

## INTERACTING FOCK SPACE VERSUS FULL FOCK MODULE\*

LUIGI ACCARDI AND MICHAEL SKEIDE

ABSTRACT. We present several examples where moments of creators and annihilators on an *interacting Fock space* may be realized as moments of creators and annihilators on a *full Fock module*. Motivated by this experience we answer the question, whether such a possibility exists for arbitrary interacting Fock spaces, in the affirmative sense. We treat the problem in full algebraic generality. As a by-product, we find a new notion of positivity for  $*$ -algebras which allows to construct tensor products of Hilbert modules over  $*$ -algebras.

Finally, we consider a subcategory of interacting Fock spaces which are embeddable into a usual full Fock space. We see that a creator  $a^*(f)$  on the interacting Fock space is represented by an operator  $\varkappa \ell^*(f)$ , where  $\ell^*(f)$  is a usual creator on the full Fock space and  $\varkappa$  is an operator which does not change the number of particles. In the picture of Hilbert modules the one-particle sector is replaced by a two-sided module over an algebra which contains  $\varkappa$ . Therefore,  $\varkappa$  may be absorbed into the creator, so that we are concerned with a usual creator. However, this creator does not act on a Fock space, but rather on a Fock module.

## 1. Introduction

In [4] Accardi, Lu, and Volovich proposed the following definition. An *interacting Fock space* over a Hilbert space is the usual *full* (or *boltzmanian*) *Fock space*  $\mathcal{F}(H) = \bigoplus_{n \in \mathbb{N}_0} H^{\otimes n}$  over a Hilbert space  $H$  where, however, direct sum and tensor products are understood algebraically, and where the (semi-)inner product on the  $n$ -particle sector  $H^{\otimes n}$  is rather arbitrary. The creators  $a^*(f)$  ( $f \in H$ ) are the usual ones. Restrictions to the semiinner product arise by the requirement that each creator  $a^*(f)$  should have an adjoint  $a(f)$  with respect to the new inner product. This implies that the creators (and also the annihilators) respect the kernel of the semiinner product.

This definition was suggested by the observation that in the *stochastic limit* of an electron coupled to the electro magnetic field as computed in Accardi, Lu [3] the limit distribution of the field operators in the vacuum state of the field and some state on the system space of the electron can be understood as the vacuum expectation of creators and annihilators on an interacting Fock space. In the meantime, we know many other examples of interacting Fock spaces; see e.g. Section 2. We mention the representation space of the *renormalized square of*

---

2000 *Mathematics Subject Classification*. Primary 46L53; Secondary 46L08, 60J25, 81S25.

*Key words and phrases*. Quantum probability, interacting Fock space, full Fock module, Hilbert modules over general involutive algebras, positivity in involutive algebras.

\* This work is supported the Italian MUR (PRIN 2007). MS is supported by the Dipartimento S.E.G.e S. of the University of Molise.

*white noise* (see [6, 7, 17, 2]) as proposed in Accardi, Lu and Volovich [5]. The list of examples can be continued arbitrarily. However, it is not our goal to give an account of the history of interacting Fock space. The few examples in Sections 2 and 3 are chosen from the didactic point of view.

In analogy with the full Fock space over a Hilbert space (i.e. a two-sided Hilbert  $\mathbb{C}$ -module) we may construct the *full Fock module* over a two-sided Hilbert module; see Pimsner [13] and Speicher [18].

*Wigner's semi-circle distribution* is the *central limit distribution of free independence* and it may be realized by moments of creators and annihilators on the full Fock space in the vacuum expectation; see Voiculescu [19]. The same is true for *operator-valued free independence* (Voiculescu [20]). More precisely, the central limit distribution of operator-valued free independence may be realized by moments of creators and annihilators on a suitable full Fock module in the *vacuum conditional expectation*; see [18].

Let us return to the QED-example. Already in [3] the idea arose to use the language of Hilbert modules to understand better the underlying structure. In fact, the idea is very natural. The limit computed in [3], actually, is the limit of the *vacuum conditional expectation* from the algebra of operators on  $\Gamma(L^2(\mathbb{R}^d)) \otimes S$  onto the algebra of operators on  $S$ , where  $S$  denotes the Hilbert space of the electron. It is well-known that any completely positive mapping admits a GNS-representation on a Hilbert module. However, in [3] the limit of the vacuum conditional expectation was computed only weakly. In [14] Skeide showed that the limit conditional expectation exists. He pointed out that that GNS-module of the vacuum conditional expectation is a full Fock module and the moments of the limits of the field operators are those of creators and annihilators in the vacuum conditional expectation of this Fock module.

The basis for the construction of a full Fock module is a two-sided Hilbert module; see Section 3. In fact, it was the correct left multiplication discovered in [14] which allowed to identify the GNS-module as a full Fock module. We demonstrate in Examples 3.17, 3.18, and 3.20 the influence of different choices for a left multiplication.

Motivated by the examples we ask, whether it is possible in general to represent operators on an interacting Fock space by operators on a full Fock module and, thus, to glue together the theory of interacting Fock spaces and the theory of full Fock modules. In Section 4 we answer this question in the affirmative sense by an explicit construction (Theorems 4.1, 4.2, and 4.6). We obtain in full algebraic generality that the algebra generated by creators and annihilators on an interacting Fock space is determined by the module generalization of the Cuntz relations. In Example 3.19 we show that it is also possible to associate with a given Fock module an interacting Fock space. In Example 4.7 we explain that the construction in 3.19 reverses the construction in Section 4.

We obtain a clearer picture of what the construction actually does, if we restrict to the subcategory of interacting Fock spaces which are embeddable (via an isometry) into a usual full Fock space. In Section 5 we show that a creator  $a^*(f)$  on an embeddable interacting Fock space may be represented as a modified creator  $\varkappa \ell^*(f)$  on a full Fock space (Theorem 5.5). Here  $\varkappa$  is in the relative commutant

of the number operator and, in other words,  $\varkappa$  leaves invariant the number of particles. In the module picture the one-particle sector of the Fock space is replaced by a two-sided module, precisely, over the algebra of such operators. Therefore, in the module picture it is possible to ‘absorb’ the operator  $\varkappa$  into the creator on the full Fock module over the one-particle module (Theorem 5.9). We also provide two criteria which show that among the interacting Fock space there are plenty of embeddable ones (Theorems 5.2, and 5.3).

In the first few sections we introduce the necessary structures and present several examples for these structures. In Section 2 we define what we understand by interacting Fock space. The definition slightly differs from the definition in [4]. The difference consists, however, only in that we divided out the kernel of the semiinner product of [4] in order to have an inner product. Then we describe some examples of interacting Fock spaces.

In Section 3 we present a generalization of the notion of Hilbert module and full Fock module to Hilbert modules over arbitrary  $*$ -algebras (not only  $C^*$ -algebras) with a new positivity structure (called  $P^*$ -algebras in [16]). This notion of positivity, still algebraic but much more flexible than just algebraic positivity, allows for tensor products of Hilbert modules. It shows to be crucial also in other applications like in [6]. In our context the generalizations are necessary in view of Example 2.7 [1] where a relation between orthogonal polynomials and interacting Fock spaces is pointed out. Being the motivation for the remaining sections, we show for some examples how distributions of creators and annihilators on an interacting Fock space may be realized as distributions of creators and annihilators on a suitable full Fock module.

In Section 6 we explain all aspects from the first sections in the example of the symmetric Fock space. We point out the origin of the complications and explain why the symmetric Fock space is a “bad” example for an interacting Fock space.

**Notations and conventions.** All constructions, like direct sum and tensor product, are purely algebraic. By  $\mathcal{L}$  we denote spaces of linear mappings (without any restriction like boundedness), whereas  $\mathcal{L}^a$  denotes spaces of adjointable mappings between or on spaces with ( $\mathbb{C}$ -valued or  $*$ -algebra-valued) inner product. With one exception (in the proof Theorem 5.2 where we consider the square root of a positive self-adjoint densely defined operator) the adjoint of an operator between pre-Hilbert spaces  $G$  and  $H$  is always a mapping  $H \rightarrow G$ . For details about Hilbert modules and GNS-representation we refer the reader to [12, 10, 15, 14]. A fairly complete account of what is necessary can be found in [8, 16].

## 2. Interacting Fock Space

The definition of interacting Fock space used here differs slightly from the definition in [4]. The difference is that we divide out the kernel of the inner product. The benefits from this approach are a positive definite inner product and absence of the condition that the operators have to respect some kernel (cf. the introduction). Of course, we lose the tensor product structure of the  $n$ -particle sector. Instead of a tensor product  $H^{\otimes n}$  we are concerned with rather arbitrary pre-Hilbert spaces

$H_n$ . However, the  $H_n$  are required to be spanned by the range of all creators. Let us introduce some notation.

**Definition 2.1.** Let  $H$  be some pre-Hilbert space. By  $\mathcal{L}^a(H)$  we denote the space of all *adjointable* mappings  $A: H \rightarrow H$  (i.e. there exists a mapping  $A^*: H \rightarrow H$  such that  $\langle f, Ag \rangle = \langle A^*f, g \rangle$  for all  $f, g \in H$ ).

Let  $(H_n)_{n \in \mathbb{N}_0}$  be a family of pre-Hilbert spaces. Denote by  $\mathcal{H} = \bigoplus_{n \in \mathbb{N}_0} H_n$  their algebraic direct sum. Setting  $H_n = \{0\}$  for  $n < 0$ , we define for each  $m \in \mathbb{Z}$  the space

$$\mathcal{L}_m^a(\mathcal{H}) = \{A \in \mathcal{L}^a(\mathcal{H}) : AH_n \subset H_{n+m} \ (n \in \mathbb{N}_0)\}.$$

**Definition 2.2.** Let  $(H_n)_{n \in \mathbb{N}_0}$  be a family of pre-Hilbert spaces with  $H_0 = \mathbb{C}\Omega$ . Let  $H$  be another pre-Hilbert space. We say  $\mathcal{I} = \bigoplus_{n \in \mathbb{N}_0} H_n$  is an *interacting Fock space* (based on  $H$ ), if there exists a mapping  $a^*: H \rightarrow \mathcal{L}_1^a(\mathcal{I})$ , fulfilling  $\text{span}(a^*(H)H_n) = H_{n+1}$ .

*Remark 2.3.* Usually, we identify  $H$  and  $H_1$  by requiring  $a^*(f)\Omega = f$  for all  $f \in H$ . However, in Example 2.7 we will have  $a^*(f)\Omega = \sqrt{\omega_1}f$  for a fixed real number  $\omega_1 > 0$ . Another problem may appear, if  $a^*(f)\Omega = 0$  although  $f \neq 0$ . Therefore, it is important to keep the freedom to choose  $H$  and  $H_1$  differently.

The operators  $a^*(f)$  are called *creators*. Their adjoints  $a(f)$  are called *annihilators*. Observe that the linear span of all  $a^*(f_n) \dots a^*(f_1)\Omega$  ( $f_i \in H$ ) is  $H_n$ . By  $\mathcal{A}(\mathcal{I})$  we denote the  $*$ -algebra generated by all  $a^*(f)$  ( $f \in H$ ).

