

Dynamical q -deformation in quantum theory and the stochastic limit

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Abstract

A model of particle interacting with quantum field is considered. The model includes as particular cases the polaron model and non-relativistic quantum electrodynamics. We show that the field operators obey q -commutation relations with q depending on time. After the stochastic (or van Hove) limit, due to the nonlinearity, the atomic and field degrees of freedom become *entangled* in the sense that the field and the atomic variables no longer commute but give rise to a new algebra with new commutation relations replacing the Boson ones. This new algebra allows to give a simple proof of the fact that the non crossing half-planar diagrams give the dominating contribution in a weak coupling regime and to calculate explicitly the correlations associated to the new algebra. The above results depend crucially on the fact that we do not introduce any dipole or multipole approximation.

1 Introduction

In recent years it has been great interest to q -deformed commutational relations, see for example [1]-[8]. In many works q -deformed relations are considered as an *ad hoc* deformation of the ordinary commutation relations or as a hidden symmetry algebra.

In this work we show that the so called collective operators $a_\lambda(t, k)$ in a model of particle interacting with quantum field satisfy the q -deformed commutation relations (see (14), (15), (21) below) where the parameter q depends on time. The collective operators are natural objects in the stochastic (van Hove) limit of the model describing interaction of particle with quantum field. The stochastic limit is used to derive the long time behavior of the system interacting with reservoir, in particular to derive the master equation [9]-[10]. The main result of this work is that in the stochastic limit the q -deformed commutation relations give rise to the generalized quantum Boltzmann commutational relations.

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We investigate the model describing interaction of non-relativistic particle with quantum field. This model is widely studied in elementary particle physics, solid state physics, quantum optics, see for example [11]-[14]. We consider the simplest case in which matter is represented by a single particle, say an electron, whose position and momentum we denote respectively by $q = (q_1, q_2, q_3)$ and $p = (p_1, p_2, p_3)$ and satisfy the commutation relations $[q_j, p_n] = i\delta_{jn}$. The electromagnetic field is described by Boson operators $a(k) = (a_1(k), a_2(k), a_3(k))$; $a^\dagger(k) = (a_1^\dagger(k), \dots, a_3^\dagger(k))$ satisfying the canonical commutation relations $[a_j(k), a_n^\dagger(k')] = \delta_{jn}\delta(k - k')$. The Hamiltonian of a free non relativistic atom interacting with a quantum electromagnetic field is

$$H = H_0 + \lambda H_I = \int \omega(k) a^\dagger(k) a(k) dk + \frac{1}{2} p^2 + \lambda H_I \quad (1)$$

where λ is a small constant, $\omega(k) = |k|$ and

$$H_I = \int d^3k (g(k) p \cdot a^\dagger(k) e^{ikq} + \bar{g}(k) p \cdot a(k) e^{-ikq}) + h.c. \quad (2)$$

Here $p \cdot a(k) = \sum_{j=1}^3 p_j a_j(k)$, $p^2 = \sum_{j=1}^3 p_j^2$, $a^\dagger(k) a(k) = \sum_{j=1}^3 a_j^\dagger(k) a_j(k)$, $kq = \sum_{j=1}^3 k_j q_j$.

The general idea of the stochastic limit is to make the time rescaling $t \rightarrow t/\lambda^2$ in the solution of the Schrödinger equation in interaction picture $U_t^{(\lambda)} = e^{itH_0} e^{-itH}$, associated to the Hamiltonian H , i.e.

$$\frac{\partial}{\partial t} U_t^{(\lambda)} = -i\lambda H_I(t) U_t^{(\lambda)} \quad , \quad U_0^{(\lambda)} = 1 \quad (3)$$

with $H_I(t) = e^{itH_0} H_I e^{-itH_0}$ (the *evolved interaction Hamiltonian*). This leads to the rescaled equation

$$\frac{\partial}{\partial t} U_{t/\lambda^2}^{(\lambda)} = -\frac{i}{\lambda} H_I(t/\lambda^2) U_{t/\lambda^2}^{(\lambda)} \quad (4)$$

and one wants to study the limits, in a topology to be specified,

$$\lim_{\lambda \rightarrow 0} U_{t/\lambda^2}^{(\lambda)} = U_t \quad (5)$$

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} H_I \left(\frac{t}{\lambda^2} \right) = H_t = \int d^3k (g(k) p \cdot b^\dagger(t, k) + \bar{g}(k) p \cdot b(t, k) + h.c.) \quad (6)$$

