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LIE ALGEBRAS ASSOCIATED WITH THE RENORMALIZED HIGHER POWERS OF WHITE NOISE

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Abstract. We recall the recently established (cf. [1] and [2]) connection between the renormalized higher powers of white noise (RHPWN) $*$-Lie algebra and the Virasoro –Zamolodchikov– w_∞ $*$-Lie algebra of conformal field theory (cf. [10]). Motivated by this connection, with the goal of investigating a possible connection with classical independent increments processes, we begin a systematic study of the sub-$*$–Lie algebras of the (1–mode) full oscillator algebra. This program has two additional motivations:
(i) the full oscillator algebra is a fundamental object of mathematics and the structure of its subalgebras deserves deep investigation;
(ii) the no–go theorems show that the current algebras over some Lie sub–algebras of Lie algebras may have a Fock representation individually without this being true for the Lie algebra generated by them. The problem of classifying which sub–algebras of the full oscillator algebra have this property is open and a preliminary step towards its analysis is the classification of the "natural" sub–algebras of the full oscillator algebra.

We construct two hierarchies of such sub–algebras, parametrized by the natural integers. One of these hierarchies begins with the Virasoro algebra. Another possibility to bypass the no–go theorems is to consider different renormalizations of the higher powers of white noise commutation relations. This approach is developed in Section 2, where we show with examples that some of them lead to known (i.e., first or second order) commutation relations. This fact is probably related with the gaussianization phenomenon discussed in [7].

1. Introduction

1.1. The White Noise algebra. In the present paper a Lie algebra is defined in terms of generators and commutation relations. The standard boson white noise $*$–Lie algebra is defined by its generators, $a_t, a_s^\dagger, 1$ satisfying the (first order white noise) commutation relations

$$[a_t, a_s^\dagger] = \delta(t-s)\cdot 1, \quad [a_t, 1] = [a_t^\dagger, 1] = [a_t^\dagger, a_s^\dagger] = [a_t, a_s] = 0 \tag{1.1}$$

as well as the adjoint conditions $(a_t^\dagger)^\dagger = a_t$ and $1^\dagger = 1$, where $t, s \in \mathbb{R}$ (or $\mathbb{R}^d$), $t, s \geq 0$, 1 is the (unique) central element, and $\delta(t)$ is the Dirac delta function. Here and in what follows, when no confusion is possible, we identify a Lie algebra with some of its representations, and the associated brackets $[x, y] := xy - yx$ with the usual operator commutator, either meant weakly on a domain or strongly on an

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invariant domain. The "full oscillator "algebra is the $*$–Lie algebra FOA defined by the self–adjoint family of generators $\{a_s, a_t^\dagger, 1 : s,t \in \mathbb{R}\}$ with adjoint relations $(a_t^\dagger)^\dagger = a_t, 1^\dagger = 1$ and commutation relations

$$[a_s, a_t^\dagger] = \delta_{s,t}1,\ [a_t, 1] = [a_t^\dagger, 1] = [a_t^\dagger, a_s^\dagger] = [a_t, a_s] = 0 \tag{1.2}$$

where $\delta_{s,t}$ is the Kronecker delta. In particular,

Definition 1.1. The 1–mode full oscillator algebra FOA(1) is the universal enveloping $*$–Lie algebra of the 1–mode Heisenberg $*$–Lie algebra

$$[a, a^\dagger] = 1, [a, a] = [a^\dagger, a^\dagger] = [a, 1] = [a^\dagger, 1] = 0$$

with adjoint conditions $(a^\dagger)^\dagger = a$ and $1^\dagger = 1$.

A concrete realization of FOA(1) is obtained by considering the Schrödinger representation of the Heisenberg algebra. Then the $*$–Lie algebra of the powers of a, $a^\dagger$ (we identify a, $a^\dagger, 1$ with their images in the Schrödinger representation) with the usual commutator (which is well defined on the dense domain of the number vectors) is isomorphic to FOA(1). Commutation relations (1.1) can be considered as a continuous generalization of the commutation relations (1.2). In the past years a considerable effort has been devoted to the extension of this discrete–continuous transition to the full oscillator algebra (cf. [4] and [5]). This is equivalent to the possibility of giving a meaning to the higher powers of white noise, i.e., to the symbolic expressions $a_t^n, {a_s^\dagger}^k$, where $n,k \in \{0,1,2,....\}$, which is an old problem of quantum field theory. That involves giving meaning to the powers of the Dirac delta function. The assignment of such a meaning will be referred to as a "renormalization".

1.2. A first renormalization. . For $n,k \in \{0,1,2,...\}$ we introduce the notation $\epsilon_{n,k} := 1 - \delta_{n,k}$, where $\delta_{n,k}$ is Kronecker's delta, and we use "falling" factorial powers $x^{(y)}$ defined by $x^{(y)} = x(x-1)\cdots(x-y+1)$ with $x^{(0)} = 1$. As shown in [4], for all $t,s \in \mathbb{R}_+$ and $n,k,N,K \geq 0$

$$\begin{aligned} [{a_t^\dagger}^n a_t^k, {a_s^\dagger}^N a_s^K] &= \epsilon_{k,0}\epsilon_{N,0} \sum_{l\geq 1} \binom{k}{l} N^{(l)}\, {a_t^\dagger}^n\, {a_s^\dagger}^{N-l}\, a_t^{k-l}\, a_s^K\, \delta^l(t-s) \\ &- \epsilon_{K,0}\epsilon_{n,0} \sum_{L\geq 1} \binom{K}{L} n^{(L)}\, {a_s^\dagger}^N\, {a_t^\dagger}^{n-L}\, a_s^{K-L}\, a_t^k\, \delta^L(t-s). \end{aligned} \tag{1.3}$$

