MARKOVIAN COCYCLES

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ABSTRACT

We show that the modular automorphism groups of quantum Markov chains are characterized by 'markovian cocycles'. We also give some characterization of a quantum Markov chain in terms of the corresponding modular automorphism group or of the associated 'potential'.

0. Introduction

Classical Markov chains are defined through an intrinsic statistical property (the Markov property) which allows the explicit form of their finite dimensional joint expectations (correlation functions) to be determined. This explicit structure of the correlation functions can in turn be generalized and gives rise to a strictly larger class of stochastic processes (generalized classical Markov chains). Moreover, it is well known that any classical Markov chain defines, uniquely up to boundary terms, a *potential function* which determines completely the conditional probability matrices of the chain [8].

Quantum Markov chains were introduced in [1] through an intrinsic definition which allowed the explicit structure of their correlation functions to be determined, and an extension of the resulting construction led to the introduction of the strictly larger class of 'generalized quantum Markov chains'. It was shown that one could use 'potentials' to construct explicit examples of quantum (and generalized quantum) Markov chains, e.g. the Ising (resp. Heisenberg) potential gives rise to quantum (resp. generalized quantum) Markov chains [2].

It would be desirable to show that in analogy with the classical case, all the quantum, and even the generalized quantum, Markov chains arise from a potential. In the present paper we establish this result for quantum Markov chains and show that the examples constructed with the Ising potentials are 'essentially' (see Theorems (3.2) and (4.2) for a precise statement) all the possible quantum Markov chains. We also give several intrinsic characterizations of the quantum Markov chains in terms of the associated modular automorphism group. In particular we show that there is a constructive one-to-one correspondence between the modular automorphism groups of quantum Markov chains and 'markovian cocycles' (cf section 5) and that for a quantum Markov state ϕ, just as the analytical continuation of the 1-parameter modular automorphism groups relative to the restrictions of ϕ on the $A_{[0,n]}$s determine the density matrices of the $\phi_{[0,n]}$s, the analytic continuation of the markovian cocycle determines the corresponding *conditional density matrices*.

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In particular our results show that, unlike the classical case, the quantum Markov chains are not sufficient to account for all the nearest neighbours potentials of interest for quantum statistical mechanics and that to this goal one must resort to the strictly larger class of generalized quantum Markov chains (at least!) However, the validity, for this latter class of states, of structural results analogous to the present ones, is at the moment still an open problem.

1. Notation

In the following N will denote the set of the natural integers and M will denote the algebra of $q \times q$ complex matrices (for a fixed arbitrary integer q); $A \equiv \otimes_N M$ the infinite C^*-tensor product of N-copies of M; $J_n : M \to A$ the natural embedding of M onto the nth factor of the product $\otimes_N M$; $J^*_n : J_n(M) \subseteq A \to M$ the left inverse of $J_n (n \in N)$. For any sub-set $I \subseteq N$ we denote A_I the C^*-subalgebra of A generated by $\{J_n(M) : n \in I\}$; we will write simply A_n for $A_{\{n\}}$. A *localization* on A is a family $\{A_I : I \in \mathscr{F}\}$ of subalgebras of A, where $\mathscr{F}$ is an increasing net of sub-sets of N whose union coincides with N. In the following we will only be concerned with the localization $(A_{[0,n]})_{n \in N}$.

A quasi-conditional expectation with respect to the triple $A_{[0,n-1]} \subseteq A_{[0,n+1]}$ is a completely positive identity preserving map $E_{n+1,n} : A_{[0,n+1]} \to A_{[0,n]}$, such that

$$E_{n+1,n}(ba) = bE_{n+1,n}(a) \ ; \ \forall b \in A_{[0,n-1]}; \ \forall a \in A_{[0,n+1]}. \tag{1.1}$$

Equivalently $E_{n+1,n}$ can be characterized as a completely positive identity preserving map $A_{[0,n+1]} \to A_{[0,n]}$ whose fixed point algebra contains $A_{[0,n-1]}$ (cf.[3], § 4).

Condition (1.1) implies that

$$E_{n+1,n} \ (A_{[n,n+1]}) \subseteq A_n \tag{1.2}$$

and we shall refer to (1.3) as 'the quantum Markov property', the terminology being justified by the fact that if the A_I's are abelian algebras and $E_{n+1,n}$ is a conditional expectation in the usual sense, then (1.2) is an equivalent formulation of the classical Markov property.

For any $I \subseteq N$ we denote Aut(A_I) (resp. Int (A_I)) the group of W^*-automorphisms (resp. inner automorphisms) of A_I and, if $\mathscr{U}$ is an unitary in A_I we write Ad($\mathscr{U}$) $\in$ int(A_I) for the map

$$\mathrm{Ad}(\mathscr{U})(a) = \mathscr{U}^* \, a \, \mathscr{U} \ ; \ a \in A_I. \tag{1.3}$$

Finally if (α_t) is a 1—parameter group of automorphisms of A_I, by an α_t—KMS state on A_I we mean a KMS state at $\beta = -1$.