**Example 2.4. The full Fock space.** The usual full Fock space  $\mathcal{F}(H)$  over a pre-Hilbert space  $H$  is obtained by setting  $H_n = H^{\otimes n}$  equipped with natural inner product of the  $n$ -fold tensor product. The creators on the full Fock space are denoted by  $\ell^*(f)$ . The action of  $\ell^*(f)$  is defined by setting  $\ell^*(f)f_n \otimes \dots \otimes f_1 = f \otimes f_n \otimes \dots \otimes f_1$  and  $\ell^*(f)\Omega = f$ . It is easy to see that  $\ell^*(f)$  has an adjoint  $\ell(f)$ , fulfilling  $\ell(f)f_n \otimes \dots \otimes f_1 = \langle f, f_n \rangle f_{n-1} \otimes \dots \otimes f_1$  and  $\ell(f)\Omega = 0$ . The  $*$ -algebra  $\mathcal{A}(\mathcal{F}(H))$  generated by  $\ell^*(H)$  is determined by the Cuntz relations  $\ell(f)\ell^*(g) = \langle f, g \rangle$ .

*Remark 2.5.* In the following examples we construct several interacting Fock spaces as described in [4]. In other words, we start with a full Fock space and then change the inner product on the  $n$ -particle sector and divide out the kernel. In these cases we always choose the creators  $a^*(f)$  to be the images of the usual ones  $\ell^*(f)$  on the quotient. Necessarily, the  $\ell^*(f)$  have to respect the kernel of the new inner product. By giving an explicit adjoint of  $\ell^*(f)$ , this condition is fulfilled, automatically. Clearly, the image of this adjoint on the quotient is the the unique adjoint  $a(f)$  of  $a^*(f)$ .

**Example 2.6. The Lu-Ruggeri interacting Fock space.** In [11] the  $n$ -particle sectors of the full Fock space  $\mathcal{F}(L^2(\mathbb{R}))$  had been equipped with a new inner product by setting

$$\begin{aligned} &\langle f_n \otimes \dots \otimes f_1, g_n \otimes \dots \otimes g_1 \rangle \\ &= \int_0^\infty dx_1 \int_{x_1}^\infty dx_2 \dots \int_{x_{n-1}}^\infty dx_n \overline{f_1(x_1)} \dots \overline{f_n(x_n)} g_n(x_n) \dots g_1(x_1). \end{aligned}$$

Notice that this is nothing but the integral over the  $n$ -simplex  $\{x_n \geq \dots \geq x_1 \geq 0\}$ . An adjoint of the creator  $\ell^*(f)$  is given by

$$[\ell(f)g \otimes g_n \otimes \dots \otimes g_1](x_n, \dots, x_1) = \int_{x_n}^{\infty} dx \overline{f(x)}g(x)g_n(x_n) \dots g_1(x_1) \text{ and } \ell(f)\Omega = 0.$$

Choosing  $H_n$  as the pre-Hilbert space obtained from  $L^2(\mathbb{R})^{\otimes n}$  by dividing out the length-zero elements of the new semiinner product, we get an interacting Fock space.

**Example 2.7. The one-mode interacting Fock space and orthogonal polynomials.** Let  $\mu$  be a symmetric probability measure on  $\mathbb{R}$  with compact support so that all moments  $\int x^n \mu(dx)$  ( $n \in \mathbb{N}_0$ ) exist. It is well-known that there exists a sequence  $(\omega_n)_{n \in \mathbb{N}}$  of non-negative real numbers and a sequence  $(P_n)_{n \in \mathbb{N}_0}$  of (real) polynomials, such that  $P_0 = 1, P_1 = x,$

$$xP_n = P_{n+1} + \omega_n P_{n-1} \quad (n \geq 1),$$

and

$$\langle P_n, P_m \rangle := \int P_m(x)P_n(x)\mu(dx) = \delta_{mn}\omega_n \dots \omega_1.$$

Let us consider the *one-mode Fock space*  $\mathcal{F}(\mathbb{C})$ . Denote by  $e_n$  the basis vector of  $\mathbb{C}^{\otimes n}$  and equip the  $n$ -particle sector with a new (semi-)inner product by setting  $\langle e_n, e_n \rangle = \omega_n \dots \omega_1$ . Of course,  $\ell^*(e_1)$  has an adjoint. Dividing out the kernel of the new inner product (which is non-trivial, if and only if some of the  $\omega_n$  are 0) we obtain the *one-mode interacting Fock space*  $\mathcal{I}_\omega$ . In [1] Accardi and Bozejko (see also [9]) showed that the mapping  $e_n \mapsto P_n$  establishes a unitary  $U$  from the completion of  $\mathcal{I}_\omega$  onto  $L^2(\mathbb{R}, \mu)$ . Moreover, denoting  $a^* = a^*(e_1)$ , one obtains  $Ua^*U^*P_n = P_{n+1}$  and  $U(a^*+a)U^* = x$ . The last equation means that the operator of multiplication by  $x$  on  $L^2(\mathbb{R}, \mu)$  is represented on the one-mode interacting Fock space by the sum  $a^* + a$ .

For later use in Example 4.8 and as a motivation for Section 5 we present a variant of the preceding discussion. Assume that all  $\omega_n$  are different from 0. (This means that the support of  $\mu$  contains infinitely many points.) Let us use the normalized polynomials  $Q_n = \frac{1}{\sqrt{\omega_n \dots \omega_1}}P_n$ . The recursion formula becomes

$$xQ_n = \sqrt{\omega_{n+1}}Q_{n+1} + \sqrt{\omega_n}Q_{n-1} \quad (n \geq 1),$$

with  $Q_0 = 1$  and  $Q_1 = \frac{x}{\sqrt{\omega_1}}$ . Then the mapping  $e_n \mapsto Q_n$  establishes a unitary  $V$  from the usual full Fock space  $\mathcal{F}(\mathbb{C})$  onto  $L^2(\mathbb{R}, \mu)$ . Moreover, denoting by  $\ell^* = \ell^*(e_1)$  the usual creator, one obtains  $V\ell^*V^*Q_n = Q_{n+1}$  and  $V(\sqrt{\omega_N}\ell^* + \ell\sqrt{\omega_N})V^* = x$ . By  $\sqrt{\omega_N}$  we mean the function  $n \mapsto \omega_n$  of the number operator  $N: e_n \mapsto ne_n$ . In other words, instead of the symmetric part of the creator  $a^*$  on the interacting Fock space, we obtain the symmetric part of the modified creator  $\sqrt{\omega_N}\ell^*$  on the usual full Fock space. It is easy to see that  $a^* \mapsto \sqrt{\omega_N}\ell^*$  still defines a  $*$ -algebra monomorphism  $\mathcal{A}(\mathcal{I}) \rightarrow \mathcal{L}^a(\mathcal{F}(\mathbb{C}))$ , if some  $\omega_n$  are 0. In this case one just has to use the partial isometry  $V$  defined as above as long as  $\omega_n \neq 0$ , and

mapping  $e_n$  to 0 for all  $n \geq n_0$  where  $n_0$  is the smallest  $n$  for which  $\omega_n = 0$ . It is noteworthy, that  $V^*$  always is an isometry.

### 3. Pre-Hilbert Modules and Full Fock Modules Over $*$ -Algebras

Usually, semi-Hilbert modules are modules with a semiinner product which takes values in a  $C^*$ -algebra  $\mathfrak{Z}$ . In a  $C^*$ -algebra the positive elements and the positive functionals can be characterized in many equivalent ways. For instance, we can say an element  $z \in \mathfrak{Z}$  is positive, if it can be written in the form  $w^*w$  for suitable  $w \in \mathfrak{Z}$ . We can also give a weak definition and say that  $z$  is positive, if  $\varphi(z) \geq 0$  for all positive functionals  $\varphi$ . Here we can say a functional on  $\mathfrak{Z}$  is positive, if  $\varphi(z^*z) \geq 0$  for all  $z \in \mathfrak{Z}$ , but also if  $\|\varphi\| = \varphi(\mathbf{1})$ . Also if we want to divide out the length-zero elements in a semi-Hilbert module in order to have a pre-Hilbert module, we can either use the generalized Cauchy-Schwarz inequality

$$\langle x, y \rangle \langle y, x \rangle \leq \|\langle y, y \rangle\| \langle x, x \rangle$$

or we can use a weak Cauchy-Schwarz inequality

$$\varphi(\langle x, y \rangle) \varphi(\langle y, x \rangle) \leq \varphi(\langle y, y \rangle) \varphi(\langle x, x \rangle)$$

( $\varphi$  running over all positive functionals), because the positive functionals separate the points in a  $C^*$ -algebra. It is easy to equip the tensor product of Hilbert modules with an inner product and to show that it is positive.

In our applications we have to consider Hilbert modules over more general  $*$ -algebras, where the preceding characterizations of positive elements and positive functionals lead to different notions of positivity. The algebraic definition, where only elements of the form  $z^*z$  or sums of such are positive, excludes many good candidates for positive elements and, in fact, is too restrictive to include our applications. The weak definition, where  $z \geq 0$ , if  $\varphi(z) \geq 0$  for all positive functionals  $\varphi$ , allows for many positive elements. However, in many cases, for instance, if we want to show positivity of the inner product on the tensor product, this condition is uncontrollable. Here we give an extended algebraic definition where we put by hand some distinguished elements to be positive and consider a certain convex cone which is generated by these elements. Of course, a suitable choice of these distinguished elements depends highly on the concrete application.

If we want to divide out the length-zero elements, we should require that the positive functionals separate the points of the  $*$ -algebra. However, also here it turns out that we cannot consider all positive functionals (i.e. functionals  $\varphi$  on  $\mathfrak{Z}$  for which  $\varphi(z^*z) \geq 0$  for all  $z \in \mathfrak{Z}$ ), because we cannot guarantee that these functionals send our distinguished positive cone into the positive reals (see Remark 3.2).

In these notes, with one but important exception in 3.16 where we consider pre-Hilbert  $\mathfrak{Z}$ - $\mathfrak{C}$ -modules (i.e. a representation of  $\mathfrak{Z}$  on a pre-Hilbert space), we are concerned exclusively with two-sided modules over the same (unital)  $*$ -algebra  $\mathfrak{Z}$ . Also our definition of positivity of the inner product involves left multiplication. Therefore, we include the two-sided structure into the definitions. We remark that it is very well possible to generalize the definitions to  $\mathfrak{W}$ - $\mathfrak{Z}$ -modules, where  $\mathfrak{W}$  may be a  $*$ -algebra different from  $\mathfrak{Z}$  (with its own distinguished positive elements).

Restricting this to the case  $\mathfrak{W} = \mathbb{C}$  (where  $\mathbb{C}$  has the obvious positivity structure), we arrive at the notion of right pre-Hilbert  $\mathfrak{Z}$ -module (generalizing the well-known notion where  $\mathfrak{Z}$  is a  $C^*$ -algebra).

**Definition 3.1.** Let  $\mathfrak{Z}$  be a unital  $*$ -algebra. We say a subset  $P$  of  $\mathfrak{Z}$  is a  $\mathfrak{Z}$ -cone, if  $z \in P$  implies  $w^*zw \in P$  for all  $w \in \mathfrak{Z}$ . A *convex  $\mathfrak{Z}$ -cone* is a  $\mathfrak{Z}$ -cone  $P$  which is stable under sums (i.e.  $z, w \in P$  implies  $z + w \in P$ ).

Let  $S$  be a distinguished subset of  $\mathfrak{Z}$ . Then by  $P(S)$  we denote the convex  $\mathfrak{Z}$ -cone *generated* by  $S$  (i.e. the set of all sums of elements of the form  $w^*zw$  with  $z \in S, w \in \mathfrak{Z}$ ). If  $S$  contains  $\mathbf{1}$  and consists entirely of self-adjoint elements, then we say the elements of  $P(S)$  are  *$S$ -positive*. (In [16, Appendix C] we call such algebras with a positive  $\mathfrak{Z}$ -cone  *$P^*$ -algebras* and analyze them in more detail.)

