

# The Semi-Martingale Property of the Square of White Noise Integrators

LUIGI ACCARDI Centro Vito Volterra, Università degli Studi di Roma Tor Vergata, Roma 00133, Italy

ANDREAS BOUKAS American College of Greece, Athens 15342, Greece

**Abstract.** The abstract commutation relations of the algebra of the square of white noise of Accardi, Lu, and Volovich are shown to be realized by operator processes acting on the Fock space of Accardi and Skeide which is very closely related to the Finite Difference Fock space of Boukas and Feinsilver. The processes are shown to satisfy the necessary conditions for inclusion in the framework of the representation free quantum stochastic calculus of Accardi, Fagnola, and Quaegebeur. The connection between the Finite-Difference operators and the creation, annihilation, and conservation operators on usual symmetric Boson Fock space is further studied.

## 1. Introduction

The "square of white noise" or "SWN" algebra was defined in [ALV 99] as the Lie algebra generated by elements  $B(f)$ ,  $B^\dagger(f)$ , and  $N(f)$  satisfying the commutation relations:

$$\begin{aligned} [B(f), B^\dagger(g)] &= 2c \int_0^\infty \bar{f}g ds + 4N(\bar{f}g) \\ [N(\phi), B(f)] &= -2B(\bar{\phi}f) \\ [N(\phi), B^\dagger(f)] &= 2B^\dagger(\phi f) \\ [B^\dagger(f), B^\dagger(g)] &= [N(\phi), N(\psi)] = 0 \end{aligned}$$

where  $c > 0$  and  $f, g, \phi, \psi$  are suitable functions.

Let  $D = \{z \in \mathbb{C} / |z| < 1/2\}$  and let  $S(R_+, D)$  denote the set of step functions defined on  $R_+$  with values in  $D$  i.e  $f \in S(R_+, D) \Leftrightarrow f = \sum_{i=1}^n a_i \chi_{I_i}$ ,  $a_i \in D, I_i \subset [0, +\infty), I_i \cap I_j = \emptyset$  for  $i \neq j, i, j = 1, 2, \dots, n \in \mathbb{N}$ .

A good candidate for a Fock space, on which the above commutation relation can be realized in the operator sense, was defined in [AS 99] as follows:

DEFINITION 1.1. Let  $c > 0$ . The SWN Fock space  $\Gamma$  is the Hilbert space completion of the linear span of "exponential vectors"  $\psi(f)$ ,  $f \in S(R_+, D)$ , under the inner product

$$\langle \psi(f), \psi(g) \rangle = e^{-\frac{c}{2} \int_0^\infty \ln(1 - 4\bar{f}g) ds}$$

After the rescaling  $c \rightarrow 2$  and  $a_i \rightarrow \frac{a_i}{2}$  the SWN inner product defined above is seen to agree with that of the Finite-Difference Fock space of [Bou 88] and [Fei 87].

To realize the above commutation relations on  $\Gamma$  we define the SWN operators  $B(f)$ ,  $B^\dagger(f)$ ,  $N(f)$  by their action on the exponential vectors  $\psi(g)$  of  $\Gamma$  as follows:

DEFINITION 1.2. Let  $f, g \in S(R_+, D)$ . Then

$$\begin{aligned} B^\dagger(f)\psi(g) &= \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \psi(g + \epsilon f) \\ B(f)\psi(g) &= (2c \int_0^\infty \bar{f}g ds + 4B^\dagger(\bar{f}g^2))\psi(g) \\ N(f)\psi(g) &= 2 \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \psi(e^{\epsilon f} g) \end{aligned}$$

REMARK 1.3 For  $\epsilon$  sufficiently close to zero,  $f, g \in S(R_+, D)$  implies that  $g + \epsilon f$  and  $e^{\epsilon f} g$  are also in  $S(R_+, D)$ .

## 2. Matrix elements of the SWN operators

PROPOSITION 2.1 Let  $f, \phi, g \in S(R_+, D)$ . Then

$$\begin{aligned} \langle \psi(\phi), B^\dagger(f)\psi(g) \rangle &= 2c \int_0^\infty \frac{f(s)\bar{\phi}(s)}{1 - 4\bar{\phi}(s)g(s)} ds \langle \psi(\phi), \psi(g) \rangle \\ \langle \psi(\phi), N(f)\psi(g) \rangle &= 4c \int_0^\infty \frac{f(s)\bar{\phi}(s)g(s)}{1 - 4\bar{\phi}(s)g(s)} ds \langle \psi(\phi), \psi(g) \rangle \\ \langle \psi(\phi), B(f)\psi(g) \rangle &= 2c \int_0^\infty \frac{\bar{f}(s)g(s)}{1 - 4\bar{\phi}(s)g(s)} ds \langle \psi(\phi), \psi(g) \rangle \end{aligned}$$

PROOF:

$$\begin{aligned} \langle \psi(\phi), B^\dagger(f)\psi(g) \rangle &= \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \langle \psi(\phi), \psi(g + \epsilon f) \rangle \\ &= \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} e^{-\frac{c}{2} \int_0^\infty (1 - 4\bar{\phi}(g + \epsilon f)) ds} \\ &= 2c \int_0^\infty \frac{f\bar{\phi}}{1 - 4\bar{\phi}g} ds e^{-\frac{c}{2} \int_0^\infty (1 - 4\bar{\phi}g) ds} \\ &= 2c \int_0^\infty \frac{f\bar{\phi}}{1 - 4\bar{\phi}g} ds \langle \psi(\phi), \psi(g) \rangle \end{aligned}$$

$$\begin{aligned} \langle \psi(\phi), N(f)\psi(g) \rangle &= 2 \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \langle \psi(\phi), \psi(e^{\epsilon f} g) \rangle \\ &= 2 \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} e^{-\frac{c}{2} \int_0^\infty (1 - 4\bar{\phi}e^{\epsilon f} g) ds} \\ &= 4c \int_0^\infty \frac{\bar{\phi}fg}{1 - 4\bar{\phi}g} ds e^{-\frac{c}{2} \int_0^\infty (1 - 4\bar{\phi}g) ds} \\ &= 4c \int_0^\infty \frac{\bar{\phi}fg}{1 - 4\bar{\phi}g} ds \langle \psi(\phi), \psi(g) \rangle \end{aligned}$$

$$\begin{aligned} \langle \psi(\phi), B(f)\psi(g) \rangle &= \langle \psi(\phi), (2c \int_0^\infty \bar{f}g ds + \\ &\quad 4B^\dagger(\bar{f}g^2))\psi(g) \rangle \\ &= 2c \int_0^\infty \bar{f}g ds \langle \psi(\phi), \psi(g) \rangle \\ &\quad + 4 \langle \psi(\phi), B^\dagger(\bar{f}g^2)\psi(g) \rangle \\ &= (2c \int_0^\infty \bar{f}g ds + \\ &\quad 8c \int_0^\infty \frac{\bar{f}g^2\bar{\phi}}{1 - 4\bar{\phi}g} ds) \\ &\quad \langle \psi(\phi), \psi(g) \rangle \\ &= 2c \int_0^\infty \frac{\bar{f}g}{1 - 4\bar{\phi}g} ds \langle \psi(\phi), \psi(g) \rangle \end{aligned}$$

