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MATHEMATICS

Transformations of Gaussian Measures Generated by the Levy Laplacian, and Generalized Traces

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In [1], we introduced the cylindrical Gaussian measure on a vector subspace of the (infinite-dimensional) Hilbert space that corresponds to the fundamental solution to the Cauchy problem for the heat equation that contains the Levy Laplacian [2, 3]; it is possible to define this measure by the correlation functional whose value on every vector from its domain is equal to the Cesaro mean of the sequence of squares of its coordinates in an appropriate (orthonormal) basis. This correlation functional is a scalar product that corresponds, in a natural sense (defined below), to the linear functional on a certain operator algebra, which is called the Levy trace. The Levy trace, which vanishes at completely continuous operators and equals 1 at the identity operator, defines, in its turn, the multiplicative functional that we call the Levy determinant. This paper deals with a deduction of the Cameron–Martin–Girsanov–Maruyama and Reimer formulas for the Gaussian measures generated by the Levy Laplacian, including the measure that corresponds to the Levy–Wiener process (its definition is adduced below); in these formulas, we use both the Levy trace and the Levy determinant.[1] At first, we describe some properties of logarithmic derivatives of cylindrical measures (we do not assume them to be countably additive); in this part of our paper, there are several points in common with the results of [4]. We also note that the recent substantial increase of interest in the Levy Laplacian is connected with the discovery [5] of the equivalence of the Yang–Mills equations to the equation that contains the Levy Laplacian. For previous results concerned with the Levy Laplacian, see [6].

[1] In [1], we used the term "the Brownian Levy motion"; we hope that there is no great danger of confusing this object with the random field that has the same name.

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1. LOGARITHMIC DERIVATIVES

Throughout the rest of this paper, we use terminology and notations from [7, 8], as a rule, without explanations; we assume that all vector spaces under consideration are real, unless otherwise stipulated. For every two vector spaces E and G in duality, we denote by $\mathfrak{A}_E(G)$ the algebra of G-cylindrical subsets of E; we denote by $\sigma_E(G)$ the σ-algebra generated by this algebra; if, further, F is a locally convex space (LCS), then $\mathfrak{M}_E(G, F)$ is the vector space of all F-valued G-cylindrical measures on E [i.e., on $\mathfrak{A}_E(G)$] that have bounded variation in every continuous seminorm on F. We use the symbol $\mathfrak{M}_E(G)$ rather than $\mathfrak{M}_E(G, R^1)$. Note that if T is one more vector space dual to the space G, then the algebras of sets $\mathfrak{A}_E(G)$ and $\mathfrak{A}_T(G)$ [and, consequently, the vector spaces $\mathfrak{M}_E(\cdot)$ and $\mathfrak{M}_T(\cdot)$] are (canonically) isomorphic. If G is an LCS, E is a space of (certain) linear continuous functionals on G, and ψ is a function on E that is the Fourier transform (FT) of a countable additive measure $\nu \in \mathfrak{M}_G(E)$ that admits a (unique) extension to a Radon measure on G [if the measure $\nu \in \mathfrak{M}_G(E, F)$, then its FT $\Phi\nu$ is defined by the equality $\Phi\nu(z) = \int \exp(iz(x))\nu(dx)]$, $\eta \in \mathfrak{M}_E(G)$, and the function $\Phi\nu\colon G \to \mathbb{C}$ is continuous, then the integral of the function ψ with respect to the measure η (which is not countably additive, generally speaking) is defined by the following formula (cf. [9]):

$$\int_E \psi(x)\eta(dx) = \int_G \Phi\eta(z)\nu(dz).$$

A function $g\colon G \to \mathbb{C}$ is called E-cylindrical if there exist $n \in N$, n elements $a_1, \ldots, a_n$ of the space E, and a function $\varphi\colon R^n \to \mathbb{C}$ such that $g(x) = \varphi(\langle a_1, x\rangle, \ldots \langle a_n, x\rangle)$. If φ is a polynomial, then the cylindrical function is called polynomial.

If G is a LCS, then $\mathfrak{M}^{\sigma}_G(E)$ is the subspace of the space $\mathfrak{M}_G(E)$ that consists of all countably additive measures that admit a unique extension to a Radon measure on G, with respect to which all polynomial

E-cylindrical functions are integrable. We also note that integrals of G-cylindrical functions with respect to measures from $\mathfrak{M}_E(G)$, as well as products of such functions and measures, are defined in a natural way (though, of course, they do not always exist); with the help of such products, one defines integrals of products of G-cylindrical functions and Fourier transforms of measures from $\mathfrak{M}_G^\sigma(E)$ with respect to measures from $\mathfrak{M}_E(G)$.

