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MATHEMATICS

Representations of Levy Laplacians and Related Semigroups and Harmonic Functions

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In this paper, we consider (some) solutions to the heat, Schrödinger, and Laplace equations containing Laplace–Levy operators (Levy Laplacians) in function spaces on rigged Hilbert spaces and describe the relation of such operators to the quantum theory of random processes. Unexpectedly, in many cases, the properties of the Laplace–Levy operator are similar to those of the more traditional (infinite-dimensional) Laplace–Volterra operator; substantial distinctions arise only when the functions on which these operators act are defined on a Hilbert space (cf. [2, 7]). The approach to studying the Laplace–Levy operators used in this paper develops the methods suggested in [1–4]. Some applications of the Laplace–Levy operators are described in [10, 11].

THE LAPLACE OPERATORS

Unless otherwise specified, the vector spaces are assumed to be real. For locally convex spaces E and G, $L(E, G)$ denotes the space of all linear continuous mappings from E to G; instead of $L(E, E)$, we use the symbol $L(E)$. A mapping F: $E \to G$ is called (Hadamard) differentiable at a point $x \in E$ if there exists an element $F'(x) \in L(E, G)$, which is called the derivative of the mapping F at the point x, such that, if $r_x(h) = F(x + h) - F(x) - F'(x)h$, then $t_n^{-1} r_x(t_n h_n) \to 0$ as $n \to \infty$, whatever a convergent sequence $(h_n) \subset E$ and a sequence $(t_n) \subset \mathbb{R}$ converging to zero be. The higher order derivatives are defined by induction; the spaces $L(E, G)$, $L(E, L(E, G))$, etc., are endowed with the topologies of convergence on sequentially compact subsets. A mapping F: $E \to G$ is called a C^n-mapping for $n \in \mathbb{N}$ if, first, it is everywhere n times differentiable and, second, F and all mappings $F^{(k)}$: $E \to L(E, \ldots, L(E, G), \ldots)$ with $k = 1, 2, \ldots, n$ are continuous. The vector space of all C^n-mappings of E to G is denoted by $C^n(E, G)$; instead of $C^n(E, \mathbb{R})$, we write $C^n(E)$; the vector space of all real functions on E is denoted by $\mathcal{F}(E)$.

For any vector subspace E_1 of a space E^*, the notation $\mathcal{M}(E, E_1)$ stands for the space of all bounded E_1-cylindrical (complex-valued) measures on E endowed with the topology $\sigma(\mathcal{M}(E, E_1), C_{cyl}(E, E_1))$, where $C_{cyl}(E, E_1)$ denotes the space of all bounded continuous (complex-valued) E_1-cylindrical functions on E; by $\mathcal{A}_{cyl}(E, E_1)$, we denote the set of all E_1-cylindrical subsets in E; and, for any $\nu \in \mathcal{M}(E, E_1)$ and $k \in E$, ν_k is the shift of the measure by the vector k; i.e., $\nu_k = [\mathcal{A}_{cyl}(E, E_1) \ni A \mapsto \nu(A + k)] \in \mathcal{M}(E, E_1)$. A measure $\nu \in \mathcal{M}(E, E_1)$ is called (n times) differentiable (along E) if the function $E \ni h \mapsto \nu_h \in \mathcal{M}(E, E_1)$ (n is (n times) differentiable at zero and all its derivatives are measures absolutely continuous with respect to ν.

Definition 1 [1, 4]. Let S_0 be a linear functional defined on a vector subspace $\mathrm{dom}S_0$ of the vector space $L(E, E^*)$. The homogeneous linear differential operator of the second order (specified by the functional S_0) in the function space on E is the linear mapping Δ_{S_0} of a subspace of the space $C^2(E)$ to the space $\mathcal{F}(E)$ defined as follows: $\mathrm{dom}\Delta_{S_0} = \{g \in C^2(E)$: $\forall x \in E$, $g''(x) \in \mathrm{dom}S_0\}$ and $(\Delta_{S_0} g)(x) = S_0(g''(x))$ for $f \in \mathrm{dom}\Delta_{S_0}$.

The (nonhomogeneous) linear differential operators of an arbitrary order in $C^n(E)$ are defined similarly [4].

A linear mapping of a vector subspace of $\mathcal{F}(E)$ to $\mathcal{F}(E)$ whose restriction to $C^n(E)$ is a differential operator is also said to be a differential operator.

Example 1. Suppose that E is a Hilbert space, $A \in L(E)$, and $S_0^A(B) = \mathrm{tr}(BA)$ for suitable $B \in L(E, E^*)$ $[= L(E)]$; then, $\Delta_{S_0^A}$ is called a Laplace–Volterra operator, or a Volterra Laplacian (generated by the operator A). Such operators were considered by Yu.L. Daletskii and L. Gross, who additionally assumed that A is a is self-adjoint positive nuclear operator.

With the use of appropriate notions of traces for the elements of the space $L(E, E^*)$, similar operators in non-Hilbert spaces can be introduces. In what follows, we write Δ_A instead of $\Delta_{S_0^A}$.