NOTE: By Proposition 2.1  $\langle \psi(\phi), B^\dagger(f)\psi(g) \rangle = \langle B(f)\psi(\phi), \psi(g) \rangle$  i.e.  $B^\dagger(f)$  and  $B(f)$  are one the adjoint of the other on the exponential domain  $\epsilon$ . Similarly  $\langle \psi(\phi), N(f)\psi(g) \rangle = \langle N(\bar{f})\psi(\phi), \psi(g) \rangle$  i.e.  $N(\bar{f})$  is the adjoint of  $N(f)$ . Moreover (in the sense of matrix elements)  $B(f)\psi(0) = N(f)\psi(0) = 0$

PROPOSITION 2.2 Let  $g, \gamma, f, \phi \in S(\mathbb{R}_+, D)$ . Then

$$\begin{aligned} \langle B^\dagger(g)\psi(f), B^\dagger(\gamma)\psi(\phi) \rangle &= (2c \int_0^\infty \frac{\bar{g}(s)\gamma(s)}{(1-4\bar{f}(s)\phi(s))^2} ds + \\ &4c^2 \int_0^\infty \frac{\bar{g}(s)\phi(s)}{1-4\bar{f}(s)\phi(s)} ds \cdot \\ &\int_0^\infty \frac{\bar{f}(s)\gamma(s)}{1-4\bar{f}(s)\phi(s)} ds) \langle \psi(f), \psi(\phi) \rangle \end{aligned}$$

$$\begin{aligned} \langle N(g)\psi(f), N(\gamma)\psi(\phi) \rangle &= (16c^2 \int_0^\infty \frac{\bar{f}(s)\phi(s)\bar{g}(s)}{1-4\bar{f}(s)\phi(s)} ds \int_0^\infty \frac{\bar{f}(s)\phi(s)\gamma(s)}{1-4\bar{f}(s)\phi(s)} ds \\ &+ 8c \int_0^\infty \frac{\gamma(s)\bar{f}(s)\phi(s)\bar{g}(s)}{(1-4\bar{f}(s)\phi(s))^2} ds) \langle \psi(f), \psi(\phi) \rangle \end{aligned}$$

$$\begin{aligned} \langle B(g)\psi(f), B(\gamma)\psi(\phi) \rangle &= (4c^2 \int_0^\infty \frac{g(s)\bar{f}(s)}{1-4\bar{f}(s)\phi(s)} ds \int_0^\infty \frac{\bar{\gamma}(s)\phi(s)}{1-4\bar{f}(s)\phi(s)} ds \\ &+ 32c \int_0^\infty \frac{\bar{\gamma}(s)g(s)\phi^2(s)\bar{f}^2(s)}{(1-4\bar{f}(s)\phi(s))^2} ds) \langle \psi(f), \psi(\phi) \rangle \end{aligned}$$

$$\begin{aligned} \langle B^\dagger(g)\psi(f), N(\gamma)\psi(\phi) \rangle &= (8c^2 \int_0^\infty \frac{\bar{g}(s)\phi(s)}{1-4\bar{f}(s)\phi(s)} ds \int_0^\infty \frac{\bar{f}(s)\gamma(s)\phi(s)}{1-4\bar{f}(s)\phi(s)} ds \\ &+ 4c \int_0^\infty \frac{\bar{g}(s)\gamma(s)\phi(s)}{(1-4\bar{f}(s)\phi(s))^2} ds) \langle \psi(f), \psi(\phi) \rangle \end{aligned}$$

$$\begin{aligned} \langle B^\dagger(g)\psi(f), B(\gamma)\psi(\phi) \rangle &= (4c^2 \int_0^\infty \frac{\bar{g}(s)\phi(s)}{1-4\phi(s)\bar{f}(s)} ds \int_0^\infty \frac{\bar{\gamma}(s)\phi(s)}{1-4\phi(s)\bar{f}(s)} ds \\ &+ 8c \int_0^\infty \frac{\bar{g}(s)\bar{\gamma}(s)\phi^2(s)}{(1-4\bar{f}(s)\phi(s))^2} ds) \langle \psi(f), \psi(\phi) \rangle \end{aligned}$$

$$\begin{aligned} \langle B(g)\psi(f), N(\gamma)\psi(\phi) \rangle &= (8c^2 \int_0^\infty \frac{\bar{f}(s)\gamma(s)\phi(s)}{1-4\bar{f}(s)\phi(s)} ds \int_0^\infty \frac{g(s)\bar{f}(s)}{1-4\bar{f}(s)\phi(s)} ds \\ &+ 16c \int_0^\infty \frac{\gamma(s)\bar{f}^2(s)\phi(s)g(s)}{(1-4\bar{f}(s)\phi(s))^2} ds) \langle \psi(f), \psi(\phi) \rangle \end{aligned}$$

PROOF

$$\begin{aligned} \langle B^\dagger(g)\psi(f), B^\dagger(\gamma)\psi(\phi) \rangle &= \frac{\partial^2}{\partial \epsilon \partial \delta} \Big|_{\epsilon=\delta=0} \langle \psi(f + \epsilon g), \psi(\phi + \delta \gamma) \rangle \\ &= \frac{\partial^2}{\partial \epsilon \partial \delta} \Big|_{\epsilon=\delta=0} e^{-\frac{\epsilon}{2} \int_0^\infty (1-4(\bar{f} + \epsilon \bar{g})(\phi + \delta \gamma)) ds} \\ &= (2c \int_0^\infty \frac{\bar{g}\gamma}{(1-4\bar{f}\phi)^2} ds + \\ &4c^2 \int_0^\infty \frac{\bar{g}\phi}{1-4\bar{f}\phi} ds \int_0^\infty \frac{\bar{f}\gamma}{1-4\bar{f}\phi} ds) \\ &\langle \psi(f), \psi(\phi) \rangle \end{aligned}$$

$$\begin{aligned} \langle N(g)\psi(f), N(\gamma)\psi(\phi) \rangle &= 4 \frac{\partial^2}{\partial \epsilon \partial \delta} \Big|_{\epsilon=\delta=0} \langle \psi(e^{\epsilon g} f), \psi(e^{\delta \gamma} \phi) \rangle \\ &= 4 \frac{\partial^2}{\partial \epsilon \partial \delta} \Big|_{\epsilon=\delta=0} e^{-\frac{\epsilon}{2} \int_0^\infty (1-4e^{\epsilon \bar{g} + \delta \gamma} \bar{f}\phi) ds} \\ &= (16c^2 \int_0^\infty \frac{\bar{f}\phi\bar{g}}{1-4\bar{f}\phi} ds \int_0^\infty \frac{\bar{f}\phi\gamma}{1-4\bar{f}\phi} ds \\ &+ 8c \int_0^\infty \frac{\gamma\bar{f}\phi\bar{g}}{(1-4\bar{f}\phi)^2} ds) \langle \psi(f), \psi(\phi) \rangle \end{aligned}$$