Moreover one wants to prove that U_t is the solution of the equation

$$\partial_t U_t = -iH_t U_t \quad ; \quad U_0 = 1 \quad (7)$$

The interest of this limit equation is in the fact that many problems become explicitly integrable. The stochastic limit of the model (1)-(2) has been considered in [9], [10], [15], [16].

The rescaling $t \rightarrow t/\lambda^2$ is equivalent to consider the simultaneous limit $\lambda \rightarrow 0$, $t \rightarrow \infty$ under the condition that $\lambda^2 t$ tends to a constant (interpreted as a new *slow scale* time). This limit captures the main contributions to the dynamics in a regime, of *long times and small coupling* arising from the cumulative effects, on a large time scale, of small interactions ($\lambda \rightarrow 0$). The physical idea is that, looked from the slow time scale of the atom, the field looks like a very chaotic object: a *quantum white noise*, i.e. a δ -correlated

(in time) quantum field $b_j^\dagger(t, k), b_j(t, k)$ also called a *master field*. If one introduces the dipole approximation the master field is the usual Boson Fock white noise. Without the dipole approximation the master field is a completely new type of white noise whose algebra is described by the relations [10]

$$b_j(t, k)p_n = (p_n + k_n)b_j(t, k) \quad (8)$$

$$b_j(t, k)b_n^\dagger(t', k') = 2\pi\delta(t - t')\delta(\tilde{\omega}(k) + kp)\delta(k - k')\delta_{jn} \quad (9)$$

$$\tilde{\omega}(k) := \omega(k) + \frac{1}{2}k^2 \quad (10)$$

Recalling that p is the atomic momentum, we see that the relation (8) shows that the atom and the master field are not independent even at a kinematical level. This is what we call *entanglement*. The relation (9) is a generalization of the algebra of free creation–annihilation operators with commutation relations

$$A_i A_j^\dagger = \delta_{ij}$$

and the corresponding statistics becomes a generalization of the Boltzmannian (or Free) statistics. This generalization is due to the fact that the right hand side is not a scalar but an operator (a function of the atomic momentum). This means that the relations (8), (9) are *module commutation relations*. For any fixed value \bar{p} of the atomic momentum we get a copy of the free (or Boltzmannian) algebra. Given the relations (8), (9), (10), the statistics of the master field is uniquely determined by the condition

$$b_j(t, k)\Psi = 0 \quad (11)$$

where Ψ is the vacuum of the master field, via a module generalization of the free Wick theorem (this is our Theorem 2 in section (4) below).

In Section 2 the dynamically q -deformed commutation relations (14), (15), (21) are obtained and the stochastic limit for collective operators is evaluated. In Section 3 the n -point correlation functions of the collective operators are computed. Finally in Section 4 the stochastic limit of n -point correlation functions is calculated.

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2 Dynamical q -deformation

In order to determine the limit (3) one rewrites the rescaled interaction Hamiltonian in terms of some rescaled fields $a_{\lambda,j}(t, k)$:

$$\frac{1}{\lambda} H_I \left(\frac{t}{\lambda^2} \right) = \int d^3k p(\bar{g}(k)a_\lambda(t, k) + g(k)a_\lambda^\dagger(t, k)) + h.c. \quad (12)$$

where

$$a_{\lambda,j}(t, k) := \frac{1}{\lambda} e^{i\frac{t}{\lambda^2} H_0} e^{-ikq} a_j(k) e^{-i\frac{t}{\lambda^2} H_0} = \frac{1}{\lambda} e^{-i\frac{t}{\lambda^2} (\tilde{\omega}(k) + kp)} e^{-ikq} a_j(k) \quad (13)$$

