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Lévy Laplacian Acting on Operators

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1. INTRODUCTION

A general definition of a second-order homogeneous differential operator acting on a (vector) space $\mathcal{F}(E, G)$ of mappings of a locally convex space E into a locally convex space G can be formulated as follows (see [6–9] and the references therein). Let γ be a linear mapping from a subspace of the space $\mathcal{L}(E(\mathcal{L}(E,G))$ of (continuous) linear mappings from the space E to the space $\mathcal{L}(E,G)$ of (continuous) linear mappings from E to G; then by a second-order homogeneous differential operator on $\mathcal{F}(E,G)$ related to γ we mean the mapping of $\mathcal{F}(E,G)$ into itself denoted by Δ^γ and defined by

$$(\Delta^\gamma(f))(x) = \gamma(f''(x)).$$

If $G = \mathbb{R}$ and if γ is a positive functional on $\mathcal{L}(E,G)$, then $(\Delta^\gamma(f))$ is called a *Laplacian*; if, for the same G, the functional γ vanishes on the elements of $\mathcal{L}(E,G)$ having finite-dimensional range, then the operator $(\Delta^\gamma(f))$ is called a *Levy Laplacian* (for the Volterra(-Gross) Laplacians, see [6, 8]).

A special case of Levy Laplacian was introduced and studied by P. Lévy in his famous book [1]. This operator, which we denote by Δ_L, is defined on a space of real-valued functions on a (separable) Hilbert space H by

$$\Delta_L = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2}; \tag{1.1}$$

here $\dfrac{\partial^2}{\partial x_n^2}$, for each $n \in N$, stands for the second derivative along the direction defined by the nth element of a sequence $\{e_n\}$ in H; thus, in this case we have $E = H$, $G = \mathbb{R}$, $\mathcal{L}(E(\mathcal{L}(E,G)) = H$ and

$$\gamma\Big(\sum_n a_n e_n\Big) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n.$$

We call this operator the *Lévy Laplacian* associated with the sequence $\{e_n\}$. If $\{e_n\}$ is an orthonormal basis, then the operator Δ_L is the classical Lévy Laplacian (see the appendix).

In a similar way one can define the Levy Laplacian for matrix-valued functions; namely,

$$\Delta_L = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \operatorname{tr} \frac{\partial^2}{\partial x_n^2}, \tag{1.2}$$

where tr stands for the trace in the space of matrixes.

The most important application of Levy Laplacians, namely, that to the theory of gauge fields, was discovered in [2–4] (see also [5]); in these papers it has been shown that gauge fields satisfying Yang–Mills equations are harmonic functions with respect to the Levy Laplacians defined on proper spaces of vector-valued functions. In particular, it has been proved in these papers that a connection on $\mathbb{R}^d$ satisfies the Euclidean Yang–Mills equations if and only if the associated parallel transport U_γ, when regarded as a function of the curve γ along which the parallel transport takes place, is a harmonic function for the Levy–Laplace equation

$$\Delta_L U_\gamma = 0. \tag{1.3}$$

On the flat manifold $\mathbb{R}^d$, the parallel transport U_γ is a matrix valued function with domain given by an appropriate function space, namely, the space of paths. Thus, the study of the Yang–Mills equations is equivalent to the study of a Levy–Laplace equation with respect to a matrix-valued function.

On the other hand, Schrödinger equations and heat equations involving Levy Laplacians are related to some problems of quantum statistical physics; finally, there exist interesting relations between the Levy Laplacian and the squared quantum white noise (see [29, 9, 10] and the references therein).

However, the study of fixed points for the heat semigroup generated by Δ_L,

$$e^{t\Delta_L} U_\gamma = U_\gamma, \qquad \forall t > 0, \tag{1.4}$$

constitutes a generalization of equation (1.3). Since the construction of the heat semigroup $\exp t\Delta_L$ is equivalent to the solution of the Levy heat equation

$$\begin{aligned} \partial_t U_\gamma(t) &= \Delta_L U_\gamma, \\ U_\gamma(0) &= id, \end{aligned} \tag{1.5}$$

where id stands for the identity matrix of size $d \times d$, we see that the study of the most general class of solutions of equation (1.3) naturally leads to the problem of constructing the heat semigroup for matrix-valued functions

$$U : (t, \gamma) \in \mathbb{R} \times \{\text{Path space}\} \to U_\gamma(t) \in \text{ Linear operators on } \mathbb{R}^d$$

In a similar way one can define a Levy-type Laplacians acting on functions of two (infinite-dimensional) variables; if those functions are symbols (in some sense) of linear operators, then one can regard these Levy-type Laplacians as acting on operators and investigate the corresponding Levy heat equations, which one can call *Levy heat equations with respect to operators*. This is just the aim of the present paper.

Recently, using functions on a nuclear space, Accardi–Ouerdiane [11] solved a heat equation associated with the Lévy Laplacian by means of an analytic one parameter semi-group, see also Accardi–Smolyanov and Accardi—Roselli—Smolyanov [6–9], Saito—Tsoi [28], and Obata [29].

In the present paper, employing the recent framework of infinite-dimensional holomorphic function due to Gammoun–Hachaichi–Ouerdiane–Rezgui [16], the theory of operators defined on a space of holomorphic functions due to Ben Chnouda–Ouerdiane and the convolution calculus studied by Ben Chouda–El Oued–Ouerdiane, we investigate the operator version of Lévy Laplacian Δ_L^Q, i.e., a Δ_L^Q acting on a infinite-dimensional algebras of operators (in the sense described above). As was already mentioned above, our method to solve the Cauchy problem

$$\begin{aligned} \frac{\partial A}{\partial t} &= \Delta_L^Q A, \\ A(0) &= A_0, \end{aligned} \tag{1.6}$$

where the initial condition A_0 is a linear operator on the nuclear space E, is the following. We consider a space of operators which are in one-to-one correspondence with their (distribution) kernels $A(x, y)$. Then we interpret equation (1.6) as an equation on the kernels and solve it by the Fourier transform method. If the kernel of A_0 is the Fourier transform of a measure μ on E' (E' stands for the strong dual of a nuclear space E) which is invariant under a certain shift operator, then a solution of (1.6) can explicitly be obtained by the method of Accardi–Roselli–Smolyanov [7]. Thus, one can say that the main results of the paper are related to solving the Cauchy problem (1.6) in some new spaces of entire functions (introduced in [15]). An example of a nuclear Fréchet space with a continuous norm, which is not a countably normed space in the sense of Gelfand–Shilov (see Theorem 1 below) is of independent interest; for the definition of countably normed spaces, see [30] and Remark 1 below.

In the appendix we give a review of some general properties of Levy Laplacians; though the results of the appendix are not used in the main part of the paper, the acquaintance with the appendix can help to put the results of the paper into a more general frame.

2. ENTIRE FUNCTIONS OF θ-EXPONENTIAL GROWTH

Let $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, convex, increasing function satisfying the condition

$$\lim_{x\to\infty} \theta(x)\overline{x} = +\infty \qquad \text{and} \qquad \theta(0) = 0. \tag{2.1}$$

Such a function is called a *Young function* and admits the representation

$$\theta(x) = \int_0^x p(t)\,dt, \qquad n \geqslant 0,$$

where

$$p\colon [0, +\infty[\to [0, +\infty[\qquad \text{has the following properties:}$$

p is

i) right continuous,
ii) increasing,
iii) $p(0) \geqslant 0$,
iv) $\lim_{t\to\infty} p(t) = \infty$.

For a Young function θ we write

$$\theta^*(x) = \sup_{t\geqslant 0}\{tx - \theta(x)\}. \tag{2.2}$$

The function θ^* is called the *polar function associated with* θ. One can show that θ^* is again a Young function and that $(\theta^*)^* = \theta$.

Definition 1. Let X be a locally convex space over the field $\mathbb{C}$ of complex numbers. A function $f\colon X \to \mathbb{C}$ is said to be a *Gâteaux entire function* if, for each ξ, $h \in X$, the $\mathbb{C}$-valued function of one complex variable,

$$\zeta \to f(\xi + \zeta\eta),$$

is holomorphic at every point $\zeta \in \mathbb{C}$. A Gâteaux entire function $f\colon X \to \mathbb{C}$ is said to be an *entire function* if it is continuous on X, or, equivalently if it is locally bounded, i.e., every point of X is contained in a neighborhood on which f is bounded.

Consider now a complex Banach space $(B, \|\ \|)$. We classify entire functions on B by means of their growth rate at infinity. Let θ be a Young function. An entire function $f\colon B \to \mathbb{C}$ is said to have θ-exponential growth of finite type $m > 0$ if

$$\|f\|_{\theta,m} \equiv \sup_{\zeta\in B} |f(\zeta)| e^{-\theta(m\|\zeta\|)} < +\infty. \tag{2.3}$$

Let $\exp(B, \theta, m)$ be the space of all such functions; this space becomes a Banach space when equipped with the norm $\|\cdot\|_{\theta,m}$.

2.1. *Entire Functions on Nuclear Spaces in Two Variables*

For two Banach spaces $(B_1, \|\cdot\|_1)$ and $(B_2, \|\cdot\|_2)$, denote by $H(B_1 \times B_2)$ the space of all entire functions on $B_1 \times B_2$, i.e., of all continuous functions $f\colon B_1 \times B_2 \to \mathbb{C}$ such that the mapping

$$(\zeta_1, \zeta_2) \to f(a_1 + \zeta_1 b_1, a_2 + \zeta_2 b_2)$$

is an entire function on $\mathbb{C} \times \mathbb{C}$ for every a_1, $b_1 \in B_1$ and a_2, $b_2 \in B_2$. For a given pair of functions (θ, φ) and a pair of positive numbers $m_1 > 0$, $m_2 > 0$, introduce the Banach spaces

$$\begin{gathered} \exp(B_1 \times B_2, (\theta, \varphi), (m_1, m_2)) \equiv \{f \in H(B_1 \times B_2), \|f\|_{(\theta,\varphi),(m_1,m_2)} < \infty\}, \\ \exp(B_1, \theta, m_1) \equiv \{f \in H(B_1), \|f\|_{\theta,m_1} < \infty\}, \end{gathered}$$

where

$$\|f\|_{(\theta,\varphi),(m_1,m_2)} = \sup_{\substack{z_1\in B_1\\ z_2\in B_2}} |f(z_1, z_2)| e^{-\theta m_1\|z_1\|_1 - \varphi m_2\|z_2\|_2},$$

$$\|f\|_{\theta,m_1} = \sup_{z\in B_1} |f(z)| e^{-\theta m_1\|z\|_1}.$$

We now introduce new spaces of entire functions of two variables.

