k, z and its dual $\bar{z}$ are commuting operators (such that $z + \bar{z} + k$ is invertible and λ, k are self-adjoint) acting on an initial Hilbert space, then $U(t) = e^{iE(t)}$ also satisfies (3.6).

4. The Sufficient Conditions: An Example

We will show how one can obtain an example of coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ satisfying the unitarity conditions of Theorem 2.1. In what follows we use small letters for sesquilinear forms on S derived from analytic functions as described in (1.4), (1.5) and capital letters for operators on H_0. Let the coefficients $\mathcal{C}$ and $\mathcal{D}$ in Theorem 2.1 be of the form

$$\mathcal{C} = L \otimes k , \tag{4.1}$$

$$\mathcal{D} = W \otimes m , \tag{4.2}$$

where $m \equiv m$. Then (2.9) and (2.8) imply respectively

$$\mathcal{B} = -L^* \otimes \bar{k} , \tag{4.3}$$

$$\mathcal{A} + \mathcal{A}^* = (L^*)^2 \otimes \bar{k}^2\alpha_-^- + L^2 \otimes k^2\alpha_+^+ - L^*L \otimes \bar{k}k\alpha_-^+ - LL^* \otimes \bar{k}k\alpha_+^-$$
$$+ L^*W^* \otimes \bar{k}m\alpha_-^0 + WL \otimes mk\alpha_0^+ - LW^* \otimes km\alpha_+^0$$
$$- WL^* \otimes mk\alpha_0^- - WW^* \otimes m^2\alpha_0^0 . \tag{4.4}$$

Replacing (4.4) in (2.12) and using $\alpha_-^0 = \alpha_0^-, \alpha_+^0 = \alpha_0^+$, we obtain

$$[L^*, L] \otimes \bar{k}k\alpha_+^- + [L^*, \mathrm{Re}W] \otimes 2\bar{k}m\alpha_0^- + [\mathrm{Re}W, L] \otimes mk\alpha_0^+$$
$$+ [W, W^*] \otimes m^2\alpha_0^0 = 0 \tag{4.5}$$

which is satisfied if

$$[L^*, L] = [L^*, \mathrm{Re}W] = [\mathrm{Re}W, L] = [W, W^*] = 0 , \tag{4.6}$$

where $[x, y] = xy - yx$ and Re denotes the real part. Returning to (4.4) we notice that if

$$L = L^* \tag{4.7}$$

and k is chosen so that $\bar{k}\alpha_-^0 = k\alpha_0^+$, i.e.

$$\frac{\bar{k}}{k} = \frac{\partial_t^*}{\partial_t}, \tag{4.8}$$

then

$$\mathcal{A} + \mathcal{A}^* = L^2 \otimes k^2 \left(\left(\frac{\partial_t^*}{\partial_t} \right)^2 \alpha_-^- + \alpha_+^+ - \left(\frac{\partial_t^*}{\partial_t} \right) (\alpha_-^+ + \alpha_+^-) \right) - WW^* \otimes m^2 \alpha_0^0. \tag{4.9}$$

If $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ satisfy (4.1)–(4.3) and (4.9), then conditions (2.8) and (2.9) are satisfied. Replacing (4.1)–(4.3) and (4.9) in (2.10) and using the fact that

$$\beta_-^+ = \beta_+^-, \tag{4.10}$$

$$\frac{\bar{k}^2 \beta_-^-}{k^2 \beta_+^+} = 1, \tag{4.11}$$

$$\frac{\bar{k} k \beta_-^+}{k^2 \beta_+^+} = 1, \tag{4.12}$$

$$k\beta_+^0 - \bar{k}\beta_-^0 = \bar{k}\beta_0^- - k\beta_0^+, \tag{4.13}$$

(2.10) becomes

$$(LW^* + WL) \otimes m(k\beta_+^0 - \bar{k}\beta_-^0) + WW^* \otimes m^2 \beta_0^0 = 0 \tag{4.14}$$

which is satisfied if we choose m, L, W so that

$$m = \frac{\bar{k}\beta_-^0 - k\beta_+^0}{\beta_0^0}, \tag{4.15}$$

$$LW^* + WL = WW^*. \tag{4.16}$$

An easy computation shows that indeed $m = \bar{m}$. Using

$$k\gamma_+^0 - \bar{k}\gamma_-^0 = \bar{k}\gamma_0^- - k\gamma_0^+ = 0 \tag{4.17}$$