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Let H be a separable Hilbert space with scalar product $(\cdot, \cdot)_H$, which is identified with its Hilbert dual H^*, and let E be a locally convex space being a dense subspace in H such that the embedding of E into H is continuous. We suppose that the locally convex topology of E^* is consistent with the duality between E^* and E; then, the mapping $H^* (= H) \to E$ adjoint to the embedding $E \to H$ is continuous and injective, and its image is dense in E^*. Thus, $E \subset H = H^* \subset E^*$ is a rigged Hilbert space; if $x \in E$ and $g \in H \subset E^*$, then $(x, g)_H = \langle q, x\rangle$ $[\equiv g(x)]$ in natural notation. Let $\mathbf{e} = (\mathbf{e}_n)$ be an orthonormal basis in H (we assume it to be fixed in what follows) formed by elements of E. Unless otherwise specified, we assume that the linear hull $E_\mathbf{e}$ of the basis $\mathbf{e}$ is dense in E.

Example 2. Suppose that D is a (strictly) positive self-adjoint operator in H whose inverse is a Hilbert–Schmidt operator, $(\mathbf{e}_n)$ is the orthonormal basis in H formed by eigenvectors of the operator D, and $E = H_D = \bigcap_n D^n H$; we assume the space E to be endowed with the topology generated by the family of (Hilbert) norms $\{\|\cdot\|_n$: $n \in \mathbb{N}$; $\forall h \in E$, $\|h\|_n^2 = (D^n h D^n h)_H\}$. Then, E is a Fréchet (reflexive) nuclear space and for every $g \in E^*$ the series $\sum_{n=1}^{\infty} \langle g, \mathbf{e}_n\rangle \mathbf{e}_n$ converges in E^*, even with respect to the strong topology.

Next, for every $n \in \mathbb{Z}$ H_n is the completion of the space $(H_D, \|\cdot\|_n)$, $\mathcal{F}_\mathbf{e}(E)$ is the set of all functions on E twice differentiable with respect to the directions of the vectors from the basis $\mathbf{e}$, and S is a positive linear functional on a subspace $\mathrm{dom}S$ of the space $\mathbb{R}^\infty$ containing $l_1 (= l_1(\mathbb{N}))$.

Definition 2. The Laplace operator in $\mathcal{F}_\mathbf{e}(E)$ (specified by the basis $\mathbf{e}$ and the functional S) is the mapping Δ_S^n: $\mathrm{dom}\Delta_S \to \mathcal{F}(E)$, where $\mathrm{dom}\Delta_S = \{f \in \mathcal{F}_\mathbf{e}(E)$: $f''_{jj}(x) \in \mathrm{dom}S\}$ $\left(\text{and } f''_{jj}(x) = \left.\frac{d^2}{dt^2}\right|_{t=0} f(x + t\mathbf{e}_j)\right)$ defined as follows: $(\Delta_S f)(x) = S(f''_{jj}(x))_{j=1}^{\infty})$. If $l_1 \subset \ker S$, then Δ_S is called a (weighed) Laplace–Levy operator (and the functional S is called as a Levy functional); if $S((x_n)) = \sum a_n x_n$ (where $a_n \geq 0$), then Δ_S is a Laplace–Volterra operator (according to the definition given in Example 1). The Laplace operators on $\mathcal{M}(E, E_1)$ are defined similarly.

The exotic Laplacian on $\mathcal{F}_\mathbf{e}(E)$ of order n ($\in N$) specified by the basis $\mathbf{e}$ and the functional S is the mapping Δ_S^n: $\mathrm{dom}\Delta_S^n \to \mathcal{F}(E)$, where $\mathrm{dom}\Delta_S^n = \{f \in \mathcal{F}_\mathbf{e}(E)$: $((\|\mathbf{e}_j\|_n^{-2} f''_{jj}(x)) \in \mathrm{dom}S\}$ $\left(\text{here, } f''_{jj}(x) = \left.\frac{d^2}{dt^2}\right|_{t=0} f(x + t\mathbf{e}_j)\right)$, defined by $(\Delta_S^n f)(x) = S(((\|\mathbf{e}_j\|_n^{-2} f''_{jj}(x))_{j=1}^{\infty})$.

If $l_1 \subset \ker S$, then the operator Δ_S^n is called a (weighed) exotic Levy Laplacian of order n. If S_c is the functional defined by the relations $\mathrm{dom}S_c = \left\{(a_n) \in \mathbb{R}^\infty: \exists \lim_n \frac{1}{n}\sum_{j=1}^{n} a_j\right\}$ and $S_c((a_n)) = \lim_n \frac{1}{n}\sum_{j=1}^{n} a_j$ (i.e., S_c is the Cesaro mean of the corresponding sequence), then the operator $\Delta_{S_c}^n$ is called the (classical) Levy Laplacian corresponding to the basis $\mathbf{e}$ and denoted by $\Delta_\mathcal{L}$; the operator $\Delta_{S_c}^n$ is called the (classical) exotic Levy Laplacian of order n and denoted by $\Delta_\mathcal{L}^n$. In what follows, we considered only the operator $\Delta_\mathcal{L}^2$; precisely these exotic Levy Laplacians arise in the theory of gauge fields (see [10, 11]).

