Renormalization and Central Extensions*

L. Accardi** and A. Boukas***

Centro Vito Volterra, Università di Roma Tor Vergata
via Columbia 2, 00133 Roma, Italy

Abstract—The stochastic limit of quantum theory [1] motivated a new approach to the renormalization program. Subsequent investigations brought to light unexpected connections with conformal field theory and some subtle relationships between renormalization and central extensions. In the present paper we review the path that has lead to these connections at the light of some recent results.

Dedicated to Igor V. Volovich with friendship and admiration for his scientific achievements

1. INTRODUCTION

The stochastic limit of quantum theory [1] has led to a multiplicity of developments in physics and mathematics. In particular the quite nontrivial *identification of classical and quantum stochastic equations with first order white noise* **Hamiltonian** *equations* naturally rose the question of the meaning of higher order white noise Hamiltonian equations.

Due to the identification of quantum white noise with the free Boson field (in momentum representation) this problem is equivalent to the problem of giving a meaning to nonlinear functions of the local quantum fields, i.e. to the old standing *renormalization problem*.

The equivalence of this problem with that of constructing a *continuous analogue* of the *-Lie-algebra of differential operators with polynomial coefficients acting on the space $\mathcal{C}^\infty(\mathbb{R}^n; \mathbb{C})$ and of its unitary representations has been discussed in the paper [9] (*continuous analogue* means that the space $\mathbb{R}^n \equiv$ {functions $\{1, \ldots, n\} \to \mathbb{R}$} is replaced by a space of test functions {functions $\mathbb{R} \to \mathbb{R}$}).

In the present paper we will discuss the connections of this problem with that of central extensions of *-Lie-algebras.

Since the simplest power higher that the first is the square, it was natural to choose this as the starting point to attack the general problem.

In the case of finitely many degrees of freedom quadratic Hamiltonians are easily diagonlized by a Boglyubov transformation but, as pointed out in the paper [16], when one tries to apply this technique to the field case, a constraint appears in the form of an inequality which excludes the simplest example one would like to be able to deal with: the square of the local quantum field, i.e. of classical white noise.

This remark convinced us that, for a successful attack the renormalization problem, a radically new approach to the problem was required.

Such a new approach was proposed by Accardi, Lu and Volovich in the paper [15] and its basic new idea can be formulated in the following problem:
first renormalize the Lie algebra structure (i.e. the commutation relations), thus obtaining a new *-Lie algebra, then construct (nontrivial) Hilbert space representations.

In the same paper [15] the concrete realizability of the new approach was proved by explicitly constructing *the Fock representation for the Renormalized Square of White Noise (RSWN)*.

---

2. QUADRATIC SECOND QUANTIZATION

Nowadays the theory of *quadratic second quantization* is rather well understood (see [14]) even if the full picture is far from being complete.

The most interesting open problem concerning the quadratic case being is to enlarge the class of non Fock representations of the Renormalized Square of White Noise (RSWN) $*$-Lie algebra. In fact there are several indications that the class constructed in [13] constitutes a tiny fraction of the representations that can be of interest for physics.

Since it illustrates well and in a simpler framework the problems which arise in the higher powers case, it is worth to quickly review the situation.

Recall that the usual commutation relations which define the non relativistic free Bose field on $\mathbb{R}^d$ $*$-Lie algebra are defined, in the sense of operator valued distributions on $\mathbb{R}^d$, by the generators:
— b_s annihilation densities
— $b_t^\dagger$ creation densities
satisfying the commutation relations

$$[b_t, b_s^\dagger] = \delta(t - s), \quad [b_t^\dagger, b_s^\dagger] = [b_t, b_s] = 0; \qquad (b_s)^* = b_s^\dagger. \tag{1}$$

The Fock representation is characterized by the existence of a unit vector Φ, called vacuum, satisfying the condition

$$b_t \Phi = 0$$

(when no confusion is possible we often identify the densities $b_t^\dagger$, b_t with their images in a given representation). The field operator is defined by

$$w_t = b_t^\dagger + b_t$$

and its vacuum distribution identifies it with a classical white noise.

If one applies formally (1) one finds the expression

$$[b_s^2, b_t^{+2}] = 4\delta(t - s)b_s^+ b_t + 2\delta(t - s)^2 \tag{2}$$

which is not well defined, even as an operator valued distribution, because of the appearance of the term $\delta(t - s)^2$.

Any rule to give a meaning to such an expression is called a *renormalization rule*.

Accardi, Lu and Volovich in [15] chose the following renormalization rule, first introduced by Ivanov (for a discussion of its precise meaning see [20] and the survey paper [11] for other possibilities to give a meaning to powers of the δ-function):

$$\delta^2(t) = c\,\delta(t) \quad ; \quad c\text{—arbitrary constant.} \tag{3}$$

Using (3), the formal expression (2) and its analogue for the number density (which does not require renormalization), become **the renormalized commutation relation with renormalization constant** c.

$$[b_s^2, b_t^{+2}] = 2c\delta(t - s)1 + 4\delta(t - s)b_s^+ b_t \tag{4}$$

$$[b_s^2, b_t^+ b_t] = 2\delta(t - s)b_t^2 \tag{5}$$

which now have a meaning in the sense of operator valued distributions.

