

# Cesàro Hilbert Space and the Lévy Laplacian

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## Abstract

In this paper we introduce a new scalar product on distribution spaces based on the Cesàro mean of a sequence. We then use this scalar product to construct a family of separable Hilbert spaces  $\mathcal{H}_C$ , called Cesàro Hilbert spaces and naturally associated to the Lévy Laplacian. Finally we use the essentially infinite dimensional character of the Lévy Laplacian to construct a class of solutions of the Lévy heat equation which has no finite dimensional (or “regular” infinite dimensional) analogue.

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# Introduction

The Lévy Laplacian  $\Delta_L$ , introduced by P. Lévy [18], has recently attracted much attention for its peculiar and unexpected properties which have been the origin of what is nowadays called “essentially infinite dimensional analysis”. This Laplacian has been studied by several authors within the framework of white noise analysis, initiated by T. Hida [13], e.g [5], [10], [15], [16], [19], [20], [22], [23], [24]. See also [9] for a Dirichlet form approach. On the other hand Accardi, Gibilisco and Volovich [1], [2] proved that a parallel transport is associated to a connection 1-form satisfying Yang-Mills equation if and only if it is a harmonic function for the Lévy Laplacian (defined on an appropriate space). In other contexts, the Schrödinger and the heat equations with the Lévy Laplacian are related to some problems of quantum statistical physics. Finally interesting relations between Lévy Laplacian and square of quantum white noise have been suggested in [3], [21] and [22].

During the study of the above mentioned developments, a key role has been played by the Cesàro mean of sequences. However, the following basic problem has remained open for a long time: to find a natural and non-trivial vector space where the Cesàro mean defines a pre-scalar product and whose completion for this pre-scalar product is separable. The main purpose of this paper is to solve this problem. Namely, extending the idea in example 3 of [8], we construct a new family of separable Hilbert spaces of distributions on a standard triple [20] whose scalar product, called the Cesàro scalar product, is defined in terms of a Cesàro mean and it is naturally related to the notion of Cesàro trace.

The paper is organized as follows. In the first section we summarize some basic definitions and results in white noise analysis. In the second section we introduce the notion of Cesàro scalar product. Then, we construct the Hilbert spaces  $\mathcal{H}_C(S, \nu)$  and we give some of their properties. In the third section we construct some solutions for heat equation with Lévy Laplacian and we investigate special representations using the Cesàro trace on a suitable Hilbert space.

## 1 Preliminaries

In this section, following [20], we introduce some basic notions of Hida white noise theory.

## 1.1 Standard white noise triples

Let  $H$  be a real separable (infinite dimensional) Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $|\cdot| =: |\cdot|_0$ . Let  $D \geq 1$  be a positive self-adjoint operator in  $H$  with Hilbert-Schmidt inverse. Then there exist a sequence of positive numbers  $1 < \lambda_1 \leq \lambda_2 \leq \dots$  and a complete orthonormal basis of  $H$ ,  $e := \{e_n\}_{n=1}^\infty \subseteq \text{Dom}(D)$  such that

$$De_n = \lambda_n e_n, \quad |e_n|_0 = 1, \quad \sum_{n=1}^{\infty} \lambda_n^{-2} = \|D^{-1}\|_{HS}^2 < \infty$$

For every  $p \in \mathbb{R}$  we define:

$$|\xi|_p^2 := \sum_{n=1}^{\infty} \langle \xi, e_n \rangle^2 \lambda_n^{2p} = |D^p \xi|_0^2, \quad \xi \in H$$

The fact that, for  $\lambda > 1$ , the map  $p \mapsto \lambda^p$  is increasing implies that:

(i) for  $p \geq 0$ , the space  $H_p$ , of all  $\xi \in H$  with  $|\xi|_p < \infty$ , is a Hilbert space with norm  $|\cdot|_p$  and, if  $p \leq q$ , then  $H_q \subseteq H_p$ ;

(ii) denoting  $H_{-p}$  the  $|\cdot|_{-p}$ -completion of  $H$  ( $p \geq 0$ ), if  $0 \leq p \leq q$ , then  $H_{-p} \subseteq H_{-q}$ .