$$\begin{aligned} \langle B(g)\psi(f), B(\gamma)\psi(\phi) \rangle &= \langle (2c \int_0^\infty \bar{g}f ds + 4B^\dagger(\bar{g}f^2))\psi(f), \\ &(2c \int_0^\infty \bar{\gamma}\phi ds + 4B^\dagger(\bar{\gamma}\phi^2))\psi(\phi) \rangle \\ &= 8c \int_0^\infty \bar{g}f ds \langle \psi(f), B^\dagger(\bar{\gamma}\phi^2)\psi(\phi) \rangle \\ &+ 4c^2 \int_0^\infty \bar{g}f ds \int_0^\infty \bar{\gamma}\phi ds \langle \psi(f), \psi(\phi) \rangle \\ &+ 8c \int_0^\infty \bar{\gamma}\phi ds \langle B^\dagger(\bar{g}f^2)\psi(f), \psi(\phi) \rangle \\ &+ 16 \langle B^\dagger(\bar{g}f^2)\psi(f), B^\dagger(\bar{\gamma}\phi^2)\psi(\phi) \rangle \\ &= (8c \int_0^\infty \bar{g}f ds \cdot 2c \int_0^\infty \frac{\bar{\gamma}\phi^2\bar{f}}{1-4\bar{f}\phi} ds \\ &+ 4c^2 \int_0^\infty \bar{g}f ds \int_0^\infty \bar{\gamma}\phi ds \\ &+ 8c \int_0^\infty \bar{\gamma}\phi ds \cdot 2c \int_0^\infty \frac{g\bar{f}^2\phi}{1-4\bar{f}\phi} ds \\ &+ 64c^2 \int_0^\infty \frac{\bar{f}^2\phi g}{1-4\bar{f}\phi} ds \int_0^\infty \frac{\bar{f}\phi^2\bar{\gamma}}{1-4\bar{f}\phi} ds) \end{aligned}$$

$$\begin{aligned}
& +32c \int_0^\infty \frac{\bar{\gamma}\phi^2 g \bar{f}^2}{(1-4\bar{f}\phi)^2} ds \langle \psi(f), \psi(\phi) \rangle \\
= & (4c^2 \int_0^\infty \frac{\bar{\gamma}\phi}{1-4\bar{f}\phi} ds \int_0^\infty \frac{\bar{f}g}{1-4\bar{f}\phi} ds \\
& +32c \int_0^\infty \frac{\bar{\gamma}g\phi^2 \bar{f}^2}{(1-4\bar{f}\phi)^2} ds) \langle \psi(f), \psi(\phi) \rangle
\end{aligned}$$

$$\begin{aligned}
\langle B^\dagger(g)\psi(f), N(\gamma)\psi(\phi) \rangle &= \frac{\partial^2}{\partial \epsilon \partial \delta} \Big|_{\epsilon=\delta=0} \langle \psi(f+\epsilon g), \psi(\epsilon^\delta \gamma \phi) \rangle \\
&= \frac{\partial^2}{\partial \epsilon \partial \delta} \Big|_{\epsilon=\delta=0} e^{-\frac{\epsilon}{2}} \int_0^\infty (1-4(\bar{f}+\epsilon \bar{g})e^{\delta \gamma} \phi) ds \\
&= (8c^2 \int_0^\infty \frac{\bar{g}\phi}{1-4\bar{f}\phi} ds \int_0^\infty \frac{\bar{f}\gamma\phi}{1-4\bar{f}\phi} ds \\
&+4c \int_0^\infty \frac{\bar{g}\gamma\phi}{(1-4\bar{f}\phi)^2} ds) \langle \psi(f), \psi(\phi) \rangle
\end{aligned}$$

$$\begin{aligned}
\langle B^\dagger(g)\psi(f), B(\gamma)\psi(\phi) \rangle &= \langle B^\dagger(g)\psi(f), (2c \int_0^\infty \bar{\gamma}\phi ds + 4B^\dagger(\bar{\gamma}\phi^2))\psi(\phi) \rangle \\
&= 2c \int_0^\infty \bar{\gamma}\phi ds \langle B^\dagger(g)\psi(f), \psi(\phi) \rangle \\
&+4 \langle B^\dagger(g)\psi(f), B^\dagger(\bar{\gamma}\phi^2)\psi(\phi) \rangle \\
&= \{2c \int_0^\infty \bar{\gamma}\phi ds \cdot 2c \int_0^\infty \frac{\bar{g}\phi}{1-4\bar{f}\phi} ds \\
&+4(2c \int_0^\infty \frac{\bar{g}\bar{\gamma}\phi^2}{(1-4\bar{f}\phi)^2} ds \\
&+4c^2 \int_0^\infty \frac{\bar{g}\phi}{1-4\bar{f}\phi} ds \int_0^\infty \frac{\bar{f}\bar{\gamma}\phi^2}{1-4\bar{f}\phi} ds) \langle \psi(f), \psi(\phi) \rangle \\
&= (4c^2 \int_0^\infty \frac{\bar{g}\phi}{1-4\bar{f}\phi} ds \int_0^\infty \frac{\bar{\gamma}\phi}{1-4\bar{f}\phi} ds \\
&+8c \int_0^\infty \frac{\bar{g}\bar{\gamma}\phi^2}{(1-4\bar{f}\phi)^2} ds) \langle \psi(f), \psi(\phi) \rangle
\end{aligned}$$

$$\begin{aligned}
\langle B(g)\psi(f), N(\gamma)\psi(\phi) \rangle &= \langle (2c \int_0^\infty \bar{g}f ds + 4B^\dagger(\bar{g}f^2))\psi(f), N(\gamma)\psi(\phi) \rangle \\
&= 2c \int_0^\infty \bar{g}f ds \langle \psi(f), N(\gamma)\psi(\phi) \rangle \\
&+4 \langle B^\dagger(\bar{g}f^2)\psi(f), N(\gamma)\psi(\phi) \rangle
\end{aligned}$$

$$\begin{aligned}
&= \{2c \int_0^\infty \bar{g}f ds \cdot 4c \int_0^\infty \frac{\bar{f}\gamma\phi}{1-4\bar{f}\phi} ds \\
&+4(8c^2 \int_0^\infty \frac{\bar{g}\bar{f}^2\phi}{1-4\bar{f}\phi} ds \int_0^\infty \frac{\bar{f}\gamma\phi}{1-4\bar{f}\phi} ds \\
&+4c \int_0^\infty \frac{\bar{g}\bar{f}^2\gamma\phi}{(1-4\bar{f}\phi)^2} ds)\} \langle \psi(f), \psi(\phi) \rangle \\
&= (8c^2 \int_0^\infty \frac{\bar{f}\gamma\phi}{1-4\bar{f}\phi} ds \int_0^\infty \frac{\bar{g}\bar{f}}{1-4\bar{f}\phi} ds \\
&+16c \int_0^\infty \frac{\bar{\gamma}\bar{f}\phi^2 g}{(1-4\bar{f}\phi)^2} ds) \langle \psi(f), \psi(\phi) \rangle
\end{aligned}$$

In view of Propositions 2.1 and 2.2 we can extend the definition of  $B(f)$ ,  $B^\dagger(f)$ , and  $N(f)$  to  $f \in L^1_{loc}(R_+, C) = \{f : R_+ \rightarrow C / \int_0^t |f(s)| ds < +\infty, \forall t \geq 0\}$  as follows:

DEFINITION 2.3 If  $\sigma = \sum_{i=1}^n a_i x_{I_i}$  is a step function with  $a_i \in C, i = 1, 2, \dots, n$  we define

$$\begin{aligned}
B(\sigma) &= \lambda \sum_{i=1}^n a_i B\left(\frac{1}{\lambda} x_{I_i}\right) \\
B^\dagger(\sigma) &= \lambda \sum_{i=1}^n a_i B^\dagger\left(\frac{1}{\lambda} x_{I_i}\right) \\
N(\sigma) &= \lambda \sum_{i=1}^n a_i N\left(\frac{1}{\lambda} x_{I_i}\right)
\end{aligned}$$

where  $\lambda > 2$  is arbitrary. Notice that  $B(\frac{1}{\lambda} x_{I_i})$ ,  $B^\dagger(\frac{1}{\lambda} x_{I_i})$ , and  $N(\frac{1}{\lambda} x_{I_i})$  are defined as in Definition 1.2

DEFINITION 2.4: Let  $f \in L^1_{loc}(R_+, C)$ . We define:

$$\begin{aligned}
B(f) &= \lim_n B(\sigma_n) \\
B^\dagger(f) &= \lim_n B^\dagger(\sigma_n) \\
N(f) &= \lim_n N(\sigma_n)
\end{aligned}$$

where convergence is in the sense of matrix elements, and  $\{\sigma_n\}_{n=1}^\infty$  is any sequence of step functions converging to  $f$  in  $L^1_{loc}(R_+, C)$ .