Fixing the test function space to be the space of complex valued step functions on $\mathbb{R}$ with finitely many values, the associated smeared commutation relations take the form:

$$[b_\varphi, b_\psi^+] = c\langle \varphi, \psi \rangle + n_{\overline{\varphi}\psi}$$

$$[n_\varphi, b_\psi] = -2b_{\overline{\varphi}\psi}$$

$$[n_\varphi, b_\psi^+] = 2b_{\varphi\psi}^+$$

$$(b_\varphi^+)^+ = b_\varphi; \qquad n_\varphi^+ = n_{\overline{\varphi}}.$$

Accardi, Lu and Volovich in [15] proved that these commutation relations effectively define a $*$-Lie algebra and that there exists a $*$-representation π_F of this $*$-Lie algebra on a Hilbert space $\mathcal{H}$ (see [9] for the definition of this notion) uniquely characterized by the properties:
(i) There exists a unit vector $\Phi \in \mathcal{H}$ cyclic for the representation.
(i) For any choice of the test functions

$$\pi_F(b_\psi)\Phi = \pi_F(n_\psi)\Phi = 0.$$

Such representation was called *the quadratic Fock representation*.

The stochastic process associated to this representation is called **the renormalized square of white noise** (RSWN).

3. CENTRAL EXTENSIONS AND RENORMALIZATION

The connections between central extensions and renormalization are very well illustrated by the RSWN.

Recall that, given a complex Lie algebra L, a Lie algebra $\widetilde{L}$ is called a one-dimensional *central extension* of L with *central element E* if, as a vector space, $\widetilde{L}$ is the direct sum of L and $\mathbb{C}\,E$ and the Lie algebra structure on $\widetilde{L}$ is uniquely determined by the prescriptions that, for all $l_1, l_2 \in L$, one has:

$$[l_1, l_2]_{\widetilde{L}} = [l_1, l_2]_L + \phi(l_1, l_2)\, E, \tag{6}$$

$$[l_1, E]_{\widetilde{L}} = 0, \tag{7}$$

where $[\cdot, \cdot]_{\widetilde{L}}$ and $[\cdot, \cdot]_L$ are the Lie brackets in $\widetilde{L}$ and L, respectively, and $\phi : L \times L \mapsto \mathbb{C}$ is a 2-*cocycle* on L, i.e. a bilinear form on L satisfying the additional conditions:

$$\phi(l_1, l_2) = -\phi(l_2, l_1) \qquad \text{(skew-symmetry)} \tag{8}$$

$$\phi([l_1, l_2]_L, l_3) + \phi([l_2, l_3]_L, l_1) + \phi([l_3, l_1]_L, l_2) = 0 \qquad \text{(2-cocycle identity)}. \tag{9}$$

If $f : L \mapsto \mathbb{C}$ is a linear function and ϕ is defined by

$$\phi(l_1, l_2) = f([l_1, l_2]_L) \tag{10}$$

then ϕ is a 2-cocycle. A 2-cocycle of the form (10) is called a 2-*coboundary* and the corresponding central extension is called *trivial*.

Fixing a set $I \subset \mathbb{R}^d$ with Lebesgue measure 1, denoting χ_I its characteristic function ($= 1$ on I and $= 0$ on its complement) and introducing the 1-mode sub-algebra of RSWN (see section (4.4.3) for a more detained discussion):

$$B^- := b_{\chi_I}; \quad B^+ := b_{\chi_I}^+; \quad M := n_{\chi_I}$$

the RSWN commutation relations restricted to this sub-algebra become

$$[B^-, B^+] = c\mathbf{1} + M,$$

$$[M, B^-] = -2B^-,$$

where we denote 1 the central element.

Recalling that $sl(2, \mathbb{R})$ is the real three-dimensional $*$-Lie algebra with generators $\{B^+, B^-, M\}$ and relations

$$[B^-, B^+] = M, \quad [M, B^\pm] = \pm 2B^\pm \tag{11}$$

$$(B^-)^* = B^+, \qquad M^* = M$$

one recognizes that the 1-mode sub-algebra of RSWN is a central extension of $sl(2, \mathbb{R})$.

This central extension is trivial (like all those of $sl(2,\mathbb{R})$ which is simple), but its role is essential because without it, i.e. putting $c = 0$ in the RSWN commutation relations, the Fock representation reduces to the zero representation. We then recognize two roles:

(i) The central extension which, even if trivial (but non zero), implies the non triviality of the Fock representation.

(ii) The introduction of test functions, i.e. the transition from the 1-mode algebra to its second quantization which, in the Lie algebra framework, manifests itself as current algebra over $\mathbb{R}^d$ of the 1-mode algebra.

The analysis of step (ii) has brought to light a new phenomenon consisting in an obstruction to the existence of some special representations (generalizing in different ways the Fock one) occurring in the transition from the one mode to the second quantized case. This obstruction manifests itself in the fact that certain kernels which are positive definite in the discrete case, lose this property in the transition to the continuous case.

The non positive definiteness of certain kernels also occurs in the physical literature where it is called *emergence of ghosts*, however the two phenomena although probably related, are deeply different because in the physical literature the emergence of ghosts takes place at the mode level and has a purely algebraic root, while in the white noise literature the phenomenon only occurs in the transition from discrete to continuum and its roots are measure theoretical, i.e. due to the non atomicity of the Lebesgue measure (see [5] and references therein).

Now we consider separately these two aspects in the more general framework of the Renormalized Higher Powers of White Noise (RHPWN) algebra.