THE ANALYTICAL PROPERTIES THE LEVY LAPLACIANS

Definition 3. The Levy trace is the functional $\mathrm{tr}_\mathcal{L}$ on a vector subspace in $L(E, E^*)$ defined by the equality $\mathrm{tr}_\mathcal{L} A = S_c((\langle A\mathbf{e}_j, \mathbf{e}_j\rangle))$. The Levy trace on $L(E, L(E, G))$ is defined similarly.

If $f \in C^2(E)$, then $\Delta_{\mathrm{tr}_\mathcal{L}} f(x) = \Delta_{S_c} f(x) = \Delta_\mathcal{L} f(x) = \mathrm{tr}_\mathcal{L}(f''(x))$.

Definition 4. The Levy scalar product $\langle \cdot, \cdot \rangle_\mathcal{L}$ is defined as follows: $\mathrm{dom}\langle \cdot, \cdot \rangle_\mathcal{L} = \{(a, b) \in E^* \times E^*$; $(\langle a, \mathbf{e}_j\rangle \cdot \langle b, \mathbf{e}_j\rangle) \in \mathrm{dom}S_c\}$ and, if $(a, b) \in \mathrm{dom}\langle \cdot, \cdot \rangle_\mathcal{L}$, then $\langle a, b\rangle_\mathcal{L} = S_c((\langle a, \mathbf{e}_j\rangle \cdot \langle b, \mathbf{e}_j\rangle))$ (this product also depends on the choice of the basis, and the set E_0^* of all $f \in E^*$ for which $\langle f, f\rangle_\mathcal{L}$ exists may be not a vector space).

Example 3 (a vector subspace $E_\mathcal{L}$ in E^* for which $E_\mathcal{L} \times E_\mathcal{L} \subset \mathrm{dom}\langle \cdot, \cdot \rangle_\mathcal{L}$). Suppose that the assumptions of Example 2 hold. For $\lambda \in (0, \pi)$, we set $s_\lambda = \sum \sin(\lambda n)\, \mathbf{e}_n$ $(\in E^*)$. Let ν be an σ-additive Borelian measure on $(0, \pi)$; for each function $\varphi \in L_1((0, 1), \nu) \cap L_2((0, 1), \nu)$, we set

$$s_\varphi = \int_0^\pi \varphi(\lambda) s_\lambda \nu(d\lambda) \quad (\in E^*).$$

Let S_ν denote the image of the space $L_1(0, \pi)$ under the mapping $f \mapsto s_f$. Then $S_\nu \times S_\nu \subset \mathrm{dom}\langle\cdot, \cdot\rangle_{\mathcal{L}}$, and the restriction of $\langle\cdot, \cdot\rangle_{\mathcal{L}}$ to $S_\nu \times S_\nu$ is a scalar product in S_ν. The pre-Hilbert space $(S_\nu, \langle\cdot, \cdot\rangle_{\mathcal{L}})$ is separable; if the support of ν is infinite, then this space is not complete. Note that

$$\langle s_\varphi, s_\psi\rangle_{\mathcal{L}} = \int_0^\pi \varphi(\lambda)\psi(\lambda)\nu(d\lambda).$$

Proposition 1 (chain rule). *If* $g \in C^2(E)$ *and* $f \in C^2(\mathbb{R}^1)$, *then*

$$\Delta_{\mathcal{L}}(f \circ g)(x) = f''(g(x))\langle g'(x), g'(x)\rangle_{\mathcal{L}} + f'(g(x))(\Delta_{\mathcal{L}} g)(x).$$

If $E = H$, *then*

$$\Delta_A(f \circ g)(x) = f''(g(x))(Ag'(x), g'(x))_H + f'(g(x))(\Delta_A g)(x).$$

Proposition 2 (Leibnitz formula). *If* $g, f \in C^2(E)$, *then*

$$\Delta_{\mathcal{L}}(f \cdot g)(x) = (g \cdot \Delta_{\mathcal{L}} f)(x) + (f \cdot \Delta_{\mathcal{L}} g)(x) + 2\langle f'(x), g'(x)\rangle_{\mathcal{L}}.$$

If $E = H$, *then*

$$\Delta_A(f \cdot g)(x) = (g \cdot \Delta_A f)(x) + (f \cdot \Delta_A g)(x) + 2(Af'(x), g'(x))_H.$$

SOLUTION OF EQUATIONS WITH LEVY LAPLACIANS

Definition 5. If ν is an E-cylindrical measure on E^*, then the functions $\tilde{\nu}_{\mathrm{La}}(\cdot)$ and $\tilde{\nu}_F(\cdot)$ (on E) defined by the equalities $\tilde{\nu}_{\mathrm{La}}(x) = \int e^{\langle f, x\rangle} \nu(df)$ and $\tilde{\nu}_F(x) = \int e^{i\langle f, x\rangle} \nu(df)$ (if the corresponding integrals exist) are called the (two-sided) Laplace and Fourier transforms, respectively.