Let M and N be two nuclear countably Hilbert spaces in the sense of [30] (cf. the comments at the end of this section) with the defining Hilbertian norms $\{|\cdot|_{M,p}\}$ and $\{|\cdot|_{N,p}\}$, respectively. For each $p \geqslant 0$, let M_p (N_p) denote the Hilbert space obtained by completing M (N) relative to $|\cdot|_{M,p}$ (to $|\cdot|_{N,p}$, respectively). Then

$$M = \operatorname*{limproj}_{p\to\infty} M_p \left(= \bigcap_{p=0}^{\infty} M_p \text{ as sets} \right),$$

$$N = \operatorname*{limproj}_{p\to\infty} N_p \left(= \bigcap_{p=0}^{\infty} N_p \text{ as sets} \right).$$

Denote by M_{-p} (N_{-p}) the dual space of M_p (N_p, respectively). By the general duality theory, the strong dual spaces M' and N' can be obtained by

$$M' = \operatorname*{limind}_{p\to\infty} M_{-p} \left(\doteq \bigcup_{p\in\mathbb{N}} M_{-p} \text{ as sets} \right),$$

$$N' = \operatorname*{limind}_{p\to\infty} N_{-p} \left(= \bigcup_{p\in N} N_{-p} \text{ as sets} \right).$$

Remark 1. According to [30], a nuclear, countably Hilbert space is a nuclear Fréchet space N whose topology can be defined by an increasing sequence of Hilbert semi-norms (p_n) satisfying the following properties:

(i) for each n, the embedding

$$n \in \mathbb{N} \subseteq (N, p_{n+1}) \to n \in N \subseteq (N, p_n)$$

has a Hilbert–Schmidt extension (nuclearity),

(ii) the Hilbert–Schmidt extension mentioned in (i) is injective (countably Hilbert). Equivalently, if a Cauchy sequence for p_{n+1} converges to zero in the p_n-norm, then it also converges to zero in the p_{n+1}-norm (in this case the norms p_n and p_{n+1} are usually said to be *compatible*).

The following counterexample shows that there exists a nuclear Fréchet space whose topology can be defined by a sequence of Hilbert norms which cannot be chosen to be compatible.

Let E be the vector subspace of $\mathcal{D}_{[0,1]}$ consisting of all functions $\varphi \in \mathcal{D}_{[0,1]}$ whose supports do not intersect the set $\{0, 1\}$.

Let $t_n = \frac{1}{2^n}$ for any $n \in \mathbb{N}$, and let $\{n_i^k : k, i \in \mathbb{N}\}$ be a set of elements of [0,1]; we assume that

$$\bigcup_{k=1}^{\infty} \{t_{n_j^k} : j \in N\} = \{t_n : n \in N\}$$

and that

$$\{t_{n_j^k} : j \in N\} \cap \{t_{n_j^k} : j \in N\} = \varnothing$$

for any $k, i \in N$, $k \neq i$ (we identify sequences of pairwise different elements (in [0,1]) with the sets of their elements).

Let E be the space of all infinitely differentiable functions on [0,1], and let, for any $r \in N$, the Hilbert norm p_r on E be defined by

$$(p_r(\varphi))^2 = \sum_{k=1}^{r} \int_0^1 |\varphi^{(k)}(t)|^2 dt + \sum_{k=1}^{r} \sum_{i=1}^{\infty} c_i^r |\varphi^{(i)}(t_{n_i^k})|^2,$$

where $c_i^r = 2^{r^i}$.

Then $p_1 \leqslant p_2 \leqslant p_3 \leqslant \dots$, but those norms are not compatible.

Let τ be the locally convex topology on E defined by the sequence $P = \{p_1, p_2, \dots\}$ of norms, and let $\overline{E}$ be the completion of (E, τ). Then the space $\overline{E}$ is a nuclear Fréchet space. Below we denote (continuous) norms on E and continuous extensions of these norms to $\overline{E}$ by the same symbols.

Theorem 1. *There is no sequence of pairwise compatible norms on E defining the topology τ.*

Proof. First, let us show that, if $s_1, s_2 \in N$, and $s_1 \neq s_2$, then the norms p_{s_1} and p_{s_2} are not compatible. Indeed, let $s_2 > s_1$; then there exists a sequence $\{\varphi_j\}$ of functions in E having the following properties: $\{\varphi_j\}$ is the Cauchy sequence with respect to both norms p_{s_1} and p_{s_2}, $p_{s_1}(\varphi_j) \to 0$, and $\varphi_j(t_{n_{s_2}^{s_2}}) \to \alpha \neq 0$.

(Then $\varphi_j \to 0$ in $\overline{E}_{p_{s_2}}$ and $\varphi_j \to \varphi \neq 0$ in $\overline{E}_{p_{s_1}}$).

Now let $Q = \{q_r\}$ be any sequence of norms on E defining the same topology τ.

Let $q_1 \in Q$. Then there exists a $p_{s_1} \in P$ such that $q_1 \leqslant c p_{s_1}$ for some $c > 0$; for simplicity, here and below, we assume that c and some other similar constants are equal to one.

Further, there exist $q_2 \in Q$ and $p_{s_2} \in P$ such that

$$q_1 \leqslant p_{s_1} \leqslant p_{s_1+1} \leqslant q_2 \leqslant p_{s_2}.$$

Then one can find (by using the definition of the norms p_r) a sequence $\{\varphi_j\} \subset E$ having the following properties.

1. $\{\varphi_j\}$ is a Cauchy sequence with respect to p_{s_1}, p_{s_1+1}, and p_{s_2};
2. $p_{s_1}(\varphi_j) \to 0$;
3. $p_{s_1+1}(\varphi_j) \not\to 0$.

Then

(1') $\{\varphi_j\}$ is a Cauchy sequence with respect to both q_1 and q_2,
(2') $q_1(\varphi_j) \to 0$,
(3') $q_2(\varphi_j) \not\to 0$.

The theorem is proved.

The space of entire functions on $M' \times N'$ of (θ, φ)-exponential growth of minimal type is defined by

$$\mathcal{F}_{(\theta,\varphi)}(M' \times N') = \operatorname*{limproj}_{\substack{p\to\infty \\ m_1>0, m_2>0}} \exp(M_{-p} \times N_{-p}, (\theta, \varphi), (m_1, m_2)).$$

Similarly, the space of entire functions on $M \times N$ of (θ, φ)-exponential growth of finite type is defined by

$$\mathcal{G}_{(L,\varphi)}(M \times N) = \operatorname*{limind}_{\substack{p\to\infty \\ m_1, m_2\to\infty}} \exp(M_p \times N_p, (\theta, \varphi), (m_1, m_2)).$$

Proposition 1. *1) A function $f\colon (M \times N)' \to \mathbb{C}$ belongs to $\mathcal{F}_{(\theta,\varphi)}(M' \times N')$ which is isomorphic to $\mathcal{F}_{(\theta,\varphi)}((M \oplus N)')$ if and only if*

$$\sup_{\substack{y\in M \\ \zeta\in M}} |f(\omega, \zeta)| e^{-\theta(m|\omega|_{M,-p}) - \varphi(m|\zeta|_{N-p})} < \infty$$

for any pair $p \geqslant 0$ and $m > 0$.

2) A function $g\colon M \times N \to \mathbb{C}$ belongs to $\mathcal{G}_{\theta,\varphi}(M \times N)$ if and only if there exist a pair $p \geqslant 0$ and $m > 0$ such that

$$\sup_{\substack{\omega\in M \\ \zeta\in M}} (g(\omega, \zeta)| e^{-\theta(m|\omega|_{M,p}) - \varphi(m|\zeta|_{N,p})} < \infty.$$

Theorem 2. *There exists a unique topological isomorphism*

$$\mathcal{F}_{(\theta,\varphi)}((M \oplus N)') \to \mathcal{F}_\theta(M') \hat{\otimes} \mathcal{F}_\varphi(N')$$

which extends the correspondence

$$e^{(\xi,\eta)} \longleftarrow e^\xi \otimes e^\eta; \quad \xi \in M \text{ and } \eta \in N,$$

where the exponential function $e^{(\xi,\eta)} = e^{\xi\oplus\eta}$ *is defined by*

$$e^{(\xi,\eta)} : M' \times N' \to \mathbb{C},$$
$$(\omega,\zeta) \to e^{(\xi,\eta)}(\omega,\eta)(\omega,\zeta) := \exp(\langle\omega,\xi\rangle + \langle\zeta,\eta\rangle),$$

and where

$$\mathcal{F}_\theta(M') = \operatorname*{limproj}_{\substack{p\to\infty\\ m\to 0}} \exp(M_{-p},\theta,m)$$

and

$$F_\varphi|N'| = \operatorname*{limproj}_{\substack{m\to\infty\\ p\to 0}} \exp(N_{-p},\varphi,m).$$

We also use the notation

$$e^{(\xi,0)} =: e^\xi; \qquad e^{(0,\eta)} =: e^\eta.$$

The main property of the exponential functions $e^{(\xi,\eta)}$ $(e^\xi$*, and* $e^\eta)$ *is that they are dense in the space* $\mathcal{F}_{(\theta,Q)}((M\oplus N)')$ *(in the spaces* $\mathcal{F}_{(\theta,Q)}((M\oplus N)')$*, and* $\mathcal{F}_Q(N')$*,* $\mathcal{F}_\varphi(M')$*, respectively) and linearly independent.*

This enables one to extend the notion of Laplace transform to these spaces and to prove its invertibility.

Theorem 3. *Let M and N be complex nuclear Fréchet spaces, and let θ and φ be Young functions. Then the Laplace transform*

$$\mathcal{L} : \mathcal{F}^*_{\theta,\varphi}(M'\oplus N') \to \mathcal{G}_{(\theta^*,\varphi^*)}(M\oplus N),$$
$$T \to \mathcal{L}T = \hat{T},$$

defined by

$$\hat{T}(\xi,\eta) = \langle T, e^{\xi\oplus\eta}\rangle,$$

is a topological isomorphism.

Proof. This can be established by direct calculations.

3. CHARACTERIZATION OF OPERATORS

We denote by $\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}_\theta)$ the algebra of continuous linear operators acting on $\mathcal{F}_\theta(N')$ and equipped with the topology of bounded convergence.

Definition 2. The *symbol* of an operator $A \in \mathcal{L}(\mathcal{F}_\theta,\mathcal{F}_\theta)$ is a complex-valued function denoted by $\sigma(A)$ and defined by

$$\sigma(A)(\xi,\eta) = A(e^\xi)(e^\eta) = \langle Ae^\xi, e^\eta\rangle; \qquad \xi\in N', \quad \eta \in N.$$

Remark 2. This definition of symbol differs from that used in the theory of pseudodifferential operators.