4. THE WHITE NOISE $*$-LIE ALGEBRAS

Starting from the first order commutation relations (1), the formal application of Leibniz's rule to the polynomial algebra generated by the creation and annihilation densities $a_s^\dagger$, a_s leads to expressions of the form (see [12]):

$$[a_t^{\dagger n} a_t^k, a_s^{\dagger N} a_s^K] = \epsilon_{k,0}\, \epsilon_{N,0} \sum_{L \geq 1} \binom{k}{L} N^{(L)}\, a_t^{\dagger n} a_s^{\dagger N-L} a_t^{k-L} a_s^K\, \delta^L(t-s)$$

$$- \epsilon_{K,0}\, \epsilon_{n,0} \sum_{L \geq 1} \binom{K}{L} N^{(L)}\, a_s^{\dagger N} a_t^{\dagger n-L} a_s^{K-L} a_t^k\, \delta^L(t-s), \tag{12}$$

where:
$-\ n, k \geq 0$,
$-\ \delta_{n,k}$ is Kronecker's delta

$$\epsilon_{n,k} := 1 - \delta_{n,k}$$

$$x^{(y)} = x(x-1)\cdots(x-y+1); \quad x^{(0)} := 1; \quad x^{(1)} := x \tag{13}$$

$$(x)_y = x(x+1)\cdots(x+y-1); \quad (x)_0 := 1; \quad (x)_1 := x \tag{14}$$

are the decreasing and increasing factorials (Pochhammer symbols), respectively. As one can see, these expressions involve formal powers of Dirac's delta function.

To give a mathematical meaning to expressions such as (12), is equivalent to give a meaning to the powers of the delta function.

4.1. The *-Lie Algebra $RPQWN_c$

Applying the following natural generalization of Ivanov's renormalization prescription (3):

$$\delta^l(t) := c^{l-1}\,\delta(t); \qquad l = 2, 3, ... ; \ c > 0 \text{ arbitrary constant} \tag{15}$$

the smeared operators, heuristically defined by

$$B_k^n(f; c) := \int_{\mathbb{R}} f(t)\, a_t^{\dagger n}\, a_t^k\, dt \tag{16}$$

satisfy the commutation and duality relations

$$[B_k^n(f; c), B_K^N(g; c)] = \sum_{L=1}^{(k\wedge N)\vee(K\wedge n)} \theta_L(n, k; N, K)\, c^{L-1}\, B_{k+K-L}^{n+N-L}(fg; c), \tag{17}$$

$$\left(B_k^n(f; c)\right)^* = B_n^k(\bar{f}; c), \tag{18}$$

where

$$\theta_L(n, k; N, K) := \epsilon_{k,0}\,\epsilon_{N,0} \begin{pmatrix} k \\ L \end{pmatrix} N^{(L)}\,N^{(L)} - \epsilon_{K,0}\,\epsilon_{n,0} \begin{pmatrix} K \\ L \end{pmatrix} N^{(L)}\,n^{(L)}$$

and, here and in the following, we use the convention that, whenever $a > b$,

$$\sum_{L=a}^{b} = 0.$$

The following result shows that the above described renormalization rule does not destroy the *-Lie algebra structure.

Theorem 1. *(Accardi, Boukas [3])*
*Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^d$. For any real number $c > 0$, there exists a unique *-Lie algebra with:*
– generators given by

$$\{B_k^n(f) := B_k^n(f; c) \ : \ f \in \mathcal{S}(\mathbb{R}^d)\, ;\, k, n \in \mathbb{N}\} \tag{19}$$

such that the maps $f \in \mathcal{S}(\mathbb{R}^d) \mapsto B_k^n(f)$ are complex linear for $n \geq k$, – Lie bracket defined by (17),
– involution defined by (18).

This result allows to apply a natural extension of a standard procedure used in distribution theory, i.e. to take the result of the formal manipulations described above as the definition of a new mathematical object:

Definition 1. The *-Lie algebra defined in Theorem (1) will be called the Ivanov-Renormalized Powers of Quantum White Noise Lie algebra with renormalization constant c and denoted $RPQWN_c$.

4.2. The *-Lie Algebra $RPQWN_*$, of Renormalized Higher Powers of White Noise with Convolution Type Renormalization

Motivated by a detailed analysis of the no–go theorems, the following, convolution type, renormalization was introduced by Accardi and Boukas in [3, 12] and [4]:

$$\delta^l(t - s) = \delta(s)\,\delta(t - s); \qquad l = 2, 3, ... \tag{20}$$

where the distribution on the right hand side is is the usual convolution of distributions.
The new renormalization leads to the commutation relations:

$$[B_k^n(g), B_K^N(f)] = (k\,N - K\,n)\, B_{k+K-1}^{n+N-1}(gf). \tag{21}$$

Theorem 2. *(Accardi, Boukas [3])*
Let $\mathcal{S}_0(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^d$ that vanish at zero. There exists a unique $*$-Lie algebra with:
— generators given by

$$\{B_k^n(f) \,:\, f \in \mathcal{S}_0(\mathbb{R}^d)\,;\, k, n \in \mathbb{N}\} \tag{22}$$

such that the maps $f \in \mathcal{S}_0(\mathbb{R}^d) \mapsto B_k^n(f)$ are complex linear for $n \geq k$, — Lie bracket defined by (22),
— involution defined by (18), i.e.

$$(B_k^n(f))^* = B_n^k(\bar{f}).$$

Definition 2. The $*$-Lie algebra defined in Theorem (22) will be called the convolution-Renormalized Powers of Quantum White Noise Lie algebra and denoted $RPQWN_*$.