The simple renormalization rule

$$\delta^2(t-s) = c\,\cdot\,\delta(t-s) \tag{1.4}$$

and its obvious generalization to $n \geq 2$,

$$\delta^n(t-s) = c^{n-1}\,\cdot\,\delta(t-s) \tag{1.5}$$

where $c > 0$ is an arbitrary real number, was introduced in [6]. For a test function $f : \mathbb{R} \to \mathbb{C}$ and $n,k \geq 0$, we define the symbols

$$B_k^n(f) := \int_{\mathbb{R}} f(s)\, {a_s^\dagger}^n\, a_s^k\, ds$$

with involution $(B_k^n(f))^* = B_n^k(\bar{f})$ and $B_0^0(f) := \int_{\mathbb{R}} f(s)\,ds$. After multiplying both sides of (1.3) by $f(t)\,g(s)$ and formally integrating the resulting identity (i.e., taking $\int\int \ldots ds\,dt$), we obtain the Lie algebra (see [5] for a proof of the fact that this is indeed a Lie algebra) commutation relations

$$[B_K^N(g), B_k^n(f)] = \sum_{L\geq 1} \theta_L(N,K;n,k)\, c^{L-1}\, B_{K+k-L}^{N+n-L}(gf) \tag{1.6}$$

where, using the notation $\binom{y,z}{x} := \binom{y}{x} z^{(x)}$, we have defined

$$\theta_L(N,K;n,k) := H(L-1)\,\left(\epsilon_{K,0}\,\epsilon_{n,0}\binom{K,n}{L} - \epsilon_{k,0}\,\epsilon_{N,0}\binom{k,N}{L}\right)$$

where $H(x)$ is the Heaviside function (i.e., $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ otherwise). Notice that if L exceeds $(K \wedge n) \vee (k \wedge N)$ then $\theta_L(N,K;n,k) = 0$. The symbol $\binom{y,z}{x}$ satisfies the "Heisenberg–Weyl transformation formula" (see [8])

$$\sum_j \binom{b, n+a-j}{N-j}\binom{m,n}{j} = \sum_j \binom{m+b-j, n}{N-j}\binom{b,a}{j} \tag{1.7}$$

which can be used for a direct proof of the Jacobi identity for (1.6). Commutation relations (1.6) contain the commutation relations of the first order white noise operators (CCR) B_1^0, B_0^1 and B_1^1

$$[B_1^0(\bar{g}), B_0^1(f)] = \langle g, f\rangle,\ [B_1^0(g), B_1^1(f)] = B_1^0(gf),\ [B_1^1(g), B_0^1(f)] = B_0^1(gf)$$

as well as the commutation relations of the Renormalized Square of White Noise (RSWN) operators B_2^0, B_0^2 and B_1^1 (associated with the RSWN quantum stochastic calculus of [6])

$$[B_2^0(\bar{g}), B_0^2(f)] = 4\,B_1^1(\bar{g}f) + 2c\,\langle g, f\rangle,\ [B_2^0(g), B_1^1(f)] = 2\,B_2^0(gf),$$

$$[B_1^1(g), B_0^2(f)] = 2\,B_0^2(gf),$$

where $\langle g, f\rangle$ denotes the usual L_2 inner product of f and g. The relation between commutation relations (1.6) and the general notion of current representations over $\mathbb{R}$ based on a Lie algebra was first discussed in [5]. The picture emerging from [4] and [5] is that this renormalization works well for the second power but does not work equally well for the higher powers, in the sense that the corresponding Lie algebra admits no Fock representation. This motivated the investigation of different forms of renormalization.

1.3. Recent developments in the direction of conformal field theory. It was recently shown in [1] and [2] that, using the renormalization

$$\delta^l(t-s) = \delta(s)\,\delta(t-s),\ \ l = 2, 3, ... \tag{1.8}$$

instead of (1.5) and by choosing test functions that vanish at zero, commutation relations (1.6) are replaced by

$$[B_k^n(g), B_K^N(f)]_{RHPWN} := (k\,N - K\,n)\ B_{k+K-1}^{n+N-1}(gf).$$

With this new renormalization, whose motivation is discussed in detail in [2], for $n, N \geq 2$ and $k, K \in \mathbb{Z}$ the white noise operators

$$\hat{B}_k^n(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(a_t - a_t^\dagger)} \left(\frac{a_t + a_t^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(a_t - a_t^\dagger)}\, dt$$

satisfy the commutation relations

$$[\hat{B}_k^n(g), \hat{B}_K^N(f)]_{w_\infty} := ((N-1)\,k - (n-1)\,K)\ \hat{B}_{k+K}^{n+N-2}(gf)$$

of the Zamolodchikov–w_∞ Lie algebra of conformal field theory with involution

$$\left(\hat{B}_k^n(f)\right)^* = \hat{B}_{-k}^n(\bar{f}).$$

In particular, for $n = N = 2$ we obtain

$$[\hat{B}_k^2(g), \hat{B}_K^2(f)]_{w_\infty} = (k - K)\ \hat{B}_{k+K}^2(gf)$$

which are the commutation relations of the Virasoro algebra. The analytic continuation $\{\hat{B}_z^n(f)\,;\, n \geq 2, z \in \mathbb{C}\}$ of the Virasoro–Zamolodchikov–w_∞ Lie algebra, and the Lie algebra of the Renormalized Higher Powers of White Noise (RHPWN) with commutator $[\cdot,\cdot]_{RHPWN}$, have recently been identified (cf. [3]). This result opens a broad landscape of connections with a multiplicity of key topics of contemporary mathematical and physical research.