For each $\beta > 0$, consider a σ-finite σ-additive nonnegative measure μ_β on S_ν (see Example 3) concentrated on the sphere S_ν^β of radius β, the space $L_2(S_\nu^\beta)$ of complex-valued functions on E^* square integrable with respect to the measure μ_β (such functions can be identified with their restrictions to S_ν^β), and the space H_β (H_β^F) of Laplace (Fourier) transforms of the measures being the products of functions from $L_2(S_\nu^\beta)$ by the measure μ_β; we endow these spaces with the Hilbert space structures determined by the scalar product in $L_2(S_\nu^\beta)$. Finally, suppose that γ is a σ-finite nonnegative measure on $(0, \infty)$ and $\mathbf{H}$ ($\mathbf{H}^F$) is the continuous Hilmbertm sum of the Hilbert spaces H_β (H_β^F) generated by this measure.

Proposition 3. *For every* $\beta > 0$, *the Laplace (Fourier) transform of the measure* μ_β *is an eigenfunction of the Levy Laplacian with eigenvalue* β^2 ($-\beta^2$). *The restriction of* $\Delta_{\mathcal{L}}$ *to* $\mathbf{H}$ (*to* $\mathbf{H}^F$) *determines an essentially self-adjoint operator* $\Delta_{\mathcal{L}H}$ *in the corresponding space, and, if* $f(\cdot) \in \mathrm{dom}\Delta_{\mathcal{L}H}$ *and* $f = \int_0^\infty f(\beta)\gamma(d\beta)$, *then*

$$\Delta_{\mathcal{L}H} f = \int_0^\infty \beta^2 f(\beta)\gamma(d\beta)$$

$$\left(\textit{respectively,}\quad \Delta_{\mathcal{L}H} f = -\int_0^\infty \beta^2 f(\beta)\gamma(d\beta)\right).$$

Proposition 4. *If, for every* $n \in \mathbb{N}$, $\mathbb{F}_n$ *is the set of all* n*-frames in* E^* *orthonormal with respect to the Levy scalar product (endowed with the natural topology),* ν *is a* σ*-additive (finite) Borel measure on* $\mathbb{F}_n$, *and* ψ: $\mathbb{R}^n \to \mathbb{R}^1$ *is a harmonic function, then the function* F_ν: $E \to \mathbb{R}^1$ *defined by* $x \mapsto \int_{\mathbb{F}_n} \psi(f_1(x), f_2(x), \ldots, f_n(x))\nu(df_1, df_2, \ldots, df_n)$ *is harmonic.*

Theorem 1. *Let be a (finite) Borel* σ*-additive (alternating) measure on a (Borel) vector subspace* E_1^* *of* E^* *contained in* E_0^*. *Then,*

$$e^{t\Delta_{\mathcal{L}H}} \tilde{\nu}_F(x) = \int_{E_1^*} e^{-t\frac{\langle f, f\rangle_{\mathcal{L}}}{2}} e^{i\langle f, x\rangle} \nu(df);$$

$$e^{it\Delta_{\mathcal{L}H}} \tilde{\nu}_F(x) = \int_{E_1^*} e^{-it\frac{\langle f, f\rangle_{\mathcal{L}}}{2}} e^{i\langle f, x\rangle} \nu(df).$$

If $\int_{E_1^*} e^{-t\frac{\langle f, f\rangle_{\mathcal{L}}}{2}} e^{\langle f, x\rangle} \|\nu\|(df) < \infty$ *for all* x ($\|\nu\|$ *is the variation of* ν), *then*

$$e^{t\Delta_{\mathcal{L}H}} \tilde{\nu}_{\mathrm{La}} = \int_{E_1^*} e^{t\frac{\langle f, f\rangle_{\mathcal{L}}}{2}} e^{\langle f, x\rangle} \nu(df).$$

If $\int_{E_1^*} e^{\langle f, x\rangle} \|\nu\|(df) < \infty$, *then*

$$e^{it\Delta_{\mathcal{L}H}} \tilde{\nu}_{\mathrm{La}} = \int_{E_1^*} e^{it\frac{\langle f, f\rangle_{\mathcal{L}}}{2}} e^{\langle f, x\rangle} \nu(df).$$

If G *is a measurable space,* η *is a measure on* G, *and* $G \ni \alpha \mapsto B_\alpha \in L(H)$ *is a measurable function such that the function* $\hat{\eta}(x) = \int_G e^{(B_\alpha x, x)_H} \eta(d\alpha)$ *is defined for all* $x \in E$, *then*

$$e^{t\Delta_{\mathcal{L}H}} \hat{\eta}(x) = \int_G e^{t\gamma \mathrm{tr}_{\mathcal{L}} B_\alpha} e^{(B_\alpha x, x)_H} \eta(dx);$$

$$e^{it\Delta_{\mathcal{L}H}} \hat{\eta}(x) = \int_G e^{it\gamma \mathrm{tr}_{\mathcal{L}} B_\alpha} e^{(B_\alpha x, x)_H} \eta(dx).$$