### 3. Commutation relations of the SWN operators

PROPOSITION 3.1. Let  $p, \sigma, f, g \in S(R_+, D)$  and let  $c > 0$  be as in Definition 1.1. Then, in the sense of equality of matrix elements:

$$\begin{aligned} [B(f), B^\dagger(g)] &= 2c \int_0^\infty \bar{f}(s)g(s)ds + 4N(\bar{f}g) \\ [N(p), B^\dagger(f)] &= 2B^\dagger(pf) \\ [N(p), B(f)] &= -2B(\bar{p}f) \\ [B^\dagger(f), B^\dagger(g)] &= [N(p), N(\sigma)] = 0 \end{aligned}$$

Moreover if  $\alpha\beta = 0$  then  $[L(\alpha), M(\beta)] = 0$  for all  $L, M \in \{B^\dagger, B, N\}$

PROOF: Let  $\phi, \gamma \in S(R_+, D)$ . Then, using Proposition 2.2,

$$\begin{aligned} &\langle [B(f), B^\dagger(g)] \psi(\phi), \psi(\gamma) \rangle \\ &= \langle B^\dagger(g)\psi(\phi), B^\dagger(f)\psi(\gamma) \rangle - \langle B(f)\psi(\phi), B(g)\psi(\gamma) \rangle \\ &= (2c \int_0^\infty \frac{\bar{g}f}{(1-4\bar{\phi}\gamma)^2} ds + 4c^2 \int_0^\infty \frac{\bar{g}\gamma}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{\bar{\phi}f}{1-4\bar{\phi}\gamma} ds \\ &\quad - 4c^2 \int_0^\infty \frac{f\bar{\phi}}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{\bar{g}\gamma}{1-4\bar{\phi}\gamma} ds - 32c \int_0^\infty \frac{\bar{g}f\gamma^2\bar{\phi}^2}{(1-4\bar{\phi}\gamma)^2} ds) \\ &\quad \langle \psi(\phi), \psi(\gamma) \rangle \\ &= (2c \int_0^\infty \frac{\bar{g}f}{(1-4\bar{\phi}\gamma)^2} ds - 32c \int_0^\infty \frac{\bar{g}f\gamma^2\bar{\phi}^2}{(1-4\bar{\phi}\gamma)^2} ds) \\ &\quad \langle \psi(\phi), \psi(\gamma) \rangle \\ &= 2c \int_0^\infty \frac{\bar{g}f(1+4\bar{\phi}\gamma)}{1-4\bar{\phi}\gamma} ds \langle \psi(\phi), \psi(\gamma) \rangle \end{aligned}$$

Moreover, by Proposition 2.1,

$$\begin{aligned} &\langle (2c \int_0^\infty \bar{f}g ds + 4N(\bar{f}g))\psi(\phi), \psi(\gamma) \rangle \\ &= 2c \int_0^\infty \bar{f}g ds \langle \psi(\phi), \psi(\gamma) \rangle + 4 \langle N(\bar{f}g)\psi(\phi), \psi(\gamma) \rangle \\ &= (2c \int_0^\infty \bar{f}g ds + 16c \int_0^\infty \frac{f\bar{g}\bar{\phi}\gamma}{1-4\bar{\phi}\gamma} ds) \langle \psi(\phi), \psi(\gamma) \rangle \\ &= 2c \int_0^\infty \frac{\bar{g}f(1+4\bar{\phi}\gamma)}{1-4\bar{\phi}\gamma} ds \langle \psi(\phi), \psi(\gamma) \rangle \end{aligned}$$

Thus

$$[B(f), B^\dagger(g)] = 2c \int_0^\infty \bar{f}g ds + 4N(\bar{f}g)$$

Similarly

$$\begin{aligned} &\langle [N(p), B^\dagger(f)] \psi(\phi), \psi(\gamma) \rangle \\ &= \langle B^\dagger(f) \psi(\phi), N(\bar{p})\psi(\gamma) \rangle - \langle N(p)\psi(\phi), B(f)\psi(\gamma) \rangle \\ &= (8c^2 \int_0^\infty \frac{\bar{f}\gamma}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{\bar{\phi}\bar{p}\gamma}{1-4\bar{\phi}\gamma} ds + 4c \int_0^\infty \frac{\bar{f}\bar{p}\gamma}{(1-4\bar{\phi}\gamma)^2} ds \\ &\quad - 8c^2 \int_0^\infty \frac{\gamma\bar{p}\bar{\phi}}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{\bar{f}\gamma}{1-4\bar{\phi}\gamma} ds - 16c \int_0^\infty \frac{\bar{p}\gamma^2\bar{\phi}f}{(1-4\bar{\phi}\gamma)^2} ds) \\ &\quad \langle \psi(\phi), \psi(\gamma) \rangle \\ &= (4c \int_0^\infty \frac{\bar{f}\bar{p}\gamma}{(1-4\bar{\phi}\gamma)^2} ds - 16c \int_0^\infty \frac{\bar{p}\gamma^2\bar{\phi}f}{(1-4\bar{\phi}\gamma)^2} ds) \\ &\quad \langle \psi(\phi), \psi(\gamma) \rangle \\ &= 4c \int_0^\infty \frac{\bar{f}\bar{p}\gamma}{1-4\bar{\phi}\gamma} ds \langle \psi(\phi), \psi(\gamma) \rangle \\ &= 2 \langle \psi(\phi), B(pf)\psi(\gamma) \rangle \\ &= \langle 2B^\dagger(pf)\psi(\phi), \psi(\gamma) \rangle \end{aligned}$$

Similarly

$$\begin{aligned} &\langle [N(p), B(f)] \psi(\phi), \psi(\gamma) \rangle \\ &= \langle B(f)\psi(\phi), N(\bar{p})\psi(\gamma) \rangle - \langle N(p)\psi(\phi), B^\dagger(f)\psi(\gamma) \rangle \\ &= (8c^2 \int_0^\infty \frac{\bar{\phi}\bar{p}\gamma}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{f\bar{\phi}}{1-4\bar{\phi}\gamma} ds + 16c \int_0^\infty \frac{\bar{p}\bar{\phi}^2\gamma f}{(1-4\bar{\phi}\gamma)^2} ds \\ &\quad - 8c^2 \int_0^\infty \frac{f\bar{\phi}}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{\gamma\bar{p}\bar{\phi}}{1-4\bar{\phi}\gamma} ds - 4c \int_0^\infty \frac{f\bar{p}\bar{\phi}}{(1-4\bar{\phi}\gamma)^2} ds) \\ &\quad \langle \psi(\phi), \psi(\gamma) \rangle \\ &= (16c \int_0^\infty \frac{\bar{p}\bar{\phi}^2\gamma f}{(1-4\bar{\phi}\gamma)^2} ds - 4c \int_0^\infty \frac{f\bar{p}\bar{\phi}}{(1-4\bar{\phi}\gamma)^2} ds) \langle \psi(\phi), \psi(\gamma) \rangle \\ &= -4c \int_0^\infty \frac{f\bar{p}\bar{\phi}}{1-4\bar{\phi}\gamma} ds \langle \psi(\phi), \psi(\gamma) \rangle \\ &= -2 \langle \psi(\phi), B^\dagger(\bar{p}f)\psi(\gamma) \rangle \\ &= \langle -2B(\bar{p}f)\psi(\phi), \psi(\gamma) \rangle \end{aligned}$$