2. Markov states on $\otimes_N M$

Definition (2.1). A state ϕ on A is called a Markov state with respect to the localization $\{A_{[0,n]}\}$ if for each $n \in N$ there exists a quasi-conditional expectation with respect to the triple $A_{[0,n-1]} \subseteq A_{[0,n]} \subseteq A_{[0,n+1]}$ such that

$$\phi(a) = \phi_{[0,n]} \ (E_{n+1,n}(a)) \ : \ \forall \ a \in A_{[0,n+1]}. \tag{2.1}$$

In this case we shall say that the quasi-conditional expectation $E_{n+1,n}$ is *compatible* with the state ϕ.

In the following we shall simply say that ϕ is a Markov state on A without explicitly mentioning the localization $\{A_{[0,n]}\}$.

Let ϕ be a Markov state on A and let $(E_{n+1,n})$ be the associated family of quasi-conditional expectations.

Define, for each $n \in N$,

$$\mathscr{E}_n = J^*_n \cdot E_{n+1,n} \cdot (J_n \otimes J_{n+1}).$$

Then, because of the Markov property, $\mathscr{E}_n$ maps $M \otimes M$ into M and is completely positive and normalized, i.e.

$$\mathscr{E}_n (1 \otimes 1) = 1. \tag{2.2}$$

Moreover, because of (2.1)

$$\begin{aligned}\phi(a_{[0,n-1]}J_n(a_n)J_{n+1}(a_{n+1})) &= \phi(a_{[0,n-1]}E_{n+1,n}(J_n(a_n)J_{n+1}(a_{n+1})) \\ &= \phi(a_{[0,n]}E_{n+1,n}(J_n(a_n)E_{n+2,n+1} \ (J_{n+1}(a_{n+1})))\end{aligned}$$

for each $a_{[0,n-1]} \in A_{[0,n-1]}$, a_n, $a_{n+1} \in M$; that is

$$\phi_{[0,n]}(a_{[0,n-1]}J_n \cdot \mathscr{E}_n(a_n \otimes a_{n+1})) = \phi_{[0,n]}(a_{[0,n-1]}J_n \cdot \mathscr{E}_n(a_n \otimes \mathscr{E}_{n+1}(a_{n+1} \otimes 1))). \tag{2.3}$$

Denoting

$$S_{n+1}(a_{n+1}) = \mathscr{E}_{n+1}(a_{n+1} \otimes 1) \tag{2.4}$$

we will express the fact that (2.3) holds for each $a_{[0,n-1]} \in A_{[0,n-1]}$ by writing simply

$$\mathscr{E}_n(a \otimes b) = \mathscr{E}_n(a \otimes S_{n+1}(b)), \ \mathrm{mod} \ \{\phi_0, (\mathscr{E}_k)\} \tag{2.5}$$

for each $a, b \in M$.

Denote ϕ_0 the restriction of ϕ on A_0 (in the following, when no confustion is possible, this state will be identified with the state $\phi_0 \cdot j_0$ on M).

The state ϕ is completely determined by the pair $\{\phi_0; (\mathscr{E}_n)\}$ through the relations:

$$\begin{aligned}\phi(J_0(a_0) \ldots J_n(a_n)) &= \phi_0(\mathscr{E}_0(a_0 \otimes \mathscr{E}_1(a_1 \otimes \ldots \otimes \mathscr{E}_n(a_{n-1} \otimes a_n) \ldots))) \\ &= \phi_0(\mathscr{E}_0(a_0 \otimes \mathscr{E}_1(a_1 \otimes \ldots \otimes \mathscr{E}_n(a_n \otimes 1) \ \ldots)\end{aligned} \tag{2.6}$$

$\forall \, a_0, \ldots, a_n \in M$, $\forall \, n \in N$.

Conversely, let $\{\phi_0 \cdot (\mathscr{E}_n)\}$ be a pair satisfying (2.2) and (2.5), with $\mathscr{E}_n : M \otimes M \to M$ completely positive and ϕ_0 — a state on M.

Then, for each $n \in N$, the right hand side of (2.6) defines a state — $\phi_{[0,n]}$ — on $A_{[0,n]}$.

The family of states $\phi_{[0,n]}$ is projective, in the sense that

$$\phi_{[0,n+1]} \restriction A_{[0,n]} = \phi_{[0,n]}. \tag{2.7}$$

Therefore there exists a unique state ϕ on A whose restriction on $A_{[0,n]}$ is $\phi_{[0,n]}(n \in N)$. Let us prove that ϕ is a Markov chain.