By the Grothendieck–Schwartz kernel theorem, if we take two nuclear Fréchet spaces X and $\mathcal{D}$, then the canonical correspondence $A \in \mathcal{L}(X,D) \to A^K$ given by

$$\langle Au, v\rangle = \langle A^K, u\otimes v\rangle \qquad u\in X, \qquad v\in D', \quad A\in\mathcal{L}(X,D),$$

yields a topological isomorphism between the spaces $\mathcal{L}(X,D)$ and

$$X'\hat{\otimes}D \equiv X'\hat{\otimes}D'' \equiv (X\hat{\otimes}D')'$$

(cf. [34] concerning these identifications for nuclear Fréchet spaces). In particular, if we take $X = D = \mathcal{F}_\theta(N')$, which is a nuclear Fréchet space, then

$$\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta(N')) \cong \mathcal{F}^*_\theta\hat{\otimes}\mathcal{F}_\theta.$$

Proposition 2. *The symbol $\sigma(A)$ of an operator $A\in\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}_\theta)$ is the Laplace transform of the kernel A^k,*

$$\sigma(A)(\zeta,\xi) = A^k(e_\xi\otimes\delta_\eta) \qquad \zeta\in N', \quad \xi\in N,$$

and the symbol mapping yields a topological isomorphism between $\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}_\theta)$ and $\mathcal{F}_\theta(N')\hat{\otimes}\mathcal{G}_{\theta^*}(N)$. More precisely, we have the following isomorphisms:

$$\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}_\theta) \xrightarrow{\sigma} \mathcal{F}_\theta\hat{\otimes}\mathcal{G}_{\theta^*} \xrightarrow{s.t} F_\theta\hat{\otimes}G_\theta,$$
$$A \to \sigma(A)(\zeta,\xi) = \sum_{l,m\geqslant 0}\langle\sigma_{l,m},\zeta^{\otimes l}\otimes\xi^{\otimes m}\rangle \to \vec{\sigma} = (\sigma_{l,m})_{l,m\geqslant 0}.$$

Proof. This can be checked by direct calculations.

Theorem 4. *The symbol mapping $A\to\hat{A} = \sigma(A)$ is a topological isomorphism*

$$\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}^*_\theta(N')) \to \mathcal{G}_{\theta^*,\theta^*}(N\oplus N) \simeq \mathcal{G}_{\theta^*}(N)\hat{\otimes}\mathcal{G}_{\theta^*}(N).$$

Remark 3. The Dirac mass δ_ζ, for every $\zeta\in N'$, is defined by $\langle\langle\zeta_2,\varphi\rangle\rangle = \varphi(z)$ for every test function $\varphi\in\mathcal{F}_\theta(N')$; for the function $\zeta\mapsto\delta_\zeta$ we have the following expansion in power series:

$$\delta_\zeta = \left(\frac{\zeta^{\otimes n}}{n!}\right)_{n\geqslant 0}; \qquad \zeta\in N'.$$

4. ELLIPTIC OPERATORS OF LÉVY TYPE

Let E_1 and E_2 be two real nuclear Fréchet spaces as before. A function $E\colon E_1\times E_2\to\mathbb{R}$ is called a function of class $\mathbb{C}^2(E_1\times E_2)$ if there exist two continuous mappings

$$(\xi_1,\xi_2)\to F'(\xi_1,\xi_2)\in E_1'\times E_2'$$

and

$$(\xi_1,\xi_2)\to F''(\xi_1,\xi_2)\in\mathcal{L}(E_1\times E_2, E_1'\times E_2')$$

such that

$$F((\xi_1,\xi_2)+(\eta_1,\eta_2)) = F(\xi_1+\eta_1),(\xi_2+\eta_2)) = F(\xi_1,\xi_2)$$
$$+\langle F'(\xi_1,\xi_2),(\eta_1,\eta_2)\rangle + \frac{1}{2}\langle F''(\xi_1,\xi_2)(\eta_1,\eta_2),(\eta_1,\eta_2)\rangle + \varepsilon(\eta_1,\eta_2),$$

where the error term satisfies the condition

$$\lim_{t\to 0}\frac{\varepsilon(t\eta_1,t\eta_2)}{t^2}\to 0$$

for any $(\eta_1,\eta_2)\in E_1\times E_2$; applying the kernel theorem, we use the common symbol $F''(\xi_1,\xi_2)$ for all

$$\langle F''(\xi_1,\xi_2)(\eta_1,\eta_2),(\eta_1,\eta_2)\rangle = \langle F''(\xi_1,\xi_2),(\eta_1\otimes\eta_1,\eta_2\otimes\eta_2)\rangle$$
$$= F''(\xi_1,\xi_2)((\eta_1,\eta_2),(\eta_1,\eta_2)) = D_{(\eta_1,\eta_2)}D_{(\eta_1,\eta_2)}F(\xi_1,\xi_2),$$

where $D_{(\eta_1,\eta_2)}$ is the derivative in the direction (η_1,η_2), i.e.,

$$(D_{(\eta_1,\eta_2)}F)(\xi_1,\xi_2) = \lim_{\lambda\to 0}\frac{F(\xi_1+\lambda\eta_1,\xi_2+\lambda\eta_2)-F(\xi_1,\xi_2)}{\lambda} = \frac{d}{dt}(F(\xi_1\oplus\xi_2+t(\eta_1\oplus\eta_2))_{t=0}.$$

Let $\{e_{n_1}\oplus f_{n_2}\}_{\substack{n_1\in\mathbb{N}\\ n_2\in\mathbb{N}}}$ be a sequence in $E_1\oplus E_2$.

Definition 3. Let $F\colon E_1 \oplus E_2 \to \mathbb{C}$ be an element of $C^2(E_1 + E_2)$. Then the elliptic operator Δ_L of Lévy type associated with the sequence $\{e_{n_1} \oplus f_{n_2}\}_{\substack{n_1\in\mathbb{N}\\ n_2\in\mathbb{N}}}$ is defined by

$$(\Delta_L F)(\xi_1,\xi_2) = \lim_{\substack{N_1\to\infty\\ N_2\to\infty}} \frac{1}{N_1N_2}\left(\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\langle F''(\xi_1,\xi_2),(e_{n_1}\oplus f_{n_2})^{\otimes 2}\rangle\right) \tag{5.1}$$

if this limit exists for every $\xi_1 \in E_1$ and $\xi_2 \in E_2$.

Lemma 1. *If $E_2 = \{0\}$, then the elliptic Levy operator Δ_L defined by* (5.1) *coincides with the Lévy Laplacian associated with $\{e_{n_1}\}$.*

Proof. Indeed, if we set $E_2 = \{0\}$ therein, then we see that

$$E_1 \oplus E_2 = E_1, \qquad \text{where } F(\xi_1,\xi_2) \equiv F(\xi_1,0) \text{ is denoted by } F(\xi_1),$$

which implies that

$$(\Delta_L F)(\xi_1) = \lim_{N_1\to\infty N_2\to\infty} \frac{1}{N_1N_2}\left(\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\langle F''(\xi_1),e_{n_1}\otimes e_{n_1}\rangle\right).$$

Since $f_{n_2} = 0$ for every integer n_2, it follows that

$$\Delta_L F(\xi_1) = \lim_{\substack{N_1\to\infty\\ N_2\to\infty}} \frac{1}{N_1N_2}\left(N_2\sum_{n_1=1}^{N_1}\langle F''(\xi_1),e_{n_1}\otimes e_{n_1}\rangle\right) = \lim_{N_1\to\infty}\frac{1}{N_1}\sum_{n_1=1}^{N_1}\langle F''(\xi_1),e_{n_1}\otimes e_{n_1}\rangle.$$

Definition 4. 1) We denote by $((E_1\oplus E_2)\otimes(E_1\oplus E_2))'_L$ the set of all elements $a \in (E_1'\oplus E_2')^{\otimes 2}$ for which the limit

$$\begin{aligned}
\langle a\rangle_L = \langle (a_1\oplus a_2)\otimes(a_1'\oplus a_2')\rangle_L &= \lim_{\substack{N_1\to\infty\\ N_2\to\infty}} \frac{1}{N_1N_2}\left(\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\langle a,(e_{n_1}\oplus f_{n_2})^{\otimes 2}\rangle\right)\\
&= \lim_{\substack{N_1\to\infty\\ N_2\to\infty}} \frac{1}{N_1N_2}\left(\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\langle a_1\oplus a_2,e_{n_1}\oplus f_{n_2}\rangle\langle a_1'\oplus a_2',e_{n_1}\oplus f_{n_2}\rangle\right)\\
&= \lim_{\substack{N_1\to\infty\\ N_2\to\infty}} \frac{1}{N_1N_2}\left(\sum_{n_1=1}^{N_1}\sum_{n_2=2}^{N_2}(\langle a_1,e_{n_1}\rangle+\langle a_2,f_{n_2}\rangle)(\langle a_1',e_{n_1}\rangle+\langle a_2',f_{n_2}\rangle)\right)
\end{aligned}$$

exists.

2) Let $(E_1\oplus E_2)'_L$ be the set of all $a = a_1\oplus a_2 \in E_1'\oplus E_2'$ such that the limit

$$\begin{aligned}
\langle a\otimes a\rangle_L &= \langle (a_1\oplus a_2)\otimes(a_1\oplus a_2)\rangle_L\\
&= \lim_{\substack{N_1\to\infty\\ N_2\to\infty}} \frac{1}{N_1N_2}\left(\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}(\langle a_1,e_{n_1}\rangle+\langle a_2,f_{n_2}\rangle)^2\right)
\end{aligned} \tag{5.2}$$

exists. We also write

$$\|a_1\oplus a_2\|_L^2 = \langle (a_1\oplus a_2)\otimes(a_1\oplus a_2\rangle_L.$$

Using the sequences $\{e_{n_1}\}$ and $\{f_{n_2}\}$, one defines the symbols $\langle a_1\rangle_L$ and $\langle a_2\rangle_L$.

Lemma 2. *For any fixed sequence $\{e_{n_1}\oplus f_{n_2}\}\subset E_1\oplus E_2$, we have the following equality:*

$$\|a_1\oplus a_2\|_L^2 = \|a_1\|_{1,L}^2 + \|a_2\|_{2,L}^2 + 2\left(\lim_{N_1\to\infty}\frac{1}{N_1}\left(\sum_{n_1=1}^{N_1}\langle a_1,e_{n_1}\rangle\right)\right)\left(\lim_{N_2\to\infty}\frac{1}{N_2}\left(\sum_{n_2=1}^{N_2}\langle a_2,f_{n_2}\rangle\right)\right).$$

Proof. Indeed, using (5.2), we have

$$\|a_1\oplus a_2\|_L^2 = \lim_{\substack{N_1\to\infty\\ N_2\to\infty}} \frac{1}{N_1N_2}\left(\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}(\langle a_1,e_{n_1}\rangle^2+\langle a_2,f_{n_2}\rangle^2+2\langle a_1,e_{n_1}\rangle\langle a_2,f_{n_2}\rangle)\right),$$

and hence

$$\begin{aligned}
&\|a_1\oplus a_2\|_L^2\\
&= \lim_{\substack{N_1\to\infty\\ N_2\to\infty}} \frac{1}{N_1N_2}\left[N_2\sum_{n_1=1}^{N_1}\langle a_1,e_{n_1}\rangle^2 + N_1\sum_{n_2=1}^{N_2}\langle a_2,f_{n_2}\rangle^2 + 2\left(\sum_{n_1=1}^{N_1}\langle a_1,e_{n_1}\rangle\right)\left(\sum_{n_2=1}^{N_2}\langle a_2,f_{n_2}\rangle\right)\right]\\
&= \lim_{N_1\to\infty}\frac{1}{N_1}\sum_{n_1=1}^{N_1}\langle a_1,e_{n_1}\rangle^2 + \frac{1}{N_2}\lim_{N_2\to\infty}\sum_{n_2=1}^{N_2}\langle a_2,f_{n_2}\rangle^2\\
&\quad + 2\left(\lim_{N_1\to\infty}\frac{1}{N_1}\cdot\sum_{n_1=1}^{N_1}\langle a_1,e_n\rangle\right)\left(\lim_{N_2\to\infty}\frac{1}{N_2}\cdot\sum_{n_2=1}^{N_2}\langle a_2,f_{n_2}\rangle\right),
\end{aligned}$$

which proves the lemma.