1.4. Contents of the present paper. Our first key point is that some renormalization procedures break the original commutation relations so that not all Lie algebras considered in this paper are current algebras over $\mathbb{R}$ based on a given Lie algebra. We always start from the universal enveloping algebra $\mathcal{U}$ of the (1–mode) full oscillator algebra, but only with the renormalization used in subsection 1.2 is it true that the RHPWN algebra is a current algebra over $\mathbb{R}$ based on $\mathcal{U}$. Clearly, one can define the Lie subalgebra, corresponding to the fixed choice $\chi_{[0,1]}$ of the test function, to be the 1–mode Lie algebra underlying the given renormalization. However, this neither simplifies the proofs nor gives more insight. We have therefore used the 1–mode approach when the 1–mode Lie algebra is a well known one, such as the FOA(1), and the test function approach when the underlying 1–mode algebra, by effect of the renormalization, becomes a new one.

In Section 2, we give examples of this situation by producing examples of two different renormalization procedures under which ${a_t^\dagger}^n$, a_t^n, and $a_t^\dagger\, a_t$ form a Lie algebra isomorphic, in one case, to the usual 1–st order white noise algebra (i.e., the CCR algebra with the addition of the number operator) and, in the other case, to the renormalized square of white noise algebra. These examples show how some renormalization prescriptions may break the original commutation relations thus making unavailable the nice appeal to the general theory of current representations of Lie algebras. In the two just mentioned examples, the fact that the resulting structure is effectively a Lie algebra is apparent, but for other renormalizations (such as (1.8)) this must be verified by direct computations which in some cases may turn out to be rather lengthy.

Finally in Section 3, we initiate our program to study some interesting sub–algebras of the full oscillator algebra FOA(1), for which we know that a Fock representation exists. Finally we prove that this family of sub–algebras include the standard boson representation of the Virasoro algebra (cf. [9]).

2. Truncated Commutators Associated With The RHPWN

Intuitively, the renormalization constant c appearing in (1.4) is equal to $\delta(0)$. Therefore c must be thought of as a very large positive number. Moreover, the RHPWN commutator $[B_K^N(g), B_k^n(f)]$ of (1.6) is a polynomial in c of degree $(K\wedge n)\vee(k\wedge N)-1$. In this section we consider the truncation of (1.6), keeping the single most dominant or the two most dominant c–terms, and we study the resulting algebraic structures. As it turns out, the truncated commutation relations are of either CCR or RSWN type. Because of the deepness and obscurity of the phenomenon of "renormalization", the discovery that there exist nontrivial renormalizations which lead to definite, although well-studied, algebraic structures is very interesting from the mathematical point of view. In particular it shows that each renormalization rule must be analyzed in its implications case by case.

Definition 2.1. For $n,k,N,K\in\{0,1,2,...\}$ with $m:=(K\wedge n)\vee(k\wedge N)\geq 1$, we define the truncated commutator

$$[B_K^N(g), B_k^n(f)]_1 := \theta_m(N,K;n,k)\, c^{m-1}\, B_{K+k-m}^{N+n-m}(gf)$$

i.e., $[B_K^N(g), B_k^n(f)]_1$ is the leading term in the expansion of $[B_K^N(g), B_k^n(f)]$ as a polynomial in c.

Definition 2.2. For $n,k,N,K\in\{0,1,2,...\}$ with $m:=(K\wedge n)\vee(k\wedge N)\geq 1$, we define the truncated commutator

$$[B_K^N(g), B_k^n(f)]_2 := \sum_{i=0}^{1} \theta_{m-i}(N,K;n,k)\, c^{m-1-i}\, B_{K+k-m+i}^{N+n-m+i}(gf)$$

i.e, $[B_K^N(g), B_k^n(f)]_2$ is the sum of the two leading terms in the expansion of $[B_K^N(g), B_k^n(f)]$ as a polynomial in c.