Of course, in a reasonable choice  $S$  should contain only self-adjoint elements. If  $S = \{\mathbf{1}\}$ , then  $P(S)$  contains all sums over elements of the form  $z^*z$ , i.e. we recover the usual algebraic definition of positivity. In a reasonable choice for  $S$  these elements should also be positive. Therefore, we require  $\mathbf{1} \in S$ . In our applications we will identify  $\mathfrak{Z} \subset \mathcal{L}^a(G)$  as a  $*$ -algebra of operators on a pre-Hilbert space  $G$ , and  $S$  will typically be a set of elements  $z$  which can be written as a sum over  $Z^*Z$  where  $Z \in \mathcal{L}^a(G)$  but not necessarily  $Z \in \mathfrak{Z}$ . In other words, we have algebraic positivity in a bigger algebra.

*Remark 3.2.* Notice that even contradictory choices of  $S$  are possible. Consider, for instance, the  $*$ -algebra generated by one self-adjoint indeterminate  $x$ . Then both  $S = \{\mathbf{1}, x\}$  and  $S = \{\mathbf{1}, -x\}$  are possible choices. Indeed, there exist representations of this algebra which send either  $x$  or  $-x$  to a positive operator on a Hilbert space which, of course, cannot be done simultaneously. Notice that  $x \mapsto -x$  extends to an isomorphism which does not preserve either of the notions of positivity.

**Definition 3.3.** Let us fix a set  $S$  of self-adjoint elements in  $\mathfrak{Z}$  containing  $\mathbf{1}$ . A *pre-Hilbert  $\mathfrak{Z}$ -module* is a  $\mathfrak{Z}$ - $\mathfrak{Z}$ -module  $E$  with a sesquilinear *inner product*  $\langle \bullet, \bullet \rangle: E \times E \rightarrow \mathfrak{Z}$ , fulfilling the following requirements

$$\begin{aligned} \langle x, x \rangle = 0 &\Rightarrow x = 0 && \text{(definiteness),} \\ \langle x, yz \rangle &= \langle x, y \rangle z && \text{(right } \mathfrak{Z}\text{-linearity),} \\ \langle x, zy \rangle &= \langle z^*x, y \rangle && \text{(*-property),} \end{aligned}$$

and the positivity condition that for all choices of  $z \in S$  and of finitely many  $x_i \in E$  there exist finitely many  $z_k \in S$  and  $z_{ki} \in \mathfrak{Z}$ , such that

$$\langle x_i, z x_j \rangle = \sum_k z_{ki}^* z_k z_{kj}.$$

If definiteness is missing, then we speak of a *semiinner product* and a *semi-Hilbert module*.

Since  $\mathbf{1} \in S$ , the inner product is  *$S$ -positive* (i.e.  $\langle x, x \rangle \in P(S)$ ), and since  $S$  consists only of self-adjoint elements, the inner product is *symmetric* (i.e.  $\langle x, y \rangle = \langle y, x \rangle^*$ ) and *left anti-linear* (i.e.  $\langle xz, y \rangle = z^* \langle x, y \rangle$ ).

**Observation 3.4.** It is sufficient to check positivity on a subset  $E_g$  of  $E$  which generates  $E$  as a right module. Indeed, for finitely many  $x_i \in E$  there exist finitely many  $y_\ell \in E_g$  and  $z_{\ell i} \in \mathfrak{Z}$  such that  $x_i = \sum_\ell y_\ell z_{\ell i}$  for all  $i$ . It follows that for  $z \in S$

$$\langle x_i, zx_j \rangle = \sum_{\ell, m} z_{\ell i}^* \langle y_\ell, zy_m \rangle z_{mj} = \sum_{\ell, m, k} z_{\ell i}^* w_{k\ell}^* z_k w_{km} z_{mj} = \sum_k v_{ki}^* z_k v_{kj},$$

where  $v_{ki} = \sum_\ell w_{k\ell} z_{\ell i}$ .

**Proposition 3.5.** Let  $E$  and  $F$  be semi-Hilbert  $\mathfrak{Z}$ -modules. Their tensor product over  $\mathfrak{Z}$

$$E \odot F = E \otimes F / \{xz \otimes y - x \otimes zy\}$$

is turned into a semi-Hilbert  $\mathfrak{Z}$ -module by setting

$$\langle x \odot y, x' \odot y' \rangle = \langle y, \langle x, x' \rangle y' \rangle.$$

*Proof.* We only check the positivity condition, because the remaining conditions are clear. By Observation 3.4 it is sufficient to check it for elementary tensors  $x \odot y$ . So let  $x_i \odot y_i$  be finitely many elementary tensors in  $E \odot F$ , and let  $z \in S$ . Then

$$\langle x_i \odot y_i, zx_j \odot y_j \rangle = \langle y_i, \langle x_i, zx_j \rangle y_j \rangle = \sum_k \langle z_{ki} y_i, z_k z_{kj} y_j \rangle = \sum_{k, \ell} w_{\ell(ki)}^* z'_\ell w_{\ell(kj)},$$

where for each  $k$  we have finitely many elements  $w_{\ell(ki)} \in \mathfrak{Z}$  corresponding to the finitely many elements  $z_{ki} y_i$  in  $F$ .  $\square$

**Observation 3.6.** Also here it is sufficient to consider elementary tensors  $x_i \odot y_i$  where  $x_i$  and  $y_i$  come from (right) generating subsets of  $E$  and  $F$ , respectively. This follows, because any element in  $F$ , in particular an element of the form  $zy$ , can be written as sum over  $y_i z_i$ , and elements of the form  $x_i \odot zy = x_i z \odot y$ , clearly, span  $E \odot F$ .

So far we were concerned with semi-Hilbert modules. For several reasons it is desirable to have a strictly positive inner product. For instance, contrary to a semiinner product, an inner product guarantees for uniqueness of adjoints. In the sequel, we provide a quotienting procedure, which allows to construct a pre-Hilbert module out of a given semi-Hilbert module, if on  $\mathfrak{Z}$  there exists a separating set  $S^*$  of positive functionals which is compatible with the positivity structure determined by  $S$ . However, before we proceed we show a simple example of a  $*$ -algebra in which the points are not separated by states.

**Example 3.7.** Let us consider the complex numbers  $\mathbb{C}$  as a two-dimensional real algebra with basis  $\{1, \iota\}$ . The complexification of  $\mathbb{C}$  (i.e.  $\{\mu 1 + \nu \iota \mid \mu, \nu \in \mathbb{C}\}$ ) becomes a (complex)  $*$ -algebra, if we define 1 and  $\iota$  to be self-adjoint. In this  $*$ -algebra the element  $-1 = -(1^2) = \iota^2$  is negative and positive, so that  $\varphi(-1) \leq 0$  and  $\varphi(-1) \geq 0$ , i.e.  $\varphi(-1) = 0$  for all states  $\varphi$ . Of course,  $-1 \neq 0$ , so that in this  $*$ -algebra the states do not separate the points.

Another example is the  $*$ -algebra of differentials of complex polynomials in a real indeterminate  $t$ . Here we have  $dt^2 = 0$ . Since  $dt$  is self-adjoint, we conclude that  $\varphi(dt) = 0$  for all states  $\varphi$ . The next example is the Ito algebra of differentials of stochastic processes.



**Definition 3.8.** Let  $\mathfrak{A}$  be a unital  $*$ -algebra with a subset  $S$  of self-adjoint elements containing  $\mathbf{1}$ . We say a functional  $\varphi$  on  $\mathfrak{A}$  is  $S$ -positive, if  $\varphi(z) \geq 0$  for all  $z \in P(S)$ . Let  $S^*$  be a set of  $S$ -positive functionals. We say  $S^*$  separates the points (or is separating), if  $\varphi(z) = 0$  for all  $\varphi \in S^*$  implies  $z = 0$ .

**Observation 3.9.** If  $S^*$  is a set of functionals such that  $\varphi(z) \geq 0$  for all  $\varphi \in S^*$  and  $z \in S$ , and such that  $\varphi \in S^*$  implies that  $\varphi(z^* \bullet z) \in S$  for all  $z \in \mathfrak{A}$ , then the elements of  $S^*$  are  $S$ -positive. We can interpret the second property in the following way. If we construct the GNS-representation  $\pi$  for an element  $\varphi \in S^*$  (which is possible, because  $\varphi$  has just been established as  $S$ -positive, i.e. in particular, as positive in the usual sense), then for any vector  $g$  in the representation space  $G$  the functional  $\langle g, \bullet g \rangle$  is an element of  $S^*$ , too.

**Proposition 3.10.** Let  $\mathfrak{A}$  be a unital  $*$ -algebra with a subset  $S$  of self-adjoint elements containing  $\mathbf{1}$ , and let  $S^*$  be a separating set of  $S$ -positive functionals on  $\mathfrak{A}$ . Let  $E$  be a semi-Hilbert  $\mathfrak{A}$ -module. Then the set  $\mathcal{N} = \{x \in E: \langle x, x \rangle = 0\}$  is a two-sided  $\mathfrak{A}$ -submodule of  $E$ . Moreover, the quotient module  $E_0 = E/\mathcal{N}$  inherits a pre-Hilbert  $\mathfrak{A}$ -module structure by setting  $\langle x + \mathcal{N}, y + \mathcal{N} \rangle = \langle x, y \rangle$ .

*Proof.* As  $S^*$  is separating, we have  $x \in \mathcal{N}$ , if and only if  $\varphi(\langle x, x \rangle) = 0$  for all  $\varphi \in S^*$ . Let  $\varphi \in S^*$ . Then the sesquilinear form  $\langle x, y \rangle_\varphi = \varphi(\langle x, y \rangle)$  on  $E$  is positive. By Cauchy-Schwarz inequality we find that  $\langle x, x \rangle_\varphi = 0$  implies  $\langle y, x \rangle_\varphi = 0$  for all  $y \in E$ . Consequently,  $x, y \in \mathcal{N}$  implies  $x + y \in \mathcal{N}$ . Obviously,  $x \in \mathcal{N}$  implies  $xz \in \mathcal{N}$  for all  $z \in \mathfrak{A}$ . And by the  $*$ -property and Cauchy-Schwarz inequality we find  $x \in \mathcal{N}$  implies  $zx \in \mathcal{N}$  for all  $z \in \mathfrak{A}$ . Therefore,  $\mathcal{N}$  is a two-sided  $\mathfrak{A}$ -submodule of  $E$  so that  $E/\mathcal{N}$  is a two-sided  $\mathfrak{A}$ -module. Once again, by Cauchy-Schwarz inequality we see that  $\langle x + \mathcal{N}, y + \mathcal{N} \rangle$  is a well-defined element of  $\mathfrak{A}$ .  $\square$

**Observation 3.11.** Notice that an operator  $A \in \mathcal{L}^a(E)$  respects  $\mathcal{N}$ , automatically. In this case any adjoint  $A^* \in \mathcal{L}^a(E)$  gives rise to a unique adjoint of  $A$  in  $\mathcal{L}^a(E_0)$ .

*Remark 3.12.* Also if we consider pre-Hilbert  $\mathfrak{A}$ -modules  $E$  and  $F$ , their tensor product  $E \odot F$  may be only a semi-Hilbert module. However, if  $\mathfrak{A}$  is a  $C^*$ -algebra and  $E$  is complete, then the semiinner product on the tensor product becomes inner; see [10].

Now we are ready to define the full Fock module in our algebraic framework. The original definition in the framework of  $C^*$ -algebras is due to Pimsner [13] and Speicher [18]. By Remark 3.12 the quotient in the following definition can be avoided, if we restrict to Hilbert modules  $E$  (which does not mean that we complete the tensor products).