Remark 4. 1) If $a = a_1\oplus 0\in E_1$, then

$$\|a\|_L^2 = \|a_1\|_L^2 = \|a_1\|_{1,L}^2, \quad \text{and if} \quad a = 0\oplus a_2, \quad \text{then}$$
$$\|a\|_L^2 = \|a_2\|_L^2 = \|a_2\|_{2,L}^2.$$

2) If the sequence $\{e_{n_1}\oplus f_{n_2}\}$ defining the Lévy operator Δ_L is "diagonal," i.e.,

$$e_{n_1}\oplus f_{n_2} = \delta_{n_1,n_2}(e_{n_1}\oplus f_{n_2}) = \begin{cases} 0 & \text{if} \quad n_1\neq n_2\\ e_n\oplus f_n & \text{if} \quad n_1=n_2=n,\end{cases}$$

then we obtain the following identity:

$$\begin{aligned}
\|a_1\oplus a_2\|_L^2 &=: \lim_{\substack{N_1\to\infty\\ N_2\to\infty}} \frac{1}{N_1N_2}\left[\sum_{n_1=1}^{N_2}\sum_{n_2=1}^{N_2}(\langle a_1,e_{n_1}\rangle+\langle a_2,f_{n_1}\rangle|^2\right]\\
&= \lim_{N_1\to\infty}\frac{1}{N_1}\cdot\sum_{n_1=1}^{N_1}(\langle a_1,e_{n_1}\rangle^2+\langle a_2,f_{n_1}\rangle^2+2\langle a_1,e_{n_1}\rangle\langle a_2,f_{n_1}\rangle)\\
&= \|a_1\|_L^2+\|a_2\|_L^2+2\lim_{N_1\to\infty}\frac{1}{N_1}\cdot\sum_{n_1=1}^{N_1}\langle a_1,e_{n_1}\rangle\langle a_2,f_{n_1}\rangle.
\end{aligned}$$

3) If the sequence $e_{n_1}\oplus f_{n_2}\subset E_1\oplus E_2$ is of the form $e_{n_1}\oplus 0$ or $0\oplus f_{n_2}$, then, for any partition $\mathbb{N}_1\cup\mathbb{N}_2$ of $\mathbb{N}$, i.e., $\mathbb{N}=\mathbb{N}_1\cup\mathbb{N}_2$ and $\mathbb{N}_1\cap\mathbb{N}_2=\varnothing$, we define a diagonal basis of $E_1\oplus E_2$ as follows: if $n\in\mathbb{N}_1$, then $k_n = e_n$ and if, $n\in\mathbb{N}_2$, then $k_n=f_n$.

Remark 5. Let k_n be a given "diagonal basis" of

$$E_1\oplus E_2\,, \quad \text{i.e.,} \quad k_n = \frac{1}{\mathbb{N}_1}(n)e_n \oplus \frac{1}{\mathbb{N}_2}(n)f_n$$

for the partition $\mathbb{N}_1$, $\mathbb{N}_2$ of $\mathbb{N}$, then we have the following "Pythagorean" equality holds:

$$\|a_1\oplus a_2\|_L^2 = \|a_1\|_L^2+\|a_2\|_L^2.$$

Proposition 3. *Let $p \in D_L(E_1 \oplus E_2)$ be the space of all functions $F \in C^2(E_1 \oplus E_2)$ for which $(\Delta_L F)(\xi_1, \xi_2)$ exists for any $\xi_1 \in E_1$ and $\xi_2 \in E_2$ and suppose that $p'(\xi_1 \oplus \xi_2) \in (M+N)'_L$, where $M = E_1 + iE_1$ and $N = E_2 + iE_2$. Then $e^{p(\xi_1 \oplus \xi_2)} \in D_L(E_1 \oplus E_2)$ and*

$$(\Delta_L e^p)(\xi_1, \xi_2) = (\langle p''(\xi_1, \xi_2)\rangle_L + \|p'(\xi_1, \xi_2)\|_L^2) e^{p(\xi_1, \xi_2)};$$

in particular,

$$(\Delta_L e^{a_1 \oplus a_2})(\xi_1, \xi_2) = \|a_1 \oplus a_2\|_L^2 e^{a_1 \oplus a_2}.$$

Proof. Write $h(\xi_1 \oplus \xi_2) = e^{p(\xi_1, \xi_2)}$. We have

$$\langle h''(\xi_1, \xi_2), (e_{n_1} \oplus f_{n_2}) \otimes (e_{n_1} \otimes f_{n_2})\rangle = D_{e_{n_1} \oplus f_{n_2}} D_{e_{n_1} \oplus f_{n_2}} h(\xi_1, \xi_2) = \frac{d^2}{dt^2}(h(\xi_1 \oplus \xi_2 + t(e_{n_1} \oplus f_{n_2})|_{t=0},$$

but

$$\frac{d}{dt} h(\xi_1 \oplus \xi_2 + te_{n_1} \oplus f_{n_2}) = \frac{d}{dt} e^{p(\xi_1 \oplus \xi_2 + t(e_{n_1} \oplus f_{n_2})} = e^{p(\xi_1 \oplus \xi_2)} \cdot \frac{d}{dt}[p(\xi_1 \oplus \xi_2 + t(e_{n_1} \oplus f_{n_2}))$$
$$= e^{p(\xi_1 \oplus \xi_2 + t(e_{n_1} \oplus f_{n_2})} \cdot \langle p'(\xi_1 \oplus \xi)2, e_{n_1} \oplus f_{n_2}\rangle,$$

$$\frac{d^2}{dt^2} h(\xi_1 \oplus \xi_2 + te_n \oplus f_n)|_{t=0} = e^{p(\xi_1 \oplus \xi_2}(\langle p'(\xi_1 \oplus \xi_2, e_{n_1} \oplus f_{n_2}\rangle^2 + \langle p''(\xi_1, \xi_2), (e_{n_1} \oplus f_{n_2})^{\otimes 2}\rangle.$$

As a corollary, we obtain the following assertion.

Theorem 5. *For $(a_1, a_2) \in M'_L \times N'_L$ we have*

$$\Delta_L e^{\langle (a_1, a_2), (\xi_1, \xi_2)\rangle} = \langle (a_1, a_2) \otimes (a_1, a_2)\rangle_L e^{\langle a_1, \xi_1\rangle + \langle a_2, \xi_2\rangle} = \|a_1 \oplus a_2\|_L^2 e^{\langle a_1, \xi_1\rangle + \langle a_2, \xi_2\rangle}$$

Theorem 6. *For an operator $A \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N'))$ we have $\sigma(A)(\zeta, \xi) \in \mathcal{F}_\theta(N') \hat{\otimes} \mathcal{G}_{\theta^*}(N)$; in particular, $\sigma(A) \in C^2(N' \times N)$.*

Proof. The proof follows by the direct computation.

In Definition 3 we introduced the action of Δ_L on functions. The following definition translates this definition to the language of operators.

Definition 5. Let $D_L^q(N' \times N)$ denote the space of all operators $A \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$ such that $\sigma(A) \in D_L(N' \times N)$ and $\Delta_L \sigma(A) \in \mathcal{F}_\theta(N') \hat{\otimes} \mathcal{G}_{\theta^*}(N)$. Then the Lévy type operator Δ_L^Q is defined by

$$\Delta_L^Q A := \sigma^{-1} \Delta_L \sigma(A).$$

The following theorem claims that the above operator is well defined.

Theorem 7. *For the operators $A \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ we have*

$$\sigma(A)(\zeta, \xi) \in \mathcal{G}_{\theta^*}(N) \hat{\otimes} \mathcal{G}_{\theta^*}(N);$$

if $\sigma(A) \in D_L(N \times N)$ and

$$\Delta_L \sigma(A) \in \mathcal{G}_{\theta^*}(N) \hat{\otimes} \mathcal{G}_{\theta^*}(N),$$

then the Lévy type operator Δ_L^Q is defined by

$$\Delta_L^Q A = \sigma^{-1} \Delta_L \sigma(A).$$

5. THE OPERATOR HEAT EQUATION

Consider the Cauchy problem associated to the quantum Lévy Laplacian Δ_L^Q,

$$\begin{cases} \dfrac{\partial A}{\partial t} &= \Delta_L^\varphi A \\ A(0) &= A_0 \end{cases} \qquad A_0 \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta) \quad (\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*), \text{ respectively}).$$

Theorem 8. *For each $A \in \mathcal{L}(F_\theta, \mathcal{F}_\theta^*)_+$, there exists a unique positive Radon measure μ_A on $E' \times E'$ such that*

$$\langle A\varphi, \psi\rangle = \langle A^k, \varphi \otimes \psi\rangle = \int_{E'^2} \varphi(x + i0)\psi(y + io)d\mu(x, y) \tag{6.1}$$

for any φ and $\psi \in \mathcal{F}_\theta(N')$. In that case, there exist numbers $p > 0$, $q > 0$, $m_1 > 0$, and $m_2 > 0$ such that μ_A is carried by the space $E_{-p} \times E_{-q}$ and

$$\int_{E_{-p} \times E_{-q}} e^{\theta(m_1|x|_{-p}) + \theta(m_2|y|_{-q})} d\mu(x, y) < \infty.$$

Conversely, a positive finite measure of this kind on the space $E' \times E'$ defines a positive operator $A \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)_+$ by formula (6.1).

Remark 2. Note that the Fourier transform of μ_A is defined by

$$\mathcal{F}\mu_A(\xi_1, \xi_2) = \int_{E'^2} e^{i(\langle x, \xi_1\rangle + \langle y, \xi_2\rangle)} d\mu_A(x, y) := \langle A^k, e^{i\xi_1} \otimes e^{i\xi_2}\rangle = \sigma(A)(i\xi_1, i\xi_2).$$

Proof of Theorem 8. Let

$$A \in \mathcal{L}(\mathcal{F}_\theta(M'), \mathcal{F}_\theta^*(N')) \simeq \mathcal{F}_\theta^*(M') \otimes \mathcal{F}_\theta^*(N') \simeq \mathcal{F}_{\theta,\theta}^*(M' \oplus N')$$

A function $\varphi(\xi_1, \xi_2) \in \mathcal{F}_{\theta,\theta}(M' \oplus N')$ is said to be *positive* if $\varphi(x_1 + oi, x_2 + io) \geqslant 0$ for any $x_1 \in E_1$ and $x_2 \in E_2$, where $M = E_1 + iE_1$ and $N = E_2 + iE_2$.