The map $E_{n+1,n} : A_{[0,n+1]} \to A_{[0,n]}$ defined (for each n) by extension of

$$J_0(a_0) \ldots J_{n+1}(a_{n+1}) \to J_0(a_0) \ldots J_{n-1}(a_{n-1})\, J_n\, \mathscr{E}_n(a_n \otimes a_{n+1}) \qquad (2.8)$$

is clearly a quasi-conditional expectation with respect to the triple $A_{[0,n-1]} \subseteq A_{[0,n]} \subseteq A_{[0,n+1]}$.

The state ϕ defined above satisfies the equalities $(a_0, ..., a_n \in M)$

$$\begin{aligned}
&\phi(E_{n+1,n}(J_0(a_0) \ldots J_{n+1}(a_{n+1})))\\
&= \phi_0(\mathscr{E}_0(a_0 \otimes \ldots \otimes \mathscr{E}_{n-1}(a_{n-1} \otimes \mathscr{E}_n(\mathscr{E}_n(a_n \otimes a_{n+1}) \otimes 1))) \ldots\ldots)\\
&= \phi_0(\mathscr{E}_0(a_0 \otimes \ldots \otimes \mathscr{E}_{n-1}(a_{n-1} \otimes S_n\, \mathscr{E}_n(a_n \otimes a_{n+1}))) \ldots)\\
&= \phi_0(\mathscr{E}_0(a_0 \otimes \ldots \otimes \mathscr{E}_{n-1}(a_{n-1} \otimes \mathscr{E}_n(a_n \otimes a_{n+1})) \ldots))\\
&= \phi_0(\mathscr{E}_0(a_0 \otimes \ldots \otimes \mathscr{E}_{n-1}(a^{n-1} \otimes \mathscr{E}_n(a_n \otimes \mathscr{E}_{n+1}(a_{n+1} \otimes 1)) \qquad (2.9)
\end{aligned}$$

where, in the last two equalities, we used (2.5) $(n \in N)$.

Because of the definition of ϕ (2.9) implies that

$$\phi(E_{n+1,n}(a)) = \phi(a);\ \forall a \in A_{[0,n+1]}. \qquad (2.10)$$

Hence ϕ is a Markov state on A.

Definition (2.2). Let ϕ_0 be a state on M. A family of linear maps $\mathscr{E}_n : M \otimes M \to M$ such that for each n

$$\mathscr{E}_n \text{ is completely positive} \qquad (2.11)$$

$$\mathscr{E}_n(1 \otimes 1) = 1 \qquad (2.12)$$

$$\mathscr{E}_n(a \otimes b) = \mathscr{E}_n(a \otimes \mathscr{E}_{n+1}(b \otimes 1)),\ a,b\, M \text{ mod. } \{\phi_0, (\mathscr{E}_k)\} \qquad (2.13)$$

will be called a family of transition expectation with initial distribution ϕ_0.

Theorem (2.3). *Every Markov state ϕ on A is determined by a pair $\{\phi_0, (\mathscr{E}_n)\}$ such that ϕ_0 is a state on M and $(\mathscr{E}_n)$ is a family of transition expectations with initial distribution ϕ_0. Conversely every such family defines a unique Markov state on A.*

PROOF. From the considerations above.

If the transition expectation $\mathscr{E}_n$ is defined by

$$\mathscr{E}_n(x) = \bar{\tau}_2(K_n^* x K);\ x \in M \otimes M \qquad (2.14)$$

where $K_n \in M \otimes M$ and $\bar{\tau}_2(a \otimes b) = a\tau(b)$ $(a,b\, M)$, then we shall say that K_n is a *conditional density* for $\mathscr{E}_n$ (cf.[1] for the concept of conditional density).

Remark. From the proof of Theorem (2.3) it is clear that any family $\mathscr{E}_n : M \otimes M \to M$ of completely positive maps such that $\mathscr{E}_n(1 \otimes 1) = 1$ defines, through the second equality in (2.6), a state ϕ on A. In this case, however, property (2.10) might fail so that ϕ might not be a Markov state on A.

Since the structure of joint expectations for such states is very similar to that of Markov states, we referred to these states in [1] as (generalized) Markov *chains*.

The following theorem shows that, up to an 'operator-renormalization' every generalized Markov chain is a Markov state.

Theorem (2.4). *If $\mathscr{E}_n : M \otimes M \to M$ $(n \in N)$ is a family of completely positive linear maps such that $\mathscr{E}_n(1 \otimes 1) = 1$, then there exists a state ϕ_0 on M and a family $T_n : M \to M$ $(n \in N)$ of completely positive linear maps such that*

(i) *for each n, T_n belongs to the closed convex hull of the set $\{S_n^k : k \in N\}$ where*

$$S_n(a) = \mathscr{E}_n(a \otimes 1); \qquad (2.15)$$

(ii) *the family $(\mathscr{L}_n)$ defined by*

$$\mathscr{L}_n(a \otimes b) = \mathscr{E}_n(a \otimes T_n b) \qquad (2.16)$$

is a family of transition expectations with initial distribution ϕ_0.