4.3. The 1-Mode Reduction of the White Noise Algebras

By inspection of (17), (18) and (21) one verifies that, for any $*$-Lie sub-algebra either of $RPQWN_c$ or of $RPQWN_*$, fixing an open set $I \subset \mathbb{R}^d \setminus \{0\}$ with

$$|I| := \text{Lebesgue measure of } I < \infty \tag{23}$$

and restricting the test function space to the single function

$$f(x) = g(x) = \chi_I(x) := \begin{cases} 0 & \text{if } x \notin I \\ 1 & \text{if } x \in I \end{cases}$$

one obtains a $*$-Lie sub-algebra of the corresponding $*$-Lie algebra.

Notice that, for both $*$—Lie algebras, the generators with indices n, k satisfying the condition

$$n + k > 2$$

define a $*$-Lie sub-algebra of the corresponding $*$-Lie algebra. In the case of this sub-algebra, when I varies among all subsets of $\mathbb{R}^d$ ($\mathbb{R}^d \setminus \{0\}$ in the case of $RPQWN_*$) not necessarily satisfying condition (23), the corresponding $*$-Lie algebras are isomorphic.

This defines the *one mode $*$-Lie algebra $RPQWN_*$*

$$[B_k^n, B_K^N] = (k\,N - K\,n)\; B_{k+K-1}^{n+N-1}, \tag{24}$$

$$(B_k^n)^* = B_n^k. \tag{25}$$

5. THE CONFORMAL $*$-LIE ALGEBRAS

The conformal $*$-Lie algebras were introduced in conformal quantum field theory, as generalizations of the Virasoro algebra, in the attempt to construct a quantum theory of gravity.

5.1. The w_∞ $*$-Lie Algebra

Generalizing previous results of A.B. Zamolodchikov, V.A. Fateev and S. Lukyanov, [18, 24], Bakas ([17]) introduced the w_∞ $*$-Lie algebra, defined by generators $(\hat{B}_k^n)$ where

$$k \in \mathbb{Z}; \qquad n \in \mathbb{N}\,,\, n \geq 2$$

with commutation and involution relations,

$$[\hat{B}_k^n, \hat{B}_K^N] = (k\,(N-1) - K\,(n-1))\; \hat{B}_{k+K}^{n+N-2} \tag{25}$$

$$\left(\hat{B}_k^n\right)^* = \hat{B}_{-k}^n. \tag{26}$$

Due to the relation

$$2 + 2 - 2 = 2$$

the set of generators

$$\{L_k := \hat{B}_k^2 \ : \ k \in \mathbb{Z}\} \qquad (27)$$

defines a sub-$*$-Lie algebra of w_∞.

$$[\hat{B}_k^2, \hat{B}_K^2] = (k - K)\,\hat{B}_{k+K}^2; \qquad \left(\hat{B}_k^2\right)^* = \hat{B}_{-k}^2). \qquad (28)$$

In the notation (27) the commutation relations (28) take the form

$$[L_k, L_K] = (k - K)\,L_{k+K}; \qquad (L_k)^* = L_{-k} \qquad (29)$$

which defines the Witt (or centerless Virasoro) $*$-Lie algebra.

5.2. The W_∞ $*$-Lie Algebra

Generalizing Bakas' result, C.N. Pope, L.J. Romans and X. Shen, in the papers [23] (see also [22], and [21])), Pope, Romans and Shen introduced the W_∞ Lie algebra as the inductive limit of the family of algebras (W_N) which appear in conformal field theory (W_3 is Zamolodchikov's algebra, see [24]).

W_∞ is a Lie algebra with generators (called *conformal currents*)

$$\{V_n^j \ : \ n, j \in \mathbb{Z}, j \geq 2\} \qquad (30)$$

and commutation relations

$$[V_m^i, V_n^j] = \sum_{l \geq 0} g_{2l}^{ij}(m, n)\,V_{m+n}^{i+j-2l} + c_i(m)\,\delta_{i,j}\,\delta_{m+n,0}, \qquad (31)$$

where

$$c_i(m) = m(m^2 - 1)(m^2 - 4)\cdots(m^2 - (i+1)^2)\,c_i \qquad (32)$$

and the constants c_i, called *central charges* are given by

$$c_i = \frac{2^{2i-3}\,i!\,(i+2)!}{(2i+1)!!\,(2i+3)!!}\,c \quad (c \in \mathbb{R} \text{ arbitrary})$$

(here, for an odd positive integer n, the *double factorial* sign $n!!$ denotes the product of all odd values up to n)

$$g_l^{ij}(m, n) = \frac{1}{2(l+1)!}\,\phi_l^{i,j},\,N_l^{i,j}(m, n)$$

$$N_l^{i,j}(m, n) = \sum_{k=0}^{l+1} (-1)^k \begin{pmatrix} l+1 \\ k \end{pmatrix}$$

$$\times (2i + 2 - l)_k (2j + 2 - k)^{(l+1-k)}(i + 1 + m)^{(l+1-k)}(j + 1 + n)^{(k)}$$

$$\phi_l^{i,j} = {}_4F_3 \begin{pmatrix} -1/2, & 3/2, & -l/2 - 1/2, & -l/2 \\ & & & ; 1 \\ -i - 1/2, & -j - 1/2, & i + j - l + 5/2, & -l/2 \end{pmatrix}.$$

6. RENORMALIZED WHITE NOISE REPRESENTATION OF w_∞

In the paper [6] Accardi and Boukas proved that the closures, in appropriate topologies, of the $*$-Lie-algebras w_∞ and $RPQWN_*$ coincide.