Here $\tilde{\omega}(k) = \omega(k) + \frac{1}{2}k^2$. It is now easy to prove that operators $a_{\lambda,j}(t, k)$ satisfy the following q -deformed module relations,

$$a_{\lambda,j}(t, k)a_{\lambda,n}^\dagger(t', k') = a_{\lambda,n}^\dagger(t', k')a_{\lambda,j}(t, k) \cdot q_\lambda(t - t', kk') + \frac{1}{\lambda^2} q_\lambda(t - t', \tilde{\omega}(k) + kp)\delta(k - k')\delta_{jn} \quad (14)$$

$$a_{\lambda,j}(t,k)p_n = (p_n + k_n)a_{\lambda,j}(t,k) \quad (15)$$

where

$$q_\lambda(t-t',x) = e^{-i\frac{t-t'}{\lambda^2}x} \quad (16)$$

is an oscillating exponent. This shows that the module q -deformation of the commutation relations arise here as a result of the dynamics and are not put artificially *ab initio*. Now let us suppose that the master field

$$b_j(t,k) = \lim_{\lambda \rightarrow 0} a_{\lambda,j}(t,k) \quad (17)$$

exist. Then it is natural to conjecture that its algebra shall be obtained as the stochastic limit ($\lambda \rightarrow 0$) of the algebra (14), (15). Notice that the factor $q_\lambda(t-t',x)$ is an oscillating exponent and one easily sees that

$$\lim_{\lambda \rightarrow 0} q_\lambda(t,x) = 0, \quad \lim_{\lambda \rightarrow 0} \frac{1}{\lambda^2} q_\lambda(t,x) = 2\pi\delta(t)\delta(x) \quad (18)$$

Thus it is natural to expect that the limit of (15) is

$$b_j(t,k)p_n = (p_n + k_n)b_j(t,k) \quad (19)$$

and the limit of (14) gives the module free relation

$$b_j(t,k)b_n^\dagger(t',k') = 2\pi\delta(t-t')\delta(\tilde{\omega}(k) + kp)\delta(k-k')\delta_{jn} \quad (20)$$

Operators $a_{\lambda,j}(t,k)$ also obey the relation

$$a_{\lambda,j}(t,k)a_{\lambda,n}(t',k') = a_{\lambda,n}(t',k')a_{\lambda,j}(t,k)q_\lambda^{-1}(t-t',kk') \quad (21)$$

In what follows we will not write indexes j, n explicitly. It is clear that the relation (21) should disappear after the limit. In fact, if the relation (21) would survive in the limit then, because of (18), it should give $b(t,k)b(t',k') = 0$, hence also $b^\dagger(t,k)b^\dagger(t',k') = 0$, so all the n -particle vectors with $n \geq 2$ would be zero. But we shall prove that this is not the case.

An accurate proof of vanishing of relation (21) looks as follows. In fact the subject of the stochastic limit is not the algebra of observables, but the quantum (or algebraic) probability space. Quantum probability space is a pair (algebra, state on this algebra). In the quantum probability space, defined by the algebra (21), (14), (15) and the vacuum expectation, we can omit the relation (21) even before the limit.

Let us explain this fact for simplicity on the example of bosonic algebra. Consider the algebra \mathcal{A} with generators a_i, a_j^\dagger and the relations

$$[a_i, a_j^\dagger] = \delta_{ij} \quad (22)$$

and the state $\langle \cdot \rangle$ on this algebra, equal to the vacuum expectation in the Fock representation.

It is easy to prove the following lemma.

LEMMA 1. *In the GNS representation of the algebra \mathcal{A} with respect to the state $\langle \cdot \rangle$ we have the extra relation*

$$[a_i, a_j] = 0 \quad (23)$$

In the language of quantum probability spaces this means that the quantum probability space $(\mathcal{A}, \langle \cdot \rangle)$ is isomorphic to the quantum probability space $(\mathcal{A}', \langle \cdot \rangle)$ where the algebra \mathcal{A}' is a factor of \mathcal{A} by the relations (23). The isomorphism of quantum probability spaces means the coincidence of all correlators.

We get, that in the language of quantum probability spaces the relation (21) follows from relations (14), (15) and the fact, that we use the vacuum expectation. For this quantum probability space we investigate the stochastic limit ($\lambda \rightarrow 0$). In the stochastic limit we have to keep the limits of relations (14), (15). The relation (21) vanishes in the limit, because the GNS representation of the limiting algebra is realised not in the symmetric but in the free (or full) Fock space.