Also

$$\langle [B^\dagger(f), B^\dagger(g)] \psi(\phi), \psi(\gamma) \rangle$$

$$\begin{aligned}
&= \langle B^\dagger(g)\psi(\phi), B(f)\psi(\gamma) \rangle - \langle B^\dagger(f)\psi(\phi), B(g)\psi(\gamma) \rangle \\
&= (4c^2 \int_0^\infty \frac{\bar{g}\gamma}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{\bar{f}\gamma}{1-4\bar{\phi}\gamma} ds + 8c \int_0^\infty \frac{\bar{g}\bar{f}\gamma^2}{(1-4\bar{\phi}\gamma)^2} ds \\
&\quad - 4c^2 \int_0^\infty \frac{\bar{f}\gamma}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{\bar{g}\gamma}{1-4\bar{\phi}\gamma} ds - 8c \int_0^\infty \frac{\bar{f}\bar{g}\gamma^2}{(1-4\bar{\phi}\gamma)^2} ds) \\
&\quad \langle \psi(\phi), \psi(\gamma) \rangle \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\langle [N(p), N(\sigma)] \psi(\phi), \psi(\gamma) \rangle \\
&= \langle N(\sigma)\psi(\phi), N(\bar{p})\psi(\gamma) \rangle - \langle N(p)\psi(\phi), N(\bar{\sigma})\psi(\gamma) \rangle \\
&= (16c^2 \int_0^\infty \frac{\bar{\phi}\gamma\bar{\sigma}}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{\bar{\phi}\gamma\bar{p}}{1-4\bar{\phi}\gamma} ds + 8c \int_0^\infty \frac{\bar{p}\bar{\phi}\gamma\bar{\sigma}}{(1-4\bar{\phi}\gamma)^2} ds \\
&\quad - 16c^2 \int_0^\infty \frac{\bar{\phi}\gamma\bar{p}}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{\bar{\phi}\gamma\bar{\sigma}}{1-4\bar{\phi}\gamma} ds - 8c \int_0^\infty \frac{\bar{\sigma}\bar{\phi}\gamma\bar{p}}{(1-4\bar{\phi}\gamma)^2} ds) \\
&\quad \langle \psi(\phi), \psi(\gamma) \rangle \\
&= 0
\end{aligned}$$

Finally, suppose e.g. that  $L = B^\dagger$  and  $M = B$ . Then

$$\begin{aligned}
&\langle [L(\alpha), M(\beta)] \psi(\phi), \psi(\gamma) \rangle \\
&= \langle [B^\dagger(\alpha), B(\beta)] \psi(\phi), \psi(\gamma) \rangle \\
&= \langle B(\beta)\psi(\phi), B(\alpha)\psi(\gamma) \rangle - \langle B^\dagger(\alpha)\psi(\phi), B^\dagger(\beta)\psi(\gamma) \rangle \\
&= (4c^2 \int_0^\infty \frac{\beta\bar{\phi}}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{\bar{\alpha}\gamma}{1-4\bar{\phi}\gamma} ds + 32c \int_0^\infty \frac{\bar{\alpha}\beta\gamma^2\bar{\phi}^2}{(1-4\bar{\phi}\gamma)^2} ds \\
&\quad - 2c \int_0^\infty \frac{\bar{\alpha}\beta}{(1-4\bar{\phi}\gamma)^2} ds - 4c^2 \int_0^\infty \frac{\bar{\alpha}\gamma}{1-4\bar{\phi}\gamma} ds \int_0^\infty \frac{\bar{\phi}\beta}{(1-4\bar{\phi}\gamma)^2} ds) \\
&\quad \langle \psi(\phi), \psi(\gamma) \rangle \\
&= 2c \int_0^\infty \frac{\bar{\alpha}\beta(1+4\gamma\bar{\phi})}{4\bar{\phi}\gamma-1} ds \langle \psi(\phi), \psi(\gamma) \rangle \\
&= 0
\end{aligned}$$

since  $\alpha, \beta \in S(R_+, D)$  and  $\alpha\beta = 0$  imply  $\bar{\alpha}\beta = 0$

#### 4. Inclusion of the SWN calculus in the representation free quantum stochastic calculus of Accardi, Fagnola, and Quaegebeur

A general, representation free, quantum stochastic calculus, which included all known examples, was developed in [AFQ92]. We will show that the SWN calculus is also included. For quick reference, we provide some basic facts of the AQF calculus related to what will follow (cf. [AFQ92], sections 1 and 2).

Denoting by  $H$ : a complex separable Hilbert space,  $B(H)$ : the algebra of all bounded operators on  $H$ ,  $D$ : a total subset of  $H$ ,  $(A_t)_{t>0}$ : an increasing family of  $W^*$ -algebras of operators on  $H$ ,  $H_{t_1}(\xi)$ : the closure of the subspace  $\{\alpha\xi/\alpha \in A_{t_1}\}$ , where  $\xi \in D$ ,  $A'_{t_1}$ : the commutant of  $A_{t_1}$  in  $B(H)$ ,  $D'_t$ : the linear span of  $A_{t_1}D$ ,  $L(D, H)$ : the vector space of all linear operators  $F$  with domain containing  $D$  such that the adjoint operator  $F^*$  also has  $D$  in its domain, a "random variable"  $F$  is defined to be an element of  $L(D, H)$ , and a "stochastic process in  $H$ , indexed by  $R_+$ " is a family  $(F(t))_{t \geq 0}$  of random variables such that for each  $n \in D$  the map  $t \in R_+ \rightarrow F(t)n$  is Borel measurable. A random variable  $F$  is "t-adapted to  $A_{t_1}$ " if  $\text{domain}(F) = D'_t$ ,  $\text{domain}(F^*) \supseteq D'_t$  and  $F\alpha'_{t_1}\xi = \alpha'_{t_1}F\xi$ ,  $F^*\alpha'_{t_1}\xi = \alpha'_{t_1}F^*\xi$  for all  $\alpha'_{t_1} \in A'_{t_1}$  and  $\xi \in D$ . A stochastic process  $(F(t))_{t \geq 0}$  is "adapted to the filtration  $(A_t)_{t \geq 0}$ " if  $F(t)$  is t-adapted to  $A_{t_1}$  for all  $t \geq 0$ . A "simple" adapted process  $(F(t))_{t \geq 0}$  is a process which can be written in the form  $F(t) = \sum_{k=1}^n F(t_k)\chi_{[t_k, t_{k+1})}$  for some  $n \in \mathbb{N}$  and  $0 \leq t_1 \leq t_2 \leq \dots \leq t_{n+1} \leq \infty$ . An "additive process" is a family  $M = (M(s, t))_{0 \leq s \leq t}$  of random variables such that, for all  $s \leq t$ ,  $M(s, t)$  is t-adapted to  $A_{t_1}$  and, for all  $r, s, t$  with  $r \leq s \leq t$ ,  $M(r, t) = M(r, s) + M(s, t)$  and  $M(t, t) = 0$  on  $D$ . With every additive process we associate the adapted process  $M(t) = M(0, t)$ . An additive process is said to be "regular" if, for all  $\xi \in D$  and  $0 \leq r \leq s \leq t$ ,  $H_{r_1}(\xi) \subseteq \text{domain}(\overline{M^\#(s, t)})$  and  $M^\#(s, t)D \subseteq D'_s$  where "-" denotes closure and  $\overline{M^\#}$  is either  $\overline{M}$  or  $\overline{M^*}$ . If  $M$  is a regular additive process and  $F = (F(t))_{t \geq 0}$  is a simple adapted process then we define the "left stochastic integral" of  $F$  with respect to  $M$  over  $[0, t]$  as an operator on  $D'_t$  by