Suppose that $\varepsilon > 0$, F is a linear measurable functional on E^*, and $\mu_\beta\{x: |F(x)|^\varepsilon = \beta^2\} > 0$ for every $\beta > 0$; suppose also that, for every $\lambda \in R^1$, η_λ is the measure on S_ν concentrated on $S_\nu^{|\lambda|^\varepsilon} \cap \{x: F(x) = \lambda\}$ and coinciding with the measure μ_β on this set. Let γ_0 be a nonnegative σ-finite measure on R^1, and let $\mathbf{H}_0$ ($\mathbf{H}_0^F$) be the continuous sum generated by this measure of the Hilbert spaces H_λ^0 (H_λ^{0F}) determined by the measures η_λ in the same way as the spaces H_β (H_β^F) are determined by the measures μ_β above.

Theorem 2. *The restriction of the Levy Laplacian to* $\mathbf{H}_0$ (*to* $\mathbf{H}_0^F$) *determines an essentially self-adjoint operator* $\Delta_{\mathcal{L}H_0}$ *in this space, and, if* $f = \int_{-\infty}^\infty f(\lambda)\gamma_0(d\lambda) \in \mathrm{dom}\exp(t\Delta_{\mathcal{L}H})$, *then*

$$e^{t\Delta_{\mathcal{L}H_0}} f = \int_{-\infty}^\infty e^{t|\lambda|^\varepsilon} f(\lambda)\gamma_0(d\lambda)$$

$$\left(\textit{respectively,}\quad e^{t\Delta_{\mathcal{L}H_0}} f = \int_{-\infty}^\infty e^{-t|\lambda|^\varepsilon} f(\lambda)\gamma_0(d\lambda)\right).$$

As the measures η_λ, the surface measures generated by some smooth measure on E^* can be taken. Applying Theorem 2 to a suitable Gaussian measure, we can derive some results that go back to Saito (see [9] and the references cited therein) and are based on the use of Hida's white-noise analysis. At $\varepsilon \in [1, 2]$ evaluating the last two integrals reduces to determining a mathematical expectation with respect to a stable distribution of order ε.

QUANTUM PROBABILITY AND LEVY LAPLACIANS

Below, we use the assumptions and notation of Example 3 and omit some analytical assumptions. Suppose that ν_0 is a canonical Gaussian H-cylindrical measure on H [i.e., $\tilde{\nu}_0(x) = e^{-\frac{1}{2}(x, x)_H}$] and ν is the Gaussian E-cylindrical measure on E^* being its image under the embedding $H \to E^*$; we denote the Lebesgue extension of this measure by the same symbol.

Let $\mathbb{S}$ be an operator in $\mathcal{H} = L_2(E^*, \nu)$ such that, for suitable $g \in \mathcal{F}(E^*)$ and $f \in \mathcal{H}$, $\mathbb{S}_0 g(x) = \mathrm{tr}(-D^{-1}g''(x) + (D^{-1}x \otimes x)g(x) - \mathrm{Id} \cdot g(x))$ and

$$(\mathbb{S}f)(x) = e^{\frac{(x, x)_H}{4}}\left(\mathbb{S}_0(f(\cdot)\, e^{\frac{(\cdot, \cdot)_H}{4}}\right)(x).$$

Then, $\mathbb{S}$ generates a (strictly) positive self-adjoint operator whose inverse is a Hilbert–Schmidt operator. Note that $\mathbb{S}$ is the particle number operator (also known as the Ornstein–Uhlenbeck operator) in $L_2(E^*, \nu)$ generated by the operator D. If $\mathcal{E} = \mathcal{H}_{\mathbb{S}}$ (Example 2), then $\mathcal{E} \subset \mathcal{H} \subset \mathcal{E}^*$ is a rigged Hilbert space. In particular, if $H = L_2(\mathbb{R})$, E is the Schwartz test function space $S(\mathbb{R})$, and D is the standard particle number operator in $S(\mathbb{R})$, then the obtained rigged Hilbert space is isomorphic to the Hida–Kubo–Takenaka space [8]. In what follows, we omit the subscript $\mathbb{S}$ if D is the operator mentioned above.

Remark 1. Replacing ν with some non-Gaussian measure ν_I and $\mathbb{S}$ with a positive (in the natural sense) operator $\mathbb{S}_I$ such that $\mathbb{S}_I\nu_I = 0$, we can define an analog of the Fock space describing systems with interaction (known as the interacting Fock space).

Remark 2. An $\mathcal{E}^*$-valued function on a subset of $\mathbb{R}$ is called a generalized random process (see [8]). A generalized quantum random process is a function defined on a subset of $\mathbb{R}$ and taking values in $L(\mathcal{E}, \mathcal{E}^*)$. If $g(\cdot)$ is a generalized quantum random process and $z \in \mathcal{E}$, then the function $g(\cdot)z$ is a generalized random process.