To finish the proof we have to prove the existence of the stochastic limit of n -point correlators. This is the subject of the next section.

3 Calculation of the n -point correlator

In the present section we prove the existence of the limit of the q -deformed correlators

$$\langle a_\lambda^{\varepsilon_1}(t_1, k_1) \dots a_\lambda^{\varepsilon_N}(t_N, k_N) \rangle \quad (24)$$

where a^ε means either a or a^\dagger ($\varepsilon = 0$ for a , $\varepsilon = 1$ for a^\dagger) and $\langle \cdot \rangle$ denotes vacuum expectation, exist. Then according to the previous section the limit of this correlator must be equal to the corresponding correlator of the master field:

$$\langle b^{\varepsilon_1}(t_1, k_1) \dots b^{\varepsilon_N}(t_N, k_N) \rangle \quad (25)$$

Let us enumerate annihilators in the product $a_\lambda^{\varepsilon_1}(t_1, k_1) \dots a_\lambda^{\varepsilon_N}(t_N, k_N)$ as $a_\lambda(t_{m_j}, k_{m_j})$, $j = 1, \dots, J$, and enumerate creators as $a_\lambda^\dagger(t_{m'_j}, k_{m'_j})$, $j = 1, \dots, I$, $I + J = N$. This means that if $\varepsilon_m = 0$ then $a_\lambda^{\varepsilon_m}(t_m, k_m) = a_\lambda(t_{m_j}, k_{m_j})$ for $m = m_j$ (and the analogous condition for $\varepsilon_m = 1$).

Let us prove the following lemma.

LEMMA 2.

$$\begin{aligned} & a_\lambda(t, k) a_\lambda^{\varepsilon_1}(t_1, k_1) \dots a_\lambda^{\varepsilon_N}(t_N, k_N) - \\ & - \prod_{i=1}^I q_\lambda^{-1}(t - t_{m_i}, k k_{m_i}) \prod_{j=1}^J q_\lambda(t - t_{m'_j}, k k_{m'_j}) a_\lambda^{\varepsilon_1}(t_1, k_1) \dots a_\lambda^{\varepsilon_N}(t_N, k_N) a_\lambda(t, k) = \\ & = \sum_{j=1}^I \delta(k - k_{m'_j}) \frac{1}{\lambda^2} q_\lambda(t - t_{m'_j}, \tilde{\omega}(k) + kp) \prod_{m_i < m'_j} q_\lambda(t - t_{m'_j}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda^{-1}(t - t_{m'_j}, k k_{m'_i}) \\ & \prod_{m_i < m'_j} q_\lambda^{-1}(t - t_{m_i}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda(t - t_{m'_i}, k k_{m'_i}) a_\lambda^{\varepsilon_1}(t_1, k_1) \dots \hat{a}_\lambda^\dagger(t_{m'_j}, k_{m'_j}) \dots a_\lambda^{\varepsilon_N}(t_N, k_N) \end{aligned} \quad (26)$$

Here the notion \hat{a}_λ^\dagger means that we omit the operator a_λ^\dagger in this product.

Proof The proof of this lemma is by induction over N . The first step of induction is the relation (14) or (21). Given the formula (26) for N , we will prove this formula for $N + 1$. We consider two cases.

1) The first case: $\varepsilon_{N+1} = 0$. In this case using (26) for N and (21) we get