$$\int_0^t dM(s)F(s) = \sum_{k=1}^n \overline{M}(t_k \wedge t, t_{k+1} \wedge t)F(t_k)|_{D'_t}$$

and the "right stochastic integral" by

$$\int_0^t F(s)dM(s) = \sum_{k=1}^n F(t_k)M(t_k \wedge t, t_{k+1} \wedge t)$$

An additive regular process  $M$  is called an "integrator of scalar type" if for each  $\xi \in D$  there exists a finite set  $J(\xi) \subseteq D$  such that for each simple process  $F$  and for each  $0 < T < +\infty$

$$\begin{aligned} \left\| \int_0^T dM^*(r) F^*(r) \xi \right\|^2 &\leq c_{T,\xi} \int_0^T d\mu_\xi(r) \sum_{n \in J(\xi)} \|F^*(r)n\|^2 \\ \left\| \int_0^T F(r) dM(r) \xi \right\|^2 &\leq C_{T,\xi} \int_0^T d\mu_\xi(r) \sum_{n \in J(\xi)} \|F(r)n\|^2 \end{aligned}$$

for some constant  $c_{T,\xi} > 0$  and some positive, locally finite, non atomic measure  $\mu_\xi$ , and

$$J(n) \subseteq J(\xi)$$

for all  $n \in J(\xi)$ .

If  $M$  is an integrator of scalar type and, for all  $\xi \in D$ , the measures  $\mu_\xi$  are absolutely continuous with respect to Lebesgue measure, then the stochastic integrals defined above can be extended to any  $F \in L^2_{loc}(R_+, dM)$ , the space of all adapted processes  $F$  such that

$$\int_0^t (\|F(s)n\|^2 + \|F^*(s)n\|^2) d\mu_\xi(s) < +\infty$$

for all  $\xi \in D, n \in J(\xi)$ , and  $0 \leq t < +\infty$ . Once the stochastic integrals have been extended, one can discuss Itô tables, stochastic differential equations, unitarity conditions for their solutions e.t.c. (c.f [AFQ 92] for a complete treatment).

We will show that the SWN processes  $B, B^\dagger, N$  are integrators of scalar type, thus including SWN calculus in AFQ calculus. In the above notation, letting  $H_0$  be an "initial" complex separable Hilbert space, we take  $H = H_0 \otimes \Gamma$  where  $\Gamma$  is as in Definition 2.1,  $D = \{u \otimes \psi(f) / u \in H_0, \psi(f) \in \Gamma\}$ , and we take  $A_{t_j}$  to be the algebra of all operators of the form  $L_{t_j} \otimes I_t$  where  $L_{t_j}$  is a bounded operator acting on vectors of the form  $u \otimes \psi(f\chi_{[0,t]})$  and  $I_t$  is the identity operator acting on vectors  $\psi(f\chi_{(t,+\infty)})$ .

DEFINITION 4.1 For  $0 \leq s \leq t$ , we define

$$\begin{aligned} B(s,t) &= B(\chi_{[s,t]}) & B^\dagger(s,t) &= B^\dagger(\chi_{[s,t]}) & N(s,t) &= N(\chi_{[s,t]}) \\ B(t) &= B(0,t) & B^\dagger(t) &= B^\dagger(0,t) & N(t) &= N(0,t) \\ B &= (B(s,t))_{0 \leq s \leq t} & B^\dagger &= (B^\dagger(s,t))_{0 \leq s \leq t} & N &= (N(s,t))_{0 \leq s \leq t} \end{aligned}$$

By Propositions 2.1 and 2.2,  $B, B^\dagger$ , and  $N$  are regular additive processes.

DEFINITION 4.2. For  $t \geq 0$  we define the "stochastic differentials"  $dB, dB^\dagger$ , and  $dN$  by

$$dB(t) = B(t, t+dt)dB^\dagger(t) = B^\dagger(t, t+dt)dN(t) = N(t, t+dt) \quad (1)$$

PROPOSITION 4.3. Let  $F = (F(t))_{t \geq 0}$  be a simple adapted process on  $H$  and let  $M \in \{B^\dagger, B, N\}$ .

Then for all  $\xi = u \otimes \psi(f) \in D$  and  $0 < t < +\infty$ :

$$\begin{aligned} \left\| \int_0^t dM^*(s) F^*(s) \xi \right\|^2 &\leq c_{t,\xi} \int_0^t \|F^*(s)\xi\|^2 ds \\ \left\| \int_0^t F(s) dM(s) \xi \right\|^2 &\leq c_{t,\xi} \int_0^t \|F(s)\xi\|^2 ds \end{aligned}$$

where

$$\begin{aligned} c_{t,\xi} &= [\lambda(t, f) \cdot t^{\frac{1}{2}} + \sqrt{\lambda^2(t, f) \cdot t + \theta(t, f)}]^2 \\ \theta(t, f) &= 8c^2 m(t, f)(1 + 2m(t, f))t + 42c \cdot m(t, f)^2 \\ \lambda(t, f) &= 8c \cdot m(t, f) \\ m(t, f) &= \max_{[0,t]} \left( \frac{1}{1 - 4|f|^2} \right) \end{aligned}$$

i.e., with  $J(\xi) = \{\xi\}$ ,  $M$  is an integrator of scalar type.

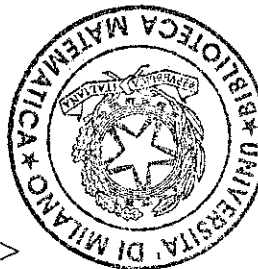
PROOF. We will only prove the second inequality. The proof of the first one is similar. Let

$$F(t) = \sum_{j=1}^n F(s_j) \chi_{[s_j, s_{j+1})}(t)$$

where  $0 \leq s_1 < s_2 < \dots < s_{n+1} \leq t < +\infty$ . Then

$$\begin{aligned} &\left\| \int_0^t F(s) dM(s) \xi \right\|^2 \\ &= \left\| \int_0^t F(s) dM(s) u \otimes \psi(f) \right\|^2 \\ &= \left\| \sum_{j=1}^n F(s_j) M(s_j, s_{j+1}) u \otimes \psi(f) \right\|^2 \\ &= \sum_{j,k=1}^n \langle F(s_j) M(s_j, s_{j+1}) u \otimes \psi(f), F(s_k) M(s_k, s_{k+1}) u \otimes \psi(f) \rangle \\ &= \sum_{j=1}^n \langle F(s_j) M(s_j, s_{j+1}) u \otimes \psi(f), F(s_j) M(s_j, s_{j+1}) u \otimes \psi(f) \rangle \\ &\quad + 2 \operatorname{Re} \sum_{1 \leq j < k \leq n} \langle F(s_j) M(s_j, s_{j+1}) u \otimes \psi(f), F(s_k) M(s_k, s_{k+1}) u \otimes \psi(f) \rangle \end{aligned}$$