This construction gives a decreasing chain of Hilbert spaces  $\{H_p\}_{p \in \mathbb{R}}$  with natural continuous inclusions  $H_q \rightarrow H_p$  ( $p \leq q$ ). Defining the countably Hilbert nuclear space (see e.g. [12]):

$$E := \text{projlim}_{p \rightarrow \infty} H_p \cong \bigcap_{p \geq 0} H_p$$

the strong dual space  $E^*$  of  $E$  is:

$$E^* := \text{indlim}_{p \rightarrow \infty} H_{-p} \cong \bigcup_{p \geq 0} H_{-p}$$

and the triple

$$E \subset H \equiv H^* \subset E^* \tag{1.1}$$

is called a real standard triple. The complexifications of  $H_p$ ,  $E$  and  $H$  respectively will be denoted

$$N_p := H_p + iH_p; \quad N := E + iE; \quad K := H + iH \tag{1.2}$$

Notice that  $e = \{e_n\}_{n=1}^\infty$  is also a complete orthonormal basis of  $K$ . Thus the complexification of the standard triple (1.1) is:

$$N \subset K \subset N^*$$

When dealing with complex Hilbert spaces, we will always assume that the scalar product is linear in the second factor and the duality  $\langle N^*, N \rangle$ , also denoted  $\langle \cdot, \cdot \rangle$ , is defined so to be compatible with the inner product of  $K$ . Thus the natural embedding  $x \in N \mapsto x^* \in N^*$  is antilinear.

A typical example of the structure described above is the standard white noise triple (1.1):

$$E \equiv S(\mathbb{R}) \subset H = L^2(\mathbb{R}, dt) \subset E^* = S'(\mathbb{R})$$

where:  $S(\mathbb{R})$  is the space of rapidly decreasing functions on  $\mathbb{R}$ ;  $L^2(\mathbb{R}, dt)$  is the Hilbert space of  $\mathbb{C}$ -valued square-integrable functions on  $\mathbb{R}$ ;  $S'(\mathbb{R})$  is the space of tempered distributions;  $D$  is the number operator  $D = -\frac{d^2}{dt^2} + t^2 + 1$  and  $\{e_n\}_{n=1}^\infty$  is the orthonormal basis of  $L^2(\mathbb{R}, dt)$  constituted by the Hermite functions [15]

$$e_n(t) = \frac{1}{\sqrt{\sqrt{\pi} 2^n n!}} H_n(t) e^{-t^2/2}$$

where  $H_n(t) = (-1)^n e^{t^2} \left(\frac{d}{dt}\right)^n e^{-t^2}$  is the  $n$ -th Hermite polynomial. Then

$$De_n = (2n + 2) e_n, \text{ i.e., } \lambda_n = 2n + 2, \quad n = 0, 1, 2, \dots \quad (1.3)$$

Moreover,

$$\|D^{-p}\|_{HS}^2 = \sum_{n=0}^{\infty} \frac{1}{(2n + 2)^{2p}} < \infty, \quad \text{for } p > \frac{1}{2}$$

and for every  $\xi \in L^2(\mathbb{R}, dt)$ , its Hilbertian norm is given by

$$|\xi|_p = \left( \sum_{n=0}^{\infty} (2n + 2)^{2p} \langle \xi, e_n \rangle^2 \right)^{\frac{1}{2}}$$

## 1.2 Lévy Laplacian

Let  $E_1$  be any (real or complex) nuclear Fréchet space. A function  $F : E_1 \rightarrow \mathbb{R}$  is called twice differentiable at  $\xi \in E_1$  if there exist  $F'(\xi) \in E_1^*$  and  $F''(\xi) \in \mathcal{L}(E_1, E_1^*)$ , (the space of linear continuous operators from  $E_1$  into  $E_1^*$ ), such that

$$F(\xi + \eta) = F(\xi) + \langle F'(\xi), \eta \rangle + \frac{1}{2} \langle F''(\xi) \eta, \eta \rangle + o(\eta), \quad \eta \in E_1$$

where the error term satisfies  $\lim_{t \rightarrow 0} o(t\eta) / t^2 = 0$ . Let  $C^2(E_1)$  denote the space of everywhere twice differentiable functions  $F : E_1 \rightarrow \mathbb{R}$  such that both  $\xi \mapsto F'(\xi) \in E_1^*$  and  $\xi \mapsto F''(\xi) \in \mathcal{L}(E_1, E_1^*)$  are continuous.

There is a natural embedding of  $E_1^* \otimes E_1^*$  (algebraic tensor product) into  $\mathcal{L}(E_1, E_1^*)$  which identifies a generic element  $e_1^* \otimes e_2^* \in E_1^* \otimes E_1^*$  to the linear operator

$$|e_1^*\rangle\langle e_2^*| = e_1^* \otimes e_2^* : \eta \in E_1 \mapsto \langle e_2^*, \eta \rangle e_1^* \in E_1^* \quad (1.4)$$

where  $\langle \cdot, \cdot \rangle$  denote the duality  $\langle E_1^*, E_1 \rangle$ . Notice that, if  $e_1^* \in E_1 \subseteq E_1^*$ , then  $e_1^* \otimes e_2^* \in \mathcal{L}(E_1, E_1)$ . The operator  $e_1^* \otimes e_2^*$  will also be denoted  $|e_1^*\rangle\langle e_2^*|$ . In some cases this embedding is in fact an identification (this is the content of the nuclear kernel theorem [12]).