$$\begin{aligned}
& a_\lambda(t, k) a_\lambda^{\varepsilon_1}(t_1, k_1) \dots a_\lambda^{\varepsilon_N}(t_{N+1}, k_{N+1}) - q_\lambda^{-1}(t - t_{N+1}, k k_{N+1}) \\
& \prod_{i=1}^I q_\lambda^{-1}(t - t_{m_i}, k k_{m_i}) \prod_{j=1}^J q_\lambda(t - t_{m'_j}, k k_{m'_j}) a_\lambda^{\varepsilon_1}(t_1, k_1) \dots a_\lambda(t_{N+1}, k_{N+1}) a_\lambda(t, k) = \\
& = \sum_{j=1}^I \delta(k - k_{m'_j}) \frac{1}{\lambda^2} q_\lambda(t - t_{m'_j}, \tilde{\omega}(k) + kp) \prod_{m_i < m'_j} q_\lambda(t - t_{m'_j}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda^{-1}(t - t_{m'_j}, k k_{m'_i}) \\
& \prod_{m_i < m'_j} q_\lambda^{-1}(t - t_{m_i}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda(t - t_{m'_i}, k k_{m'_i}) a_\lambda^{\varepsilon_1}(t_1, k_1) \dots \hat{a}_\lambda^\dagger(t_{m'_j}, k_{m'_j}) \dots a_\lambda^{\varepsilon_N}(t_N, k_N)
\end{aligned}$$

that is exactly (26) for $N + 1$.

2) The second case: $\varepsilon_{N+1} = 1$. In this case using (26) for N and (14) we get

$$\begin{aligned}
& a_\lambda(t, k) a_\lambda^{\varepsilon_1}(t_1, k_1) \dots a_\lambda^{\varepsilon_N}(t_{N+1}, k_{N+1}) - \\
& - \prod_{i=1}^I q_\lambda^{-1}(t - t_{m_i}, k k_{m_i}) \prod_{j=1}^J q_\lambda(t - t_{m'_j}, k k_{m'_j}) a_\lambda^{\varepsilon_1}(t_1, k_1) \dots a_\lambda^{\varepsilon_N}(t_N, k_N) \\
& \left(a_\lambda^\dagger(t_{N+1}, k_{N+1}) a_\lambda(t, k) q_\lambda(t - t_{N+1}, k k_{N+1}) + \delta(k - k_{N+1}) \frac{1}{\lambda^2} q_\lambda(t - t_{N+1}, \tilde{\omega}(k) + kp) \right) = \\
& = \sum_{j=1}^I \delta(k - k_{m'_j}) \frac{1}{\lambda^2} q_\lambda(t - t_{m'_j}, \tilde{\omega}(k) + kp) \prod_{m_i < m'_j} q_\lambda(t - t_{m'_j}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda^{-1}(t - t_{m'_j}, k k_{m'_i}) \\
& \prod_{m_i < m'_j} q_\lambda^{-1}(t - t_{m_i}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda(t - t_{m'_i}, k k_{m'_i}) \\
& a_\lambda^{\varepsilon_1}(t_1, k_1) \dots \hat{a}_\lambda^\dagger(t_{m'_j}, k_{m'_j}) \dots a_\lambda^{\varepsilon_N}(t_N, k_N) a_\lambda^\dagger(t_{N+1}, k_{N+1})
\end{aligned}$$

Moving the term

$$\delta(k - k_{N+1}) \frac{1}{\lambda^2} q_\lambda(t - t_{N+1}, \tilde{\omega}(k) + kp)$$

to the right hand side of this formula and commuting it with creators and annihilators using (15) we get (26) for $N + 1$. This finishes the proof of Lemma 2.

The next theorem describes the form of N -point correlator.

THEOREM 1.

i) If the number of creators is not equal to the number of annihilators, then the correlator (24) is equal to zero;

ii) if the number of creators is equal to the number of annihilators ($N = 2n$), then the correlation function is equal to the following sum over pair partitions

$$\begin{aligned}
& \sum_{\sigma(\varepsilon)} \prod_{h=1}^n \delta(k_{m_h} - k_{m'_h}) \frac{1}{\lambda^2} q_\lambda \left((t_{m_h} - t_{m'_h}), \left(\tilde{\omega}(k_{m_h}) + k_{m_h} p + \sum_{m_\alpha < m_h < m'_\alpha} k_{m_\alpha} \cdot k_{m_h} \right) \right) \\
& \prod_{(m_j, m'_j), (m_i, m'_i); i, j=1, \dots, n: m_j < m_i < m'_j < m'_i} q_\lambda(t_{m_i} - t_{m'_j}, k_{m_i} \cdot k_{m'_j})
\end{aligned} \quad (27)$$

where $\sigma(\varepsilon) = \{(m_j < m'_j) : j = 1, \dots, n\}$ is a partition of $\{1, \dots, 2n\}$ associated with $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{2n})$.