**Definition 3.13.** Let  $\mathfrak{A}$  be a unital  $*$ -algebra with a subset  $S$  of self-adjoint elements containing  $\mathbf{1}$ , and let  $S^*$  be a separating set of  $S$ -positive functionals on  $\mathfrak{A}$ . Let  $E$  be pre-Hilbert  $\mathfrak{A}$ -module. The full Fock module over  $E$  is the pre-Hilbert  $\mathfrak{A}$ -module

$$\mathcal{F}(E) = \left( \bigoplus_{n \in \mathbb{N}_0} E^{\odot n} \right) / \mathcal{N}$$

where we set  $E^{\odot 0} = \mathfrak{A}$  and  $E^{\odot 1} = E$ .

For  $x \in E$  we define the *creator*  $\ell^*(x) \in \mathcal{L}^a(\mathcal{F}(E))$  by setting

$$\ell^*(x)x_n \odot \dots \odot x_1 = x \odot x_n \odot \dots \odot x_1 \text{ and } \ell^*(x)z = xz.$$

The adjoint of  $\ell^*(x)$  is the *annihilator*  $\ell(x)$  defined by setting

$$\ell(x)x_n \odot \dots \odot x_1 = \langle x, x_n \rangle x_{n-1} \odot \dots \odot x_1 \text{ and } \ell(x)z = 0.$$

By  $\mathcal{A}(\mathcal{F}(E))$  we denote the unital  $*$ -subalgebra of  $\mathcal{L}^a(\mathcal{F}(E))$  generated by  $\ell^*(E)$  and  $\mathfrak{Z}$ , where  $\mathfrak{Z}$  acts canonically on  $\mathcal{F}(E)$  by left multiplication.

*Remark 3.14.* Like in the case of the usual full Fock space, the  $*$ -algebra  $\mathcal{A}(\mathcal{F}(E))$  is determined by the generalized Cuntz relations  $\ell(x)\ell^*(y) = \langle x, y \rangle$ ; see [13].

**3.15. Convention.** If  $\mathfrak{Z}$  is a  $C^*$ -algebra, then, unless stated otherwise explicitly, we always assume that  $\mathfrak{Z}$  comes along with its usual positivity structure. It is routine to check that in this case the usual definition of a two-sided pre-Hilbert module and ours coincide. In particular, our definition of full Fock module reduces to the original definition in [13, 18].

Before we come to our examples of full Fock modules, we describe a well-known construction which allows to relate a pre-Hilbert modules to spaces of operators between pre-Hilbert spaces which carry representations of  $\mathfrak{Z}$ . However, also here we have to pay attention to the problem of positivity.

**Example 3.16. Concrete pre-Hilbert modules.** Let  $\pi$  be a representation of  $\mathfrak{Z}$  on a pre-Hilbert space  $G$ . In other words,  $G$  is a  $\mathfrak{Z}$ - $\mathbb{C}$ -module and we can ask, whether (equipping  $\mathbb{C}$  with convex  $\mathbb{C}$ -cone generated by  $\mathbf{1}$  as positive elements, and extending our definition of pre-Hilbert modules to two-sided modules over different algebras in an obvious fashion)  $G$  with its natural inner product is a pre-Hilbert module. For this it is necessary and sufficient that  $\langle g, \pi(z)g \rangle \geq 0$  for all  $g \in G$  and all  $z \in S$ . (Indeed, for  $g_i \in G$  ( $i = 1, \dots, n$ ) we find  $\sum_{i,j} \bar{c}_i \langle g_i, \pi(z)g_j \rangle c_j = \sum_{i,j} \langle c_i g_i, \pi(z)g_j c_j \rangle \geq 0$  for all  $(c_i) \in \mathbb{C}^n$ . Therefore, the matrix  $(\langle g_i, \pi(z)g_j \rangle)$  is positive in  $M_n$  and, henceforth, can be written in the form  $\sum_k \bar{d}_{ki} d_{kj}$  for suitable  $(d_{ki}) \in M_n$ .) If  $\pi$  has this property, then we say it is *S-positive*.

Now let  $\pi$  be *S-positive* and let  $E$  be a pre-Hilbert  $\mathfrak{Z}$ -module. Then our tensor product construction goes through as before (notice that  $\text{id}_{\mathbb{C}}$  constitutes a separating set of positive functionals on  $\mathbb{C}$ ) and we obtain a pre-Hilbert  $\mathfrak{Z}$ - $\mathbb{C}$ -module  $H = E \odot G$ . In other words,  $H$  is a pre-Hilbert space with a representation  $\rho$  of  $\mathfrak{Z}$  which acts as  $\rho(z)(x \odot g) = zx \odot g$ . Additionally, we may interpret elements  $x$  as mappings  $L_x: g \mapsto x \odot g$  in  $\mathcal{L}^a(G, H)$  with adjoint  $L_x^*: y \odot g \mapsto \pi(\langle x, y \rangle)g$ . Of course,  $L_{zxz'} = \rho(z)L_x\pi(z')$ . Observe that  $x \mapsto L_x$  is one-to-one, if  $\pi$  is faithful. In this case also  $\rho$  is faithful.

Obviously, also  $\rho$  is *S-positive* so that we may continue the construction. For a tensor product of pre-Hilbert  $\mathfrak{Z}$ -modules  $E$  and  $F$  we find  $L_{x \odot y} = L_x L_y \in \mathcal{L}^a(G, E \odot F \odot G)$  where  $L_y \in \mathcal{L}^a(G, F \odot G)$  and  $L_x \in \mathcal{L}^a(F \odot G, E \odot F \odot G)$ . We recover the facts, well-known for Hilbert modules over  $C^*$ -algebras, that a two-sided pre-Hilbert module may be interpreted as a functor which sends one representation of  $\mathfrak{Z}$  to another, and that the composition of two such functors is the tensor product, at least within the category of *S-positive* representations.

**Example 3.17. Centered pre-Hilbert modules.** Let  $H$  and  $G$  be pre-Hilbert spaces, and let  $\mathfrak{Z} \subset \mathcal{L}^a(G)$  be a  $*$ -algebra of operators on  $G$  with positivity structure determined by the set  $S = \{1\}$ . Then  $E = H \otimes \mathfrak{Z}$  with the obvious operations becomes a pre-Hilbert  $\mathfrak{Z}$ -module with inner product  $\langle h \otimes z, h' \otimes z' \rangle = \langle h, h' \rangle z^* z'$ . (In fact,  $E$  is the exterior tensor product of the pre-Hilbert  $\mathbb{C}$ - $\mathbb{C}$ -module  $H$  and the pre-Hilbert  $\mathfrak{Z}$ - $\mathfrak{Z}$ -module  $\mathfrak{Z}$ ; see [10, 14].)  $E$  is an example for what we call a centered pre-Hilbert module in [14]. (Centered modules are two-sided modules which are generated as a module by those elements which commute with all algebra elements.) In [15] it is shown that all centered pre-Hilbert modules over a  $C^*$ -algebra may be embedded into a suitable completion of  $E$  for a suitable choice of the Hilbert space  $H$ . In [8] it is shown that under some normality and closure conditions any two-sided Hilbert  $\mathcal{B}(\overline{G})$ -module is of the form  $\mathcal{B}(\overline{G}, \overline{H \otimes G})$  for a suitable Hilbert space  $H$ .

One easily checks that  $E \odot E = H \otimes H \otimes \mathfrak{Z}$ . Continuing this procedure, we find  $\mathcal{F}(E) = \mathcal{F}(H) \otimes \mathfrak{Z}$  with inner product

$$\begin{aligned} \langle (h_n \otimes z_n) \odot \dots \odot (h_1 \otimes z_1), (h'_n \otimes z'_n) \odot \dots \odot (h'_1 \otimes z'_1) \rangle = \\ \langle h_n \otimes \dots \otimes h_1, h'_n \otimes \dots \otimes h'_1 \rangle z_1^* \dots z_n^* z'_n \dots z'_1. \end{aligned}$$

Again one can show that the full Fock module over an arbitrary centered pre-Hilbert module is contained in a suitable completion of  $\mathcal{F}(E)$ .

**Example 3.18. A non-centered example.** Let  $E$  be as in the preceding example. We perform the construction like in 3.16. If  $\mathfrak{Z}$  acts non-degenerately on  $G$ , then we find  $E \odot G = H \otimes G$  and the representation  $\rho$  of  $\mathfrak{Z}$  on  $H \otimes G$  is just  $1 \otimes \text{id}$ . Of course, also any  $S$ -positive representation of  $\mathfrak{Z}$  on  $H$  gives rise to a proper left action of  $\mathfrak{Z}$  on  $E$ . With such a left action  $E$  is no longer centered.

As a special example we set  $H = G$  and for  $\rho$  we choose  $\rho = \text{id} \otimes 1$ . This means that an element of  $\mathfrak{Z}$  now acts from the left as operator on the left factor of  $G \otimes G$ . (Here  $E = G \otimes \mathfrak{Z}$  may be understood as the exterior tensor product of the pre-Hilbert  $\mathfrak{Z}$ - $\mathbb{C}$ -module  $G$  and the pre-Hilbert  $\mathbb{C}$ - $\mathfrak{Z}$ -module  $\mathfrak{Z}$ .) Also this time the full Fock module over  $E$  may be identified with  $\mathcal{F}(G) \otimes \mathfrak{Z}$  where, however, now the inner product is given by

$$\begin{aligned} \langle (g_n \otimes z_n) \odot \dots \odot (g_1 \otimes z_1), (g'_n \otimes z'_n) \odot \dots \odot (g'_1 \otimes z'_1) \rangle = \\ \langle g_n \otimes (z_n g_{n-1}) \otimes \dots \otimes (z_2 g_1), g'_n \otimes (z'_n g'_{n-1}) \otimes \dots \otimes (z'_2 g'_1) \rangle z_1^* z'_1. \end{aligned} \quad (3.1)$$

Of course, an element  $z \in \mathfrak{Z}$  acts from the left on  $g_n \otimes \dots \otimes g_1 \otimes z'$  as an operator on the factor  $g_n$ .

**Example 3.19. Interacting Fock spaces arising from full Fock modules.** Let  $E$  be pre-Hilbert  $\mathfrak{Z}$ -module and suppose that  $\mathfrak{Z} \subset \mathcal{B}^a(G)$  acts ( $S$ -positively) on a pre-Hilbert space  $G$ . Let  $\Omega$  be a fixed unit vector in  $G$  and suppose that the state  $\langle \Omega, \bullet \Omega \rangle$  separates the elements of  $E$  in the sense that  $\langle \Omega, \langle x, x \rangle \Omega \rangle = 0$  implies  $x = 0$ . We set  $H_0 = \mathbb{C}\Omega$ . Referring again to the construction in the proof of Proposition 3.16, we denote by  $H_n = E^{\odot n} \odot \Omega$  ( $n \in \mathbb{N}$ ) the subspaces of  $E^{\odot n} \odot G$  consisting of all elements  $L_x \Omega$  ( $x \in E^{\odot n}$ ).

Then  $\mathcal{I} = \bigoplus_{n \in \mathbb{N}_0} H_n$  is an interacting Fock space based on  $H_1$ . Let  $G_\Omega = \mathfrak{Z}G$ . It is easy to see that  $H_n = E^{\odot n} \odot G_\Omega$  ( $n \in \mathbb{N}$ ). Thus,  $\mathcal{I}$  is just  $\mathcal{F}(E) \odot G_\Omega \ominus (\mathbf{1} - |\Omega\rangle\langle\Omega|)G_\Omega$ . The creators are given by  $a^*(h) = \ell^*(x) \odot \text{id} \upharpoonright \mathcal{I}$ , where  $x$  is the unique element in  $E$ , fulfilling  $L_x\Omega = h \in H_1$ . By construction,  $\ell^*(x) \odot \text{id}$  leaves invariant the subspace  $\mathcal{I}$  of  $\mathcal{F}(E) \odot G$ . Defining the projection  $p_\Omega = |\Omega\rangle\langle\Omega| \oplus \bigoplus_{n \in \mathbb{N}} \text{id}_{H_n}$  on  $\mathcal{F}(E) \odot G$ , we have  $a^*(h) = (\ell^*(x) \odot \text{id})p_\Omega$ , so that the adjoint of  $a^*(h)$  is given by  $a(h) = p_\Omega(\ell(x) \odot \text{id}) \upharpoonright \mathcal{I}$ . It is easy to see that  $a(h)$  maps into  $\mathcal{I}$ .