The proof is constructive, giving explicit representations of the generators of each of the two $*$-Lie-algebras in terms of infinite series of generators of the other one converging in the above mentioned topology. The following result will be used in the present paper.

Theorem 3. *If the higher powers of the delta function are renormalized with the generalized Ivanov renormalization prescription (15), then the QWN operators*

$$\hat{B}_k^n(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(a_t - a_t^\dagger)} \left(a_t + a_t^\dagger \right)^{n-1} e^{\frac{k}{2}(a_t - a_t^\dagger)}\, dt, \tag{33}$$

where $n, k \in \mathbb{Z}$ with $n \geq 2$, and the operators (33) satisfy the involution condition (26) and the commutation relations

$$[\hat{B}_k^n(f), \hat{B}_K^N(g)] = \sum_{m=0}^{n-1} \sum_{l=0}^{N-1} \beta_{m,l}(n, k; N, K; c)\, \hat{B}_{k+K}^{m+l+1}(fg), \tag{34}$$

where by definition $0^0 := 1$ and the remaining structure constants are given by (38).

Theorem 4. *Let $n \geq 2$ and $k \in \mathbb{Z}$. Then, in the sense of formal series, for all test functions f,*

$$\hat{B}_k^n(f) = \sum_{m=0}^{n-1} \sum_{m'=0}^{n-1-m} \sum_{p,q=0}^{\infty} \binom{n-1}{m} \binom{n-1-m}{m'}$$

$$\times (-1)^p \frac{k^{p+q}}{p!\, q!}\, \phi_m(c, k)\, B_{n-1-m-m'+q}^{m'+p}(f), \tag{35}$$

where

$$\phi_m(c, k) := \begin{cases} 0 & \text{if } m \text{ is odd} \\ \left(\delta_{m,0} + \epsilon_{m,0} \Pi_{i=0}^{i=\frac{m}{2}-1} (m - 2i - 1)\, c^{m/2} \right) e^{-ck^2/2} & \text{if } m \text{ is even or zero} \end{cases}$$

and the case $k = 0$ (only $p = q = 0$ survives and we use $0^0 = 1$) is interpreted as

$$\hat{B}_0^n(f) = \sum_{m=0}^{n-1} \sum_{m'=0}^{n-1-m} \binom{n-1}{m} \binom{n-1-m}{m'}\, \phi_m(c, 0)\, B_{n-1-m-m'}^{m'}(f).$$

7. CONTRACTIONS OF $*$-LIE ALGEBRAS

Definition 3. A family

$$(C_{\alpha,\beta}^\gamma)_{\alpha,\beta,\gamma \in T}$$

of structure constants, defining a Lie-algebra (or $*-$Lie$-$algebra) $\mathcal{L}$ is called *locally finite* if, for each pair $\alpha, \beta \in T$, one has:

$$C_{\alpha,\beta}^\gamma \neq 0$$

only for a finite number of $\gamma \in T$. A set $(\ell_\alpha)_{\alpha \in T}$ of generators of a $*$-Lie-algebra $\mathcal{L}$ is called *locally finite* if such is the associated family of structure constants.

Definition 4. Let I be a topological space and T be a set. A family of structure constants

$$\{C_{\alpha,\beta}^\gamma(c) \ : \ \alpha, \beta, \gamma \in T\}; \qquad \forall c \in I$$

defining a family of Lie$-$algebras (or $*-$Lie$-$algebras) $(\mathcal{L}_c)_{c \in I}$ is said to be *convergent as $c \to c_0$* if:

$$\lim_{c \to c_0} C_{\alpha,\beta}^\gamma(c) =: C_{\alpha,\beta}^\gamma; \qquad \forall \alpha, \beta, \gamma \in T \tag{36}$$

in the sense that the limit exists and defines the right hand side.

If this is the case it is not difficult to verify that also

$$\{C_{\alpha,\beta}^{\gamma} \; : \; \alpha,\beta,\gamma \in T\}; \qquad \forall c \in I$$

is a family of structure constants of some Lie-algebra (or $*$-Lie-algebra) $\mathcal{L}$.

Moreover, condition (36) implies that, if the family $(C_{\alpha,\beta}^{\gamma}(c))$ is locally finite, the same is true for the limit family $(C_{\alpha,\beta}^{\gamma})$ because in the limit the family of nonzero structure constants can only decrease.

Definition 5. In the notations of Definition 4 the Lie-algebra (or $*$-Lie-algebra) $\mathcal{L}$ is called *a contraction* of the family of Lie-algebras (or $*$-Lie-algebras) $(\mathcal{L}_c)_{c \in I}$ as $c \to c_0$.

8. CONTRACTION OF CONFORMAL ALGEBRAS: $W_\infty \to w_\infty$

C. N. Pope, L.J. Romans and X. Shen proved in [23] that by rescaling the generators of W_∞ according to the rule:

$$W_m^i \to q^{-i} W_m^i, \tag{37}$$

where $q > 0$ is a parameter, the w_∞ algebra can be obtained as a contraction of the W_∞ algebra, as $q \to 0$.