Proof The proof of this theorem is by induction over n . The first step of induction is obvious. Let us assume the correlator (24) is expressed by the formula (27) for $N = 2n - 2$ and prove that the same is true for $N = 2n$. We consider $2n$ -point correlator

$$\langle a_\lambda^{\varepsilon_1}(t_1, k_1) \dots a_\lambda^{\varepsilon_{2n}}(t_{2n}, k_{2n}) \rangle$$

It is easy to see that if this correlator is not equal to zero then the first operator is annihilator and the last is creator. Without loss of generality we can consider the case when the correlator is as follows

$$\langle a_\lambda(t_{m_1}, k_{m_1}) a_\lambda^{\varepsilon_2}(t_2, k_2) \dots a_\lambda^{\varepsilon_{2n-1}}(t_{2n-1}, k_{2n-1}) a_\lambda^\dagger(t_{m'_n}, k_{m'_n}) \rangle \quad (28)$$

From the Lemma 2 follows the following formula for this correlator

$$\begin{aligned} (28) &= \sum_{j=1}^n \delta(k_{m_1} - k_{m'_j}) \frac{1}{\lambda^2} q_\lambda(t_{m_1} - t_{m'_j}, \tilde{\omega}(k_{m_1}) + k_{m_1} p) \prod_{m_i < m'_j < m'_i} q_\lambda(t_{m_1} - t_{m'_j}, k_{m_1} k_{m_i}) \\ &\prod_{m_i < m'_j} q_\lambda^{-1}(t_{m_1} - t_{m_i}, k_{m_1} k_{m_i}) \prod_{m'_i < m'_j} q_\lambda(t_{m_1} - t_{m'_i}, k_{m_1} k_{m'_i}) \langle \hat{a}_\lambda(t_{m_1}, k_{m_1}) \dots \hat{a}_\lambda^\dagger(t_{m'_j}, k_{m'_j}) \dots \rangle \end{aligned} \quad (29)$$

The product $\prod_{m_i < m'_j < m'_i} q_\lambda(t_{m_1} - t_{m'_j}, k_{m_1} k_{m_i})$ in (29) arise from the products

$$\prod_{m_i < m'_j} q_\lambda(t - t_{m'_j}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda^{-1}(t - t_{m'_j}, k k_{m'_i})$$

in (26) due to cancellation of corresponding terms because of $\delta(k_{m_i} - k_{m'_i})$ in the correlator (27) for $N = 2n - 2$. We have

$$\begin{aligned} &\prod_{m_i < m'_j} q_\lambda(t - t_{m'_j}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda^{-1}(t - t_{m'_j}, k k_{m'_i}) = \\ &= \prod_{m'_i < m'_j} q_\lambda(t - t_{m'_j}, k k_{m_i}) \prod_{m_i < m'_j < m'_i} q_\lambda(t - t_{m'_j}, k k_{m_i}) \prod_{m'_i < m'_j} q_\lambda^{-1}(t - t_{m'_j}, k k_{m_i}) = \\ &= \prod_{m_i < m'_j < m'_i} q_\lambda(t - t_{m'_j}, k k_{m_i}) \end{aligned}$$

Let us prove now that the (29) is equal in fact to (27). This will give a proof of the theorem. We have

$$\begin{aligned} &\prod_{m_i < m'_j} q_\lambda^{-1}(t_{m_1} - t_{m_i}, k_{m_1} k_{m_i}) \prod_{m'_i < m'_j} q_\lambda(t_{m_1} - t_{m'_i}, k_{m_1} k_{m'_i}) = \\ &= \prod_{m'_i < m'_j} q_\lambda^{-1}(t_{m_1} - t_{m_i}, k_{m_1} k_{m_i}) \prod_{m_i < m'_j < m'_i} q_\lambda^{-1}(t_{m_1} - t_{m_i}, k_{m_1} k_{m_i}) \prod_{m'_i < m'_j} q_\lambda(t_{m_1} - t_{m'_i}, k_{m_1} k_{m'_i}) \end{aligned} \quad (30)$$

because $m_i < m'_i$. From (27) for $2n - 2$ we have $k_{m_i} = k_{m'_i}$. By using this and combining the first product with the third we get

$$\prod_{m'_i < m'_j} q_\lambda(t_{m_i} - t_{m'_i}, k_{m_1} k_{m_i}) \prod_{m_i < m'_j < m'_i} q_\lambda^{-1}(t_{m_1} - t_{m_i}, k_{m_1} k_{m_i}) \quad (31)$$