In Example 4.7 we will see that by this construction an arbitrary interacting Fock space based on  $H_1$  can be recovered from a full Fock module. If  $a^*(h)\Omega = 0$  implies  $a^*(h) = 0$ , then the whole construction also works for interacting Fock spaces based on more general pre-Hilbert spaces  $H$ .

**Example 3.20. The full Fock module related to the Anderson model [4].** The one-particle module is the centered module  $E$  from Example 3.17 with  $H = G = L^2(\mathbb{R})$  and the algebra  $\mathfrak{Z}$  is the  $C^*$ -algebra  $L^\infty(\mathbb{R})$  with its usual positivity structure. On the  $n$ -particle sector  $E^{\odot n}$  (in the notations of [4]) the (semi-)inner product is given by Equation (3.1). This means that it cannot be understood as the canonical inner product on the  $n$ -fold tensor product.

However, Example 3.18 tells us that the inner product immediately becomes the canonical inner product of the  $n$ -fold tensor product  $E^{\odot n}$ , if we put on  $E$  the correct left multiplication. The centered left action of an element  $z \in L^\infty(\mathbb{R})$  as suggested in [4] maps a function  $f(s)z'(t) \in L^2(\mathbb{R}) \otimes L^\infty(\mathbb{R})$  to the function  $f(s)z(t)z'(t)$ . The correct left action from Example 3.18, instead, maps  $f \otimes z'$  to the function  $z(s)f(s)z'(t)$ .

In [4] the creators  $A^*(x)$  ( $x \in E$ ) on the interacting Fock module had been defined by setting

$$A^*(x)x_n \otimes \dots \otimes x_1 = x \otimes x_n \otimes \dots \otimes x_1.$$

We remark that the relation  $zA^*(x) = A^*(zx) = A^*(x)z$  (where  $E$  is equipped with the centered left multiplication) cannot be fulfilled on the interacting Fock module whose inner product is defined by (3.1). Indeed, let  $x = \chi_{[0, \frac{1}{2}]} \otimes \mathbf{1}$  and  $z = \chi_{[0, \frac{1}{2}]} - \chi_{[\frac{1}{2}, 1]}$ . Then

$$\langle x \otimes x, A^*(xz)A^*(x)\mathbf{1} \rangle = \langle x \otimes x, xz \otimes x \rangle = \int_0^1 dr \int_0^1 ds (\chi_{[0, \frac{1}{2}]} - \chi_{[\frac{1}{2}, 1]})(s) = 0,$$

but

$$\begin{aligned} \langle x \otimes x, A^*(x)A^*(zx)\mathbf{1} \rangle &= \langle x \otimes x, x \otimes zx \rangle \\ &= \int_0^1 dr \int_0^1 ds (\chi_{[0, \frac{1}{2}]} - \chi_{[\frac{1}{2}, 1]})(t) = (\chi_{[0, \frac{1}{2}]} - \chi_{[\frac{1}{2}, 1]})(t) \neq 0. \end{aligned}$$

**Example 3.21. The full Fock module for the Lu-Ruggeri interacting Fock space.** In principle, our goal is to recover the inner product of elements in the interacting Fock space from Example 2.6 by the inner product of suitable elements in the full Fock module from the preceding example. Before we can do this it is necessary to modify this module slightly. On the one hand,  $L^2(\mathbb{R}) \otimes L^\infty(\mathbb{R})$  does

not yet contain the elements which we need. On the other hand, we will have to evaluate  $L^\infty$ -functions on the fixed point 0. This does not make sense in a function space whose functions are determined only almost everywhere. For these reasons our one-particle module is

$$E = \text{span}\{f \boxtimes z : (s, t) \mapsto f(s)\chi_{[0,s]}(t)z(t) \mid f \in L^2(\mathbb{R}^+), z \in \mathcal{C}_b(\mathbb{R}^+)\}$$

where  $\mathbb{R}^+ = [0, \infty)$ . One may understand the  $\boxtimes$ -sign as a time-ordered tensor product. Observe that not one of the non-zero functions in  $E$  is simple. Clearly,  $E$  is invariant under the left multiplication  $z(f \boxtimes z') = (zf) \boxtimes z'$  and the right multiplication  $(f \boxtimes z')z = f \boxtimes (z'z)$  by elements  $z \in \mathcal{C}_b(\mathbb{R}^+)$ . Moreover, the inner product

$$\langle f \boxtimes z, f' \boxtimes z' \rangle(t) = \int ds \overline{(f \boxtimes z)(s, t)} (f' \boxtimes z')(s, t) = \int_t^\infty ds \overline{f(s)z(t)} f'(s)z'(t)$$

maps into the continuous bounded functions on  $\mathbb{R}^+$  so that  $E$  becomes a pre-Hilbert  $\mathcal{C}_b(\mathbb{R}^+)$ -module.

Define the state  $\varphi(z) = z(0)$  on  $\mathcal{C}_b(\mathbb{R}^+)$ . One easily checks that

$$\varphi(\langle (f_n \boxtimes \mathbf{1}) \odot \dots \odot (f_1 \boxtimes \mathbf{1}), (g_n \boxtimes \mathbf{1}) \odot \dots \odot (g_1 \boxtimes \mathbf{1}) \rangle) = \langle f_n \otimes \dots \otimes f_1, g_n \otimes \dots \otimes g_1 \rangle$$

where the right-hand side is the inner product from Example 2.6.

#### 4. Full Fock Modules Associated With an Interacting Fock Space

Our goal is to associate with an arbitrary interacting Fock space  $\mathcal{I}$  a full Fock module in such a way that certain  $*$ -algebras of operators on  $\mathcal{I}$  may be represented as operators on that Fock module. In particular, we want to express the moments of operators on  $\mathcal{I}$  in the vacuum expectation  $\langle \Omega, \bullet \Omega \rangle$  by moments of the corresponding operators on the Fock module in a state of the form

$$\varphi(\langle \mathbf{1}, \bullet \mathbf{1} \rangle),$$

where  $\langle \mathbf{1}, \bullet \mathbf{1} \rangle$  denotes the vacuum conditional expectation on the Fock module, and where  $\varphi$  is a state. We will see that we can achieve our goal by a simple reinterpretation of the *graduation* of  $\mathcal{L}^a(\mathcal{I})$  in Definition 2.1. Since we work in a purely algebraic framework, we cannot consider the full  $*$ -algebra  $\mathcal{L}^a(\mathcal{I})$ . It is necessary to restrict to the  $*$ -algebra  $\mathcal{A}^0(\mathcal{I}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n^a(\mathcal{I})$ ; see Remark 4.4 below. Clearly,  $\mathcal{A}^0(\mathcal{I})$  is a graded  $*$ -algebra.

Let  $\mathcal{I} = \bigoplus_{n \in \mathbb{N}_0} H_n$  be an interacting Fock space and let  $S$  be the subset of  $\mathcal{L}_0^a(\mathcal{I})$  consisting of all elements of the form  $a^*a$  where, however,  $a \in \mathcal{L}^a(\mathcal{I})$ . As  $\mathcal{L}_k^a(\mathcal{I})\mathcal{L}_\ell^a(\mathcal{I}) \subset \mathcal{L}_{k+\ell}^a(\mathcal{I})$  we find that all spaces  $\mathcal{L}_m^a(\mathcal{I})$  are  $\mathcal{L}_0^a(\mathcal{I})$ - $\mathcal{L}_0^a(\mathcal{I})$ -modules. Clearly,

$$\langle x, y \rangle = x^*y$$

fulfills our positivity condition and all other properties of an  $\mathcal{L}_0^a(\mathcal{I})$ -valued inner product so that  $\mathcal{L}_m^a(\mathcal{I})$  becomes a pre-Hilbert  $\mathcal{L}_0^a(\mathcal{I})$ -module.

One easily checks that  $\mathcal{L}_k^a(\mathcal{I}) \odot \mathcal{L}_\ell^a(\mathcal{I}) = \text{span}(\mathcal{L}_k^a(\mathcal{I})\mathcal{L}_\ell^a(\mathcal{I}))$  via the identification  $x \odot y = xy$ . (See also Remark 4.4.) We set  $E^0 = \mathcal{L}_1^a(\mathcal{I})$  and define the *maximal*

full Fock module  $\mathcal{F}^0(\mathcal{I})$  associated with the interacting Fock space  $\mathcal{I}$  by

$$\mathcal{F}^0(\mathcal{I}) = \mathcal{F}(E^0) = \bigoplus_{n \in \mathbb{N}_0} (E^0)^{\odot n} \subset \bigoplus_{n \in \mathbb{N}_0} \mathcal{L}_n^a(\mathcal{I}).$$

We explain in Remark 4.3 in which sense this module is maximal.

Let  $A \in \mathcal{L}_m^a(\mathcal{I})$ . By setting

$$ax_n \odot \dots \odot x_1 = ax_n \dots x_1 = \begin{cases} Ax_n \dots x_1 & \text{for } n + m \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

we define an element  $a$  in  $\mathcal{L}^a(\mathcal{F}(E^0))$ .

**Theorem 4.1.** *The linear extension of the mapping  $A \mapsto a$  to all element  $a$  in  $\mathcal{A}^0(\mathcal{I})$  defines a  $*$ -algebra monomorphism  $\mathcal{A}^0(\mathcal{I}) \rightarrow \mathcal{L}^a(\mathcal{F}(E^0))$ .*

*Proof.* We perform the construction as in the proof of Proposition 3.16. One easily checks that  $\mathcal{F}^0(\mathcal{I}) \odot \mathcal{I} = \mathcal{I}$  and that  $\rho(a) = A$ . Therefore,  $A \mapsto a$  is injective and, clearly, a  $*$ -homomorphism. (Cf. also the appendix of [14].)  $\square$

**Theorem 4.2.** *For all  $A \in \mathcal{A}^0(\mathcal{I})$  we have*

$$\langle \Omega, A\Omega \rangle = \langle \Omega, \langle \mathbf{1}, a\mathbf{1} \rangle \Omega \rangle.$$

*Proof.* It is sufficient to check the statement for  $A \in \mathcal{L}_m^a(\mathcal{I})$ . If  $m \neq 0$ , then both sides are 0. If  $m = 0$ , then  $a\mathbf{1} = a = A$ . (Here we made the identifications  $\mathcal{L}_0^a(\mathcal{I})(E^0)^{\odot 0} \subset (E^0)^{\odot 0} = \mathcal{L}_0^a(\mathcal{I})$ .) Therefore, both sides coincide also for  $m = 0$ .  $\square$

*Remark 4.3.* The module  $\mathcal{F}^0(\mathcal{I})$  is maximal in the sense that the vacuum  $\mathbf{1}$  is cyclic for  $\mathcal{A}^0(\mathcal{I})$  and that  $\mathcal{A}^0(\mathcal{I})$  is the biggest subalgebra of  $\mathcal{L}^a(\mathcal{I})$  which may be represented on a purely algebraic full Fock module. Cf. also Remark 4.4.