Denote by $\mathcal{F}_{\theta,\theta}(M' \oplus N')_+$ the cone of all positive functions.

An operator A is said to be *positive* if $\langle A^k, \varphi\rangle \geqslant 0$ for every positive test function

$$\varphi(\xi_1, \xi_2) \in \mathcal{F}_\theta(M' \oplus N'),$$

where $A^k \geqslant 0$ provide that A^k stands for the kernel associated to the operator A, equivalently,

$$\langle A^k, \varphi \otimes \psi\rangle = \langle A\varphi, \psi\rangle \geqslant 0$$

for every φ and $\psi \in \mathcal{F}_\theta(N')$ such that φ and ψ are positive.

Step 1. Now let $A \in \mathcal{L}(\mathcal{F}_\theta(M'), \mathcal{F}_\theta(N')_+$. Then the function

$$C_A(\xi_1 \oplus \xi_2) := \langle\langle A^k, e^{i\xi_1 \oplus \xi_2}\rangle,$$

which is the Fourier transform of A^k, is a positive definite function, i.e., $C_A(0)$ is finite, $C_A(\xi_1 \oplus \xi_2)$ is continuous, and C_A is positive definite.

Note that C_A is a characteristic function. According to the Bochner–Minlos theorem [17], there exists a unique positive Radon measure μ_A on $E'_1 \times E'_2$ such that

$$C_A(\xi_1, \xi_2) = \int_{E'_1}\int_{E'_2} e^{i\langle x \oplus y, \xi_1 \oplus \xi_2\rangle} d\mu_A(x, y) = \langle A^k, e^{i(\xi_1 \oplus \xi_2)}\rangle. \tag{6.2}$$

Since the elements $e^{i\langle \xi_1 \oplus \xi_2, \cdot \oplus \cdot\rangle}$ form a dense set in the space $\mathcal{F}_\theta(M') \otimes \mathcal{F}_\theta(N')$, we can extend equality (5.2) in a usual way to the functions

$$f \in \mathcal{F}_\theta(M') \otimes \mathcal{F}_\theta(N').$$

Indeed, let $f \in \mathcal{F}_\theta(M') \otimes \mathcal{F}_\theta(M')$, and let α_n be a sequence in

$$\mathcal{E} = \text{Vect}\ \{e^{i\langle \xi_1 \oplus \xi_2, \cdot \oplus \cdot\rangle}; \xi_1 \in E_1, \xi_2 \in E_2\}$$

such that $\alpha_n \to f$ for the topology of $\mathcal{F}_{\theta,\theta}(M' \oplus N')$. On the other hand, if we write

$$f^*(z_1, z_2) = \overline{f(\overline{z}_1, \overline{z}_2)}, \qquad (\xi_1, \xi_2) \in M' \times N',$$

then we can see that the mapping $f \to f^*$ is an involution on the algebra $\mathcal{F}_\theta(M') \otimes \mathcal{F}_\theta(N')$. Therefore,

$$\langle A^k, (\alpha_m - \alpha_n)(\alpha_m - \alpha_n)^* = \int_{E_1'} \int_{E_2'} |\alpha_m - \alpha_n|^2 d\mu_A(x, y). \tag{6.3}$$

Using the fact that the usual product on the algebra $\mathcal{F}_\theta(M') \otimes \mathcal{F}_\theta(N')$ is continuous, we see from equality (6.3) by passing to the limit that the sequence $(\alpha_n)_{n\in N}$ is a Cauchy sequence in the space $L^2(E_1' \times E_2', \mu_A)$, and it follows that $\alpha_n \to \alpha$ in $L^2(E_1' \times E_2', \mu_A)$.

However, α_n converges to f in $\mathcal{F}_\theta(M') \otimes \mathcal{F}_\theta(N')$, which implies that (α_n) converges pointwise to f, and therefore $\alpha = f$ (μ_A-almost everywhere). Then we have

$$\begin{aligned}\langle A^k, f\rangle &= \lim_{n\to\infty} \langle A^k, \alpha_n\rangle \\ &= \lim_{n\to\infty} \int_{E_1'} \int_{E_2'} \alpha_n(x, y) d\mu_A(x, y) = \int_{E_1' \times E_2'} \alpha(x, y) d\mu_A(x, y) = \int_{E_1' \times E_2'} f(x, y) d\mu_A(x, y).\end{aligned}$$

Step 2. To prove the integrability condition

$$\exists p, q, \quad m_1, m_2 > 0 \quad \text{such that} \quad \mu_A(E_{-p} \times E_{-q}) = 1,$$

it suffices to assume that

$$\int_{E_{-p} \times E_{-q}} e^{\theta(m_1|x|_{-p}) + \theta(m_2|y|_{-q})} d\mu_A(x, y) < \infty. \tag{6.4}$$

In fact if μ satisfies (6.4) and is carried by $E_{-p} \times E_{-q}$, then one can readily prove that the linear form

$$\langle A_\mu^k, f\rangle = \int_{E_1' \times E_2'} f(x_1 + i0, x_2 + i0) d\mu(x_1, x_2)$$

is continuous on $\mathcal{F}_\theta(M') \otimes \mathcal{F}_\theta(N')$, and the positivity condition for the kernel A_μ^k follows immediately from the positivity condition for the measure μ.

Conversely, the condition (6.4) is necessary. Suppose that the measure μ satisfies the relation

$$\langle A^k, f\rangle = \int_{E_1' \times E_2'} f(x_1, x_2) d\mu(x_1, x_2)$$

for a certain $A^k \geqslant 0$; then there exist some p, q, m_1, m_2 such that

$$C_{A^k}(\xi_1, \xi_2) = \langle A^k, e^{i\langle \xi_1 \oplus \xi_2, \cdot \oplus \cdot\rangle} \in \exp(M_p, \theta^*, m_1) \otimes \exp(N_q, \theta^*, x_2),$$

and by the Bochner–Minlos theorem there exist $p_1 > p$ and $q_1 > p$ such that $\mu(E_{-p_1} \times E_{-q_1}) = 1$ and, moreover, the operator $i_{p_1, q_1, p, q} \colon M_{p_1} \oplus N_{q_1} \to M_p \oplus N_q$ is of Hilbert–Schmidt type.

Let us now use two technical lemmas.

Lemma 4. *Let μ be a measure representing a positive kernel A^k. Then*

$$\int_{E_{-p_1} \times E_{-q_1}} \langle (x_1 \oplus x_2)^{\otimes n}, (\xi_1 \oplus \xi_2)^{\otimes n}\rangle d\mu(x_1, x_2) \\ \leqslant \|\mathcal{L}A^k\|_{\theta^*, -p_1 - q, m_1, m_2} (2n)! (m_1 m_2)^{2n} \theta_{2n}^* \|\xi_1 \oplus \xi_2\|^{2n}.$$

Lemma 5. *Let μ be a measure representing a positive kernel A^k. Then we have*

$$\int_{E_{-p_1} \times E_{-p_2}} \|x_1 \oplus x_2\|_{-(p_1, p_2)}^n d\mu(x_1, x_2) \leqslant (\|\mathcal{L}A^k\| \theta_{2n}^* 2n!)^{1/2} (\sqrt{e} m_1 m_2 \|i_{p_1, p_2}\|_{HS})^n$$

Finally, we can see by using the relation $(\theta^*)^* = \theta$ that it suffices to find m' and $m'' > 0$ for which

$$\int_{E_{-p_1} \times E_{-p_2}} \sup_{t_1} \{e^{-t_1 m' |x_1|_{-p_1} - \theta^*(t_1)}\} \sup_{t_2} \{e^{-t_2 m'' |x_2|_{-p_2} - \theta^*(t_2)}\} \times d\mu(x_1, x_2) < \infty;$$

really, we have

$$\begin{aligned} e^{t_1 m' |x_1|_{-p_1} - \theta^*(t_1)} &= e^{-\theta^*(t_1)} \sum_{n=0}^{\infty} \frac{(t, m^1)^n}{n!} |x_1|_{-p_1}^n, \\ e^{t_2 m'' |x_2|_{-p_2} - \theta^*(t_2)} &= e^{-\theta^*(t_2)} \sum \frac{(t_2 m'')^n}{n!} |x_2|_{-p_2}^n, \\ \sup_{t_1} \{e^{t_1 \dot{m} |x_1|_{-p_1} - \theta^*(t_1)}\} &\leqslant \sum_n \frac{(\dot{m} |x_1|_{-p_1})^n}{n! \theta_n^*}, \end{aligned}$$

and hence we obtain

$$\int_{E_{-p_1} E_{-p_2}} e^{\theta(m'(x_1|_{-p_1}) + \theta(m''|x_2|_{-p_2})} d\mu(x_1, x_2) \leqslant K(\|\mathcal{L}A^k\|)^{1/2}$$

by choosing m', m'', and $\|i_{q,0}\|_{HS}$.

Theorem 9 (the evolution of positive operators). *Let $A_0 \in \mathcal{L}(\mathcal{F}_0, \mathcal{F}_0^*)_+$. Denote by μ the corresponding Radon measure on $E' \times E'$ (in the sense of Theorem 7). Let (e_n), (f_n) be bases in E. If μ is invariant under S^* (where S stands for the shift operator on $E' \times E'$), i.e., $S(e_n, f_n) = (e_{n+1}, f_{n+1})$, then $(x, y) \in (E' \times E')_L$ for μ-a.e. x, and*

$$A_t(\xi_1, \xi_2) = \int_{E' \times E} e^{-t\|(x,y)\|_L^2} d\mu(x, y), \qquad \xi_1, \xi_2 \in E, \qquad t \geqslant 0,$$

is a solution of the quantum Cauchy problem

$$\frac{\partial A}{\partial t} = \Delta_L^Q A, \qquad A(0, (\xi_1, \xi_2)) = \sigma(A_0)(\xi_1, \xi_2).$$

Proof. The statement follows from Theorem 8 and Theorem 7.

APPENDIX

The enumeration of propositions in this appendix is independent of the enumeration in the main part of the paper.

1. *Laplace Operators*

In this section we give general definitions of Laplace operators of different types.