PROOF. Set $\mathscr{T}_0 = \{\phi_0 \in \mathscr{S}(M_0) : \phi_0 \cdot S_0 = \phi_0\}$ is not empty since $\mathscr{S}(M)$ is convex compact and $\phi \in \mathscr{S}(M) \to \phi \cdot S_0$ is continuous. Define

$$\mathscr{S}_0 = \{\phi \in \mathscr{S}(A) : \phi \upharpoonright M_0 \in \mathscr{T}_0\}.$$

Define $\mathscr{S}_n$ by induction as follows: let $\mathscr{R}_n$ be the set of states $\phi_{[0,n]} \in \mathscr{S}(M_{[0,n]})$ such that

$$\phi_{[0,n]} \upharpoonright A_{[0,n-1]} \in \mathscr{T}_{n-1}$$

$$\phi_{[0,n]}(a_{[0,n-2]}\, J_{n-1}(a_{n-1})\, J_n(a_n)) = \phi_{[0,n]}(a_{[0,n-2]} J_{n-1}\, \mathscr{E}_{n-1}(a_{n-1} \otimes T_n a_n))$$

where $T_n : M \to M$ is any map lying in the closed convex hull the set $\{S_n^k : k \in N\}$. $\mathscr{R}_n$ is compact convex and the map

$$\phi_{[0,n]} \in \mathscr{R}_n \to \phi_{[0,n]} \cdot (id_{[0,n-1]} \otimes S_n) \qquad (2.17)$$

transforms continuously $\mathscr{R}_n$ into itself. Let

$$\mathscr{T}_n = \{\phi_{[0,n]} \in \mathscr{R}_n : \phi_{[0,n]} \text{ is fixed under the map (2.17)}\},$$

$$\mathscr{S}_n = \{\phi \in \mathscr{S}(A) : \phi \upharpoonright A_{[0,n]} \in \mathscr{T}_n\}.$$

Clearly $\mathscr{S}(A) \supseteq \mathscr{S}_n \supseteq \mathscr{S}_{n+1} \neq \phi$ and the $\mathscr{S}_n$ are closed. The compactness of $\mathscr{S}(A)$ thus implies that $\bigcap_n \mathscr{S}_n \neq \phi$. Let $\phi \in \bigcap_n \mathscr{S}_n$. Then by construction

$$\phi(a_{[0,n-1]} J_n(a_n) J_{n+1}(a_{n+1})) = \phi(a_{[0,n-1]} J_n \cdot \mathscr{E}_n(a_n \otimes T_{n+1} a_{n+1}))$$

for some completely positive $T_n : M \to M$ such that $T_n(1) = 1$.

Denoting

$$\mathscr{L}_n(a \otimes b) = \mathscr{E}_n(a \otimes T_{n+1} b) \tag{2.18}$$

clearly L_n is positive and $L_n(1 \otimes 1) = 1$; moreover

$$\begin{aligned}\phi_{[0,n]}(a_{[0,n-1]}J_n \mathscr{L}_n(a_n \otimes \mathscr{L}_{n+1}(a_{n+1} \otimes 1)) &= \phi_{[0,n]}(a_{[0,n-1]}J_n \mathscr{E}_n(a_n \otimes T_{n+1}S_{n+1}a_{n+1}))\\ &= \phi_{[0,n]}(a_{[0,n-1]}J_n \mathscr{E}_n(a_n \otimes T_{n+1}a_{n+1}))\\ &= \phi_{[0,n]}(a_{[0,n-1]}J_n \mathscr{E}_n(a_n \otimes a_{n+1})).\end{aligned}$$

Therefore, by (2.3) and (2.5) this implies also that (2.13) is satisfied, hence $(\mathscr{L}_n)$ defined by (2.18) is a family of transition expectations with initial distribution $\phi_0 = \phi \upharpoonright A_0$.

Remark 1. Also in the classical case one can generalize the explicit formula for the joint expectations of a Markov chain obtaining a strictly larger class of probability measures with a structure similar to the usual Markov chains but, in general, with different statistical properties (e.g. the Markov property in the usual sense might fail for these measures). There is a close connection between generalized classical Markov chains and generalized quantum Markov chains. This connection will be discussed elsewhere.