The following somewhat lengthy remark explains to some extent why we have to restrict to  $\mathcal{A}^0(\mathcal{I})$ , and why  $\mathcal{L}_k^a(\mathcal{I}) \odot \mathcal{L}_\ell^a(\mathcal{I})$  cannot coincide with  $\mathcal{L}_{k+\ell}^a(\mathcal{I})$ . The reader who is not interested in these explanations may skip the remark.

**Remark 4.4. An excursion about duality.** In our framework, where the constructions of direct sum and tensor product are understood purely algebraically, there is a strong anti-relation between spaces which arise by such constructions and spaces of operators on them. For instance, a vector space  $V$  may be understood as the direct sum  $\bigoplus_{b \in B} (\mathbb{C}b)$  over all subspaces  $\mathbb{C}b$  where  $b$  runs over a basis  $B$  of  $V$ . To any  $b \in B$  we associate a linear functional  $\beta_b$  in the algebraic dual  $V'$  of  $V$  by setting  $\beta_b(b') = \delta_{bb'}$ . Then the direct sum  $V'_B = \bigoplus_{b \in B} (\mathbb{C}\beta_b)$  over all subspaces  $\mathbb{C}\beta_b$  of  $V'$  is a subspace of  $V'$  which depends on  $B$ , whereas the direct product over all  $\mathbb{C}\beta_b$  may be identified with  $V'$  itself. Obviously,  $V'_B$  is dense in  $V'$  with respect to the weak\* topology. Problems of this kind are weakened, when topology comes in, but they do not disappear. For instance, also the Banach space dual  $V^*$  of a Banach space  $V$ , usually, is much “bigger” than  $V$ .

As another example let us consider the space  $\mathcal{L}(V, W)$  of linear mappings between two vector spaces  $V$  and  $W$ ; cf. the appendix of [14]. Clearly,  $\mathcal{L}(V, W)$  is an  $\mathcal{L}(W)$ - $\mathcal{L}(V)$ -module. Denote by  $\mathcal{L}_f(V, W)$  the finite rank operators. Notice that

we may identify  $\mathcal{L}_f(V, W)$  with  $W \otimes V'$ . The elements of  $W' \otimes V$  act on  $\mathcal{L}(V, W)$  as linear functionals. Clearly,  $\mathcal{L}_f(V, W)$  is dense in  $\mathcal{L}(V, W)$  with respect to the locally convex Hausdorff topology coming from this duality. It is noteworthy that an element  $a \in \mathcal{L}(W)$  acts as right module homomorphism on both,  $\mathcal{L}(V, W)$  and  $\mathcal{L}_f(V, W)$ . Actually,  $a$  as an element of  $\mathcal{L}^r(\mathcal{L}(V, W))$  is uniquely determined by its action on  $\mathcal{L}_f(V, W)$  and, therefore, the algebras  $\mathcal{L}^r(\mathcal{L}(V, W))$  and  $\mathcal{L}^r(\mathcal{L}_f(V, W))$  are isomorphic; see [14].

Applying the preceding considerations in an appropriate way, one may show the following results. (Here  $\bar{\phantom{x}}$  means closure in a space of operators between pre-Hilbert spaces with respect to the weak topology.)

$$\begin{aligned} \overline{\text{span}}(\mathcal{L}_0^a(\mathcal{I})a^*(H)\mathcal{L}_0^a(\mathcal{I})) &= E^0 \\ \overline{\mathcal{L}_k^a(\mathcal{I}) \odot \mathcal{L}_\ell^a(\mathcal{I})} &= \mathcal{L}_{k+\ell}^a(\mathcal{I}). \end{aligned}$$

Finally, the action of  $\mathcal{A}^0(\mathcal{I})$  on  $\mathcal{F}^0(\mathcal{I})$  may be extended (uniquely) to an action of  $\mathcal{L}^a(\mathcal{I}) = \overline{\mathcal{A}^0(\mathcal{I})}$  on  $\overline{\mathcal{F}^0(\mathcal{I})}$ . This suggests also to introduce the closures  $\overline{E \odot \overline{F}}$  and  $\overline{\mathcal{F}(E)}$  as a *dual* tensor product and a *dual* full Fock module, respectively.

Now let us return to our original subject. So far we said what we understand by the maximal full Fock module associated with  $\mathcal{I}$ . What could be the minimal full Fock module? The answer is simple. A minimal Fock module should contain everything within the maximal Fock module, what is *cum grano salis* generated by  $a^*(H)$ , but not more.

Consequently, we restrict to the  $*$ -subalgebra  $\mathcal{A}(\mathcal{I})$  of  $\mathcal{A}^0(\mathcal{I})$  generated by  $a^*(H)$ . The graduation on  $\mathcal{A}^0(\mathcal{I})$  gives rise to a graduation on  $\mathcal{A}(\mathcal{I})$ . Using the notation

$$A^\epsilon = \begin{cases} A^* & \text{if } \epsilon = 1 \\ A & \text{if } \epsilon = -1 \end{cases}$$

we find

$$\begin{aligned} E_m &:= \mathcal{A}(\mathcal{I}) \cap \mathcal{L}_m^a(\mathcal{I}) \\ &= \text{span}\{a^{\epsilon_n}(f_n) \dots a^{\epsilon_1}(f_1) \mid f_i \in H, (\epsilon_1, \dots, \epsilon_n) \in \{-1, 1\}^n, \sum_{k=1}^n \epsilon_k = m\}. \end{aligned}$$

We set  $\mathfrak{F} = E_0 + \mathbb{C}\mathbf{1}$ . Again all  $E_m$  are pre-Hilbert  $\mathfrak{F}$ - $\mathfrak{F}$ -modules. However, now we have  $E_k \odot E_\ell = E_{k+\ell}$ . Set  $E = E_1$ . Clearly,  $E = \text{span}(\mathfrak{F}a^*(H)\mathfrak{F})$ .

**Definition 4.5.** By the *minimal* full Fock module associated with the interacting Fock space  $\mathcal{I}$  we mean  $\mathcal{F}_0(\mathcal{I}) = \mathcal{F}(E)$ .

**Theorem 4.6.** *Theorems 4.1 and 4.2 remain true when restricted to  $\mathcal{A}(\mathcal{I})$  and  $\mathcal{F}_0(\mathcal{I})$ . In particular,  $A \mapsto a$  defines a  $*$ -algebra isomorphism  $\mathcal{A}(\mathcal{I}) \rightarrow \mathcal{A}(\mathcal{F}_0(\mathcal{I}))$ .*

*Proof.* Clear. □

**Example 4.7. The converse of Example 3.19.** Let  $\mathcal{F}_0(\mathcal{I})$  be the minimal Fock module associated with an interacting Fock space based on  $H_1$ ; cf. Remark 2.3. Then the state  $\langle \Omega, \bullet \Omega \rangle$  separates the elements of  $E$ . Obviously, the pre-Hilbert

space  $\mathcal{F}_0(\mathcal{I}) \odot \Omega$  as constructed in Example 3.19, is nothing but  $\mathcal{I}$  and the creator  $\ell^*(x) \odot |\Omega\rangle\langle\Omega|$  on the former coincides with the creator  $a^*(h)$  on the latter, where  $h = x \odot \Omega$ . Therefore, the construction of the minimal Fock module is reversible.

We could ask, whether also the construction in Example 3.19 is reversible, in the sense that it is possible to recover the Fock module  $\mathcal{F}(E)$  we started with. However, as the construction only involves the subspace  $\mathcal{F}_0(\mathcal{I}) \odot \Omega$  and not the whole space  $\mathcal{F}_0(\mathcal{I}) \odot G$ , we definitely may lose information. For instance, if  $E$  is the direct sum of two  $\mathfrak{Z}_i$ -modules  $E_i$  ( $i = 1, 2$ ) with an obvious  $\mathfrak{Z}_1 \oplus \mathfrak{Z}_2$ -module structure, and if we choose a state  $\langle\Omega_1, \bullet\Omega_1\rangle$ , which is 0 on  $\mathfrak{Z}_2$ , then we lose all information about  $E_2$ .

**Example 4.8.** Let  $H$  be a pre-Hilbert space. Then the full Fock space  $\mathcal{I} = \mathcal{F}(H)$  is itself an interacting Fock space. On the minimal Fock module  $\mathcal{F}_0(\mathcal{I})$  we may represent not much more than the  $*$ -algebra  $\mathcal{A}(\mathcal{F}(H))$  which is generated by all creators  $a^*(f) = \ell^*(f)$  on the original Fock space. On the maximal Fock module  $\mathcal{F}^0(\mathcal{I})$  we may represent the full  $*$ -algebra  $\mathcal{A}^0(\mathcal{F}(H))$ . In particular, operators on  $\mathcal{F}(H)$  of the form  $z\ell^*(f)z'$  ( $f \in H; z, z' \in \mathcal{L}_0^a(\mathcal{F}(H))$ ) are represented by creators  $\ell^*(z\ell^*(f)z')$  on  $\mathcal{F}^0(\mathcal{I})$ .

For instance, in Example 2.7 we established an isometry  $\xi = V^*U: \mathcal{I}_\omega \rightarrow \mathcal{F}(\mathbb{C})$  from the one-mode interacting Fock into the one-mode full Fock space. We found  $\xi a^* \xi^* = \sqrt{\omega_N} \ell^*$ . This *squeezed* creator on the full Fock space, immediately, becomes the creator  $\ell^*(\sqrt{\omega_N} \ell^*)$  on the maximal Fock module  $\mathcal{F}^0(\mathcal{F}(\mathbb{C}))$  associated with  $\mathcal{F}(\mathbb{C})$ .

It is noteworthy that all ingredients of the construction of  $\mathcal{F}_0(\mathcal{I}_\omega)$  and  $\mathcal{F}^0(\mathcal{I}_\omega)$ , being subsets of  $\mathcal{A}^0(\mathcal{I}_\omega)$ , may be identified isometrically with ingredients of the corresponding construction of  $\mathcal{F}_0(\mathcal{F}(\mathbb{C}))$  and  $\mathcal{F}^0(\mathcal{F}(\mathbb{C}))$ , being subsets of  $\mathcal{A}^0(\mathcal{F}(\mathbb{C}))$ , via the mapping  $\Xi(\bullet) = \xi \bullet \xi^*$ .

What we did in Examples 2.7 and 4.8 for the one-mode interacting Fock space consisted in two parts. Firstly, we constructed an isometry from  $\mathcal{I}_\omega$  into  $\mathcal{F}(\mathbb{C})$ . Under this isometry the creator  $a^*$  on  $\mathcal{I}_\omega$  became the squeezed creator  $\sqrt{\omega_N} \ell^*$  on  $\mathcal{F}(\mathbb{C})$ . Secondly, after constructing the maximal Fock module  $\mathcal{F}_0(\mathcal{F}(\mathbb{C}))$  the squeezed creator became a usual creator on the maximal Fock module. In the following section we will see that these two steps are possible in general for a wide class of interacting Fock spaces.

### 5. Embeddable Interacting Fock Spaces

**Definition 5.1.** Let  $\mathcal{I} = \bigoplus_{n \in \mathbb{N}_0} H_n$  be an interacting Fock space based on  $H$ . We say  $\mathcal{I}$  is an *embeddable* interacting Fock space, if there exists an isometry  $\xi: \mathcal{I} \rightarrow \mathcal{F}(H)$ , which respects the  $n$ -particle sector, i.e.

$$\xi H_n \subset \overline{H^{\otimes n}} \text{ and } \xi \Omega = \Omega.$$

We say  $\mathcal{I}$  is *algebraically embeddable*, if  $\xi$  maps into  $\mathcal{F}(H)$ .

The following two theorems show that there exist many embeddable and many algebraically embeddable interacting Fock spaces. Actually, all known examples of interacting Fock spaces fit into the assumptions of one of these two theorems.