A  $\mathbb{C}$ -valued function  $F$  on  $E_1$  belongs to  $C^2(E_1)$  if and only if its real and imaginary parts have this property. In particular, choosing  $E_1 = N$  (the complexification of  $E$ ) one has, for  $\xi \in N$ ,

$$F'(\xi) \in N^* ; \quad F''(\xi) \in \mathcal{L}(N, N^*)$$

**Definition 1** *In the notations of the previous subsection let  $E_1$  denote either  $E$  or  $N$ . The Lévy Laplacian on  $E_1$  (with respect to the basis  $e = \{e_n\}_{n=1}^\infty$ ) is the Cesàro mean of the second derivatives along the elements of this basis. More precisely, it is the linear operator  $(\Delta_L, \mathcal{D}_L)$ :*

$$\mathcal{D}_L(E_1) := \left\{ F \in C^2(E_1) ; \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \langle F''(\xi) e_j, e_j \rangle \text{ exists for all } \xi \in E_1 \right\} \quad (1.5)$$

$$(\Delta_L F)(\xi) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \langle F''(\xi) e_j, e_j \rangle, \quad \xi \in E_1, F \in \mathcal{D}_L(E_1) \quad (1.6)$$

**Remark 2** *Note that, since the existence of Cesàro mean of a sequence of numbers  $\{a_n\}$  depends on its order, the definition of  $\Delta_L$  depends not only on the choice of the complete orthonormal basis as a set, but also on the orientation of the space, i.e., the choice of a map  $e : \mathbb{N} \ni n \mapsto e_n$ .*

### 1.3 The Cesàro trace on distribution spaces

**Definition 3** *Let  $\{e_n\}_{n=1}^\infty \subseteq E_1$  be as in the previous subsection (i.e.  $E_1 = E$  or  $E_1 = N$ ) and denote by  $\mathcal{L}(E_1, E_1^*)_C$  the set of all operators  $A \in \mathcal{L}(E_1, E_1^*)$  for which the limit*

$$Tr_C(A) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \langle A e_j, e_j \rangle \quad (1.7)$$

*exists and is finite. The map  $A \in \mathcal{L}(E_1, E_1^*)_C \mapsto Tr_C(A)$  is called the Cesàro trace on  $\mathcal{L}(E_1, E_1^*)$ .*

**Remark 4** Although not explicitly written, also the set  $\mathcal{L}(E_1, E_1^*)_C$  depends on the choice of the map  $e : n \in \mathbb{N} \mapsto e_n$ .

Definition 1 is equivalent to say that a function  $F \in C^2(E_1)$  belongs to  $\mathcal{D}_L(E_1)$  if and only if  $F''(\xi) \in \mathcal{L}(E_1, E_1^*)_C$  for all  $\xi \in E_1$ . In that case we have

$$(\Delta_L F)(\xi) = \text{Tr}_C \left( F''(\xi) \right)$$

In the notation (1.4) let  $E_C^*$  denote the subset of all  $x \in E_1^*$  such that  $x \otimes x \in \mathcal{L}(E_1, E_1^*)_C$ , i.e., the limit

$$\text{Tr}_C(x \otimes x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \langle (x \otimes x)e_j, e_j \rangle = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \overline{\langle x, e_j \rangle} \langle x, e_j \rangle \quad (1.8)$$

exists and is finite. For  $x \in E_C^*$ , we also write

$$\|x\|_C^2 := \text{Tr}_C(x \otimes x) \quad (1.9)$$

The set  $E_C^*$ , which is closed under multiplication by a scalar, is not necessarily closed under addition, hence in general it is not a vector space (see [8] for a counterexample). From the explicit form (1.8), it follows that  $x \mapsto \|x\|_C^2$  is a quadratic form on  $E_C^*$ , whose associated sesquilinear form on any vector subspace of  $E_C^*$  can be written as

$$(x, y)_C = \text{Tr}_C(x \otimes y) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \overline{\langle x, e_j \rangle} \langle y, e_j \rangle \quad (1.10)$$

With a slight abuse of language (cf. the comments at the end of this section) the map (1.10) will be called the Cesàro inner (or scalar) product.

Parseval identity implies that the Hilbert space  $H$  (resp.  $K$ ) is contained in  $E_C^*$  and that the restriction of the map  $x \mapsto \|x\|_C^2$  to  $H$  (resp.  $K$ ) is identically zero.

According to the assumption that  $E_1$  is a reflexive nuclear Fréchet space, for each  $a \in E_1^*$  the series  $\sum_{n=1}^{\infty} \langle a, e_n \rangle e_n$  converges in  $E_1^*$  even in the strong topology (which in our case coincides with the Mackey topology). In the following the symbol  $\sum_{n=1}^{\infty} a_n e_n$  will denote the (unique) element  $a \in E_1^*$  for which  $\langle a, e_n \rangle = a_n$  for all  $n$  if such element exists.