Lemma 2.3. *For $i\in\{1,2\}$, the adjoints of the truncated commutators $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ are given by*

$$[B_K^N(g), B_k^n(f)]_i^* = [B_n^k(\bar f), B_N^K(\bar g)]_i.$$

Proof. We will show the proof for $i=1$. The proof for $i=2$ is similar. By Definition 2.1 we have

$$\begin{aligned}
& [B_K^N(g), B_k^n(f)]_1^* \\
= \;& \theta_{(K\wedge n)\vee(k\wedge N)}(N,K;n,k)\, c^{(K\wedge n)\vee(k\wedge N)-1} \left(B_{K+k-(K\wedge n)\vee(k\wedge N)}^{N+n-(K\wedge n)\vee(k\wedge N)}(gf)\right)^* \\
= \;& \theta_{(K\wedge n)\vee(k\wedge N)}(N,K;n,k)\, c^{(K\wedge n)\vee(k\wedge N)-1}\, B_{N+n-(K\wedge n)\vee(k\wedge N)}^{K+k-(K\wedge n)\vee(k\wedge N)}(\bar g\,\bar f) \\
= \;& \theta_{(K\wedge n)\vee(k\wedge N)}(k,n;K,N)\, c^{(K\wedge n)\vee(k\wedge N)-1}\, B_{N+n-(K\wedge n)\vee(k\wedge N)}^{K+k-(K\wedge n)\vee(k\wedge N)}(\bar g\,\bar f) \\
= \;& [B_n^k(\bar f), B_N^K(\bar g)]_1.
\end{aligned}$$

$\square$

Remark 2.4. The truncated commutators $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ do not, in general, satisfy the conditions of a Lie algebra commutator. For example, suppressing test functions,

$$[B_1^3,[B_0^2,B_3^0]_1]_1+[B_0^2,[B_3^0,B_1^3]_1]_1+[B_3^0,[B_1^3,B_0^2]_1]_1=36\,c^2\,B_0^1+18\,c\,B_1^2\neq 0$$

and

$$[B_1^3,[B_0^2,B_3^0]_2]_2+[B_0^2,[B_3^0,B_1^3]_2]_2+[B_3^0,[B_1^3,B_0^2]_2]_2=54\,c\,B_1^2+30\,B_2^3\neq 0$$

i.e the Jacobi identity is in general not satisfied. However, $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ can give rise to Lie-algebraic structures as illustrated in the remaining of this section.

2.1. CCR–type Lie algebras.

Proposition 2.5. *For $n\geq 1$ and $1\leq k\leq n$, $B_0^n(\cdot)$, $B_n^0(\cdot)$, and $B_k^k(\cdot)$, form a Lie–algebra with respect to $[\cdot,\cdot]_1$ with*

$$\begin{aligned}
[B_n^0(g),B_0^n(f)]_1 &= n!\,c^{n-1}\int_{\mathbb{R}} g(t)\,f(t)\,dt,\\
[B_k^k(f),B_0^n(g)]_1 &= n^{(k)}\,c^{k-1}\,B_0^n(fg),\\
[B_n^0(g),B_k^k(f)]_1 &= n^{(k)}\,c^{k-1}\,B_n^0(gf).
\end{aligned}$$

Proof.

$$\begin{aligned}
[B_n^0(g),B_0^n(f)]_1 &= \theta_n(0,n;n,0)\,c^{n-1}\,B_{n+0-n}^{0+n-n}(gf)\\
&= n!\,c^{n-1}\,B_0^0(gf)\\
&= n!\,c^{n-1}\int_{\mathbb{R}} g(t)\,f(t)\,dt.
\end{aligned}$$

Moreover,

$$\begin{aligned}
[B_k^k(g),B_0^n(f)]_1 &= \theta_k(k,k;n,0)\,c^{k-1}\,B_{k+0-k}^{k+n-k}(gf)\\
&= n^{(k)}\,c^{k-1}\,B_0^n(gf)
\end{aligned}$$

from which by taking adjoints, and then replacing $\bar{f}$ and $\bar{g}$ by f and g respectively, using Lemma 2.3 we find

$$[B_n^0(f),B_k^k(g)]_1=n^{(k)}\,c^{k-1}\,B_n^0(fg).$$

To prove that the resulting structure is indeed a Lie–algebra we notice that, by defining

$$\langle g,f\rangle_n:=n!\,c^{n-1}\int_{\mathbb{R}} g(t)\,f(t)\,dt,\quad M_k^n(f):=\frac{1}{n^{(k)}\,c^{k-1}}B_k^k(f),$$

the commutation relations in the statement of this proposition, become respectively

$$[B_n^0(g),B_0^n(f)]_1=\langle g,f\rangle_n,\ [M_k^n(f),B_0^n(g)]_1=\ B_0^n(fg)$$

and

$$[B_n^0(g),M_k^n(f)]_1=\ B_n^0(gf)$$

which are the usual first order white noise algebra commutators. Since that is known to be a Lie algebra, the statement follows. □

2.2. RSWN–type Lie algebras.

Proposition 2.6. *For $n \geq 2$, $B_n^0(\cdot)$, $B_0^n(\cdot)$, and $B_1^1(\cdot)$, form a RSWN–type Lie algebra with respect to $[\cdot,\cdot]_2$, with*

$$\begin{aligned}
[B_n^0(g), B_0^n(f)]_2 &= n! \left(c^{n-1} \int_{\mathbb{R}} g(t)\, f(t)\, dt + n\ c^{n-2}\, B_1^1(gf) \right) \\
[B_1^1(g), B_0^n(f)]_2 &= n\, B_0^n(gf) \\
[B_n^0(f), B_1^1(g)]_2 &= n\, B_n^0(fg).
\end{aligned}$$

Proof.