3. Ising potentials

Definition (3.1). An *Ising potential* in A is a family $(h_{[0,n]})_{n \in N}$ of self-adjoint operators in A such that, for each $n \in N$,

$$h_{[0,n]} \in A_{[0,n]} \tag{3.1}$$

$$\exp\{-ith_{[0,n]}\} \cdot \exp\{ith_{[0,n+1]}\} \in A_{[n,n+1]};\ \forall t \in R \tag{3.2}$$

If, moreover, for each $n \in N$

$$Tr(\exp\{-h_{[0,n]}\}) = 1 \tag{3.3}$$

the Ising potential $(h_{[0,n]})$ is called *normalized*. Two Ising potentials $(h_{[0,n]})$, $(K_{[0,n]})$ are called *equivalent* if, for each $n \in N$,

$$\exp\{-itK_{[0,n]}\} \cdot \exp\{ith_{[0,n]}\} \in A_n;\ \forall t \in R. \tag{3.4}$$

Theorem (3.2). *The following statements are equivalent:*

(i) $(h_{[0,n]})_n$ *is an Ising potential in* A*;*

(ii) *there exists a sequence* $(\bar{H}_{n,n+1})$ *of self-adjoint operators in* A *such that, for each* $n \in N$*,*

$$h_{[0,n]} = \sum_{K=1}^{n} \bar{H}_{K-1,K} \tag{3.5}$$

$$\bar{H}_{n-1,n} \in A_{[n-1,n]} \tag{3.6}$$

$$[\bar{H}_{n-1,n}, \cdot\,]^K (\bar{H}_{n,n+1}) \in A_{[n,n+1]};\ K = 0,1,2,\ldots \tag{3.7}$$

where, here and in the following, we use the notation:

$[A, \cdot\,]^K(B) = [A, [A, [\ldots, [A,B] \ldots]]]$ (*K*-times); $[A, \cdot\,]^0 = id$

(iii) *there exist two sequences* (H_n) *and* $(H_{n-1,n})$ *of self-adjoint operators in* A *such that for every* $n \in N$

$$h_{[0,m]} = \sum_{K=1}^{n} H_K + \sum_{K=1}^{n} H_{K-1,K}$$

$$H_n \in A_n;\ H_{n-1,n} \in A_{[n-1,n]} \tag{3.8}$$

$$[H_{n-1,n}, [H_n, \cdot\,]^K(H_{n,n+1})\,] = 0;\ K = 0,1,2,\ldots. \tag{3.9}$$

PROOF (i) ⇒ (ii). Denoting $\forall\, n \in N$, $\forall\, t \in R$:

$$\mathscr{U}^n_t = \exp\{ith_{[0,n]}\} \cdot \exp\{-ith_{[0,n+1]}\} \tag{3.10}$$

the following norm convergent expansion holds:

$$\mathscr{U}^n_t = 1 + \sum_{K=1}^{\infty} \int \ldots \int \Delta_n(t_K) \cdot \Delta_n(t_{K-1}) \cdot \ldots \cdot \Delta_n(t_1) dt_1 \cdot \ldots \cdot dt_K\ 0 \leqslant t_1 \leqslant \ldots \leqslant t_K \leqslant t \tag{3.11}$$

where

$$\Delta_n(s) = -i\ e^{ish_{[0,n]}}(h_{[0,n+1]} - h_{[0,n]})\ e^{-ish_{[0,n]}} \tag{3.12}$$

$$= -i(h_{[0,n+1]} - h_{[0,n]}) - i \sum_{K=1}^{\infty} \frac{(is)^K}{K!} [h_{[0,n]}, \cdot\,]^K(h_{[0,n+1]}) \cdot$$

Now, if $(h_{[0,n]})$ is an Ising potential, then $\mathscr{U}^n_t \in A_{[n,n+1]}$ for each n and t, hence

$$\frac{d}{dt}\mathscr{U}^n_t = \Delta_n(t)\mathscr{U}^n_t \in A_{[n,n+1]}. \tag{3.13}$$

Thus $\Delta_n(t) \in A_{[n,n+1]}$ for each n and t and, in particular,

$$-\frac{1}{i}\Delta_n(0) = h_{[0,n+1]} - h_{[0,n]} \in A_{[n,n+1]}. \tag{3.14}$$

Denoting $\bar{H}_{n,n+1} = h_{[0,n+1]} - h_{[0,n]}$, (3.6) holds and, for each $K = 1,2,\ldots$,

$$\left.\frac{d^K}{dt^K}\right|_{t=0} \Delta_n(t) = -(i)^{K+1}[h_{[0,n]}, \cdot\,]^K(h_{[0,n+1]})$$

$$= -(i)^{K+1}[\bar{H}_{n-1,n}, \cdot\,]^K(\bar{H}_{n,n+1}) \in A_{[n,n+1]} \tag{3.15}$$

and this proves (3.7).