It is clear that  $E_1 \subset E_C^* \subset E_1^*$ . The following example shows that these inclusions are strict.

**Example.** For  $a = \sum_{n=1}^{\infty} a_n e_n \in E$ , we have

$$|a|_{-p}^2 = \sum_{n=1}^{\infty} \lambda_n^{-2p} \langle a, e_n \rangle^2 = \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n^{2p}}$$

In particular, if  $E^* = S'(\mathbb{R})$ ,  $D$  is given by (1.3), and  $a_n = n$  for all  $n$ , then  $|a|_{-p}^2 = \sum_{n=1}^{\infty} \frac{n^2}{(2n+2)^{2p}}$ . This series converges for  $p > \frac{3}{2}$ , so that  $a = \sum_{j=1}^n n e_n \in E^*$ . However,

$$\|a\|_C^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \langle a, e_j \rangle^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n j^2 = +\infty$$

which means that  $a \notin E_C^*$ .

In conclusion,  $E_C^*$  is strictly included in  $E^*$  (resp. in  $N^*$ ) and the sesquilinear form (1.10) is positive semi-definite on any vector subspace of  $E_C^*$ . However  $E_C^*$ , although closed under multiplication by a scalar, in general it is not a vector space hence we cannot use the sesquilinear form (1.10) to define a pre-Hilbert space structure on it.

In the following sections we will construct an uncountable set of vectors in  $E_C^*$  which are mutually orthogonal with respect to the pre-inner product (1.10). This gives a natural non separable Hilbert space contained in  $E_C^*$ . Subsequently we use these vectors to construct a large class of separable Hilbert space contained in  $E_C^*$  which are invariant under the action of the Lévy heat semigroup.

## 2 The Cesàro Hilbert Space

### 2.1 The Cesàro scalar product on (a subset of) $N^*$

Let  $e = (e_j)_j$  be a fixed basis of the complex Hilbert space  $K = H + iH$ . In the notation (1.4), for every  $n \in \mathbb{N}$ , define

$$P_n := \sum_{j=1}^n |e_j\rangle\langle e_j| \in \mathcal{L}(K, K) \quad (2.1)$$

Then, for  $f, g \in K$ , we have

$$\langle g, f \rangle = Tr |f\rangle\langle g| = \lim_{n \rightarrow \infty} Tr (|f\rangle\langle g| P_n) = \lim_{n \rightarrow \infty} Tr (P_n |f\rangle\langle g| P_n)$$

where, for  $T \in \mathcal{L}(K, K)$ ,

$$Tr(T) := \sum_{j=1}^{\infty} \langle e_j, T e_j \rangle \quad (2.2)$$

and  $Tr$  is the usual trace on  $\mathcal{L}(K, K)$ .

For  $a \in N$  and  $\varphi \in N^*$  we extend the notation (1.4) by defining

$$\langle a, \varphi \rangle := \overline{\langle \varphi, a \rangle}$$

and, for the linear operator  $|\varphi\rangle\langle a| \in \mathcal{L}(N^*, N^*)$ ,

$$\langle a, \varphi \rangle =: \text{Tr}(|\varphi\rangle\langle a|).$$

From the definition (2.2) and in the notation (1.4), one can prove that

$$\langle \varphi, a \rangle = \text{Tr}(|a\rangle\langle \varphi|).$$

In these notations, the operator  $P_n$ , defined by (2.1), can also be considered as an element of  $\mathcal{L}(N^*, N)$ .

Notice that, for  $f, g \in N^*$ , one has, in the notations  $|f\rangle\langle g| \in \mathcal{L}(N, N^*)$

$$P_n|f\rangle\langle g|P_n = |P_n f\rangle\langle P_n g| \in \mathcal{L}(N^*, N)$$

hence, in the notations introduced above,

$$\text{Tr}(P_n|f\rangle\langle g|P_n) = \langle P_n g, P_n f \rangle \quad (2.3)$$

in the sense that the left hand side of (2.3) is well defined for any  $n \in \mathbb{N}$  and any  $f, g \in N^*$  and the scalar product in the right hand side is meant in  $K$ .

**Remark 5** *In the above notations, the Cesàro scalar product  $\langle g, f \rangle_C$  can be written in the form:*

$$\langle g, f \rangle_C := \lim_{n \rightarrow \infty} \frac{1}{n} \text{Tr}(P_n|f\rangle\langle g|P_n), \quad f, g \in N^* \quad (2.4)$$

*in the sense that the left hand side of (2.4) exists when and only when the limit on the right hand side exists and in this case they are equal.*

Let  $N_e$  be the linear space algebraically generated by  $e = \{e_n\}$ . In our case  $N_e$  is a dense subspace of  $N$ . For a sequence of real or complex numbers  $(a_n)$ , the series  $\sum_{n=1}^{\infty} a_n e_n$  (its partial sums are considered to be elements of  $E^*$ ) converges in the topology  $\sigma(N^*, N_e)$  to an element  $f \in N^*$  if and only if for any  $n \in \mathbb{N}$ ,  $a_n = \langle e_n, f \rangle$ . Then, it follows by density that the topology  $\sigma(N^*, N_e)$  is Hausdorff. Hence, if  $\langle e_n, f \rangle = 0$  for all  $n \in \mathbb{N}$ , then  $f = 0$ .