$$\begin{aligned}
[B_n^0(g), B_0^n(f)]_2 &= \theta_n(0,n;n,0)\ c^{n-1}\, B_0^0(gf) + \theta_{n-1}(0,n;n,0)\ c^{n-2}\, B_1^1(gf) \\
&= n!\, c^{n-1}\, B_0^0(gf) + n\ \cdot\ n!\, c^{n-2}\, B_1^1(gf) \\
&= n! \left(c^{n-1}\, B_0^0(gf) + n\, c^{n-2}\, B_1^1(gf) \right) \\
&= n! \left(c^{n-1} \int_{\mathbb{R}} g(t)\, f(t)\, dt + n\, c^{n-2}\, B_1^1(gf) \right)
\end{aligned}$$

and

$$\begin{aligned}
[B_1^1(g), B_0^n(f)]_2 &= \theta_1(1,1;n,0)\ c^0\, B_0^n(gf) + \theta_0(1,1;n,0)\ c^{-1}\, B_1^{1+n}(gf) \\
&= n\, B_0^n(gf) + 0 \\
&= n\, B_0^n(gf)
\end{aligned}$$

from which by taking adjoints, and then replacing $\bar{f}$ and $\bar{g}$ by f and g respectively, using Lemma 2.3 we obtain

$$[B_n^0(f)), B_1^1(g)]_2 = n\, B_n^0(fg).$$

To prove that the resulting structure is indeed a Lie–algebra we notice that, by defining

$$D_1^1(g) := \frac{B_1^1(g)}{n},\ D_0^n(g) := \frac{1}{\sqrt{n! n^2 c^{n-1}}}\, B_0^n(g),\ D_n^0(g) := \frac{1}{\sqrt{n! n^2 c^{n-1}}}\, B_n^0(g)$$

the commutation relations in the statement of this proposition, become respectively

$$[D_n^0(g), D_0^n(f)]_2 = \frac{1}{n^2} \int_{\mathbb{R}} g(t)\, f(t)\, dt + \frac{1}{c}\, D_1^1(gf),\ [D_1^1(g), D_0^n(f)]_2 = D_0^n(gf)$$

and

$$[D_n^0(f), D_1^1(g)]_2 = D_n^0(fg)$$

which are equivalent to the defining relations of the renormalized square of white noise Lie algebra.

□

3. Lie $*$–sub–algebras of the full oscillator algebra

In this section we restrict our attention to the 1–mode full oscillator algebra FOA(1). Our goal is to study $*$–Lie subalgebras of FOA(1). To achieve this goal we begin by looking for a *canonical form* for the elements of FOA(1). The algebra FOA(1) is the linear span of the noncommutative polynomials in the variables a, a^+ and, using the Heisenberg commutation relations, it is easy to see that this coincides with the linear span of the *normally ordered monomials*

$$(a^\dagger)^m a^n = \begin{cases} (a^\dagger)^m a^m \cdot a^{n-m} \ , & \text{if} \quad n \geq m \\ (a^\dagger)^{m-n} (a^\dagger)^n a^n \ , & \text{if} \quad m \geq n \end{cases} \tag{3.1}$$

Lemma 3.1. *Let $[a, a^\dagger] = 1$. Then for $n \geq 0$*

$$(a^\dagger \, a)^n = \sum_{k=0}^{n} S_{n,k} \, (a^\dagger)^k \, (a)^k$$

$$(a^\dagger)^n \, (a)^n = \sum_{k=0}^{n} s_{n,k} \, (a^\dagger \, a)^k$$

where $s_{n,k}$ and $S_{n,k}$ are the Stirling numbers of the first and second kind respectively, with $s_{0,0} = S_{0,0} = 1$ and $s_{0,k} = s_{n,0} = S_{0,k} = S_{n,0} = 0$ for all $n, k \geq 1$.

Proof. This result is well known in the literature, see e.g [11] $\square$

Using (3.1) and Lemma (3.1), we see that FOA(1) is the $*$–linear space generated by the operators

$$N^k a^n; \ N := a^\dagger a; \ n, k \in \mathbb{N}. \tag{3.2}$$

It is therefore natural to study the structure of the $*$–Lie algebras generated by each of the generators. In order to solve this problem we need some preliminary results.

Proposition 3.2. *For $n, k, \gamma \geq 0$, using the notation $N := a^\dagger \, a$ we have*

$$[a^k, N^n] \;=\; \epsilon_{k,0} \sum_{m=0}^{n} S_{n,m} \, \epsilon_{m,0} \sum_{l \geq 1} \binom{k}{l} m^{(l)} \sum_{\gamma=0}^{m-l} s_{m-l,\gamma} \, N^\gamma \, a^k \tag{3.3}$$

$$[N^\gamma \, a^k, N^n] \;=\; \epsilon_{k,0} \sum_{m=0}^{n} S_{n,m} \, \epsilon_{m,0} \sum_{l \geq 1} \binom{k}{l} m^{(l)} \sum_{w=0}^{m-l} s_{m-l,w} \, N^{w+\gamma} \, a^k \tag{3.4}$$

and

$$[N^n, (a^\dagger)^k \, N^\gamma] = \epsilon_{k,0} \sum_{m=0}^{n} S_{n,m} \, \epsilon_{m,0} \sum_{l \geq 1} \binom{k}{l} m^{(l)} \sum_{w=0}^{m-l} s_{m-l,w} \, (a^\dagger)^k \, N^{w+\gamma}. \tag{3.5}$$

Proof.

$$\begin{aligned} [a^k, (a^\dagger a)^n] \;&=\; \epsilon_{k,0} \sum_{m=0}^{n} S_{n,m} \, \epsilon_{m,0} \sum_{l \geq 1} \binom{k}{l} m^{(l)} \, {a^\dagger}^{m-l} \, a^{k-l} \, a^m \\ &=\; \epsilon_{k,0} \sum_{m=0}^{n} S_{n,m} \, \epsilon_{m,0} \sum_{l \geq 1} \binom{k}{l} m^{(l)} \, {a^\dagger}^{m-l} \, a^{m-l} \, a^k \end{aligned}$$

from which (3.3) follows by applying Lemma 3.1. (3.4) follows from (3.3) and the Leibnitz rule:

$$[N^\gamma\, a^k, N^n] = N^\gamma\, [a^k, N^n].$$

Finally, (3.5) follows from (3.4) by taking adjoints.