9. CONTRACTION OF WHITE NOISE ALGEBRAS

9.1. Contraction of $RPQWN_c$ to $RPQWN_*$ as $c \to 0$

Theorem 5. *$RPQWN_*$ is the contraction of the family $(RPQWN_c)_{c>0}$ as $c \to 0$.*

9.2. The $W_\infty(c)$ Lie Algebras

In section (9.1) we have seen that the $*$-Lie-algebra $RPQWN_*$ is a contraction of $RPQWN_c$ as $c \to 0$.

Therefore, in view of the results of section (6) it is natural to conjecture that w_∞ is the contraction, as $c \to 0$, of a family $W_\infty(c)$, of $*$-Lie algebras contained in some natural closure of $RPQWN_c$.

Recently Accardi and Boukas have proved that this conjecture is true [2].

Theorem 6. *For each $c > 0$ there exists a unique $*$-Lie algebra, hereafter denoted by $W_\infty(c)$, with generators*

$$\{\hat{B}_k^n(f) = \hat{B}_k^n(f;c) \; : \; n, k \in \mathbb{Z} \, ; \, n \geq 2 \, ; \, f \in \mathcal{S}(\mathbb{R}^d)\}$$

involution (26), i.e.

$$\left(\hat{B}_k^n\right)^* = \hat{B}_{-k}^n$$

commutation relations (34), i.e.

$$[\hat{B}_k^n(f), \hat{B}_K^N(g)] = \sum_{m=0}^{n-1} \sum_{l=0}^{N-1} \beta_{m,l}(n,k;N,K;c) \, \hat{B}_{k+K}^{m+l+1}(fg)$$

and structure constants given by

$$\beta_{m,l}(n,k;N,K;c) = (1 - \delta_{(n-1-m)+(N-1-l),0}) \begin{pmatrix} n-1 \\ m \end{pmatrix} \begin{pmatrix} N-1 \\ l \end{pmatrix}$$
$$\times \left((-1)^{n-m-1} - (-1)^{N-l-1}\right) k^{N-l-1} K^{n-m-1} c^{n+N-(m+l)-3}. \tag{38}$$

9.3. Contraction of $(W_\infty(c))_{c>0}$ to w_∞ as $c \to 0$

Theorem 7. *w_∞ is the contraction of the family $(W_\infty(c))_{c>0}$ as $c \to 0$.*

Remark. The white noise representation of the w_∞ generators, introduced in [3] and [4] and based not on the Ivanov renormalization, as here, but on the convolution type renormalization (20) of the powers of the delta function, the QWN is

$$\hat{B}_k^n(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(a_t - a_t^\dagger)} \left(\frac{a_t + a_t^\dagger}{2} \right)^{n-1} e^{\frac{k}{2}(a_t - a_t^\dagger)}\, dt. \tag{39}$$

With these notations the structure constants become:

$$\beta_{m,l}(n, k; N, K; c) = \frac{1}{2^{n+N-2}} \left(1 - \delta_{(n-1-m)+(N-1-l),0}\right) \binom{n-1}{m} \binom{N-1}{l}$$

$$\times \left((-1)^{n-m-1} - (-1)^{N-l-1} \right) k^{N-l-1} K^{n-m-1} c^{n+N-(m+l)-3}.$$

Remark. The Witt algebra, the subalgebra of $W_\infty(c)$ generated by

$$\hat{B}_k^2(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(a_t - a_t^\dagger)} \left(a_t + a_t^\dagger \right) e^{\frac{k}{2}(a_t - a_t^\dagger)}\, dt$$

remains fixed during the expansion of w_∞ to $W_\infty(c)$.

10. CENTRAL EXTENSIONS OF $RQPWN_*$

It is known (see [10]) that, with the exception of its Heisenberg algebra sector, $RQPWN_*$ admits no nontrivial central extension. Precisely, the non-trivial central extensions of $RQPWN_*$ are given by

$$[B_k^n(f), B_K^N(g)] = (kN - Kn)\, B_{k+K}^{n+N-1}(fg) + \rho_z(n, k; N, K)\, E,$$

where E is the (self-adjoint) central element and

$$\rho_z(n, k; N, K) = \delta_{n+k,0}\delta_{N,0}\delta_{K,1}\, z + \delta_{N+K,0}\delta_{n,1}\delta_{k,0}\, \bar{z}$$

with $z \in \mathbb{C} \setminus \{0\}$ arbitrary.

The same is true, with the exception of its Virasoro algebra sector, for w_∞ whose non-trivial central extensions are given by

$$[\hat{B}_k^n(f), \hat{B}_K^N(g)] = (k(N-1) - K(n-1))\, \hat{B}_{k+K}^{n+N-2}(fg) + \delta_{n,2}\, \delta_{N,2}\, \delta_{k+K,0}\, k(k^2 - 1)\, E$$

where traditionally $E = \frac{c}{12}$, where $c > 0$ is the "central charge".

Remark. The factor $\delta_{i,j}\, \delta_{m+n,0}$ is non zero only if

$$n = -m \qquad \text{and} \qquad i = j$$

which corresponds to the sub-algebras

$$[V_m^j, V_{-m}^j] = \sum_{l \geq 0} g_{2l}^{jj}(m, -m)\, V_0^{2(j-l)} + c_j(m); \qquad j \in \{2, 3, ...\} \tag{40}$$

of which the case $j = 0$ should correspond to Virasoro. This suggests that we look for central extensions before taking the contraction $c \to 0$.