A vector field (respectively, a time-dependent vector field) on the vector space E with values in a (vector) subspace T of the space G^* is a mapping h: $E \to T$ (respectively, a mapping k: $R^1 \times E \to T$). If G is an LCS, then $\mathrm{vect}_E G$ is the vector space of all vector fields on E with values in the strong conjugate space G' to the space G that are Fourier transforms of measures from $\mathfrak{M}_G^\sigma(E)$; $\mathrm{vect}_E^v G$ is the vector space of all time-dependent vector fields k on E with values in G' such that $k(t, \cdot) \in \mathrm{vect}_E G$ for every $t \in R^1$.

Definition 1. *The differentiability subspace of the measure* $\nu \in \mathfrak{M}_E(G)$ is the (vector) subspace of the space G' that is denoted by the symbol $D(\nu)$ and defined in the following way: $h \in D(\nu)$ if and only if there exists a function $\beta_\nu(h, \cdot)$: $D(\nu) \to R^1$, which is called the *logarithmic derivative of the measure* ν *in the direction* h, for which the function $\beta_\nu(h, \cdot)\Phi\mu$ is ν-integrable, and the equality

$$\int \Phi\mu(x)\beta_\nu(h, x)\nu(dx) = -\int (\Phi\mu)'(x)h\nu(dx)$$

holds for any measure $\mu \in \mathfrak{M}_G^\sigma(E)$. In this case, the measure ν is called *differentiable relative to the subspace* $D(\nu)$, and the mapping β_ν is called *the logarithmic derivative* (or the *logarithmic gradient*) of the measure ν (cf. [10]).

Remark 1. Since $(\Phi\mu)'(\cdot)h = \Phi(i\langle h, \cdot\rangle\mu)$, the equality from Definition 1 is equivalent to the following equality:

$$\int (\Phi\mu)(x)\beta_\nu(h, x)\nu(dx) = \int (\Phi\nu)(z)i\langle h, z\rangle\mu(dz).$$

Example 1. Assume that γ is a Gaussian G-cylindrical measure on E with the correlation operator B: $G \to G^*$ and the zero mean (by definition, this means that $\Phi\gamma(z) = e^{\frac{-\langle Bz, z\rangle}{2}}$). If the function $\Phi\gamma$ is continuous, then $D\gamma = \mathrm{Im}B$ and $\beta_\gamma(h, x) = -\langle Bh, h\rangle$ for $h \in D\gamma$ (indeed, by virtue of properties of the Fourier transformation, $(\Phi\gamma)(\cdot) \cdot (-i\langle \cdot, h\rangle) = ie^{\frac{-\langle B\cdot, \cdot\rangle}{2}}(B^{-1}h))$.

2. MEASURES CONNECTED WITH THE LEVY LAPLACIAN

Assume that S_1 is the vector space of infinite sequences of real numbers defined in the following way: $(x_n) \in S_1$ if and only if there exists a continuous almost-periodic function φ: $R^1 \to R^1$ such that $x_n = \varphi(n)$ for every $n \in N$. If one equips this space with the "Levy scalar product" $\langle \cdot, \cdot \rangle_L$, which is (correctly) defined by the equality

$$\langle (x_n^1), (z_n^1)\rangle_L = \lim_{k\to\infty} \frac{1}{k}\sum_{n=1}^{k} x_n^1 z_n^1,$$

then it becomes a Hilbert space, which we denote by the same symbol. Assume also that H is the (separable) Hilbert space, $b = (e_n)$ is its orthonormal basis, E_1 is the image of the space l_1 under the embedding $l_1 \to H$, $(x_n) \mapsto \sum x_n e_n$, equipped with the norm induced by the norm of the space l_1, E is a Banach space that is a vector subspace of the space E_1, and the canonical embedding $E \to E_1$ is continuous. Under these assumptions, the injective (continuous) mapping $S_1 \mapsto (E', \sigma(E', E))$, $(x_n) \mapsto \sum x_n e_n$ is correctly defined; the image of the space S_1 under this mapping will be denoted by the symbol S, and the scalar product in S induced by the Levy scalar product will be denoted by the symbol $\langle \cdot, \cdot \rangle_{PL}$.