**Theorem 5.2.** *Let  $\mathcal{I}$  be an interacting Fock space based on  $H$  and define the surjective linear operator  $\Lambda: \mathcal{F}(H) \rightarrow \mathcal{I}$  by setting*

$$\Lambda(f_n \otimes \dots \otimes f_1) = a^*(f_n) \dots a^*(f_1)\Omega \text{ and } \Lambda(\Omega) = \Omega.$$

*Then the following two conditions are equivalent.*

- (i) *The operator  $\Lambda$  has an adjoint  $\Lambda^*$  in  $\mathcal{L}(\mathcal{I}, \overline{\mathcal{F}(H)})$ .*
- (ii) *There exists an operator  $L: \mathcal{F}(H) \rightarrow \overline{\mathcal{F}(H)}$  fulfilling  $LH^{\otimes n} \subset \overline{H^{\otimes n}}$ , such that*

$$\langle a^*(f_n) \dots a^*(f_1)\Omega, a^*(g_n) \dots a^*(g_1)\Omega \rangle = \langle f_n \otimes \dots \otimes f_1, Lg_n \otimes \dots \otimes g_1 \rangle.$$

*Moreover, if one of the conditions is fulfilled, then  $\mathcal{I}$  is embeddable.*

*Proof.* Clearly Condition (i) implies Condition (ii), because  $L = \Lambda^*\Lambda$  has the claimed properties. So let us assume that Condition (ii) is fulfilled.

Firstly, we show that  $\mathcal{I}$  is embeddable. The operator  $L$  must be positive. In particular,  $L$  is bounded below. Henceforth, by *Friedrich's theorem*  $L$  has a self-adjoint extension. Denote by  $\lambda$  the positive square root of this extension (whose domain, clearly, contains  $\mathcal{F}(H)$ ). Then the equation  $\xi a^*(f_n) \dots a^*(f_1)\Omega = \lambda f_n \otimes \dots \otimes f_1$  defines an isometry  $\xi: \mathcal{I} \rightarrow \overline{\mathcal{F}(H)}$ .

Secondly, we show existence of  $\Lambda^*$ . We have to show that for each  $I \in \mathcal{I}$  there exists a constant  $C_I > 0$ , such that  $\langle \Lambda F, I \rangle \leq \|F\| C_I$  for all  $F \in \mathcal{F}(H)$ . We may choose  $G \in \mathcal{F}(H)$  such that  $\Lambda G = I$ . Then our assertion follows from

$$\langle \Lambda F, I \rangle = \langle \Lambda F, \Lambda G \rangle = \langle F, LG \rangle \leq \|F\| \|LG\|.$$

□

**Theorem 5.3.** *Let  $\mathcal{I}$  be an interacting Fock space based on  $H$  and suppose that  $H$  has a countable Hamel basis. Then  $\mathcal{I}$  is algebraically embeddable.*

*Proof.* Let  $(e_i)_{i \in \mathbb{N}}$  denote the Hamel basis for  $H$ . We may assume this basis to be orthonormal. (Otherwise, apply the *Gram-Schmidt orthonormalization procedure*.) Enumerate the vectors  $e_k^n = e_{k_n} \otimes \dots \otimes e_{k_1}$  ( $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ ) in a suitable way. In other words, find a bijective mapping  $\sigma: \mathbb{N} \rightarrow \mathbb{N}^n$ . Then apply the orthonormalization to the total sequence  $(b_{\sigma(i)}^n)_{i \in \mathbb{N}}$  of vectors in  $H_n$  where we set  $b_k^n = a^*(e_{k_n}) \dots a^*(e_{k_1})\Omega$ . The result of orthonormalization is another sequence  $(c_i^n)_{i \in \mathbb{N}}$  of vectors, some of which are 0 and the remaining forming an orthonormal basis for  $H_n$ . Then

$$\xi c_i^n = \begin{cases} e_{\sigma(i)}^n & \text{for } c_i^n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

defines the claimed isometry. □

We remark that  $\xi$  has an adjoint  $\xi^*$  defined on the domain  $\mathcal{D}_{\xi^*} = \xi\mathcal{I} \oplus (\xi\mathcal{I})^\perp$  dense in  $\overline{\mathcal{F}(H)}$ . Clearly, this domain is mapped by  $\xi^*$  onto  $\mathcal{I}$ .

Before we show the implications of Definition 5.1, we provide a simple but useful factorization Lemma about operators on tensor products of vector spaces.

**Lemma 5.4.** *Let  $U, V, W$ , and  $X$  be vector spaces and let  $S \in \mathcal{L}(W, U)$  and  $T \in \mathcal{L}(V \otimes W, X)$  be operators, such that  $Sw = 0$  implies  $T(v \otimes w) = 0$  for all  $v \in V$ . Then there exists an operator  $R \in \mathcal{L}(V \otimes U, X)$ , such that*

$$T = R(\text{id} \otimes S).$$

*Proof.* Denote  $\mathcal{N} = \ker(S)$ . Then there exists a subspace  $\mathcal{N}^0 \subset W$ , such that  $W = \mathcal{N}^0 \oplus \mathcal{N}$  and  $S \upharpoonright \mathcal{N}^0$  is a bijective mapping onto  $SW$ . Analogously, we may find  $(SW)^0$ , such that  $U = SW \oplus (SW)^0$ . In this way we expressed  $S$  as the mapping

$$S = (S \upharpoonright \mathcal{N}^0) \oplus 0: \mathcal{N}^0 \oplus \mathcal{N} \longrightarrow SW \oplus (SW)^0.$$

Defining the mapping

$$S^{\text{inv}} = (S \upharpoonright \mathcal{N}^0)^{-1} \oplus 0: SW \oplus (SW)^0 \longrightarrow \mathcal{N}^0 \oplus \mathcal{N},$$

we find  $S^{\text{inv}}S = \mathbf{1} \oplus 0$  on  $\mathcal{N}^0 \oplus \mathcal{N}$ .

Set  $R = T(\text{id} \otimes S^{\text{inv}})$ . Then for all  $v \in V$  and  $w \in \mathcal{N}$  we have  $R(\text{id} \otimes S)(v \otimes w) = 0 = T(v \otimes w)$  and for  $w \in \mathcal{N}^0$  we find  $R(\text{id} \otimes S)(v \otimes w) = T(v \otimes S^{\text{inv}}Sw) = T(v \otimes w)$ .  $\square$

The basis for our application of Lemma 5.4 is the identification

$$\mathcal{F}(H) = H \otimes \mathcal{F}(H) \oplus \mathbb{C}\Omega. \quad (5.1)$$

If  $S$  is a mapping on  $\mathcal{F}(H)$ , then by  $\text{id} \otimes S$  we mean the mapping  $\text{id} \otimes S \oplus 0$  acting on the right-hand side of (5.1). We have the commutation relation

$$\ell^*(f)S = (\text{id} \otimes S)\ell^*(f).$$

Notice also that  $\overline{\mathcal{F}(H)} \supset H \otimes \overline{\mathcal{F}(H)} \oplus \mathbb{C}\Omega$ .

**Theorem 5.5.** *Let  $\mathcal{I}$  be an embeddable interacting Fock space based on  $H$ . Then there exists a mapping  $\varkappa: (H \otimes \mathcal{D}_{\xi^*} \oplus \mathbb{C}\Omega) \rightarrow \mathcal{D}_{\xi^*}$ , respecting the  $n$ -particle sectors, such that*

$$\varkappa\ell^*(f) = \xi a^*(f)\xi^*$$

for all  $f \in H$ . In other words, the mapping  $a^*(f) \mapsto \varkappa\ell^*(f)$  extends to a  $*$ -algebra monomorphism  $\mathcal{A}(\mathcal{I}) \rightarrow \mathcal{L}^a(\mathcal{D}_{\xi^*})$  and the vacuum expectation is mapped to the vacuum expectation.

If  $\mathcal{I}$  is even algebraically embeddable, then  $\varkappa\ell^*(f)$  is an element of  $\mathcal{L}^a(\mathcal{F}(H))$ .

*Remark 5.6.* Of course,  $\varkappa\ell^*(f)$  has an adjoint (even an adjoint which leaves invariant the domain  $\mathcal{D}_{\xi^*}$ ). However, notice that this does not imply that  $\varkappa$  has an adjoint.

*Proof of Theorem 5.5.* We have  $\Lambda\ell^*(f) = a^*(f)\Lambda$ . In particular, if  $\Lambda F = 0$  for some  $F \in \mathcal{F}(H)$ , then  $\Lambda(f \otimes F) = \Lambda(\ell^*(f)F) = a^*(f)\Lambda F = 0$  for all  $f \in H$ .

We set  $V = H$ ,  $W = \mathcal{F}(H)$ ,  $U = \xi\mathcal{I}$ , and  $X = \overline{\mathcal{F}(H)}$ . Furthermore, we define  $S = \xi\Lambda \in \mathcal{L}(W, U)$  and  $T = S \upharpoonright (H \otimes W)$ . Clearly, the assumptions of Lemma 5.4 are fulfilled. Therefore, there exists a mapping  $R \in \mathcal{L}(V \otimes U, X) = \mathcal{L}(H \otimes \mathcal{D}_{\xi^*}, \overline{\mathcal{F}(H)})$ , such that  $T(f \otimes F) = R(f \otimes SF)$  for all  $f \in H$  and all  $F \in \mathcal{F}(H)$ .

We have

$$\begin{aligned} \xi a^*(f) \xi^* (\xi \Lambda) F &= \xi a^*(f) \Lambda F = \xi \Lambda \ell^*(f) F \\ &= T(f \otimes F) = R(f \otimes SF) = R \ell^*(f) (\xi \Lambda) F. \end{aligned}$$

Since the domain of  $R \ell^*(f)$  is  $U$  and  $\xi \Lambda F$  ( $F \in \mathcal{F}(H)$ ) runs over all elements of  $U$ , we find  $\xi a^*(f) \xi^* \upharpoonright U = R \ell^*(f)$ . We define  $\varkappa \in \mathcal{L}(H \otimes \mathcal{D}_{\xi^*} \oplus \mathbb{C}\Omega, X)$  by setting

$$\varkappa(f \otimes F) = \begin{cases} R(f \otimes F) & \text{for } F \in \xi \mathcal{I} \\ 0 & \text{for } F \in (\xi \mathcal{I})^\perp \end{cases}$$

and  $\varkappa(\Omega) = 0$ . Then  $\varkappa \ell^*(f) = \xi a^*(f) \xi^*$ . Clearly, the range of  $\varkappa$  is  $\xi \mathcal{I}$ , because the range of  $\xi a^*(f) \xi^*$  is contained in  $\xi \mathcal{I}$ .  $\square$

We define  $\lambda = \xi \Lambda$  and denote by  $\lambda_n$  the restriction of  $\lambda$  to the  $n$ -particle sector. Notice that  $\lambda_n$  is a mapping  $H^{\otimes n} \rightarrow \overline{H}^{\otimes n}$ . Denote also by  $\varkappa_n$  the restriction of  $\varkappa$  to the  $n$ -particle sector of  $\mathcal{D}_{\xi^*}$ .