Using the change of variables in the second product in (31)

$$t_{m_1} - t_{m_i} = (t_{m_1} - t_{m'_j}) - (t_{m'_j} - t_{m_i})$$

and the property $k_{m_1} = k_{m'_j}$ we get that (30) equals

$$\prod_{m'_i < m'_j} q_\lambda(t_{m_i} - t_{m'_i}, k_{m_1} k_{m_i}) \prod_{m_i < m'_j < m'_i} q_\lambda^{-1}(t_{m_1} - t_{m'_j}, k_{m_1} k_{m_i}) \prod_{m_i < m'_j < m'_i} q_\lambda(t_{m_i} - t_{m'_j}, k_{m'_j} k_{m_i})$$

Substituting this into the formula (29) we get

$$\sum_{j=1}^n \delta(k_{m_1} - k_{m'_j}) \frac{1}{\lambda^2} q_\lambda(t_{m_1} - t_{m'_j}, \tilde{\omega}(k_{m_1}) + k_{m_1} p) \quad (32)$$

$$\prod_{m'_i < m'_j} q_\lambda(t_{m_i} - t_{m'_i}, k_{m_1} k_{m_i}) \prod_{m_i < m'_j < m'_i} q_\lambda(t_{m_i} - t_{m'_j}, k_{m'_j} k_{m_i}) \langle \hat{a}_\lambda(t_{m_1}, k_{m_1}) \dots \hat{a}_\lambda^\dagger(t_{m'_j}, k_{m'_j}) \dots \rangle$$

Here the notion $\langle \dots \hat{a} \dots \rangle$ means that we omit the operator \hat{a} in this correlation function. For $2n - 2$ -point correlator in (32) we use the formula (27) for $\varepsilon - \{m_1, m'_j\}$:

$$\begin{aligned} & \langle \hat{a}_\lambda(t_{m_1}, k_{m_1}) \dots \hat{a}_\lambda^\dagger(t_{m'_j}, k_{m'_j}) \dots \rangle = \\ & = \sum_{\sigma(\varepsilon - \{m_1, m'_j\})} \prod_{h=1}^{n-1} \delta(k_{n_h} - k_{n'_h}) \frac{1}{\lambda^2} q_\lambda \left((t_{n_h} - t_{n'_h}), \left(\tilde{\omega}(k_{n_h}) + k_{n_h} p + \sum_{n_\alpha < n_h < n'_\alpha} k_{n_\alpha} \cdot k_{n_h} \right) \right) \\ & \quad \prod_{(n_j, n'_j), (n_i, n'_i); i, j=1, \dots, n-1: n_j < n_i < n'_j < n'_i} q_\lambda(t_{n_i} - t_{n'_i}, k_{n_i} \cdot k_{n'_i}) \end{aligned} \quad (33)$$

where $\sigma(\varepsilon - \{m_1, m'_j\}) = \{(n_j < n'_j) : j = 1, \dots, n-1\}$ is a partition (without one pair) of $\{1, \dots, 2n\}$ associated with $\varepsilon - \{m_1, m'_j\}$. The indices n_h correspond to annihilators, n'_h correspond to creators.

Substituting (33) into (32) we get

$$\begin{aligned} & \sum_{j=1}^n \delta(k_{m_1} - k_{m'_j}) \frac{1}{\lambda^2} q_\lambda(t_{m_1} - t_{m'_j}, \tilde{\omega}(k_{m_1}) + k_{m_1} p) \\ & \quad \prod_{m'_i < m'_j} q_\lambda(t_{m_i} - t_{m'_i}, k_{m_1} k_{m_i}) \prod_{m_i < m'_j < m'_i} q_\lambda(t_{m_i} - t_{m'_j}, k_{m'_j} k_{m_i}) \\ & \quad \sum_{\sigma(\varepsilon - \{m_1, m'_j\})} \prod_{h=1}^{n-1} \delta(k_{n_h} - k_{n'_h}) \frac{1}{\lambda^2} q_\lambda \left((t_{n_h} - t_{n'_h}), \left(\tilde{\omega}(k_{n_h}) + k_{n_h} p + \sum_{n_\alpha < n_h < n'_\alpha} k_{n_\alpha} \cdot k_{n_h} \right) \right) \end{aligned} \quad (34)$$