(which using the notation  $f_\lambda = f\chi_{[0,\lambda]}, f_{(\lambda,+\infty)} = f\chi_{(\lambda,+\infty)}$ )



$$\begin{aligned}
&= \sum_{j=1}^n \langle F(s_j)u \otimes \psi(f_{s_j}), F(s_j)u \otimes \psi(f_{s_j}) \rangle \\
&\quad \langle M(s_j, s_{j+1})\psi(f_{s_j}), M(s_j, s_{j+1})\psi(f_{s_j}) \rangle \\
&\quad + 2\operatorname{Re}\sum_{1 \leq j < k \leq n} \langle F(s_j)M(s_j, s_{j+1})u \otimes \psi(f_{s_k}), F(s_k)u \otimes \psi(f_{s_k}) \rangle \\
&\quad \langle \psi(f_{s_k}), M(s_k, s_{k+1})\psi(f_{s_k}) \rangle \\
&\quad \text{(which by Propositions 3.1 and 3.2 is)} \\
&= \sum_{j=1}^n \|F(s_j)u \otimes \psi(f_{s_j})\|^2 \gamma_j(f) \cdot \|\psi(f_{s_j})\|^2 \\
&\quad + 2\operatorname{Re}\sum_{1 \leq j < k \leq n} \langle F(s_j)M(s_j, s_{j+1})u \otimes \psi(f_{s_k}), F(s_k)u \otimes \psi(f_{s_k}) \rangle \\
&\quad \cdot \delta_k(f) \cdot \|\psi(f_{s_k})\|^2
\end{aligned}$$

where for  $k, j \in \{1, 2, \dots, n\}$

$$\gamma_j(f) = \begin{cases} 4c^2 \int_{s_j}^{s_{j+1}} \frac{f}{1-4|f|^2} ds \int_{s_j}^{s_{j+1}} \frac{\bar{f}}{1-4|f|^2} ds + 32c \int_{s_j}^{s_{j+1}} \frac{|f|^4}{(1-4|f|^2)^2} ds & \text{if } M = B \\ 4c^2 \int_{s_j}^{s_{j+1}} \frac{f}{1-4|f|^2} ds \int_{s_j}^{s_{j+1}} \frac{\bar{f}}{1-4|f|^2} ds + 2c \int_{s_j}^{s_{j+1}} \frac{1}{(1-4|f|^2)^2} ds & \text{if } M = B^\dagger \\ 16c^2 (\int_{s_j}^{s_{j+1}} \frac{|f|^2}{1-4|f|^2} ds)^2 + 8c \int_{s_j}^{s_{j+1}} \frac{|f|^2}{(1-4|f|^2)^2} ds & \text{if } M = N \end{cases}$$

and

$$\delta_k(f) = \begin{cases} 2c \int_{s_k}^{s_{k+1}} \frac{f}{1-4|f|^2} ds & \text{if } M = B \\ 2c \int_{s_k}^{s_{k+1}} \frac{\bar{f}}{1-4|f|^2} ds & \text{if } M = B^\dagger \\ 4c \int_{s_k}^{s_{k+1}} \frac{|f|^2}{1-4|f|^2} ds & \text{if } M = N \end{cases}$$

Thus

$$\begin{aligned}
&\| \int_0^t F(s) dM(s) u \otimes \psi(f) \|^2 \\
&= \sum_{j=1}^n \|F(s_j)u \otimes \psi(f)\|^2 \gamma_j(f) \\
&\quad + 2\operatorname{Re}\sum_{1 \leq j < k \leq n} \langle F(s_j)M(s_j, s_{j+1})u \otimes \psi(f), F(s_k)u \otimes \psi(f) \rangle \delta_k(f) \\
&= \sum_{j=1}^n \|F(s_j)u \otimes \psi(f)\|^2 \gamma_j(f) \\
&\quad + 2\operatorname{Re}\sum_{k=1}^n \langle \int_0^t F(s) dM(s) u \otimes \psi(f), F(s_k)u \otimes \psi(f) \rangle \delta_k(f)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\| \int_0^t F(s) dM(s) u \otimes \psi(f) \|^2 \leq \sum_{j=1}^n \|F(s_j)u \otimes \psi(f)\|^2 |\gamma_j(f)| \\
&\quad + 2 \| \int_0^t F(s) dM(s) u \otimes \psi(f) \| \sum_{k=1}^n \|F(s_k)u \otimes \psi(f)\| |\delta_k(f)|
\end{aligned}$$

Since, for all  $j, k \in \{1, 2, \dots, n\}$

$$|\gamma_j(f)| \leq \theta(t, f) \cdot (s_{j+1} - s_j)$$

and

$$|\delta_k(f)| \leq \lambda(t, f) \cdot (s_{j+1} - s_j)$$

we obtain

$$\begin{aligned}
&\| \int_0^t F(s) dM(s) u \otimes \psi(f) \|^2 \\
&\leq \theta(t, f) \int_0^t \|F(s)u \otimes \psi(f)\|^2 ds \\
&\quad + 2\lambda(t, f) \| \int_0^t F(s) dM(s) u \otimes \psi(f) \| \int_0^t \|F(s)u \otimes \psi(f)\| ds \\
&\leq \theta(t, f) \int_0^t \|F(s)u \otimes \psi(f)\|^2 ds \\
&\quad + 2\lambda(t, f) \| \int_0^t F(s) dM(s) u \otimes \psi(f) \| \{ \int_0^t \|F(s)u \otimes \psi(f)\|^2 ds \}^{1/2} t^{1/2}
\end{aligned}$$

Letting

$$R(t) = \| \int_0^t F(s) dM(s) u \otimes \psi(f) \|^2$$

and

$$a(t) = [ \int_0^t \|F(s)u \otimes \psi(f)\|^2 ds ]^{1/2}$$

the above inequality becomes

$$R^2(t) \leq \theta(t, f) a^2(t) + 2\lambda(t, f) R(t) a(t) t^{1/2}$$

i.e

$$R^2(t) - 2\lambda(t, f) a(t) t^{1/2} R(t) - \theta(t, f) a^2(t) \leq 0$$

which yields

$$R^2(t) \leq [\lambda(t, f) t^{1/2} + \sqrt{t\lambda^2(t, f) + \theta(t, f)}] a^2(t)$$

i.e

$$\| \int_0^t F(s) dM(s) \xi \|^2 \leq c_{t, \xi} \int_0^t \|F(s)u \otimes \psi(f)\|^2 ds$$



## 5. On the connection between the Finite Difference operators and the creation, annihilation, and conservation operators

To obtain a realization of the commutation relations of the Finite Difference algebra of [Fei 87] and [Bou 88] in terms of creation, annihilation, and conservation operators on the usual Boson Fock space, Parthasarathy and Sinha defined in [PS91] operators  $Q(f), P(f)$ , and  $T(f)$  as follows:

DEFINITION 5.1. Let  $g : R_+ \rightarrow R$  be a step function such that  $|g| < 1$ , and let  $\alpha(g) = \{\frac{g^n}{\sqrt{n}}\}_{n=1}^{\infty} \in l_2$ . Let also  $h = L^2(R_+, l_2)$  be equipped with the inner product

$$\begin{aligned} \langle a(g), a(\phi) \rangle &= \int_0^{\infty} \sum_{n=1}^{\infty} \frac{g^n(s) \phi^n(s)}{n} ds \\ &= - \int_0^{\infty} \ln(1 - g(s)\phi(s)) ds \end{aligned}$$