**Lemma 6** *Let  $n \in \mathbb{N}$ . Then  $P_n : N^* \rightarrow N (\subseteq K)$  is continuous with respect to the topology  $\sigma(N^*, N_e)$  and for any  $f \in N^*$*

$$\langle e_j, P_n f \rangle = \langle P_n e_j, f \rangle = \langle e_j, f \rangle, \quad 1 \leq j \leq n$$



**Proof.** Let  $(g_k)_k$  be a sequence of  $N^*$  which converges to  $g \in N^*$  with respect to the topology  $\sigma(N^*, N_e)$ . Our aim is to prove that  $P_n]g_k$  converges to  $P_n]g$  with respect to any  $|\cdot|_p$ ,  $p \in \mathbb{R}$ . Since

$$P_n](g_k - g) = \sum_{j=1}^n \langle e_j, g_k - g \rangle e_j$$

for any  $p \in \mathbb{R}$ , one has

$$|P_n](g_k - g)|_p \leq \sum_{j=1}^n |\langle e_j, g_k - g \rangle| |e_j|_p \longrightarrow 0$$

as  $k \rightarrow \infty$ , where we used the fact, for any  $j$ ,  $|\langle e_j, g_k - g \rangle| \rightarrow 0$  as  $k \rightarrow \infty$ . On the other hand, for  $f \in N^*$ ,

$$\langle e_j, P_n]f \rangle = \sum_{k=1}^n \langle e_j, e_k \rangle \langle e_k, f \rangle = \langle e_j, f \rangle = \langle P_n]e_j, f \rangle$$

as desired. □

## 2.2 The Hilbert spaces $\mathcal{H}_C(S, \nu)$

From now on we fix:  $S = \mathbb{R}$  and  $\nu(d\lambda)$  a bounded positive measure on  $S$ , (it will be clear from the following that this restriction can be relaxed).

We know that, for  $\lambda \in S$ , we have

$$s_\lambda := \sum_{n=1}^{\infty} e^{in\lambda} e_n \in N^* \quad (2.5)$$

where the limit in the series (2.5) is in the sense  $\sigma(N^*, N_e)$ .

**Lemma 7** *For any  $\lambda, \lambda' \in S$  and  $n \in \mathbb{N}$  we have*

$$s_\lambda \in N_{-p} \subseteq N^* \quad (2.6)$$

$$\langle s_\lambda, s_{\lambda'} \rangle_C = \delta_{\lambda, \lambda'} \quad (2.7)$$

where  $\delta_{\lambda, \lambda'}$  is the Kronecker delta function. Moreover, for  $\lambda \in S$ , define the complex vector space

$$H_\lambda := \mathbb{C} \cdot s_\lambda \subseteq N^* \quad (2.8)$$

Then we have

$$\lambda \neq \lambda' \Rightarrow H_\lambda \cap H_{\lambda'} = \{0\}. \quad (2.9)$$

**Proof.** For any  $\lambda \in S$ , one has

$$\|s_\lambda\|_{-p}^2 := \sum_{k=1}^{\infty} \frac{|\langle s_\lambda, e_k \rangle|^2}{\lambda_k^{2p}} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{2p}} < \infty, \quad \forall p > 1/2 \quad (2.10)$$

This shows that  $s_\lambda \in N_{-p}$  for any  $p > 1/2$ . In particular  $s_\lambda \in N^*$ .

Let  $\lambda, \lambda' \in S$ . Then

$$\langle s_\lambda, s_{\lambda'} \rangle_C = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \langle s_\lambda, e_k \rangle \langle e_k, s_{\lambda'} \rangle = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{ik(\lambda - \lambda')}$$

Thus, if  $\lambda = \lambda'$ , (2.7) holds. If  $\lambda \neq \lambda'$ , then,

$$\langle s_\lambda, s_{\lambda'} \rangle_C = \lim_{n \rightarrow \infty} \frac{1}{n} \left( \frac{e^{i(n+1)(\lambda - \lambda')} - 1}{e^{i(\lambda - \lambda')} - 1} - 1 \right) = 0$$

and also in this case (2.7) holds. Finally (2.7) implies that, if  $v \in H_\lambda \setminus \{0\}$ , then  $\langle v, v \rangle_C$  exists and is  $\neq 0$ . But, again because of (2.7), if  $v \in H_\lambda \cap H_{\lambda'}$  with  $\lambda \neq \lambda'$ , then  $\langle v, v \rangle_C = 0$  hence  $v = 0$  and (2.9) holds.  $\square$