For example, suppose that $b(h)(z)(x) = z'(x)h$ for $z \in \mathcal{E}$, $x \in E^*$, and $h \in E^*$; if $h = \delta_t$, we write $b(t)$ instead of $b(\delta_t)$. It can be proved [8] that $b(t) \in L(\mathcal{E})$ and, therefore, $b(t)^* \in L(\mathcal{E}^*)$ [$= L(\mathcal{E}^*, \mathcal{E}^*)$] [the functions $b(\cdot)$ and $b(\cdot)^*$ are called birth and death processes].

Proposition 5. *If $H = L_2(0, a)$ for $a > 0$ and* **e** *is a uniformly bounded basis in H equidense on* [0, 1] [6], *then the function g defined by the equality*

$$g(t, \xi) = \int_0^1 e^{t\gamma s} e^{\int_0^s \xi(\tau)^2 d\tau} \eta(ds)$$

is a solution to the Levy heat equation.

This is a corollary to the preceding theorem; it is sufficient to set $B_s = [L_2(0, a) \ni \xi \mapsto \xi_s \in L_2(0, a)]$, where $\xi_s(t) = \xi(t)$ if $t \le s$ and $\xi_s(t) = 0$ if $t > s$ (we then have $\mathrm{tr}_{\mathcal{L}} B_s = s$).

Remark 3. The function

$$(t, \xi) \mapsto e^{\int_0^t \xi(\tau)^2 d\tau} \eta(ds)$$

is a (unique) solution to the Cauchy problem

$$\frac{df}{dt}(\xi) = \xi(t)^2 f(t)(\xi), \quad f(0) = 1$$

relative to the functions defined on $[0, a]$ and taking values in $\mathcal{F}(H)$. An analog of Proposition 5 is also valid for $H = L_2(\mathbb{R})$ and $E = S(\mathbb{R})$, and $f: [0, \infty) \to \mathcal{F}(E)$ is a solution to same problem but in the space of $\mathcal{F}(E)$-valued functions on $[0, \infty)$. Namely, the function

$$(t, \xi) \mapsto \int_0^a e^{t\gamma s} f(s)(\xi)\eta(ds)$$

is a solution to the Cauchy problem for the Levy heat equation under the assumption that the basis **e** satisfies the additional conditions

$$\lim \int_{-\infty}^{0} e_n(t)^2 = \lim \int_a^{\infty} e_n(t)^2 = 0.$$

It was shown in [8] that, if F is a solution to the Cauchy problem

$$\frac{dF}{dt} = (b(t)^2 + b(t)^{*2}) \circ F, \quad F(0) = \mathrm{Id}$$

(the symbol $\circ$ denotes normal product) and, for every t $G(t) = F(t)(z_0)$, where $z_0(x) = 1$ for $x \in E^*$ (thus, z_0 is a vacuum vector), then the so-called S-transform $C(t)$ of the function $G(t)$ is defined by the equality

$$C(t) = e^{\int_0^t \xi(\tau)^2 d\tau}.$$

Thus, the preceding paragraph contains the concluding result of [8], which establishes a relation between $C(\cdot)$ and the Levy heat equation.

In what follows, we assume $H = L_2(R^1)$ and $E = S(R_1)$. For every $a > 0$, consider the function d^a of real argument defined as $d^a(t) = 1$ if $t \in \left[-\frac{a}{2}, \frac{a}{2}\right]$ and $d^a(t) = 0$ if $t < -\frac{a}{2}$ or $t > \frac{a}{2}$. In particular, $d^0(t) = 1$ at $t = 0$ and $d^0(t) = 0$ at $t \ne 0$. For $x \in \mathbb{R}$, let d_x^0 be the function on $\mathbb{R}$ defined by $d_x^0(t) = d^0(t - x)$. Suppose also that

$$\int_{\{t=s\}} \delta_t \otimes \delta_s \, dt\, ds = \lim_{\varepsilon \to 0} \int_{\{|t-s|<\varepsilon\}} \delta_t \otimes \delta_s \, dt\, ds$$

(here and in what follows, we consider integrals of functions taking values in the space of distributions or operator-valued functions on $\mathbb{R}$) and

$$\int_{\{t=s\}} b(t)b(s)\, dt\, ds = \lim_{\varepsilon \to 0} \int_{\{|t-s|<\varepsilon\}} b(t)b(s)\, dt\, ds.$$

We stress that $\int_{\{t=s\}} (\ldots)$ is by no means an integral over the straight line $\{t = s\}$. The first definition implies that $\int_{\{t=s\}} \delta_t \otimes \delta_s dt\,ds = \int_{R^1} \delta_t \otimes d_t^0\, dt$. Note that the relation $\delta_x(x)dt = 1$ ($x \in \mathbb{R}$) from the Ito table [12] can be written as $\delta_\cdot(\cdot)dt = 1$; in its turn, this equality means that $x \in \mathbb{R}$ $\delta_x(\cdot)dt = d_x^0(\cdot)$ for each $x = t$; in particular, for $\delta_t(\cdot)dt = d_t^0(\cdot)$, we obtain the equality $\delta_t(\cdot)dt = d_t^0(\cdot)$ [it is useful to compare these relations with the heuristic equalities $\delta_\cdot(\cdot)dt = \int_{\cdot + dt} \delta(x)dx = 1$]. The last two equalities outside the parentheses imply that

$$\int_{R^1} \delta_t \otimes d_t^0\, dt = \int_{R^1} \delta_t \otimes \delta_t\, dt\, ds = \int_{R^1} \otimes\, \delta_t^2\, dt^2.$$

The formal use of the symbols $dtds$ and dt^2 does not lead to contradictions.