**Corollary 5.7.**  $\lambda$  fulfills

$$\lambda \upharpoonright (H \otimes \mathcal{F}(H)) = \varkappa(\text{id} \otimes \lambda).$$

In terms of  $n$ -particle sectors this becomes the recursion formula

$$\lambda_{n+1} = \varkappa_{n+1}(\text{id} \otimes \lambda_n) \text{ and } \lambda_0 = \text{id}_{\mathbb{C}\Omega}$$

for  $\lambda_n$ . The recursion formula is resolved uniquely by

$$\lambda_n = \varkappa_n(\text{id} \otimes \varkappa_{n-1}) \dots (\text{id}^{\otimes(n-1)} \otimes \varkappa_1) \quad (n \geq 1).$$

*Proof.* We have

$$\begin{aligned} \varkappa(\text{id} \otimes \lambda)(f_n \otimes \dots \otimes f_1) &= \varkappa \ell^*(f_n) \lambda(f_{n-1} \otimes \dots \otimes f_1) \\ &= \xi a^*(f_n) \xi^* \xi \Lambda(f_{n-1} \otimes \dots \otimes f_1) = \xi \Lambda \ell^*(f_n)(f_{n-1} \otimes \dots \otimes f_1) \\ &= \lambda(f_n \otimes \dots \otimes f_1). \end{aligned}$$

$\square$

**Corollary 5.8.** We have

$$\begin{aligned} \langle a^*(f_n) \dots a^*(f_1) \Omega, a^*(g_n) \dots a^*(g_1) \Omega \rangle \\ = \langle \varkappa_n \ell^*(f_n) \dots \varkappa_1 \ell^*(f_1) \Omega, \varkappa_n \ell^*(g_n) \dots \varkappa_1 \ell^*(g_1) \Omega \rangle. \end{aligned}$$

**Theorem 5.9.** Let  $\mathcal{I}$  be an algebraically embeddable interacting Fock space based on  $H$ . Then the mapping

$$a^*(f) \longmapsto \ell^*(\varkappa \ell^*(f))$$

extends to a  $*$ -algebra monomorphism from  $\mathcal{A}^0(\mathcal{I})$  into the  $*$ -algebra of adjointable operators on the maximal full Fock module  $\mathcal{F}^0(\mathcal{F}(H))$  associated with  $\mathcal{F}(H)$ . (Here the full Fock space  $\mathcal{F}(H)$  is interpreted as an interacting Fock space.) Also Theorem 4.2 remains true.

*Proof.*  $\varkappa \ell^*(f) = \xi a^*(f) \xi^*$  is an element of  $E^0 = \mathcal{L}_1^a(\mathcal{F}(H))$  and  $\Xi(\bullet) = \xi \bullet \xi^*$  is a  $*$ -algebra monomorphism  $\mathcal{A}^0(\mathcal{I}) \rightarrow \mathcal{A}^0(\mathcal{F}(H))$ . Validity of Theorem 4.2 follows by  $\xi \Omega = \Omega$ .  $\square$

## 6. The Symmetric Fock Space

In this section we discuss how the symmetric (or Bose) Fock space fits into the set-up of interacting Fock spaces. In particular, we identify concretely several mappings which played a crucial role in the preceding section.

Let  $H$  be a pre-Hilbert space. On  $H^{\otimes n}$  ( $n \in \mathbb{N}$ ) we define a projection  $p_n$  by setting

$$p_n f_n \otimes \dots \otimes f_1 = \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma_n} \otimes \dots \otimes f_{\sigma_1}.$$

The range of this projection is the  $n$ -fold symmetric tensor product  $H^{\otimes_s n}$ . By  $p$  we denote the projection  $\bigoplus_{n \in \mathbb{N}_0} p_n$  on the full Fock space  $\mathcal{F}(H)$  (where  $p_0 = 1$ ). The range  $\Gamma(H) = p\mathcal{F}(H)$  of  $p$  is the symmetric Fock space over  $H$ . Let  $N$  denote the number operator on  $\mathcal{F}(H)$ , i.e.  $NF_n = nF_n$  for  $F_n \in H^{\otimes n}$ . By setting  $H_n = H^{\otimes_s n}$  and  $a^*(f) = \sqrt{N}p\ell^*(f)$  we turn  $\Gamma(H)$  into an interacting Fock space based on  $H = H_1$ . Indeed, from  $p\ell^*(f)p = p\ell^*(f)$  it follows that  $a^*(f)$  has an adjoint  $\ell^*(f)p\sqrt{N}$ .

Defining  $\xi$  as the canonical embedding of  $\Gamma(H)$  into  $\mathcal{F}(H)$ , we see that  $\Gamma(H)$  is algebraically imbeddable. Notice that  $\xi^* = p$ . But also the stronger conditions of Theorem 5.2 are fulfilled (even leaving invariant the algebraic domain). Indeed, from the commutation relation  $\ell^*(f)\sqrt{N} = \sqrt{N-1}\ell^*(f)$  we find that

$$\begin{aligned} a^*(f_n) \dots a^*(f_1)\Omega &= p\sqrt{N} \dots \sqrt{N-n+1}\ell^*(f_n) \dots \ell^*(f_1) \\ &= p\sqrt{N} \dots \sqrt{N-n+1}f_n \otimes \dots \otimes f_1, \end{aligned}$$

i.e.  $\Lambda = p\sqrt{N!}$  where  $\sqrt{N!}$  denotes the operator sending  $F_n \in H^{\otimes n}$  to  $\sqrt{n!}F_n$ . Of course,  $\Lambda^* = \xi\Lambda\xi$ . So, if we are sloppy in distinguishing between  $\Gamma(H)$  and the subspace  $p\mathcal{F}(H)$  of  $\mathcal{F}(H)$ , then  $\Lambda$  is symmetric and coincides more or less with  $\lambda$ . Of course,  $L = pN!$ . The definition of  $a^*(f)$  yields directly  $\varkappa = p\sqrt{N}$ . We may verify explicitly the recursion formula in Corollary 5.7.

We easily verify the well-known CCR  $a(f)a^*(g) - a^*(g)a(f) = \langle f, g \rangle$ , or equivalently

$$a(f)a^*(g) = a^*(g)a(f) + \langle f, g \rangle$$

Here we see that the algebra  $\mathfrak{Z} = E_0$ , over which the minimal Fock module is a two-sided module, contains already the quite complicated operator  $a^*(g)a(f) + \langle f, g \rangle$  commuting with the number operator. The complications are caused by the fact that the projection  $p_n$  on the  $n$ -particle sector acts on all tensors of its argument. This is extremely incompatible with what creators on a full Fock space can do, which only act at the first tensor. Correspondingly, the additional algebraic structure which we introduce in the module description has to do a lot to repair this ‘defect’.

On the other hand, it is well-known that the symmetric Fock space over  $L^2(\mathbb{R}^+)$  is isomorphic to the space of time ordered functions (henceforth, *time ordered Fock space*) considered in Example 2.6. Also here we can write down the operator  $L$ . However, if  $F_n(t_n, \dots, t_1)$  is a time-ordered function, and if we ‘create’ a function  $f_{n+1}$ , then we find  $f_{n+1}(t_{n+1})F_n(t_n, \dots, t_1)$ . In order to project this function to the time-ordered subspace, we need only to look for the relation between  $t_{n+1}$  and  $t_n$ .

The 'deeper' time arguments are not involved by the projection. This explains why the module description of the time ordered Fock space is much more transparent and also more illuminating than the module description of the symmetric Fock space  $\Gamma(L^2(\mathbb{R}^+))$ .

We see that, although a module description is in principle always possible, we must choose carefully for which of the interacting Fock spaces we try a module description. A good criterion is to look at how complicated the algebra  $\mathfrak{J}$  is. Fortunately, in all applications there are natural choices for  $\mathfrak{J}$  and the image of  $\mathfrak{J}$  in the algebra  $\mathcal{A}(\mathcal{F}(E))$ , usually, is much 'smaller' than  $\mathcal{A}(\mathcal{F}(E))$ .

## References

1. Accardi, L., and Bozejko, M.: Interacting Fock spaces and gaussianization of probability measures, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **1** (1998) 663–670.
2. Accardi, L., Franz, U., and Skeide, M.: Renormalized squares of white noise and other non-gaussian noises as Lévy processes on real Lie algebras, *Commun. Math. Phys.*, **228** (2002) 123–150. (Rome, Volterra-Preprint 2000/0423).
3. Accardi, L., and Lu, Y.G.: The Wigner semi-circle law in quantum electro dynamics, *Commun. Math. Phys.* **180** (1996) 605–632.
4. Accardi, L., Lu, Y.G., and Volovich, I.V.: *Interacting Fock spaces and Hilbert module extensions of the Heisenberg commutation relations*, IAS Publications, Kyoto, 1997.
5. Accardi, L., Lu, Y.G., and Volovich, I.V.: White noise approach to classical and quantum stochastic calculi. Preprint, Rome, 1999. To appear in the lecture notes of the Volterra International School of the same title, held in Trento.
6. Accardi, L., and Skeide, M.: Hilbert module realization of the square of white noise and the finite difference algebra, *Math. Notes* **86** (2000) 803–818. (Rome, Volterra-Preprint 1999/0384).
7. Accardi, L., and Skeide, M.: On the relation of the square of white noise and the finite difference algebra, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **3** (2000) 185–189. (Rome, Volterra-Preprint 1999/0386).
8. Bhat, B.V.R., and Skeide, M.: Tensor product systems of Hilbert modules and dilations of completely positive semigroups, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **3** (2000) 519–575. (Rome, Volterra-Preprint 1999/0370).
9. Hora, A., and Obata, N.: *Quantum probability and spectral analysis of graphs*, Springer, 2007.
10. Lance, E.C.: *Hilbert  $C^*$ -modules*, Cambridge University Press, 1995.
11. Lu, Y.G., and Ruggeri, S.: A new example of interacting free Fock space. Preprint, Rome, 1998.
12. Paschke, W.L.: Inner product modules over  $B^*$ -algebras, *Trans. Amer. Math. Soc.* **182** (1973) 443–468.
13. Pimsner, M.V.: A class of  $C^*$ -algebras generalizing both Cuntz-Krieger algebras and crossed products by  $\mathbf{Z}$ , in D.V. Voiculescu, editor, *Free probability theory* **12** (1997) 189–212, Fields Inst. Commun.
14. Skeide, M.: Hilbert modules in quantum electro dynamics and quantum probability, *Commun. Math. Phys.* **192** (1998) 569–604. (Rome, Volterra-Preprint 1996/0257).
15. Skeide, M.: Generalized matrix  $C^*$ -algebras and representations of Hilbert modules, *Mathematical Proceedings of the Royal Irish Academy* **100A** (2000) 11–38. (Cottbus, Reihe Mathematik 1997/M-13).
16. Skeide, M.: *Hilbert modules and applications in quantum probability*. Habilitationsschrift, Cottbus, 2001. Available at [http://www.math.tu-cottbus.de/INSTITUT/lswas/\\_skeide.html](http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html).
17. Sniady, P.: Quadratic bosonic and free white noise, *Commun. Math. Phys.* **211** (2000) 615–628.

18. Speicher, R.: *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, Number 627 in Mem. Amer. Math. Soc. American Mathematical Society, 1998.
19. Voiculescu, D.V.: Dual algebraic structures on operator algebras related to free products, *J. Operator Theory* **17** (1987) 85–98.
20. Voiculescu, D.V.: Operations on certain non-commutative operator-valued random variables, *Astérisque* **232** (1995) 243–275.

LUIGI ACCARDI: CENTRO VITO VOLTERRA, UNIVERSITÀ DEGLI STUDI DI ROMA “TOR VERGATA”, VIA COLUMBIA 2, 00133 ROMA, ITALY

*E-mail address:* `accardi@volterra.mat.uniroma2.it`

*URL:* `http://volterra.mat.uniroma2.it`

MICHAEL SKEIDE: DIPARTIMENTO S.E.G.E S., UNIVERSITÀ DEGLI STUDI DEL MOLISE, VIA DE SANCTIS, 86100 CAMPOBASSO, ITALY

*E-mail address:* `skeide@unimol.it`

*URL:* `http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html`