**Lemma 8** For any  $\varphi \in L^2(S, \nu(d\lambda))$  the Bochner integral

$$s_\varphi := \int_S \varphi(\lambda) s_\lambda \nu(d\lambda) \quad (2.11)$$

exists in  $N_{-p}$  for any  $p > 1/2$  and one has, in the sense of the norm of  $N_{-p}$

$$\int_S \varphi(\lambda) s_\lambda \nu(d\lambda) = \lim_{n \rightarrow \infty} \int_S \varphi(\lambda) P_{[n]}(s_\lambda) \nu(d\lambda)$$

**Proof.** For  $\varphi \in L^2(S, \nu(d\lambda))$  and  $n \in \mathbb{N}$  define

$$s_\varphi^{(n)} := \int_S \varphi(\lambda) P_{[n]}(s_\lambda) \nu(d\lambda) \quad (2.12)$$

Then clearly  $s_\varphi^{(n)} \in N \subseteq N^*$ . Therefore, by definition of Bochner integral, it will be sufficient to prove that for any  $p > 1/2$  one has:

$$\lim_{n \rightarrow \infty} \int_S |\varphi(\lambda) s_\lambda - \varphi(\lambda) P_{[n]}(s_\lambda)|_{-p} \nu(d\lambda) = 0 \quad (2.13)$$

But, for any  $\lambda \in S$ ,

$$\begin{aligned}
|\varphi(\lambda)s_\lambda - \varphi(\lambda)P_n](s_\lambda)|_{-p} &= |\varphi(\lambda)| \cdot |s_\lambda - P_n](s_\lambda)|_{-p} = |\varphi(\lambda)| \cdot \left| \sum_{k=n+1}^{\infty} e^{ik\lambda} e_k \right|_{-p} \\
&= |\varphi(\lambda)| \cdot \left( \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^{2p}} \right)^{1/2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_S |\varphi(\lambda)s_\lambda - \varphi(\lambda)P_n](s_\lambda)|_{-p} \nu(d\lambda) &= \left( \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^{2p}} \right)^{1/2} \int_S |\varphi(\lambda)| \nu(d\lambda) \\
&\leq \left( \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^{2p}} \right)^{1/2} \pi^{1/2} \|\varphi\|_{L^2(S, \nu(d\lambda))}
\end{aligned}$$

which tends to zero because the series  $\sum_{n=1}^{\infty} \lambda_n^{-2p}$  is convergent for any  $p > 1/2$ .  $\square$

**Proposition 9** *In the notations of Lemma 8, for every  $\varphi \in L^2(S, \nu(d\lambda))$  the following estimate holds :*

$$|s_\varphi|_{-p}^2 \leq \beta_p \|\varphi\|_{L^2(S, \nu(d\lambda))}^2 \quad (2.14)$$

where

$$\beta_p = \pi \left( \sum_{k=1}^{\infty} \frac{1}{(\lambda_k)^{2p}} \right) \quad (2.15)$$

*In particular, for any  $\varphi \in L^2(S, \nu(d\lambda))$ ,  $s_\varphi \in N^*$ .*

**Proof.** For  $\varphi \in L^2(S, \nu(d\lambda))$  and

$$s_\varphi := \int_S \varphi(\lambda) s_\lambda \nu(d\lambda) = \int_S \varphi(\lambda) \left( \sum_{n=1}^{\infty} e^{in\lambda} e_n \right) \nu(d\lambda) \in \mathcal{H}_C$$

we have

$$|s_\varphi|_{-p}^2 \leq \left( \int_S |\varphi(\lambda)| |s_\lambda|_{-p} \nu(d\lambda) \right)^2$$

From the estimate (2.10) we obtain

$$|s_\varphi|_{-p}^2 \leq \pi \left( \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{2p}} \right) \|\varphi\|_{L^2(S, \nu(d\lambda))}^2 = \pi \left( \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{2p}} \right) \|s_\varphi\|_C^2$$

$\square$

**Theorem 10** *In the notation (2.11) the linear map*

$$s : \varphi \in L^2(S, \nu(d\lambda)) \mapsto s_\varphi \in N^* \quad (2.16)$$

*satisfies the identity:*

$$\langle s_\varphi, s_\psi \rangle_C = \langle \varphi, \psi \rangle_{L^2(S, \nu(d\lambda))}, \quad \forall \varphi, \psi \in L^2(S, \nu(d\lambda)) \quad (2.17)$$