Denote: K is the vector subspace of the space S^* generated by the set $E \cup S'$; we identify the spaces S and S'; thus, if $K \ni z = z_1 + z_2$, $z_1 \in E$, $z_2 \in S'(=S)$, $a \in S$ $(\subset E')$, then

$$\langle a, z\rangle = a(z_1) + \langle a, z_2\rangle_{PL}.$$

Remark 2. The duality between S^* and S (defined by the bilinear form $(a, x) \mapsto a(x)$) induces a duality between S and E, which coincides with that induced by the duality between E' and E (and, of course, also the canonical duality between S and K); these dualities will be used below.

Proposition 1. $\sigma_S(E) = \sigma_{\mathbb{S}}(S')$.

Proposition 2. *Any countably additive number-valued E-cylindrical measure on* $\mathfrak{A}_S(E)$ *has a unique extension to a Radon measure on S.*

This follows from Corollary 5 from [11, p. 74] and Proposition 1.

Definition 2. *The Levy trace* that corresponds to the basis b is the functional on some vector subspace V_L of the space of all linear mappings from E in E' denoted by the symbol tr_L and defined in the following way: $A \in V_L \Leftrightarrow (Ae_n, e_n) \in S_1$; if $A \in V_L$, then

$$\mathrm{tr}_L A = \langle (Ae_n, e_n), (1)\rangle_L,$$

where $(1) = 1, 1, 1, \ldots (\in S_1)$.

Definition 3. The Levy Laplacian that corresponds to the basis b is the mapping Δ_L of the subspace V_P of the space F of number-valued functions on S into the space F that is defined in the following way: $g \in V_P$ if only if g is Gäteaux twice differentiable relative to the subspace E at every point $x \in S$, where (in natural notations) $g''(x) \in V_L$ for all x; if $g \in V_P$, then $(\Delta_L g)(x) = \mathrm{tr}_L g''(x)$.

Remark 3. This definition is slightly different from that used in [1, 2].

Remark 4. Assume that A is an operator of trace class in S and A^* is its extension to E defined by the equality $\langle Ax, z\rangle = \langle x, A^*z\rangle_L$. Then $\mathrm{tr}_L A^* = \mathrm{tr} A$, where tr is the (usual) trace of A in S.

Definition 3. The Levy–Gauss measure [with a parameter $t \in (0, \infty)$] is the S-cylindrical measure ν_{GL} (on E, or, which is equivalent, on K) whose Fourier transform is defined by the equality

$$(\Phi\nu_{GL})(x) = \exp\left(-\frac{\langle x, x\rangle_L}{2}\right).$$

Thus, the Levy–Gauss measure is the S-cylindrical Gaussian measure whose correlation operator B is defined by the relation $S \ni x \mapsto x \in S \equiv S'(\subset S^*)$. This measure defines the Green measure of the Cauchy problem for the heat equation that contains the Levy Laplacian [1].

Proposition 3. *$D\nu_{GL} = S$ and $\beta_\nu(h, x) = -\langle h, x\rangle$ for $\nu = \nu_{GL}$, $h \in S$, $x \in K$; in particular, $\beta_\nu(h, x) = -\langle h, x\rangle_L$ for $h, x \in S$.*

Definition 4 (cf. [1]). Assume that $F([0, 1], K)$ is the vector space of all K-valued functions on [0, 1], P_S is the space of all S-valued measures on [0, 1] that have single-point supports, connected by the natural duality. *The Levy–Wiener measure* on $F([0, 1], K)$ *generated by the Levy Laplacian* is the P_S-cylindrical Gaussian measure w_L (on $F([0, 1], K)$) whose Fourier transform is defined by the equality

$$\Phi w_L(\eta) = \exp\left(-\frac{1}{2}\int\int \min(\tau, t)\langle \eta(dt)\eta(dt)\rangle_L\right).$$

Proposition 4. *The space Dw_L can be described as follows: $g \in Dw_L$ if and only if there exists an S-valued square integrable function f on* [0, 1] *such that $g(t) = \int_0^t f(\tau)d\tau$ for $t \in$* [0, 1].

3. SHIFTS ALONG INTEGRAL CURVES OF VECTOR FIELDS

Assume that $h \in \mathrm{vect}_K^V S$ and a is the mapping $R^1 \times K \to K$ into K such that $a(0, x) = x$ and $a_1'(t, x) = h(t, a(t, x))$ for every $x \in K$. Assume also that $\nu \in \mathfrak{M}_K(S)$. Then the t-shift of the measure $\nu \in \mathfrak{M}_K(S)$ along the integral curves of the vector field h is the measure ν_{th}^a that has the following property: for any measure $\mu \in \mathfrak{M}_S^\sigma(K)$,

$$\int(\Phi\mu)(a(-t, x))\nu(dx) = \int(\Phi\mu)(x)\nu_{th}(dx).$$

The logarithmic derivative of the measure $\nu \in \mathfrak{M}_K(S)$ along the vector field h is the function $\beta_\nu^h: K \to R^1$ that has the following property: for every measure $\mu \in \mathfrak{M}_S^\sigma(K)$, the equality

$$\int(\Phi\mu)'(x)h(x)\nu(dx) = -\int\Phi\mu(x)\beta_\nu^h(x)\nu(dx)$$

is valid.