Unless otherwise stated explicitly, the vector spaces are assumed to be real; if E and G are locally convex spaces (LCS), then $L(E,G)$ is the space of all linear continuous mappings from E to G; if $G=\mathbb{R}^1$, then one uses the symbol E^* instead of $L(E,G)$ and writes $L(E)$ instead of $L(E,E)$; the symbol $L_0(E)$ denotes the space of all compact operators in $L(E)$ (we use this space only if E is a Hilbert space). A mapping $F\colon E\to G$ is said to be (Hadamard) differentiable at $x\in E$ if there exists an element $F'(x)\in L(E,G)$, the so-called derivative of F at x, such that, if

$$r_x(h)=F(x+h)-F(x)-F'(x)h,$$

then

$$t_n^{-1}r_x(t_nh_n)\to 0 \quad \text{as} \quad n\to\infty$$

for any convergent sequence (h_n) of elements of E and for any sequence $(t_n)\in\mathbb{R}$ convergent to zero (see [31] and the references therein). The derivatives of higher orders are defined by induction; in this case one assumes that the spaces $L(E,G)$, $L(E,L(E,G))$, and so on are equipped with the topology of convergence on sequentially compact subsets. A mapping $F\colon E\to G$ is called a C^n-mapping ($n\in\mathbb{N}$) if it is n times differentiable everywhere and if F itself and all mappings

$$F^{(k)}\colon E\to L(E,\dots,L(E,G)\dots),\qquad k=1,2,\dots,n,$$

are continuous. The vector space of all C^n-mappings of E into G is denoted by $C^n(E,G)$; if $G=\mathbb{R}^1$, then one uses the symbol $C^n(E)$ instead of $C^n(E,G)$; the vector space of all real-valued functions on E is denoted by $\mathcal{F}(E)$.

One can show that a mapping $F\colon E\to G$ is an element of the space $C^n(E,G)$ if and only if it is continuous and Gateaux differentiable at each point and all mappings

$$F^{(k)}\colon E\to L(E,\dots,(L(E,G))\dots),\qquad k=1,2,\dots,n,$$

are continuous [22].

In a similar way one can define the differentiability and derivatives of (cylindrical) measures on a LCS (see [31, 32] and the references therein). To do this, one can observe that $f\in C^1(E)$ if and only if the function

$$E\ni h\mapsto [x\mapsto f(x+h)]\in C(E)$$

is differentiable at 0, assuming that $C(E)$ is equipped with the topology of compact convergence. Certainly, one can also equip $C(E)$ with some other topologies and investigate relations between the corresponding differentiability properties and the Hadamard differentiability. Let now, for any vector subspace E_1 of E^*, the symbol $\mathcal{M}(E,E_1)$ denote the space of all bounded E_1-cylindrical (complex valued) measures on E equipped with the topology $\sigma(\mathcal{M}(E,E_1),C_{cyl}(E,E_1))$, where the symbol $C_{cyl}(E,E_1))$ denotes the space of all bounded continuous (complex) E_1-cylindrical functions on E; let $\mathcal{A}_{cyl}(E,E_1)$ be the collection of all E_1-cylindrical subsets of E and, for any $\nu\in\mathcal{M}(E,E_1)$ and $k\in E$, let ν_k be the shift of ν on k:

$$\nu_k=[\mathcal{A}_{cyl}(E,E_1)\ni A\mapsto \nu(A+k)]\in\mathcal{M}(E,E_1).$$

A measure $\nu\in\mathcal{M}(E,E_1)$ is called (n times) differentiable (along E) if the function

$$E\ni t\mapsto \nu_h\in\mathcal{M}(E,E_1)$$

is (n times) differentiable at zero and all the derivatives are measures absolutely continuous with respect to ν (cf. [31, 32]).

The following definition gives a motivation for the definition of Laplace operators.

Definition 1. ([6]) Let S_0 be a linear functional defined on a vector subspace $domS_0$ of the vector space $L(E,E^*)$. A second-order homogeneous differential operator (defined by the functional S_0) in a space of functions on E is a linear mapping (denoted by Δ_{S_0}) from a subspace of the space $C^2(E)$ into the space $\mathcal{F}(E)$. The mapping Δ_{S_0} is defined as follows:

$$dom\Delta_{S_0}=\{g\in C^2(E):\forall x\in E,\ g''(x)\in domS_0\};$$

if $g\in dom\Delta_{S_0}$, then

$$(\Delta_{S_0}g)(x)=S_0(g''(x)).$$

In a similar way one can also define (nonhomogeneous) linear differential operators of any order [1] in $C^n(E)$.

A linear mapping from a vector subspace of $\mathcal{F}(E)$ into $\mathcal{F}(E)$ such that the restriction of this mapping to $C^n(E)$ is a differential operator is also called a differential operator.

Example 1. Let E be a Hilbert space, let $A\in L(E)$, and let $S_0^A(B)=\operatorname{tr}(BA)$ (here the symbol tr stands for the standard operator trace) for a suitable operator $B\in L(E,E^*)(=L(E))$; then $\Delta_{S_0^A}$ is called a Laplace–Volterra operator, or a Volterra Laplacian (associated with A). Such operators were considered in the sixties by Yu. L. Daletskii and L. Gross who also assumed that A is a self-adjoint positive operator, but this assumption is not essential for us now. Moreover, in the definition we even need not assume that A is continuous.

Using a proper notion of trace for elements of $L(E,E^*)$, one can introduce Laplace–Volterra operators for any LCS. Below we write Δ_A instead of $\Delta_{S_0^A}$.

Let H be a separable Hilbert space with the inner product $(\cdot,\cdot)_H$, let H be identified with its Hilbert adjoint H^*, and let E be an LCS which is a dense subspace in H as a vector space; one also assumes that the canonical embedding of E in H is continuous. Let E^* be equipped with a locally convex topology respecting the duality between E^* and E; then the mapping $H^*(=H)\to E^*$ adjoint to the embedding $E\to H$ is continuous, injective, and has a dense image in E^*. Hence, these objects form a rigged Hilbert space $E\subset H=H^*\subset E^*$; we also note that, if $x\in E$ and $g\in H\subset E^*$, then $(x,g)_H=\langle g,x\rangle$ ($\equiv g(x)$) in the natural notation. Let $\mathbf{e}=(\mathbf{e}_n)$ be an orthonormal basis in H formed by elements of E. Below one assumes that the linear span $E_{\mathbf{e}}$ of the basis $\mathbf{e}$ is dense in E.

Proposition 1. *The series*

$$\sum_{n=1}^{\infty}a_n\mathbf{e}_n$$

converges to an element $f\in E^*$ in the topology $\sigma(E^*,E_{\mathbf{e}})$ if $a_n=f(e_n)$ for any n.

Corollary 1. *The mapping $A\colon E^*\ni f\mapsto (f(\mathbf{e}_n))\in\mathbb{R}^\infty$ is a homeomorphism of the space $(E^*,\sigma(E^*,E_{\mathbf{e}}))$ onto its image in $\mathbb{R}^\infty$.*

Corollary 2. *The mapping A of Corollary 1 is a continuous bijection of E^* onto $A(E^*)$.*

Remark 1. Let the symbol

$$\sum_{n=1}^{\infty}a_n\mathbf{e}_n$$

denote a (unique) element f of E^* such that $f(e_n)=a_n$ for any n if such an element exists. According to what was said above, if the sum

$$\sum_{n=1}^{\infty}a_n\mathbf{e}_n$$

exists, then the series converges in the topology $\sigma(E^*, E_{\mathbf{e}})$ but need not converge even in the topology $\sigma(E^*, E)$ because the series

$$\sum_{n=1}^{\infty} f(\mathbf{e}_n)g(\mathbf{e}_n)$$

can be divergent (in the general case) for some $f \in E^*$ and $g \in E$ (see Remark 2 and Example 2).

Remark 2. The following statements are equivalent: (1) the series

$$\sum_{n=1}^{\infty} f(\mathbf{e}_n)g(\mathbf{e}_n)$$

converges for any $f \in E^*$ and $g \in E$; (2) the series

$$\sum_{n=1}^{\infty} f(\mathbf{e}_n)\mathbf{e}_n$$

converges in the topology $\sigma(E^*, E)$ for any $f \in E^*$; (3) the series

$$\sum_{n=1}^{\infty} g(\mathbf{e}_n)\mathbf{e}_n$$

converges in the topology $\sigma(E, E^*)$ for any $g \in E$.

Example 2. Let $H = L_2(-1,1)$ and $E = C[-1,1]$; then E^* is the space of all (signed) Borel measures on $[-1,1]$. Let

$$\mathbf{e}_n(t) = c_n \cos(\frac{\pi}{2}nt)$$

for each $n = 0, 1, 2\dots$. Let $g \in C[-1,1]$ be a function whose Fourier series diverges at zero, and let $\delta \in E^*$ be the measure of unit mass (on $[-1,1]$) concentrated at zero. Then the series

$$\sum_{j=1}^{\infty} \delta(\mathbf{e}_j)g(\mathbf{e}_j) = \sum_{j=1}^{\infty} \mathbf{e}_j(0)g(\mathbf{e}_j)$$

diverges.

Example 3. Let D be a (strictly) positive self-adjoint operator in H having a Hilbert–Schmidt inverse, let $(\mathbf{e}_n)$ be an (orthonormal) basis in H formed by eigenvectors of the operator D, and let $E = H_D = \bigcap_n D^n H$; one assumes that E is equipped with the topology defined by the following collection of Hilbert norms:

$$\{\|\cdot\|_n : n \in \mathbb{N};\ \forall h \in E,\ \|h\|_n^2 = (D^n h\ D^n h)_H\}.$$

Then E is a (reflexive) nuclear Fréchet space, and for each $g \in E^*$ the series

$$\sum_{n=1}^{\infty} \langle g, \mathbf{e}_n\rangle \mathbf{e}_n$$

converges in E^* even in the strong topology (which coincides in this case with the Mackey topology). For any $n \in \mathbb{Z}$, we denote the completion of the space $(H_D, \|\cdot\|_n)$ by H_n.

Let now $\mathcal{F}_{\mathbf{e}}(E)$ be the set of all functions $f \in \mathcal{F}(E)$ which are twice differentiable along the vectors of the basis $\mathbf{e}$. Let also S be a positive linear functional defined on a vector subspace $domS$ of the space $\mathbb{R}^\infty$ of all sequences of real numbers, where $domS$ contains the space $\ell_1 (= \ell_1(\mathbb{N}))$.

Definition 2. The Laplace operator on $\mathcal{F}_{\mathbf{e}}(E)$ (defined by the basis $\mathbf{e}$ and by the functional S) is the mapping

$$\Delta_S{}^n : dom\Delta_S \to \mathcal{F}(E)$$

defined on

$$dom\Delta_S = \{f \in \mathcal{F}_{\mathbf{e}}(E) : f''_{jj}(x)) \in domS\},$$

where

$$f''_{jj}(x) = \frac{d^2}{dt^2}\bigg|_{t=0} f(x + t\mathbf{e}_j),$$

by the rule

$$(\Delta_S f)(x) = S\big((f''_{jj}(x))_{j=1}^{\infty}\big).$$

If $\ell_1 \subset \ker S$, then the operator Δ_S is called a (weighted) Laplace–Levy operator (and the functional S is called a Levy functional); if

$$S((x_n)) = \sum_n a_n x_n$$

(where $a_n \geqslant 0$), then Δ_S is a Laplace–Volterra operator (according to the definition in Example 1). In a similar way one can define both types of Laplace operators on $\mathcal{M}(E, E_1)$.