(ii) ⇔ (iii). Let the sequence $(\bar{H}_{n,n+1})$ be as in (ii) and define, for $n \in N$

$$H_n = \bar{\tau}_{n-1}(\bar{H}_{n-1,n}) \ ; \ H_{n-1,n} = \bar{H}_{n-1,n} - H_n \tag{3.16}$$

where $\bar{\tau}_{n-1}$ denotes the partial trace over A_{n-1}. Since $\bar{\tau}_{n-1}$ is a conditional expectation onto A'_{n-1}, conditions (3.7) and (3.16) imply that

$$[\bar{H}_{n-1,n}, \cdot\ ]^{K+1}(\bar{H}_{n,n+1}) = [H_n, [\bar{H}_{n-1,n}, \cdot\ ]^K(\bar{H}_{n,n+1})]$$

or, equivalently,

$$[H_{n-1,n}, [\bar{H}_{n-1,n}, \cdot\ ]^K(\bar{H}_{n,n+1})] = 0$$

and, since K is arbitrary, this is equivalent to (3.9).

(ii) ⇔ (i). Let the sequence $(\bar{H}_{n,n+1})$ be as in (ii), and define $h_{[0,n]}$ by (3.5) ($n \in N$). The equalities (3.15), (3.12), (3.11), (3.10) show then that condition (3.7) implies that $(h_{[0,n]})$ is an Ising potential.

Remark. If M is the algebra of 2×2 complex matrices, we can show that the only translation invariant Ising potential is the potential of the usual 1-dimensional quantum Ising model. We do not know if this result remains true in dimensions $\geqslant 3$.

4. Ising potentials associated with Markov states

In the classical case one can give an intrinsic definition of Markov states, based on the properties of the conditional expectations canonically associated with the state and the given localization. In Definition (1.1) of quantum Markov states, however, following the previous approach in [1] we referred only to an (arbitrary) sequence of quasi-conditional expectations. Since we now know that one can associate also in the quantum case, in a canonical way, a (generalized) conditional expectation to a normal state on a von Neumann algebra and a subalgebra [3], it is natural to ask whether Definition (1.1) defines the same class of quantum Markov states as the definition obtained from it restricting it to the canonical quasi-conditional expectations. In the following we give several characterizations of the quantum Markov states and show, in particular, that the answer to the problem above is affirmative.

Let us first prove the following lemma which is an easy consequence of a result due to Sirugue and Winnik [7].

Lemma (4.1). *Let $\mathscr{C} \subseteq \mathscr{B} \subseteq \mathscr{A}$ be von Neumann algebras; ϕ a faithful normal state on $\mathscr{A}$; α_t, σ^0_t two 1-parameter automorphism groups respectively of $\mathscr{A}$ and $\mathscr{B}$. Assume that ϕ is an α_t–KMS state and that $\phi \restriction \mathscr{B}$ is a σ^0_t–KMS state. Then*

$$\alpha_t(\mathscr{C}) \subseteq \mathscr{B} \quad ; \quad \forall t \in R \tag{4.1}$$

if and only if

$$\alpha_t \restriction \mathscr{C} = \sigma^0_t \restriction \mathscr{C} \ ; \quad \forall t \in R. \tag{4.2}$$

PROOF. Let $c \in \mathscr{C}$, $b \in \mathscr{B}$ and consider the integral

$$\int_R \int_R f(s)g(t)\phi(\alpha_s(c)\sigma^0_t(b))\, ds dt \tag{4.3}$$

where f, g are functions whose Fourier transforms are infinitely differentiable with compact support. The KMS condition for σ^0_t implies that (4.3) is equal to

$$\int\int f(s)g(t-i)\phi(\sigma^0_t(b)\alpha_s(c))dsdt \tag{4.4}$$

and (4.4), because of the KMS conditions for α_t, is equal to

$$\int\int f(s-i)g(t-i)\phi(\alpha_s(C)\sigma^0_t(b))dsdt. \tag{4.5}$$

The lemma of Sirugue and Winnik mentioned above asserts that the equality of (4.3) and (4.5) implies that the functions $s, t \to \phi(\alpha_s(c)\cdot\sigma^0_t(b))$ ($c \in \mathscr{C}$ and $b \in \mathscr{B}$) depend only on $t - s$. In particular

$$\phi(\alpha_t(c)\sigma^0_t(b)) = \phi(c\ .\ b)$$

and, since ϕ is faithful, this implies that

$$\alpha_t(c) = \sigma^0_t(c), \quad \forall c \in \mathscr{C},$$

and this proves the statement.