Remark. As shown in section (8) w_∞ can be obtained as a contraction of W_∞ as $q \to 0$. In this limit only the Virasoro central extension survives and we obtain the w_∞ Lie algebra commutation relations (25) and their Virasoro central extension in the form

$$[V_{m;0}^i, V_{n;0}^j] = ((j+1)m - (i+1)n)\, V_{m+n;0}^{i+j} + \frac{c}{12}m(m^2 - 1)\delta_{i,0}\delta_{j,0}\delta_{m+n,0}$$

which can be put in the form of (25) by defining

$$\hat{B}_m^i = V_{m;0}^{i-2}.$$

Notice that the Witt-Virasoro algebra generators are

$$\hat{B}_m^2 = V_{m;0}^0.$$

Remark. Letting

$$\hat{B}_k^n = V_k^{n-2}; \qquad n, N = 0, 1, ...$$

we see that the W_∞ commutation relations take the form

$$[\hat{B}_k^n, \hat{B}_K^N] = \sum_{l \geq 0} g_{2l}^{(n-2)(N-2)}(k, K)\, \hat{B}_{k+K}^{n+N-2(l+1)} + c_{n-2}(k)\, \delta_{n,N}\, \delta_{k+K,0}, \tag{41}$$

i.e.

$$[\hat{B}_k^n, \hat{B}_{-n}^n] = \sum_{l \geq 0} g_{2l}^{(n-2)(n-2)}(k, -k)\, \hat{B}_0^{2(n-l-1)} + c_{n-2}(k) \tag{42}$$

while, for $c_n = 0$, we have the non-centrally extended commutation relations

$$[\hat{B}_k^n, \hat{B}_K^N] = \sum_{l \geq 0} g_{2l}^{(n-2)(N-2)}(k, K)\, \hat{B}_{k+K}^{n+N-2(l+1)}. \tag{43}$$

Remark. Letting $M = n - 1 - m$ and $L = N - 1 - l$ we see that the $W_\infty(c)$ commutation relations of Theorem (3) can be put in the form

$$[\hat{B}_k^n(f), \hat{B}_K^N(g)] = \sum_{M=0}^{n-1} \sum_{L=0}^{N-1} \hat{\beta}_{M,L}(n, k; N, K; c)\, \hat{B}_{k+K}^{n+N-(M+L+1)}(fg), \tag{44}$$

where

$$\hat{\beta}_{m,l}(n, k; N, K; c) = (1 - \delta_{(n-1-m)+(N-1-l),0}) \begin{pmatrix} n-1 \\ m \end{pmatrix} \begin{pmatrix} N-1 \\ l \end{pmatrix}$$

$$\times \left((-1)^{n-m-1} - (-1)^{N-l-1} \right) k^{N-l-1} K^{n-m-1} \times c^{n+N-(m+l)-3}.$$

We notice that, due to the presence of the

$$\left((-1)^{n-m-1} - (-1)^{N-l-1} \right) = \left((-1)^M - (-1)^L \right)$$

factor, the only non-zero contribution to the commutator $[\hat{B}_k^n, \hat{B}_K^N]$ comes from terms with M, L of different even/odd parity which, in turn, implies that $M + L + 1$ is always even. Therefore, just like in W_∞, the commutator contains only terms of the form

$$\hat{B}_{k+K}^{n+N-2(l+1)},$$

where we have set $n + N - (M + L + 1) = n + N - 2(l + 1)$ with l ranging from 0 to $n + N - 2$. We may therefore write the $W_\infty(c)$ commutation relations (44) as

$$[\hat{B}_k^n(f), \hat{B}_K^N(g)] = \sum_{l \geq 0} \hat{b}_l(n, k; N, K; c)\, \hat{B}_{k+K}^{n+N-2(l+1)}(fg), \tag{45}$$

where

$$\hat{b}_l(n, k; N, K; c) = \sum_{\substack{M, L \in \{0, 1, ..., n-1\} \\ M + L = 2l + 1}} \hat{\beta}_{M,L}(n, k; N, K; c). \tag{46}$$

In the one-mode case, i.e. over a fixed interval, commutation relations (45) become

$$[\hat{B}_k^n, \hat{B}_K^N] = \sum_{l \geq 0} \hat{b}_l(n, k; N, K; c) \, \hat{B}_{k+K}^{n+N-2(l+1)}. \tag{47}$$

Notice the similarity between the one-mode $W_\infty(c)$ commutation relations (47) and the non-centrally extended W_∞ commutation relations (43). This similarity motivates the investigation of central extensions of $W_\infty(c)$. The following section is devoted to this topic.

11. CENTRAL EXTENSIONS OF $W_\infty(c)$

Theorem 8. *The non-trivial central extensions of the $W_\infty(c)$ commutation relations (44) are given by*

$$[\hat{B}_k^n(f), \hat{B}_K^N(g)] = \sum_{l \geq 0} \hat{b}_l(n, k; N, K; c) \, \hat{B}_{k+K}^{n+N-2(l+1)}(fg) + \delta_{n,N}\, \delta_{k+K,0}\, k(k^2 - 1)\, \sigma(n, k)\, E,$$

i.e.