Definition 2a. Let the assumptions of Definition 2 be satisfied. The exotic Laplace operator on $\mathcal{F}_{\mathbf{e}}(E)$ of order n ($\in \mathbb{N}$) defined by the basis $\mathbf{e}$ and by the functional S is the mapping

$$\Delta_S^n : dom\Delta_S^n \to \mathcal{F}(E)$$

defined on

$$dom\Delta_S^n = \{f \in \mathcal{F}_{\mathbf{e}}(E) : ((\|\mathbf{e}_j\|_n^{-2} f''_{jj}(x)) \in domS\},$$

where

$$f''_{jj}(x) = \frac{d^2}{dt^2}\bigg|_{t=0} f(x + t\mathbf{e}_j),$$

by the rule

$$(\Delta_S^n f)(x) = S\big(((\|\mathbf{e}_j\|_n^{-2} f''_{jj}(x))_{j=1}^{\infty}\big).$$

If $\ell_1 \subset \ker S$, then the operator Δ_S^n is called a (weighted) *exotic Laplace–Levy operator of the order* n (we do not consider exotic Laplace-Volterra operators).

Let the functional S_c be defined by

$$domS_c = \{(a_n) \in \mathbb{R}^\infty : \exists \lim_n \frac{1}{n}\sum_{j=1}^{n} a_j\}; \qquad S_c((a_n)) = \lim_n \frac{1}{n}\sum_{j=1}^{n} a_j$$

(i.e. S_c is the Cesàro average of the corresponding sequence). The operator Δ_{S_c} is called a (classical) Levy Laplacian associated with the basis $\mathbf{e}$ and is denoted below by $\Delta_{\mathcal{L}}$; the operator $\Delta_{S_c}^n$ is called a (classical) exotic Levy Laplacian of order n and is denoted by $\Delta_{\mathcal{L}}^n$.

Remark 3. In a similar way one can also define the Cesàro average for elements of the space G^∞ (where G is an LCS), and hence the classical Levy Laplacians, both ordinary and exotic, for mappings from E to G.

Below we will see that, in full analogy with the relation between the Laplace–Volterra operator and the notion of the (usual) trace of an operator (see Example 1), the Laplace–Levy operator is related to a similar object that we call the Levy trace.

Definition 3. The Levy trace (with respect to a basis $\mathbf{e}$) is the functional $\operatorname{tr}_{\mathcal{L}}$ whose domain is a part of $L(E, E^*)$, defined by $\operatorname{tr}_{\mathcal{L}} A = S_c(((A\mathbf{e}_j, \mathbf{e}_j)))$. One can also similarly define the Levy trace on $L(E, L(E, G))$.

If $f \in C^2(E)$, then $\Delta_{\operatorname{tr}_{\mathcal{L}}} f(x) = \Delta_{S_c} f(x) = \Delta_{\mathcal{L}} f(x) = \operatorname{tr}_{\mathcal{L}}(f''(x))$.

This definition can be generalized as follows.

Definition 3a. The Levy trace is a functional $\operatorname{tr}^L_{\mathcal{L}}$ whose domain is a part of $L(E, E^*)$ containing operators in $L(H)$ of the form $c \cdot Id + K$, $K \in L_0(H)$, such that $\operatorname{tr}^L_{\mathcal{L}}(c \cdot Id + K) = c$.

If $f \in C^2(E)$, then the operator $\Delta_{\operatorname{tr}^L_{\mathcal{L}}}$ (defined by $\Delta_{\operatorname{tr}^L_{\mathcal{L}}} f(x) = ^L f''(x)$) is also called the Levy Laplacian.

Definition 4. The Levy inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ (with respect to the basis $\mathbf{e}$) is defined by the rule $dom\langle \cdot, \cdot \rangle_{\mathcal{L}} = \{(a, b) \in E^* \times E^*;\ (\langle a, \mathbf{e}_j \rangle \cdot \langle b, \mathbf{e}_j \rangle) \in dom S_c\}$; if $(a, b) \in dom\langle \cdot, \cdot \rangle_{\mathcal{L}}$, then $\langle a, b \rangle_{\mathcal{L}} = S_c((\langle a, \mathbf{e}_j \rangle \cdot \langle b, \mathbf{e}_j \rangle))$.

Remark 4. The domain of the Levy inner product need not be a vector space; moreover, even the set E_0^* of the functionals $f \in E^*$ for which the Levy inner product $\langle f, f \rangle_{\mathcal{L}}$ exists need not be a vector space.

Example 5. If $a \in E^*$ and $A = a \otimes a$ $(\in L(E, E^*))$, then $\operatorname{tr}_{\mathcal{L}} A = \langle a, a \rangle_{\mathcal{L}}$.

Now we will give an example of a vector subspace $E_{\mathcal{L}}$ in E^* for which $E_{\mathcal{L}} \times E_{\mathcal{L}} \subset dom\langle \cdot, \cdot \rangle_{\mathcal{L}}$. Let the assumptions of Example 3 be satisfied; the general case can be considered in a similar way. For each $\lambda \in (0, \pi)$, let

$$s_\lambda = \sum \sin(\lambda n)\mathbf{e}_n \qquad (\in E^*);$$

let also ν be a σ-additive Borel measure on $(0, \pi)$ and, for each function $\varphi \in L_1((0,1), \nu) \cap L_2((0,1), \nu)$, let

$$s_\varphi = \int_0^\pi \varphi(\lambda) s_\lambda\, \nu(d\lambda) \qquad (\in E^*).$$

Let finally S_ν be the image of the space $L_1(0, \pi)$ under the mapping $f \mapsto s_f$. Then $S_\nu \times S_\nu \subset dom\langle \cdot, \cdot \rangle_{\mathcal{L}}$, and the restriction of $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ to $S_\nu \times S_\nu$ is an inner product on S_ν. The space $(S_\nu, \langle \cdot, \cdot \rangle_{\mathcal{L}})$ is a separable (pre-Hilbert) space; if the support of ν is an infinite set, then this space is not complete. Let us also note that

$$\langle s_\varphi, s_\psi \rangle_{\mathcal{L}} = \int_0^\pi \varphi(\lambda)\psi(\lambda)\nu(d\lambda).$$

Remark 4a. The most part of the above constructions can be applied for the case in which the basis $\mathbf{e}$ is replaced by any orthonormal set of elements in H.

2. *Analytic Properties of Levy Laplacians*

In this section we prove a chain rule and a Leibniz formula for Laplacians (both Levy and Volterra). After that, we prove some relationships between the Levy and Volterra Laplacians. Below we assume that $A, B \in L(H)$.

Proposition 2. *(Chain rule). If $g \in C^2(E)$ and $f \in C^2(\mathbb{R}^1)$, then*

$$\Delta_{\mathcal{L}}(f \circ g)(x) = f''(g(x))\langle g'(x), g'(x) \rangle_{\mathcal{L}} + f'(g(x))(\Delta_{\mathcal{L}} g)(x).$$

If $E = H$, then

$$\Delta_A(f \circ g)(x) = f''(g(x))(Ag'(x), g'(x))_H + f'(g(x))(\Delta_A g)(x).$$

Proof. Due to the chain rule, $(f \circ g)''(x) = f''(g(x))(g'(x) \otimes g'(x)) + f'(g(x)) \cdot g''(x)$; now, to prove the first statement, it suffices to apply the mapping $\operatorname{tr}_{\mathcal{L}}$ to both sides of the first identity and, to prove the other statement, it suffices to multiply the same identity by A and then to apply the functional $\operatorname{tr}_{\mathcal{L}}$ to the identity thus obtained.

Corollary 3. *Let the assumptions of the preceding proposition be satisfied. If $\Delta_{\mathcal{L}} g = 0$ (if $E = H$ and $\Delta_A g = 0$, respectively), then*

$$\Delta_{\mathcal{L}}(f \circ g)(x) = f''(g(x))\langle g'(x), g'(x) \rangle_{\mathcal{L}}$$

and

$$\Delta_A(f \circ g)(x) = f''(g(x))(Ag'(x), g'(x))_H,$$

respectively.

On the other hand, if $\langle g'(x), g'(x) \rangle_{\mathcal{L}} = 0$ (this is the case, e.g., if $E = H$), then one obtains the following formula, which is usually presented in texts on Levy Laplacians:

$$\Delta_{\mathcal{L}}(f \circ g)(x) = (\Delta_{\mathcal{L}} g)(x) f'(g(x)).$$

If $\Delta_{\mathcal{L}} g = 0$ and $\langle g'(x), g'(x) \rangle_{\mathcal{L}} = 0$ simultaneously, then $\Delta_{\mathcal{L}}(f \circ g)(x) = 0$.

Similar results hold for the Laplace–Volterra operators (if $E = H$). Namely, if $(Ag'(x), g'(x))_H = 0$, then

$$\Delta_A(f \circ g)(x) = f'(g(x))((\Delta_A g)(x));$$

if $(Ag'(x), g'(x))_H = 0$ and $\Delta_A g = 0$, then $\Delta_A(f \circ g)(x) = 0$.

Corollary 4. *If $f \in E^*$, $\psi \in C^2(\mathbb{R}^1)$, $F(\xi) = \psi(\langle f, \xi \rangle)$, and the inner square $\langle f, f \rangle_{\mathcal{L}}$ exists, then*

$$(\Delta_{\mathcal{L}} F)(\xi) = \langle f, f \rangle_{\mathcal{L}} \psi''(\langle f, \xi \rangle).$$

In particular, if $F(\xi) = e^{\langle f, \xi \rangle}$, then $(\Delta_{\mathcal{L}} F)(\xi) = \langle f, f \rangle_{\mathcal{L}} e^{\langle f, \xi \rangle}$.

If $\Psi(\xi) = f((B\xi, \xi)_H)$, then $(\Delta_{\mathcal{L}} \Psi)(\xi) = 2(\operatorname{tr}_{\mathcal{L}} B) f'((B\xi, \xi)_H)$.

Similar statements hold for the Volterra Laplacian (if $E = H$). Namely, if $f \in H (= E^*)$, then $(\Delta_A F)(\xi) = (Af, f)_H \psi''((f, \xi)_H)$; if $F(\xi) = e^{(f,\xi)_H}$, then $(\Delta_A F)(\xi) = (Af, f)_H e^{(f,\xi)_H}$; if $(A\xi, B\xi)_H = 0$, then $(\Delta_A \Psi)(\xi) = 2\operatorname{tr}(BA) f'((B\xi, \xi)_H)$.