condition (2.11) becomes

$$(W + W^*) \otimes m + L^2 \otimes (-\bar{k}^2 \gamma_-^- - k^2 \gamma_+^+ + \bar{k}k\gamma_-^+ + \bar{k}k\gamma_+^-) + WW^* \otimes m^2 \gamma_0^0 = 0 \tag{4.18}$$

which is satisfied if we choose W, m so that

$$W + W^* = WW^* = L^2, \tag{4.19}$$

$$m + m^2 \gamma_0^0 = -\bar{k}^2 \gamma_-^- - k^2 \gamma_+^+ + \bar{k}k\gamma_-^+ + \bar{k}k\gamma_+^-. \tag{4.20}$$

Let

$$\Theta = \partial_t \partial_t^*. \tag{4.21}$$

Using (4.15), dividing by $\bar{k}$ and then using (4.8), (4.20) implies

$$\begin{aligned} k &= \frac{\frac{\beta_-^0}{\beta_0^0} - \frac{\partial_t \beta_+^0}{\partial_t^* \beta_0^0}}{\frac{\partial_t^* \gamma_-^-}{\partial_t} + \frac{\partial_t \gamma_+^+}{\partial_t^*} - \gamma_-^+ - \gamma_+^- - \frac{\partial_t^*}{\partial_t} \left(\frac{\beta_-^0}{\beta_0^0} \right)^2 \gamma_0^0 - \frac{\partial_t}{\partial_t^*} \left(\frac{\beta_+^0}{\beta_0^0} \right)^2 \gamma_0^0 + \frac{2\beta_-^0 \beta_+^0 \gamma_0^0}{\beta_0^{0\,2}}} \\ &= \frac{1 + 4\Theta}{-8\partial_t^*(1 + 2\Theta + 8\Theta^2)} \end{aligned} \tag{4.22}$$

and by (4.15)

$$m = \frac{(-1 + 4\Theta)(1 + 4\Theta)^2}{16\Theta(1 + 2\Theta + 8\Theta^2)}. \tag{4.23}$$

Using (4.22) and (4.23), (4.9) implies

$$\mathcal{A} + \mathcal{A}^* = L^2 \otimes c\, a(\partial_t, \partial_t^*), \tag{4.24}$$

where

$$a(\partial_t, \partial_t^*) = \frac{(1 + 4\Theta)^2(-1 + 4\Theta + 64\Theta^3)}{32\Theta(1 + 2\Theta + 8\Theta^2)^2}. \tag{4.25}$$

Moreover, (4.19) implies that

$$\mathrm{Re} W = 2WW^* \tag{4.26}$$

from which, using $[W, W^*] = 0$, we obtain

$$(\mathrm{Im} W)^2 = \frac{\mathrm{Re} w - 2(\mathrm{Re} W)^2}{2} = L^2 - 4L^4 \tag{4.27}$$

which implies

$$\mathrm{Im} W = (L^2 - 4L^4)^{1/2} \tag{4.28}$$

provided that

$$L^2 - 4L^4 \geq 0. \tag{4.29}$$

We may now prove the following:

Theorem 4.1. *Let L, H be self-adjoint operators in H_0 such that $L^2 \leq I/4$, let $a(\partial_t, \partial_t^*), k(\partial_t, \partial_t^*), m(\partial_t, \partial_t^*)$ be sesquilinear forms on S defined by (4.25), (4.22), (4.23), and let $h(\partial_t, \partial_t^*)$ be a sesquilinear form on S such that $h = \bar{h}$. Then the solution $U = \{U(t) \colon 0 \leq t \leq t_0 < +\infty\}$ of the initial value problem*

$$\begin{aligned} dU(t) &= \{(L^2 \otimes ca(\partial_t, \partial_t^*) + iH \otimes h(\partial_t, \partial_t^*))dt - L \otimes \bar{k}(\partial_t, \partial_t^*)dB(t) \\ &\quad + L \otimes k(\partial_t, \partial_t^*)dB^+(t) + (2L^2 + i(L^2 - 4L^4)^{1/2}) \\ &\quad \otimes m(\partial_t, \partial_t^*)dN(t)\}U(t), \end{aligned} \tag{4.30}$$

$$U(0) = I$$

is unitary.

Proof. Conditions (2.8)–(2.11) are obviously satisfied since (4.30) was constructed to satisfy them. Direct substitution of the coefficients of (4.30) into (2.12)–(2.14) shows that they are also satisfied and the result follows by Theorem 2.1. □

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