Theorem (4.2). *For a normal faithful state ϕ on A, the following conditions are equivalent:*

(i) *for each $n \in N$ the ϕ-conditional expectation (cf.[3])*

$$E_{n+1,n} : A_{[0,n+1]} \to A_{[0,n]}$$

leaves $A_{[0,n-1]}$ pointwise invariant;

(ii) *ϕ is a Markov state in the sense of Definition* (1.1);

(iii) *for each $n \in N$*

$$\sigma^{n+1}_t \restriction A_{[0,n-1]} = \sigma^n_t \restriction A_{[0,n-1]} \ ; \ \forall t \in R; \tag{4.6}$$

(iv) *the family* $(h_{[0,n]})$ *is an Ising potential;*

(v) *for each* $n \in N$

$$\exp\left\{-\frac{1}{2}h_{[0,n]}\right\}\cdot\exp\left\{\frac{1}{2}h_{[0,n+1]}\right\}\in A_{[n,n+1]}. \tag{4.7}$$

Proof (i) ⇒ (ii). This is obvious, since the ϕ-conditional expectation $E_{n+1,n}$ is a completely positive identity preserving map.

(ii) ⇒ (iii). For each $n \in N$, let there be a completely positive identity preserving map $F_{n+1,n}: A_{[0,n+1]} \to A_{[0,n]}$, whose fixed point algebra contains $A_{[0,n-1]}$, and such that, for each $n \in N$,

$$\phi = \phi \cdot F_{n+1,n}.$$

Since $F_{n+1,n}$ leaves the faithful normal state $\phi_{[0,n+1]}$ globally invariant, by the L^2-ergodic theorem for completely positive maps (cf. [5]), there is a faithful normal ϕ-invariant conditional expectation from $A_{[0,n+1]}$ onto the fixed point algebra of $F_{n+1,n}$ which therefore, by Takesaki's theorem, is σ_t^{n+1} invariant. Moreover, since the fixed point algebra of $F_{n+1,n}$ is contained in $A_{[0,n]}$ and contains $A_{[0,n-1]}$. Lemma (4.1) implies that equality (4.6) holds.

(iii) ⇒ (iv). Using the explicit form of σ^n_t, condition (iii) becomes equivalent to

$$a_{[0,n-1]} = e^{ith_{[0,n]}}e^{-ith_{[0,n+1]}}a_{[0,n-1]}e^{ith_{[0,n+1]}}\ e^{-ith_{[0,n]}}$$

$(a_{[0,n-1]} \in A_{[0,n-1]}\,;\, t \in R)$,
i.e. to the condition
$\exp\{ith_{[0,n]}\}\cdot\exp\{-ith_{[0,n+1]}\} \in A'_{[0,n-1]} \cap A_{[0,n+1]} = A_{[n-1,n]}$

which defines an Ising potential.

(iv) ⇒ (v). This is proved by analytic continuation.

(v) ⇒ (i). In fact, under our assumptions, the ϕ-conditional expectation $E_{n+1,n}: A_{[0,n+1]} \to A_{[0,n]}$ has the form $E_{[n+1,n]}(a) = E^0_{[n+1,n]}(K_n^* a K_n)$; $a \in A_{[0,n+1]}$ where $E^0_{[n+1,n]}: A_{[0,n+1]} \to A_{[0,n]}$ is the conditional expectation with respect to the trace on $A_{[0,n+1]}$ and

$$K_n = \exp\left\{-\frac{1}{2}h_{[0,n+1]}\right\}\cdot\exp\left\{\frac{1}{2}\mathrm{h}_{[0,n]}\right\}$$

(cf. [3], formula (3.28)).

Theorem (4.2) shows that for any Markov state $\phi = (\phi_{[0,n]})$ on A, denoting $\exp\{-h_{[0,n]}\}$ the density matrix of $\phi_{[0,n]}$ $(n \in N)$, the family $(h_{[0,n]})$ is a normalized Ising potential — called the Ising potential associated with the Markov state ϕ — and ϕ is a KMS state for the 1-parameter group

$$\sigma_t = \lim_n Ad(e^{ith_{[0,n]}})$$

the limit being pointwise in norm on A.

Conversely:

Theorem (4.3). *Given a (not necessarily normalized) Ising potential* $(h_{[0,n]})$, *the set of Markov states whose associated Ising potential is equivalent (in the sense of Definition (3.1)) to* $(h_{[0,n]})$ *is not empty and coincides with the set of* α_t*-KMS states on* A, *where*

$$\alpha_t = \lim_n Ad(e^{ith_{[0,n]}}). \tag{4.8}$$

Moreover, this set depends only on the equivalence class of $(h_{[0,n]})$.

Proof. Denoting $\alpha^n_t = Ad(e^{ith_{[0,n]}})$, $(n \in N)$, one has, for any $a_{[0,n-1]} \in A_{[0,n-1]}$, $\alpha_t(a_{[0,n-1]}) = \alpha^n_t(a_{[0,n-1]})$.