$$\prod_{(n_j, n'_j), (n_i, n'_i); i, j=1, \dots, n-1: n_j < n_i < n'_j < n'_i} q_\lambda(t_{n_i} - t_{n'_j}, k_{n_i} \cdot k_{n'_j})$$

It is easy to see that

$$\sum_{j=1}^n \sum_{\sigma(\varepsilon - \{m_1, m'_j\})} = \sum_{\sigma(\varepsilon)} \quad (35)$$

Using (35) and combining the first product in (34) with the third and the second product with the fourth we obtain (27).

This finishes the proof of the theorem.

Let us analyze the behavior of the n -point correlator in the stochastic limit. It is easy to see, that the stochastic limit of pairings exists and equals to the product of δ -functions of different arguments. Product of pairing cannot spoil the convergence, because this product in any case can be considered as the product of two terms. The first term can result only in shift in the δ -functions, corresponding to pairings. The second term is an oscillating exponent and vanish in the stochastic limit.

We have proved that the stochastic limit of the n -point correlator exists.

4 The QED module Wick theorem

We proved that $b(t, k)$ satisfy the following free QED module algebra relations

$$b(t, k_1) b^\dagger(\tau, k_2) = 2\pi \delta(t - \tau) \delta(\tilde{\omega}(k_1) + k_1 p) \delta(k_1 - k_2) \quad (36)$$

$$b(t, k) p = (p + k) b(t, k) \quad (37)$$

and the functional $\langle \cdot \rangle$ is the vacuum expectation. Let us prove the following module analog of the Wick theorem.

THEOREM 2. *The limit correlation functions exist always and*

i) *if the number of creators is not equal to the number of annihilators, then the correlator (25) is equal to zero (even before the limit);*

ii) *if the number of creators is equal to the number of annihilators ($N = 2n$), then the limit is equal to the following*

$$\prod_{h=1}^n \delta(k_{m_h} - k_{m'_h}) 2\pi \delta(t_{m_h} - t_{m'_h}) \delta \left(\tilde{\omega}(k_{m_h}) + k_{m_h} p + \sum_{m_\alpha < m_h < m'_\alpha} k_{m_\alpha} \cdot k_{m_h} \right) \quad (38)$$

where $\{(m_j < m'_j) : j = 1, \dots, n\}$ is the unique non-crossing partition of $\{1, \dots, 2n\}$ associated with $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{2n})$.

Proof The proof is done by computing the correlation functions using the commutation relations listed above. We investigate the correlator

$$\langle b^{\varepsilon_1}(t_1, k_1) \dots b^{\varepsilon_N}(t_N, k_N) \rangle$$

At first we simplify this correlator using (36). Obtained δ -functions we will move through $b^\varepsilon(t, k)$, using (37). We will iterate this procedure before monomial will take normally ordered form. Because the functional $\langle \cdot \rangle$ is equal to vacuum expectation, only δ -functions will survive.

The pairing $b(t_{m'_h}, k_{m'_h})b^\dagger(t_{m_h}, k_{m_h})$ equal

$$\delta(k_{m'_h} - k_{m_h})2\pi\delta(t_{m'_h} - t_{m_h})\delta(\tilde{\omega}(k_{m_h}) + k_{m_h}p) \quad (39)$$

and the relation (37) gives the term $\sum_{m_\alpha < m_h < m'_\alpha} k_{m_\alpha} \cdot k_{m_h}$ in the phase shift (an argument of the last δ -function in (38)), arising from moving of this δ -function through $b^\epsilon(t, k)$. This finishes the proof of theorem 2. The alternative proof can be given by calculation of stochastic limit of correlator given by the Theorem 1 (for dynamically q -deformed algebra).

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