*in the sense that the left hand side exists and the identity holds. In particular the range of  $s$ , denoted  $\mathcal{H}_C(S, \nu)$ , is a Hilbert space for the Cesàro scalar product (1.10) and  $s$  is a unitary isomorphism of  $L^2(S, \nu(d\lambda))$  with  $\mathcal{H}_C(S, \nu)$ . Thus  $\mathcal{H}_C(S, \nu)$  is separable.*

**Proof.** Let  $\varphi \in L^2(S, \nu(d\lambda))$ . For  $n \geq 1$ ,

$$\begin{aligned} \frac{1}{n} \text{Tr} \left( P_{[n]} \Big|_{s_\varphi} \Big|_{s_\psi} \Big|_{P_{[n]}} \right) &= \frac{1}{n} \int_S \left\langle P_{[n]} \varphi(\lambda) s_\lambda, P_{[n]} \psi(\lambda) s_\lambda \right\rangle_H \nu(d\lambda) \\ &= \frac{1}{n} \int_S \bar{\varphi}(\lambda) \psi(\lambda) \left\langle \sum_{j=1}^n e^{i\lambda_j} e_j, \sum_{k=1}^n e^{i\lambda_k} e_k \right\rangle_H \nu(d\lambda) \\ &= \int_S \bar{\varphi}(\lambda) \psi(\lambda) \nu(d\lambda) = \langle \varphi, \psi \rangle_{L^2(S, \nu(d\lambda))}. \end{aligned} \quad (2.18)$$

From this the thesis easily follows.  $\square$

**Definition 11** *A Hilbert space  $\mathcal{H}_C \equiv \mathcal{H}_C(S, \nu)$ , introduced in Theorem 10, shall be called a Cesàro Hilbert space.*

**Proposition 12** *For each  $p > 1/2$ , we have the continuous embedding*

$$\mathcal{H}_C \subseteq N_{-p} \quad (2.19)$$

*More precisely, for every  $\varphi \in L^2(\mathbb{R})$  and for  $\beta_p$  given by (2.15), the following estimate holds :*

$$\| s_\varphi \|_{-p}^2 \leq \beta_p \| \varphi \|_{\mathcal{H}_C}^2 \quad (2.20)$$

**Proof.** Keeping into account the identity (2.17), the thesis follows from the estimate (2.14).  $\square$

### 3 Lévy Heat Equation

In the paper [6], using Fourier transform techniques, a class of solutions of the standard (i.e. with constant diffusion coefficient), parabolic Lévy heat equation was constructed in analogy with the finite dimensional case.

In this section, exploiting the essentially infinite dimensional nature of the Lévy laplacian, we build a class of solutions of the same equation which has no classical analogue. The reason why this new class of solutions appears only in the essentially infinite dimensional case, can be intuitively illustrated as follows. Consider the function

$$u : (t, x) \in \mathbb{R} \times \mathbb{R}^d \mapsto u(t, x) \in \mathbb{R}$$

defined by

$$u(t, x) := e^{t\text{Tr}(A)} e^{-\frac{1}{2} \langle x, Ax \rangle} \quad (3.1)$$

where  $A$  is a linear operator acting on  $\mathbb{R}^d$ . Then

$$\partial_t u(t, x) = \text{Tr}(A) u(t, x)$$

$$\partial_x^2 u(t, x) = (A + \langle \cdot, Ax \rangle \langle Ax, \cdot \rangle) u(t, x)$$

Thus taking the usual trace of  $\partial_x^2 u$  we see that  $u(t, x)$  does not satisfy a standard heat equation. However the Cesàro trace will kill the 1-dimensional operator  $\langle \cdot, Ax \rangle \langle Ax, \cdot \rangle$ , hence a function such as (3.1) can be (for appropriate choice of  $A$ ), a solution of the Lévy heat equation. Theorem 15 below shows that this is indeed the case.

For any  $\beta > 0$ , let  $S_C^\beta$  denote the sphere of radius  $\beta$  of the Hilbert space  $\mathcal{H}_C$ ; namely

$$S_C^\beta := \{x \in \mathcal{H}_C; \|x\|_C = \beta\} \subseteq N^* \quad (3.2)$$

We know, from Uglanov's theory [25], that for each real number  $\beta > 0$ , there exist a probability measure  $\mu_\beta$  on  $(N^*, \mathcal{B})$  such that

$$\mu_\beta(S_C^\beta) = 1, \quad (3.3)$$

where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra on  $N^*$ . Moreover, the family  $(\mu_\beta)$  can be chosen so that, for any bounded Borel function  $f$  on  $(N^*, \mathcal{B})$ , the map  $\beta \mapsto \mu_\beta(f)$  is Borel measurable.

In the following, we will study the heat equation associated with the Lévy Laplacian

$$\frac{\partial F}{\partial t} = \frac{\alpha}{2} \Delta_L F, \quad F(0, \xi) = F_0(\xi) \quad (3.4)$$

where  $\alpha \in (0, +\infty)$  and the initial condition  $F_0$  is a suitable function on  $N$ . Equation (3.4) is called Lévy heat equation with parameter  $\alpha$ .