□

Proposition 3.3.

$$\begin{aligned}
&[(a^\dagger)^k\, N^\gamma, N^{\gamma'}\, a^k] = \qquad (3.6)\\
&\sum_{\alpha=0}^{\gamma}\sum_{\beta=0}^{\gamma'} S_{\gamma,\alpha}\, S_{\gamma',\beta}\, \{\epsilon_{\alpha,0}\, \epsilon_{\beta,0} \sum_{l\geq 1}\binom{\alpha}{l}\beta^{(l)} \sum_{m=0}^{\alpha+\beta+k-l} s_{\alpha+\beta+k-l,m}\, N^m\\
&-\epsilon_{\alpha+k,0}\, \epsilon_{\beta+k,0} \sum_{L\geq 1}\binom{\beta+k}{L}(\alpha+k)^{(L)} \sum_{m'=0}^{\alpha+\beta+k-L} s_{\alpha+\beta+k-L,m'}\, N^{m'}\}.
\end{aligned}$$

Proof. As in the proof of Proposition 3.2:

$$\begin{aligned}
&[(a^\dagger)^k\, N^\gamma, N^{\gamma'}\, a^k] =\\
&\sum_{\alpha=0}^{\gamma}\sum_{\beta=0}^{\gamma'} S_{\gamma,\alpha}\, S_{\gamma',\beta}\, \{\epsilon_{\alpha,0}\, \epsilon_{\beta,0} \sum_{l\geq 1}\binom{\alpha}{l}\beta^{(l)}\, {a^\dagger}^{\alpha+\beta+k-l}\, a^{\alpha+\beta+k-l}\\
&-\epsilon_{\alpha+k,0}\, \epsilon_{\beta+k,0} \sum_{L\geq 1}\binom{\beta+k}{L}(\alpha+k)^{(L)}\, {a^\dagger}^{\beta+\alpha+k-L}\, a^{\beta+\alpha+k-L}\},
\end{aligned}$$

which yields the equality (3.6) by applying Lemma 3.1 to ${a^\dagger}^{\alpha+\beta+k-l}\, a^{\alpha+\beta+k-l}$ and ${a^\dagger}^{\alpha+\beta+k-L}\, a^{\alpha+\beta+k-L}$.

□

Proposition 3.4. *Let $k \geq 1$ be fixed, and for $n \geq 0$ define $A_k(n) := N^n\, a^k$ and $A_k(n)^\dagger := (a^\dagger)^k\, N^n$, where $N = a^\dagger\, a$. Then , for any γ, γ', $n \in \mathbb{N}$:*

$$\begin{aligned}
&[A_k(\gamma)^\dagger, A_k(\gamma')] = \qquad (3.7)\\
&\sum_{\alpha=0}^{\gamma}\sum_{\beta=0}^{\gamma'} S_{\gamma,\alpha}\, S_{\gamma',\beta}\{\epsilon_{\alpha,0}\, \epsilon_{\beta,0} \sum_{l\geq 1}\binom{\alpha}{l}\beta^{(l)} \sum_{m=0}^{\alpha+\beta+k-l} s_{\alpha+\beta+k-l,m}\, N^m\\
&-\sum_{L\geq 1}\binom{\beta+k}{L}(\alpha+k)^{(L)} \sum_{m'=0}^{\alpha+\beta+k-L} s_{\alpha+\beta+k-L,m'}\, N^{m'}\}
\end{aligned}$$

$$[A_k(\gamma), N^n] = \sum_{m=0}^{n} S_{n,m}\, \epsilon_{m,0} \sum_{l\geq 1}\binom{k}{l} m^{(l)} \sum_{w=0}^{m-l} s_{m-l,w}\, A_k(w+\gamma), \qquad (3.8)$$

$$[N^n, A_k(\gamma)^\dagger] = \sum_{m=0}^{n} S_{n,m}\, \epsilon_{m,0} \sum_{l\geq 1}\binom{k}{l} m^{(l)} \sum_{w=0}^{m-l} s_{m-l,w}\, A_k(w+\gamma)^\dagger \qquad (3.9)$$

$$
\begin{aligned}
&[A_k(\gamma), A_k(\gamma')] = \qquad (3.10)\\
&\quad \sum_{m'=0}^{\gamma'} S_{\gamma',m'}\epsilon_{m',0} \sum_{l\geq 1} \binom{k}{l} m'^{(l)} \sum_{\lambda'=0}^{m'-l} s_{m'-l,\lambda'}\, A_{2k}(\gamma+\lambda')\\
&\quad - \sum_{m=0}^{\gamma} S_{\gamma,m}\, \epsilon_{m,0} \sum_{L\geq 1} \binom{k}{L} m^{(L)} \sum_{\lambda=0}^{m-L} s_{m-L,\lambda}\, A_{2k}(\gamma'+\lambda)
\end{aligned}
$$