We denote by  $\Gamma_s(h)$  the symmetric Boson Fock space over  $h$ , defined as the completion of the linear span of "exponential vectors"  $\psi(g)$  under the inner product

$$\langle \psi(g), \psi(\phi) \rangle = e^{\langle a(g), a(\phi) \rangle} = e^{- \int_0^{\infty} \ln(1 - g(s)\phi(s)) ds}$$

DEFINITION 5.2. For  $u = \{u_n\}_{n=1}^{\infty} \in l_2$  we define  $X_1 u, X_2 u \in l_2$  by

$$\begin{aligned} (X_1 u)_n &= n u_n + \sqrt{n(n+1)} u_{n+1} \\ (X_2 u)_n &= n u_n + \sqrt{n(n-1)} u_{n-1} \end{aligned}$$

DEFINITION 5.3. Let  $f : R_+ \rightarrow R$  be a step function such that  $|f| < 1$ . We define operators  $P(f), Q(f)$ , and  $T(f)$  on  $\Gamma_s(h)$  by

$$\begin{aligned} P(f) &= A(f \otimes X_2 e_1) + A^\dagger(f \otimes e_1) + \Lambda(M_f \otimes X_1) \\ Q(f) &= A(f \otimes e_1) + A^\dagger(f \otimes X_2 e_1) + \Lambda(M_f \otimes X_2) \\ T(f) &= P(f) + Q(f) + \int_0^{\infty} f(t) dt \end{aligned}$$

where

$$\begin{aligned} e_1 &= (1, 0, 0, \dots) \in l_2 \\ X_2 e_1 &= (1, \sqrt{2}, 0, \dots) \\ f \otimes e_1 &= (f, 0, 0, \dots) \\ f \otimes X_2 e_1 &= (f, \sqrt{2}f, 0, \dots) \\ X_1 a(g) &= \{\sqrt{n}(g^n + g^{n+1})\}_{n=1}^{\infty} \\ X_2 a(g) &= \{\sqrt{n}(g^n + g^{n-1})\}_{n=1}^{\infty} \\ M_f \otimes X_1 a(g) &= \{\sqrt{n}f(g^n + g^{n+1})\}_{n=1}^{\infty} \\ M_f \otimes X_2 a(g) &= \{\sqrt{n}f(g^n + g^{n-1})\}_{n=1}^{\infty} \end{aligned}$$

and  $A^\dagger, A, \Lambda$  are the creation, annihilation, and conservation operators defined in [Pa92].

It was pointed out in [PS91] that, on the exponential domain in  $\Gamma_s(h)$

$$P(f)^* = Q(f), T(f)^* = T(f)$$

and

$$\begin{aligned} \{[P(f), Q(g)] &= [P(f), T(g)] = [T(f), Q(g)] = T(fg) \\ [P(f), P(g)] &= [Q(f), Q(g)] = [T(f), T(g)] = 0 \end{aligned}$$

i.e.  $P(\cdot), Q(\cdot)$ , and  $T(\cdot)$ , realize the commutation relations of [Bou88] on  $\Gamma_s(h)$ . However, the next proposition shows that the operators  $P, Q, T$  defined in Definition 5.3 differ statistically from the operators  $P, Q, T$  of [Bou88].

PROPOSITION 5.4. In the notation of Definitions 5.1, 5.2, 5.3

$$\langle Q(f)\psi(g), \psi(\phi) \rangle = \int_0^{\infty} (fg + f\phi + f\phi^2 + \frac{fg\phi(1+\phi)}{1-g\phi}) ds \langle \psi(g), \psi(\phi) \rangle$$

(while the corresponding matrix element in [Bou88] is

$$\langle Q(f)\psi(g), \psi(\phi) \rangle = \int_0^{\infty} \frac{f\phi(1+g)}{1-g\phi} ds \langle \psi(g), \psi(\phi) \rangle$$

PROOF:

$$\begin{aligned} \langle A(f \otimes e_1)\psi(g), \psi(\phi) \rangle &= \langle \langle f \otimes e_1, a(g) \rangle \psi(g), \psi(\phi) \rangle \\ &= \langle a(g), f \otimes e_1 \rangle \langle \psi(g), \psi(\phi) \rangle \\ &= \langle (g, \frac{g^2}{\sqrt{2}}, \frac{g^3}{\sqrt{3}}, \dots), (f, 0, 0, \dots) \rangle \langle \psi(g), \psi(\phi) \rangle \\ &= \int_0^{\infty} gf ds \langle \psi(g), \psi(\phi) \rangle \end{aligned}$$

while

$$\begin{aligned}
 \langle A^\dagger(f \otimes X_2 e_1) \psi(g), \psi(\phi) \rangle &= \langle \psi(g), A(f \otimes X_2 e_1) \psi(\phi) \rangle \\
 &= \langle \psi(g), \langle f \otimes X_2 e_1, a(\phi) \rangle \psi(\phi) \rangle \\
 &= \langle f \otimes X_2 e_1, a(\phi) \rangle \langle \psi(g), \psi(\phi) \rangle \\
 &= \langle (f, \sqrt{2}f, 0, \dots), (\phi, \frac{\phi^2}{\sqrt{2}}, \frac{\phi^3}{\sqrt{3}}, \dots) \rangle \\
 &\quad \langle \psi(g), \psi(\phi) \rangle \\
 &= \int_0^\infty (f\phi + f\phi^2) ds \langle \psi(g), \psi(\phi) \rangle
 \end{aligned}$$

and

$$\begin{aligned}
 &\langle \Lambda(M_f \otimes X_2) \psi(g), \psi(\phi) \rangle \\
 &= \langle \psi(g) \Lambda(M_f \otimes X_2)^* \psi(\phi) \rangle \\
 &= \langle \psi(g), \Lambda((M_f \otimes X_2)^*) \psi(\phi) \rangle = \langle \psi(g), \Lambda(M_f \otimes X_1) \psi(\phi) \rangle \\
 &= \langle a(g), M_f \otimes X_1 a(\phi) \rangle \langle \psi(g), \psi(\phi) \rangle \\
 &\quad \langle \{ \frac{g^n}{\sqrt{n}} \}_{n=1}^\infty, \{ \sqrt{n} f(\phi^n + \phi^{n+1}) \}_{n=1}^\infty \rangle \\
 &\quad \langle \psi(g), \psi(\phi) \rangle \\
 &= \int_0^\infty \sum_{n=1}^\infty f g^n (\phi^n + \phi^{n+1}) ds \langle \psi(g), \psi(\phi) \rangle \\
 &= \int_0^\infty f(1+\phi) \sum_{n=1}^\infty g^n \phi^n ds \langle \psi(g), \psi(\phi) \rangle \\
 &= \int_0^\infty f(1+\phi) \left( \frac{1}{1-g\phi} - 1 \right) ds \langle \psi(g), \psi(\phi) \rangle \\
 &= \int_0^\infty \frac{f(1+\phi)g\phi}{1-g\phi} ds \langle \psi(g), \psi(\phi) \rangle
 \end{aligned}$$

and, in view of Definition 5.3, the result follows by addition.

REMARK 5.5. The matrix elements  $\langle Q(f)\psi(g), \psi(\phi) \rangle$  in the sense of [PS91] and [Bou 88] agree if and only if  $g + \phi^2 = 0$  e.g in the vacuum state.

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