Proposition 5. *If, for all $x \in K$, the derivative of the mapping h relative to the subspace S is an operator of trace class in S, then the logarithmic derivative of the measure ν_{GL} along the vector field h exists and is defined by the equality $\beta_{\nu_{GL}}^h(x) = \mathrm{tr}_L h'(x) - \langle x, h\rangle$.*

Theorem 1. *In the assumptions of the previous proposition the t-shift of the measure ν_{GL} along the integral curves of the vector field h (exists and) is defined by the equality*

$$(\nu_{GL})_{th}^a = \exp\left[\int_0^t \mathrm{tr}_L h'(a(\tau, x))d\tau - \frac{1}{2}((a(t, x), a(t, x)) - (x, x))\right]\nu_{GL}.$$

This theorem is deduced from a proposition that is similar to Theorem 2 from [3], and Proposition 5.

4. SHIFTS ALONG VECTOR FIELDS

If $h \in \mathrm{vect}_K S$, then the t-shift of measure $\nu \in \mathfrak{M}_K(S)$ along the vector field h is the measure ν_{th} such that for every measure $\mu \in \mathfrak{M}_S(K)$

$$\int\Phi\mu(x - \mathrm{th}(x))\nu(dx) = \int\Phi\mu(x)\nu_{th}(dx).$$

The shift along the vector field h coincides with the shift along integral curves of the auxiliary vector field k that satisfies the equality $h(x) = k(t, x + \mathrm{th}(x))$ (if this vector field exists).

Theorem 2. *Assume that $h \in \mathrm{vect}_K S$ and the conditions of Proposition 5 are fulfilled. If, for every $t \in R^1$, the mapping $\psi_t: x \mapsto x + \mathrm{th}(x)$ is invertible (and some additional conditions of analytic character are fulfilled), then*

$$(\nu_{GL})_h = \det{}_L(I + h'(x)) \times \exp\left(-\frac{\langle h(x)h(x)\rangle_L}{2} - \langle x, h(x)\rangle\right)\nu_{GL},$$

where $\det_L(I + h'(x)) = e^{\mathrm{tr}\ln(I + h'(x))}$ *is "the Levy determinant". One uses here, in particular, the following equalities*:

$$\exp\int_0^1 \mathrm{tr}_L k_2'(\tau, \psi_\tau(\cdot))d\tau$$

$$= \exp\mathrm{tr}_L\int h'(\psi_\tau^{-1}(\psi_\tau(\cdot))\psi_\tau'(\psi_\tau^{-1}(\psi_\tau(\cdot)))d\tau$$

$$= \exp\mathrm{tr}_L\int h'(x)(I + \tau h'(x))d\tau$$

$$= \exp\mathrm{tr}_L\ln(I + \tau h'(\cdot))d\tau$$

$$(Ix = x \quad \forall x \in S).$$

Now, assume that h is a vector field in $F([0, 1], K)$ that takes its values in Dw_L, and

$$h(x)(\alpha) = \int_0^1 F(x(s), s, \alpha)ds,$$

where F is continuous, and $F(x(s), s, \alpha) = 0$ for $s \geq \alpha$ or $F(x(s), s, \alpha) = 0$ for $s \leq \alpha$. Then $\mathrm{tr}\, h'(x) = \int F(x(s)), s, s)ds = 0$ and, moreover, $\mathrm{tr}(h'(x))^n = 0$, so that $\mathrm{tr}\ln(I + h'(x)) = 0$.

Theorem 3. *Under the above assumption,* $(w_L)_h = \rho w_L$, *where*

$$\rho(x) = \exp\left\{-\int_0^1 \langle (h(x))'(t)(h(x))'(t)\rangle_L dt - \frac{1}{2}\int_0^1 \langle (h(x))'(t)x'(t)\rangle_L dt\right.$$

(*this is an analog of the Girsanov–Maruyama formula for nonanticipating functionals*; *one can prove a similar formula for the general case in the same way*).

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