Example 6. Let $a \in E^*$, and let $P_a \colon E \to \mathbb{R}^1$ be a homogeneous polynomial, $P_a(\xi) = \langle a, \xi \rangle^k$. Then $\Delta_{\mathcal{L}} P_a(\xi) = k(k-1)\langle a, a \rangle_{\mathcal{L}} \cdot (\langle a, \xi \rangle)^{k-2}$ (this means that both sides of the identity exist simultaneously and are equal). In particular, if $\langle a, a \rangle_{\mathcal{L}} = 0$ (for example, if $a \in H$), then $\Delta_{\mathcal{L}} P_a(\xi) = 0$ for all $\xi \in E$. If $E = H (= E^*)$ and $P_a(\xi) = (a, \xi)_H^k$, then $\Delta_A P_a(\xi) = k(k-1)(Aa, a)_H \cdot ((a, \xi)_H)^{k-2}$. If $(Aa, a)_H = 0$, then $\Delta_A P_a(\xi) = 0$ for any $\xi \in E$.

Proposition 3 (Leibniz formula). *If $g, f \in C^2(E)$, then*

$$\Delta_{\mathcal{L}}(f \cdot g)(x) = (g \cdot \Delta_{\mathcal{L}} f)(x) + (f \cdot \Delta_{\mathcal{L}} g)(x) + 2\langle f'(x), g'(x) \rangle_{\mathcal{L}}.$$

If $E = H$, then

$$\Delta_A(f \cdot g)(x) = (g \cdot \Delta_A f)(x) + (f \cdot \Delta_A g)(x) + 2(Af'(x), g'(x))_H.$$

The proof is similar to that of Proposition 2.

Corollary 5. *If $E = H = E^*$, then $\langle f'(x), g'(x) \rangle_{\mathcal{L}} = 0$, and one obtains the well-known formula $\Delta_{\mathcal{L}}(f \cdot g)(x) = (g \cdot \Delta_{\mathcal{L}} f)(x) + (f \cdot \Delta_{\mathcal{L}} g)(x)$.*

Now we consider some relations between Levy Laplacians and Volterra Laplacians (cf. [21, 33]).

Proposition 4. *Let (A_n) be a sequence of Hilbert–Schmidt operators in H such that $\|A_n\|_{HS} \to \infty$ and $\sup \|A_n\| < \infty$. Let a linear functional P on a vector subspace of $L(H)$ be defined as follows:*

$$C \in domP \iff \exists \lim \frac{\operatorname{tr} A_n^* A_n C}{\|A_n\|_{HS}^2},$$

and let, for each $C \in domP$,

$$P(C) = \lim \frac{\operatorname{tr} A_n^* A_n C}{\|A_n\|_{HS}^2}.$$

Then P is a Levy trace. Moreover, if (C_k) is a sequence of compact operators convergent to C in the strong topology, then, for any k,

$$\lim \frac{\operatorname{tr} A_n^* A_n C_k}{\|A_n\|_{HS}^2} = 0$$

and the convergence is uniform with respect to k.

Proof. Let us first prove that P is a Levy trace. It suffices to prove that $P(Id) = 1$ and $P(K) = 0$ for any $K \in L_0(H)$. However, the first identity is obvious because $\operatorname{tr} A_n^* A_n = \|A_n\|_{HS}^2$. Let now $K \in L_0(H)$; then $K = V \cdot K_0$, where K_0 is a positive operator and V is an isometry; we denote by (e_n) a basis of H formed by eigenvectors of K_0 and by (k_n) the set of the corresponding eigenvalues. If $A \in L_2(H)$, then

$$\operatorname{tr} A^*AK = \operatorname{tr} A^*AVK_0 = \sum_{n=1}^{\infty}(K_0 e_n, VA^*Ae_n) = \sum_{n=1}^{\infty} k_n(e_n, VA^*Ae_n);$$

on the other hand,

$$\max_{n\geqslant r_0+1} |k_n| \sum_{n=r_0=1}^{\infty} (e_n, VA^*Ae_n)| \leqslant \max_{n\geqslant r_0+1} |k_n| \cdot \operatorname{tr} A^*A$$

for each $r_0 \in \mathbf{N}$. Hence,

$$\frac{1}{\|A_n\|_{HS}^2} \operatorname{tr} A_n^* A_n K = \frac{1}{\|A_n\|_{HS^2}} \sum_{r=1}^{r_0} k_r(e_r, A_n^* A_n V e_r) + \frac{1}{\|A_n\|_{HS}^2} \sum_{r=r_0=1}^{\infty} k_r(A_n^* A_n e_r, V e_r)$$
$$\leqslant \frac{r_0}{\|A_n\|_{HS}^2} \max_{1\leqslant r\leqslant r_0} k_r \cdot \sup_n \|A_n\|^2 + \frac{r_0}{\|A_n\|_{HS}^2} \max_{R\geqslant r_0=1} k_r \cdot \operatorname{tr} A_n^* A_n$$

for each $r_0 \in \mathbf{N}$. Since $\sup \|A_n\| < \infty$, for each $r_0 \in \mathbf{N}$, it follows that the first term on the right-hand side converges to zero; the second term is bounded above by $\max_{r\geqslant r_0+1} \to 0 (r_0 \to \infty)$. Thus, P is a Levy functional.

The proof of the other statement of the proposition follows from the fact that, for every $k \in \mathbf{N}$, the following identity holds:

$$\lim_{n\to\infty} \frac{\operatorname{tr} A_n A_n^* C_k}{\|A_n\|_{HS}^2} = 0,$$

where the convergence is uniform with respect to k; in turn, the uniformity follows from the above calculations.

This proposition implies that, if a sequence of operators $a_n Id + K_n, a_n \in \mathbb{R}^1, K_n \in L_0(H)$ converges in the strong topology to an operator $a \cdot Id + K$, then

$$P(a \cdot Id + K) = \lim \frac{\operatorname{tr} A_n^* A_n (a_n Id + K_n)}{\|A_n\|_{HS}^2}.$$

Proposition 5. *If $f \in C^2(H)$ and $f''(x) = a_x Id + K_x, K_x \in L_2(H)$ for each $x \in H$, then the following identity holds for the Levy Laplacian $\Delta_{\operatorname{tr}^L}$:*

$$(\Delta_{\operatorname{tr}^L} f)(x) = \lim \frac{\Delta_{Id} f \circ A_n}{\|A_n\|_{HS}},$$

where (A_n) is a sequence of Hilbert–Schmidt operators satisfying the assumptions of the preceding proposition and convergent in the strong topology to the identity (Δ_{Id} stands for the Laplace–Volterra operator associated with Id.)

Proof. Let us first note that $(f \circ A_n)'(x) = f''(A_n x)A_n$ and $(f \circ A_n)''(x) = A_n^* f''(A_n x) A_n$. According to our assumptions, $A_n x \to x$ for each $x \in H$; hence,

$$f''(A_n x) \to f''(x) \quad \text{for each} \quad x \in H.$$

Therefore, due to the second statement of Proposition 4,

$$\frac{\operatorname{tr} A_n^* f''(A_n x) A_n}{\|A_n\|_{HS}^2} \to P(f''(x)) = (\Delta_{\mathcal{L}} f)(x).$$

On the other hand, the above calculations show that

$$\operatorname{tr} A_n^* f''(A_n x) A_n = \Delta_{\operatorname{tr}^L}(f \circ A_n)(x).$$

The proposition is proved.

For some special sequences (A_n) and for some special Hilbert spaces, one can improve Propositions 4 and 5.

Let $H = L_2(0, 1)$, and let A_n be integral operators on H with integral kernels K_n. We assume that each K_n is a continuous function and that the following relation holds for any sequence of functions $\psi_n \in L_\infty(0, 1)$ convergent in $L_\infty(0, 1)$ to a limit ψ:

$$\frac{1}{\|A_n\|_{HS}^2} \int_0^1 \int_0^1 K_n(t, s)\psi_n(t) K_n(t, s)\, dtds \to \int_0^1 \psi(t)\, dt,$$

where

$$\|A_n\|_{HS}^2 = \int_0^1 \int_0^1 (K_n(T, s))^2\, dtds.$$

We assume also that the sequence (A_n) satisfies the conditions of Propositions 4 and 5. Then the conclusion of Proposition 5 holds (in which under $\Delta_{\mathcal{L}}$ one means the classical Levy Laplacian on $L_2(0, 1)$).

A typical example of a sequence (A_n) is a sequence of operators of convolution defined by functions of a proper δ-sequence (cf. [21]).

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Stationary Flow of a Viscoplastic Medium with Small Yield Stress in a Plane Confusor

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Abstract. The problem of stationary flow of a viscoplastic medium in a plane confusor is studied by using the Shvedov–Bingham–Il'yushin model in the case of small shear yield stress. The solution of the problem is constructed in the first approximation with respect to the yield stress, and this solution is generated by the solution of the corresponding Jeffery–Hamel problem on the flow of a viscous fluid. The numerical-analytic method of accelerated convergence developed by the authors is applied. In the first approximation with respect to the yield stress, the main characteristics of the viscoplastic flow are defined and commented, namely, the asymptotic domains of the first approximation; and their behavior is studied in a wide range of parameters of the system. New effects of mechanic nature (the number, the shape, and the arrangement of the asymptotic domains of the first approximation; bifurcations of the flow patterns, the behavior of the velocity of the particles of the medium at the boundaries of the asymptotic domains of the first approximation, etc.) are obtained.

1. JEFFERY–HAMEL PROBLEM

It is well known that, when trying to find a stationary solution in the problem on the viscoplastic flow under the action of the pressure at infinity in a plane diffusor or confusor in the form $v(r,\theta) = V(\theta)/r$, which is customary for viscous fluids, or in a conic diffusor or confusor in the form $v(r,\theta) = V(\theta)/r^2$, one faces contradictions and substantial difficulties [1]. The flow lines are not straight, and their family is a complicated pattern in the plane angle or inside the cone.

In the case of plane confusor, the stream function $\psi(r,\theta)$ was sought in the form [2]

$$\psi = -\sum_{n=1}^{\infty} \omega_i(\theta) r^{1-i}. \tag{1.1}$$

The basic function $\omega_1(\theta)$ in (1.1) determines the value of the outflow; the functions $\omega_2(\theta)$ and $\omega_3(\theta)$ do not influence in the outflow and only correct the shape of the profile.

A slow viscoplastic flow in conic and plane confusors was studied under a small (aperture) angle [3]. For the basic solution, we take that for the Poiseuille flow in a cylindrical tube or in a plane layer. In what follows we assume that the flow occurs in a cone with a small cone angle β rather than in a cylinder. We obtain the following formula for the outflow Q in the first approximation with respect to β:

$$Q = \frac{9\pi \tan^2\beta}{32\mu\tau_s}\left(p_1 - p_2 - \frac{2\tau_s}{\tan\beta}\log\frac{r_2}{r_1}\right)^2 \frac{r_1^3 r_2^3}{(r_2^{3/2} - r_1^{3/2})^2}, \tag{1.2}$$

where μ and τ_s stand for the dynamical viscosity and the shear yield stress of the material, p_1 and p_2 for the pressure at the input and output cross-sections, and r_1 and r_2 for the radii of these cross-sections. For a plane diffusor, the formula similar to (1.2) will be:

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