$$
\begin{aligned}
&[A_k(\gamma')^\dagger, A_k(\gamma)^\dagger] = \qquad (3.11)\\
&\quad \sum_{m'=0}^{\gamma'} S_{\gamma',m'}\, \epsilon_{m',0} \sum_{l\geq 1} \binom{k}{l} m'^{(l)} \sum_{\lambda'=0}^{m'-l} s_{m'-l,\lambda'}\, A_{2k}(\gamma+\lambda')^\dagger\\
&\quad - \sum_{m=0}^{\gamma} S_{\gamma,m}\, \epsilon_{m,0} \sum_{L\geq 1} \binom{k}{L} m^{(L)} \sum_{\lambda=0}^{m-L} s_{m-L,\lambda}\, A_{2k}(\gamma'+\lambda)^\dagger
\end{aligned}
$$

and

$$[N^\gamma, N^n] = 0. \qquad (3.12)$$

Proof. Equations (3.7)– (3.9) are a direct consequence of Propositions 3.2 and 3.3 above. To prove (3.10) we notice that

$$
\begin{aligned}
[A_k(\gamma), A_k(\gamma')] &= [N^\gamma\, a^k, N^{\gamma'}\, a^k]\\
&= N^\gamma\, a^k\, N^{\gamma'}\, a^k - N^{\gamma'}\, a^k\, N^\gamma\, a^k\\
&= N^\gamma\, ([a^k, N^{\gamma'}] + N^{\gamma'} a^k)\; a^k - N^{\gamma'}\, ([a^k, N^\gamma] + N^\gamma\, a^k)\, a^k\\
&= N^\gamma\, [a^k, N^{\gamma'}]\, a^k - N^{\gamma'}\, [a^k, N^\gamma]\, a^k
\end{aligned}
$$

and (3.10) follows from (3.3). Equation (3.11) is the adjoint of (3.10). Finally, equation (3.12) is (3.4) for $k = 0$.

□

Proposition 3.5. *Let $k \geq 1$ be fixed, and for $n, m \geq 0$ define*

$$P_n(N) := \sum_{\rho=0}^{n} \alpha_\rho\, N^\rho,\;\; Q_m(N) := \sum_{\sigma=0}^{m} \beta_\sigma\, N^\sigma$$

where $\alpha_\rho, \beta_\sigma \in \mathbb{N}_0$. If $\lambda, \mu \in \mathbb{N}_0$ then

$$[P_n(N)\, a^{\lambda\, k}, Q_m(N)\, a^{\mu\, k}] = \Phi_{\lambda,\mu}(P_n, Q_m)(N)\;\, a^{(\lambda+\mu)\, k} \qquad (3.13)$$

and

$$[{a^\dagger}^{\mu\, k}\, Q_m(N), {a^\dagger}^{\lambda\, k}\, P_n(N)] = {a^\dagger}^{(\lambda+\mu)\, k}\; \Phi_{\lambda,\mu}(P_n, Q_m)(N) \qquad (3.14)$$

where $\Phi_{\lambda,\mu}(P_n, Q_m)$ is a polynomial in N defined by

$$\Phi_{\lambda,\mu}(P_n, Q_m)(N) := \epsilon_{\lambda,0}\, D_\lambda(P_n, Q_m)(N) - \epsilon_{\mu,0}\Delta_\mu(P_n, Q_m)(N)$$

where $D_\lambda(P_n, Q_m)$ and $\Delta_\mu(P_n, Q_m)$ are polynomials in N defined by

$$D_\lambda(P_n, Q_m)(N) := \sum_{\rho=0}^{n} \sum_{\sigma=0}^{m} \alpha_\rho \, \beta_\sigma \, \pi_\lambda(\sigma, \rho)(N)$$

$$\Delta_\mu(P_n, Q_m)(N) := \sum_{\rho=0}^{n} \sum_{\sigma=0}^{m} \alpha_\rho \, \beta_\sigma \pi_\mu(\rho, \sigma)(N)$$

where, for $A, B, C \in \mathbb{N}_0$, $\pi_A(B, C)(N)$ is a polynomial in N defined by

$$\pi_A(B, C)(N) := \sum_{m=0}^{B} S_{B,m} \, \epsilon_{m,0} \sum_{l \geq 1} \binom{A\,k}{l} m^{(l)} \sum_{x=C}^{C+m-l} s_{m-l,x-C} \, N^x .$$

The degree of $\Phi_{\lambda,\mu}(P_n, Q_m)(N)$ is $\epsilon_{\lambda+\mu,0}\,(n+m-1)$, the degree of $D_\lambda(P_n, Q_m)(N)$ is $\epsilon_{\lambda,0}\,(n+m-1)$, the degree of $\Delta_\mu(P_n, Q_m)(N)$ is $\epsilon_{\mu,0}\,(n+m-1)$ and the degree of $\pi_A(B, C)(N)$ is $\epsilon_{A,0}\,(B + C - 1)$.