$$[\hat{B}_k^n(f), \hat{B}_{-k}^n(g)] = \sum_{l \geq 0} \hat{b}_l(n, k; n, -k; c) \, \hat{B}_0^{2(n-l-1)}(fg) + k(k^2 - 1)\sigma(n, k)\, E, \tag{48}$$

where, in the notation of (13) and (14),

$$\sigma(n, k) := \begin{cases} \prod_{i=2}^{n-1} \dfrac{(k - r_i - 1)^{(k-2)}}{(k + r_i + 1)^{(k-2)}} & \text{if } k \geq 0 \\[2em] \prod_{i=2}^{n-1} \dfrac{(k + r_i + 1)_{-k-2}}{(k - r_i - 1)_{-k-2}} & \text{if } k \leq 0 \end{cases} \tag{49}$$

and $r_1 = -2, r_2, r_3, \ldots, r_{n-1}$ are the roots of the Jacobi polynomial

$$P_{n-1}^{(0, -2n+1)}(-2r - 1) = {}_2F_1(1 - n, 1 - n; 1, r + 1) = \sum_{L=0}^{n-1} \binom{n-1}{L}^2 (r+1)^L.$$

REFERENCES

1. L. Accardi, Y. G. Lu and I. Volovich, *Quantum Theory and its Stochastic Limit* (Springer, New York, 2002).
2. L. Accardi and A. Boukas, "Contractions and central extensions of quantum white noise Lie algebras," submitted to: Infin. Dimen. Anal. Quant. Prob. Relat. Topics, January (2012)
3. L. Accardi and A. Boukas, "Renormalized higher powers of white noise (RHPWN) and conformal field theory," Infin. Dimen. Anal. Quant. Prob. Relat. Topics **9** (3), 353−360 (2006).
4. L. Accardi and A. Boukas, "The emergence of the Virasoro and w_∞ Lie algebras through the renormalized higher powers of quantum white noise," Int. J. Math. Comp. Sci. **1** (3), 315−342 (2006).
5. L. Accardi and A. Boukas, "Higher powers of q-deformed white noise," Meth. Funct. Anal. Top. **12** (3), 205−219 (2006).
6. L. Accardi and A. Boukas, "Renormalized higher powers of white noise and the Virasoro-Zamolodchikov-w_∞ algebra," Rep. Math. Phys. **61** (1), 1−11 (2008).
7. L. Accardi and A. Boukas, "Fock representation of the renormalized higher powers of white noise and the Virasoro-Zamolodchikov w_∞ ∗-Lie algebra," J. Phys. A: Math. Theor. **41** (2008).
8. L. Accardi and A. Boukas, "Analytic central extensions of infinite dimensional white noise ∗-Lie algebras," Stochastics **81** (3-4), 201−218 (2009).
9. L. Accardi and A. Boukas, "Quantum probability, renormalization and infinite dimensional ∗-Lie algebras," SIGMA, Sym. Integ. Geom. Meth. Appl. **5** 056, 31 (2009).
10. L. Accardi and A. Boukas, "Cohomology of the Virasoro-Zamolodchikov and renormalized higher powers of white noise ∗-Lie algebras," Infin. Dimen. Anal. Quant. Prob. Relat. Topics **12** (2), 1−20 (2009).
11. L. Accardi and A. Boukas, "Powers of the delta function," Quant. Prob. White Noise Anal. **20**, 33−44 (World Scientific, 2007).

12. L. Accardi, A. Boukas and U. Franz, "Renormalized powers of quantum white noise," Infin. Dimen. Anal. Quant. Prob. Relat. Topics **9** (1), 129–147 (2006).
13. L. Accardi, G. Amosov and U. Franz, "Second quantized automorphisms of the renormalized square of white noise (RSWN) algebra," Infin. Dimen. Anal. Quant. Prob. Relat. Topics **7** (2), 184–194 (2004); Volterra Prepr. (2002).
14. L. Accardi and A. Dhahri, "The quadratic Fock functor," J. Math. Phys. **51**, 022105 (2010).
15. L. Accardi, Y. G. Lu and I. V. Volovich, "White noise approach to classical and quantum stochastic calculi," Lect. Notes Volterra-CIRM Int. School (Trento, Italy, 1999); Volterra Prepr. **375** (1999).
16. L. Accardi and I. V. Volovich, "Quantum white noise with singular non-linear interaction," in *Developments of Infinite-Dimensional Noncommutative Analysis*, ed. N. Obata (RIMS Kokyuroku 1099), Publications of the Research Inst. for Math. Sciences, 61–70 (Kyoto, 1999); Volterra Prepr. **278** (1997), quant-ph/9704029.
17. I. Bakas, "The structure of the $W_{\inf}$ algebra," Commun. Math. Phys. **134**, 487–508 (1990).
18. V. A. Fateev and S. Lukyanov, Int. J. Mod. Phys. A **3**, 507 (1988).
19. P. J. Feinsilver and R. Schott, *Algebraic Structures and Operator Calculus. Volumes I and III* (Kluwer, 1993).
20. V. K. Ivanov, "The algebra of elementary generalized functions," Soviet Math. Dokl. **20** (3), 553–556 (1979).
21. S. V. Ketov, *Conformal Field Theory* (World Scientific, Singapore, 1995).
22. C. N. Pope, "Lectures on W algebras and W gravity," Lectures at the Trieste Summer School in High-Energy Physics (1991); hep-th/9112076v1.
23. C. N. Pope, L. J. Romans and X. Shen, "$W_{\inf}$ and the Racah-Wigner algebra," Nucl. Phys. B **339**, 191–221 (1990).
24. A. B. Zamolodchikov, "Infinite extra symmetries in two-dimensional conformal quantum field theory," Teor. Mat. Fiz. **65** (3), 347–359 (1985) [in Russian].