Hence the limit (4.3) exists pointwise in norm on A and depends only on the equivalence class of $(h_{[0,n]})$.

By a result of Powers and Sakai [6], the set of α_t–KMS states on A, at any temperature, is not empty. Let $\phi = (\phi_{[0,n]})$ be an α_t–KMS state and denote σ^n_t the modular automorphism group of $\phi_{[0,n]}$ on $A_{[0,n]}$. Lemma (4.1) then implies that, for any $n \in N$ and $a_{n-1} \in A_{[0,n-1]}$ one has

$$\sigma^n_t\,(a_{n-1}) = \alpha^n_t\,(a_{n-1})$$

and since $(h_{[0,n]})$ is an Ising potential, $\sigma^{n+1}_t(a_{n-1}) = \alpha^{n+1}_t(a_{n-1}) = \alpha^n_t(a_{n-1}) = \sigma^n_t(a_{n-1})$ i.e. the sequence (σ^n_t) satisfies condition (iii) of Theorem (4.2). Therefore ϕ is a Markov state whose associated Ising potential is equivalent, because of (3), to $(h_{[0,n]})$, and this ends the proof.

Remark. In general an Ising potential $(h_{[0,n]})$ need not determine the associated Markov state uniquely. However (cf. [4]) uniqueness holds if

$$\sup_n \| h_{[0,n+1]} - h_{[0,n]} \| < +\infty.$$

In particular uniqueness holds for any translation invariant Ising potential.

5. Markovian cocycles

Let $(h_{[0,n]})$ be an Ising potential in the sense of section (3). Define, for $n \in N$,

$$(\mathcal{U}^n_t)^* = e^{ith_{[0,n]}}\,e^{-ith_{[0,n+1]}}\,;\ t \in R \tag{5.1}$$

$$\alpha^0_t = Ad(e^{-ith_0}) = e^{-ith_0}(\,\cdot\,)\,e^{ith_0}. \tag{5.2}$$

The pair $\{\alpha^0_t, (\mathscr{U}^n_t)\}$ enjoys the following properties:

(i) $\mathscr{U}^n_t$ is unitary and $\alpha^0_t \in \mathrm{Int}(A_0)$; in particular, we can consider the action of α^0_t on the whole of A;

(ii) $\mathscr{U}^n_t \in A_{[n-1,n]}$;

(iii) defining inductively, for $n \geqslant 1$,

$$\alpha^n_t(a_{[0,n]}) = \alpha^{n-1}_t (\mathscr{U}^{n*}_t \, a_{[0,n]} \, \mathscr{U}^n_t) \tag{5.3}$$

$(a_{[0,n]} \in A_{[0,n]})$ one has

$$\mathscr{U}^n_{s+t} = \mathscr{U}^n_t \alpha^{n-1}_{-t} (\mathscr{U}^n_s) \; ; \; \forall n \; ; \; \forall t. \tag{5.4}$$

A pair $\{\alpha^0_t, (\mathscr{U}^n_t)\}$ enjoying properties (i), (ii), (iii) above will be called a *markovian cocycle on A*.

Clearly for any markovian cocycle $\{\alpha^0_t, (\mathscr{U}^n_t)\}$ on A and for each $n \in N$, α^n_t defined by (5.3) is a 1-parameter group of automorphisms of $A_{[0,n]}$ and

$$\alpha^{n+1}_t \restriction A_{[0,n-1]} = \alpha^n_t \restriction A_{[0,n-1]}. \tag{5.5}$$

In particular the limit

$$\alpha_t = \lim_n \alpha^n_t \tag{5.6}$$

exists pointwise in norm on A and defines a 1-parameter group of automorphisms of A.

The equalities (5.1) and (5.2) show that, given an Ising potential, one can construct a markovian cocycle. Conversely, let $\{\alpha^0_t, (\mathscr{U}^n_t)\}$ be a markovian cocycle on A. Then, for each $n \in N$,

$$\alpha^n_t = Ad(e^{ith_{[0,n]}})$$

for some self-adjoint $h_{[0,n]} \in A_{[0,n]}$. In these notations (5.3) is equivalent to

$$e^{-ith_{[0,n-1]}} e^{ith_{[0,n]}} = \mathscr{U}^{n*}_t \cdot \lambda_n(t) \tag{5.7}$$

for some $\lambda_n(t) \in C$, $|\lambda_n(t)| = 1$. Hence, by condition (ii), $(h_{[0,n]})$ is an Ising potential. Summing up:

Proposition (5.1). Let $\{\alpha^0_t, (\mathscr{U}^n_t)\}$ be a markovian cocycle and let α_t be defined by (5.3) and (5.6). Then any α_t–KMS state is markovian. Conversely any Markov state defines a markovian cocycle and is KMS for the associated 1-parameter group.

PROOF. This follows from the discussion above.

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