Now, let us define $\int_{R^1} b(t)^2 dt^2 \equiv \int_{R^1} b(t)b(t)dtds$ and $b^0(t)$ by the equalities $\int_{R^1} b(t)^2 dt^2 = \int_{R^1} \delta_t \otimes \delta_t dt^2 \circ \mathrm{D}^2$ and $b^0(t) = d^0(t)\mathrm{D}$, where D denotes differentiation.

Theorem 3. *The following two chains of equalities hold*:

$$\lim_{\varepsilon \to 0} \int_{\{|t-s|<\varepsilon\}} \delta_t \otimes \delta_s \, dt\, ds = \int_{R^1} \delta_t \otimes \delta_t \, dt\, ds$$

$$= \int_{R^1} \otimes\, \delta_t^2\, dt^2 = \int_{R^1} \delta_t \otimes d_t^0\, dt = \int_{\{t=s\}} \delta_t \otimes \delta_s dt\, ds;$$

$$\lim_{\varepsilon \to 0} \int_{\{|t-s|<\varepsilon\}} b(t)b(s)dt\, ds = \int_{R^1} b(t)b(t)dt\, ds$$

$$= \int_{R^1} b(t)^2 dt^2 = \int_{R^1} b(t)b^0(t)\, dt = \int_{\{t=s\}} b(t)b(s)dt\, ds.$$

Theorem 4. *The following representations of the Levy Laplacian hold*:

$$\Delta_{\mathcal{L}} = \lim_{a \to 0} \int_{R^2} d^a(t-s)b(t)b(s)dt\, ds$$

$$= \lim_{\varepsilon \to 0} \int_{\{|t-s|<\varepsilon\}} b(t)b(s)dt\, ds$$

$$= \lim_{a \to 0} \int_{R^1} \int_{-\frac{a}{2}}^{\frac{a}{2}} b(t-s)b(t+s)dt\, ds$$

$$= \lim_{k \to \infty} \frac{1}{k} \sum_{n=1}^{k} b(e_n)^2 = \lim_{k \to \infty} \int_{R^2} \frac{1}{k} \sum_{n=1}^{k} e_n(t)e_n(s)b(t)b(s)dtds$$

(in the topology of pointwise convergence on a suitable set of smooth functions)

$$= \int_{R^1} b(t)b(t)dt\, ds = \int_{R^1} b(t)^2 dt^2$$

$$= \int_{R^1} b(t)b^0(t)dt = \int_{\{t=s\}} b(t)b(s)dt\, ds.$$

The last two representations coincide with those given in [5, 12] with a reference to H.-H. Kuo (although the corresponding formulas in [12] contain misprints).

Theorem 5. *The following representations of the Volterra Laplacian hold*:

$$\Delta_{\mathrm{Id}} = \lim_{a \to 0} \int_{R^2} a^{-1} d^a(t-s)b(t)b(s)dt\, ds$$

$$= \lim_{a \to 0} \int_{|s-t| \le \frac{a}{2}} a^{-1} b(t)b(s)dt\, ds$$

$$= \lim_{a \to 0} a^{-1} \int_{R^1} \int_{-\frac{a}{2}}^{\frac{a}{2}} b(t-s)b(t+s)dt\, ds$$

$$= \sum_{n=1}^{\infty} b(e_n)^2 = \lim_{k \to \infty} \int_{R^2} \sum_{n=1}^{k} e_n(t)e_n(s)b(t)b(s)dtds$$

(in the topology of pointwise convergence on a suitable set of smooth functions)

$$= \int_{R^2} \delta(t-s)b(t)b(s)dt\, ds = \int_{R^1} b(t)b(t)dt.$$

Let θ_t denote the shift of the Heaviside function by t for every $t \in \mathbb{R}$.

Theorem 6. *In the topology of pointwise convergence on a suitable set of smooth functions,*

$$\Delta_{\mathcal{L}}^2 = \lim_{a \to 0} \int_{R^2} d^a(t-s)b(\theta_t)b(\theta_s)dt\, ds.$$