Proof. We have

$$\begin{aligned} &[P_n(N)\, a^{\lambda\, k}, Q_m(N)\, a^{\mu\, k}] \\ &= \sum_{\rho=0}^{n} \sum_{\sigma=0}^{m} \alpha_\rho \, \beta_\sigma \, [N^\rho \, a^{\lambda\, k}, N^\sigma \, a^{\mu\, k}] \\ &= \sum_{\rho=0}^{n} \sum_{\sigma=0}^{m} \alpha_\rho \, \beta_\sigma \, \left(N^\rho \, [a^{\lambda\, k}, N^\sigma] \, a^{\mu\, k} - N^\sigma \, [a^{\mu\, k}, N^\rho] \, a^{\lambda\, k} \right) \end{aligned}$$

from which (3.13) follows by applying (3.3). Equation (3.14) is the adjoint of (3.13). □

Theorem 3.6. *Let $k \in \mathbb{N}$ be fixed. The $*$–linear subspace $\mathcal{L}(k)$ of FOA(1) generated by the set*

$$\{N^m, A_{2^\alpha k}(n), A^\dagger_{2^{\alpha_k}}(n) : m, n, \alpha \in \mathbb{N}\}$$

is a $*$–Lie algebra with structure constants given by (3.7),..., (3.12).

Proof. The proof follows directly from the previous Proposition. □

3.1. Anti–normally ordered generators. The class of generators we have obtained is based on the canonical form (3.1), associated to the representation of the FOA(1) as a linear span of normally ordered products. But we can also represent FOA(1) as a linear span of anti–normally ordered products, i.e., products of the form:

$$a^n (a^\dagger)^m = \begin{cases} a^n (a^\dagger)^n a^{\dagger (m-n)} \;, & \text{if} \quad m \geq n \\ a^{n-m} a^m (a^\dagger)^m \;, & \text{if} \quad n \geq m \end{cases}$$

and, using the analogue of Lemma (3.1) for the anti–normally ordered case, we express the products $a^n (a^\dagger)^n$ as linear combinations of powers of the anti–number operator $N_a = a a^\dagger$. This leads to the class of generators

$$B_k(n) := a^k N_a^n \;; \quad B_k^\dagger(n) = N_a^n (a^\dagger)^k \;; \quad N_a = a a^\dagger \;; \quad k, n \in \mathbb{N} \tag{3.15}$$

which replaces (3.2) in the anti–normally ordered case. By (3.2) and the relation

$$N_a = aa^\dagger = [a, a^\dagger] + a^\dagger a = N + 1$$

one has

$$N_a^n = \sum_{h=0}^{n} \binom{n}{h} N^h.$$

Therefore

$$B_k(n) = \sum_{h=0}^{n} \binom{n}{h} a^k N^h = \sum_{h=0}^{n} \binom{n}{h} A_k(h).$$

This means that the operators $B_k(n)$ are linear combinations of the $A_k(h)$ so the $B_k(n)$ are in $\mathcal{L}(k)$ (where $\mathcal{L}(k)$ is as in Theorem 3.6). That is an indication that the analogue of Proposition 3.4 for the anti–normally ordered case should not be true. The following proposition (which gives the well known oscillator representation of the Virasoro algebra, cf. [9], Section 3) shows that, inside $\mathcal{L}(1)$ there are strictly smaller Lie sub–algebras. It is not clear if there are strictly smaller (i.e., not obtained by restricting the powers of the number operator to be larger than a fixed number) $*$–Lie subalgebras.

Proposition 3.7. *For $m \in \mathbb{N}$, let*

$$L_m := \frac{1}{\sqrt{2}}\, a^{2m+1} a^\dagger = \frac{1}{\sqrt{2}}\, a^{2m} N_a. \tag{3.16}$$

Then, the linear space generated by the set

$$\{L_m : m \in \mathbb{N}\}$$

is a Lie subalgebra of FOA (1), isomorphic to the Virasoro algebra:

$$[L_n, L_m] = (n - m) L_{n+m}$$

Proof. For arbitrary m, $n \in \mathbb{N}$ one has:

$$\begin{aligned}
[a^{n+1}a^\dagger, a^{m+1}a^\dagger] &= [a^{n+1}a^\dagger, a^{m+1}]a^\dagger + a^{m+1}[a^{n+1}a^\dagger, a^\dagger] \\
&= a^{n+1}[a^\dagger, a^{m+1}]a^\dagger + a^{m+1}[a^{n+1}, a^\dagger]a^\dagger \\
&= (-1)^{m+1}(m+1)a^{n+m+1}a^\dagger + a^{m+1}((n+1)a^n a^\dagger) \\
&= a^{m+n+1}a^\dagger(n + 1 + (-1)^{m+1}(m+1)).
\end{aligned}$$

Therefore, if $m = 2\mu$ and $n = 2\nu$ are even numbers, one has

$$\left[\frac{a^{2\nu+1}a^\dagger}{\sqrt{2}}, \frac{a^{2\mu+1}a^\dagger}{\sqrt{2}}\right] = \frac{1}{2}\, a^{2(\nu+\mu)+1}a^\dagger(2\nu - 2\mu) = (\nu - \mu)a^{2(\nu+\mu)+1}a^\dagger$$

Thus, using the notation (3.16), one obtains $[L_n, L_m] = (n - m)L_{n+m}$ which are the commutation relations of the Virasoro algebra. $\square$

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