Theorem 7. *Suppose that ν is a (countably additive) E-cylindrical measure on E^* and $\tilde{\nu}_F(\cdot)$ is its Fourier transform extended by continuity over the largest space H_n among those admitting such an extension. Then, $\Delta_{\mathcal{L}}\tilde{\nu}_F(\cdot) = \widetilde{-(\cdot, \cdot)_{\mathcal{L}}\nu_F}$ and $\widetilde{\Delta_{\mathcal{L}}\nu_F} = -(\cdot, \cdot)_{\mathcal{L}}\tilde{\nu}_F$. In particular, if $x \in H$, then $\Delta_{\mathcal{L}}\nu_F = 0$. If $\tilde{\nu}_F = e^{-\frac{1}{2}(Bx, x)}$, i.e., if ν is a Gaussian measure with correlation operator B, then $\Delta_{\mathcal{L}}\tilde{\nu}_F(x) = \mathrm{tr}_{\mathcal{L}}B \cdot \tilde{\nu}_F(x) = \Delta_{\mathcal{L}}\nu_F(x) + (\mathrm{tr}_{\mathcal{L}}B - (x, x)_{\mathcal{L}}\tilde{\nu}_F(x)$. If $(x, x) = 0$ (in particular, if $x \in H$) and $B =$ Id, then $\Delta_{\mathcal{L}}\widetilde{\nu_F}(x) = \widetilde{\Delta_{\mathcal{L}}\nu_F}(x) + \tilde{\nu}_F(x)$.*

These relations also hold for measures absolutely continuous with respect to the corresponding Gaussian measure (provided that their densities satisfy some additional conditions). If the Gaussian measure has identity correlation operator, then the set of elements of $\mathcal{E}^*$ generated by such measures is dense in $\mathcal{E}^*$ and these relations can be extended to the space $\mathcal{E}^*$; note that the last of them coincides with a formula given in [6].

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MATHEMATICS

Rigid Relations on Constructive Models

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Following the terminology of monographs [1–4], we call an algebraic system **A** of finite signature Σ computable if the basis set and the base relations of signature Σ are computable and the base operations of Σ are partially computable functions. Such systems are frequently called recursive algebraic systems in the literature. If the signature is infinite, then uniform computability is additionally required. We say that two computable algebraic systems **A** and **B** of signature Σ are constructively isomorphic if there exists an isomorphism φ of system **A** onto **B** such that the function φ is computable. If a computable algebraic system **B** is isomorphic to **A**, then we call **B** a computable representation of the system **A**. Obviously, any system **A** can have several computable representations but no more than countably many.

If **B** and **B**' are constructive but not constructively isomorphic representations of a system **A**, we call them nonautoequivalent representations [1, 5, 6]. As in [1, 5], for the number of nonautoequivalent representations of system **A**, we use the term algorithmic dimension of system **A** and notation $\dim_A(\mathbf{A})$. As mentioned, $\dim_A(\mathbf{A}) \leq \omega$. It is shown in [5] that the spectrum of algorithmic dimensions of algebraic systems is precisely $\omega \cup \{\omega\}$. Following Mal'tsev [6], we call algebraic systems of algorithmic dimension one autostable. The problem of describing autostable models plays an important role in studying constructive models; to date, a number of sufficient and necessary conditions for autostability have been found.

In the general case, the problem of describing autostable models is not solved completely. This problem is also interesting as applied to some classical algebraic objects. The most important results in this direction are given in [1, 3]. There are also a number of interesting results concerning algebraic systems of infinite algorithmic dimension [1–3]. The greatest difficulties are involved in characterizing systems of finite algorithmic dimensions. In this paper, we suggest an approach reducing the problem of characterizing various types of computable representations to studying special definable relations.

A relation $P \subseteq A^n$ on the elements of a system **A** is said to be definable over a set $a_1, a_2, \ldots, a_m$ from **A** if there exists a formula $f(x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_m)$ of language $L^{\Sigma}_{\omega_1\omega}$ that defines the relation P, or, equivalently, if the set P is closed with respect to the automorphisms in the enrichment $(\mathbf{A}, a_1, a_2, \ldots, a_m)$ by constants for the elements $a_1, a_2, \ldots, a_m$.

A relation P (a family $P = \{P_i | i \in I\}$ of relations) is called rigid for a constructive representation **A** of a system $\mathbf{A}^*$ if the enrichment $\mathbf{A}^P$ of the system **A** by the predicate (the family of predicates) for P (for P_i, where $i \in I$) is computable and has algorithmic dimension one and there exists a set $a_1, a_2, \ldots, a_m$ such that the relation P is definable (all relations from P are definable) over $a_1, a_2, \ldots, a_m$.

Theorem 1. *For any constuctivizable superatomic Boolean algebra, there exists a family of one-element relations rigid for its some constructivization.*

Note that all superatomic Boolean algebras, except finite algebras, are nonautostable and have infinite algorithmic dimensions.

Theorem 2. *For the Boolean algebra* $\mathbf{B}_\omega$ *of finite and cofinite subsets of* ω, *only one (to an autoequivalence) computable representation has a rigid relation.*

As to the vector spaces, it is well known that all those of finite dimension are autostable, and the countable space $\mathbf{V}^{(F)}_\infty$ of infinite dimension over a computable field F has infinite algorithmic dimension.

The results obtained in [6] directly imply the following theorem.

Theorem 3. *A constructive representation of the space* $\mathbf{V}_\infty$ *with a solvable basis has a rigid family of relations.*

However, it is not known whether or not it has a rigid relation for an arbitrary computable field F.

Theorem 4. *The constructive representations of algebraically closed (real-closed ordered) fields with recursive transcendence bases have rigid families of relations.*

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