**Proposition 13** For each  $\beta > 0$ , the map

$$f_\beta(t, \xi) := \int_{N^*} e^{\frac{-t\alpha\|x\|_C^2}{2}} e^{i\langle x, \xi \rangle} \mu_\beta(dx) \quad (3.5)$$

is a solution of the Lévy heat equation (3.4) with  $F_0 = \hat{\mu}_\beta$ , the Fourier transform of the measure  $\mu_\beta$ .

**Proof.** For  $x \in N^*$ , define

$$q_x : \xi \in N \mapsto q_x(\xi) := e^{i\langle x, \xi \rangle}$$

then, by direct computation  $q_x \in \mathcal{D}_L(N)$  and

$$\Delta_L q_x = -\langle x, x \rangle_C q_x = -\beta^2 q_x, \quad \mu_\beta - a.e. \ x \in N^*$$

Hence, using the dominated convergence Theorem, we see that  $\hat{\mu}_\beta$  is an eigenfunction of the Lévy Laplacian  $\Delta_L$  associated with the eigenvalue  $-\beta^2$

$$\Delta_L(\hat{\mu}_\beta)(\xi) = -\beta^2 \hat{\mu}_\beta(\xi), \quad \forall \xi \in N.$$

Again by dominated convergence one has

$$\frac{\partial f_\beta}{\partial t}(t, \xi) = \frac{\alpha}{2} f_\beta(t, \xi).$$

□

**Corollary 14** Let  $\nu$  be a (positive) measure on  $\mathbb{R}_+$  such that

$$\int_{\mathbb{R}_+} \beta^2 \nu(d\beta) < \infty \quad (3.6)$$

Then the function

$$F_\nu(t, \xi) := \int_{\mathbb{R}_+} f_\beta(t, \xi) \nu(d\beta)$$

is a solution of the heat equation (3.4) with initial condition

$$F_\nu(0, \xi) := \int_{\mathbb{R}_+} \hat{\mu}_\beta(\xi) \nu(d\beta)$$

**Proof.** By the condition (3.6) and the assumption (3.3) on the family  $(\mu_\beta)$ , one has :

$$\left| \int_{\mathbb{R}_+} \beta^2 \nu(d\beta) \int_{N^*} e^{-\frac{t\alpha\|x\|_C^2}{2}} e^{i\langle x, \xi \rangle} \mu_\beta(dx) \right| \leq \int_{\mathbb{R}_+} \beta^2 \nu(d\beta) < \infty$$

It follows

$$\begin{aligned} \partial_t F_\nu(t, \xi) &= \partial_t \int_{\mathbb{R}_+} f_\beta(t, \xi) \nu(d\beta) = \int_{\mathbb{R}_+} \frac{\alpha}{2} \Delta_L f_\beta(t, \xi) \nu(d\beta) \\ &= \frac{\alpha}{2} \int_{\mathbb{R}_+} -\beta^2 f_\beta(t, \xi) \nu(d\beta) = \frac{\alpha}{2} \Delta_L \int_{\mathbb{R}_+} f_\beta(t, \xi) \nu(d\beta) \\ &= \frac{\alpha}{2} \Delta_L F_\nu(t, \xi) \end{aligned}$$

□

**Theorem 15** Let  $\alpha > 0$ ,  $\beta > 0$ ,  $S = \mathbb{R}$ ,  $\nu$  a  $\sigma$ -finite measure on  $S$  and let  $s \in S \mapsto A_s \in \mathcal{L}(H)$  be a function such that  $\text{Tr}_C(A_s)$  exists and finite for each  $s \in S$ . Suppose the following properties are satisfied:

1. the integrals

$$\begin{aligned} g(t, \xi) &:= \int_S e^{t\alpha \text{Tr}_C(A_s)} e^{\langle A_s \xi, \xi \rangle} \nu(ds) \\ &\int_S \text{Tr}_C(A_s) e^{t\alpha \text{Tr}_C(A_s)} e^{\langle A_s \xi, \xi \rangle} \nu(ds) \end{aligned}$$

exist for any  $\xi \in N$

2.

$$\partial_t \int_S e^{t\alpha \text{Tr}_C(A_s)} e^{\langle A_s \xi, \xi \rangle} \nu(ds) = \int_S \partial_t e^{t\alpha \text{Tr}_C(A_s)} e^{\langle A_s \xi, \xi \rangle} \nu(ds), \quad \text{on } S$$

3.

$$\partial_\eta^2 \int_S e^{t\alpha \text{Tr}_C(A_s)} e^{\langle A_s \xi, \xi \rangle} \nu(ds) = \int_S \partial_\eta^2 e^{t\alpha \text{Tr}_C(A_s)} e^{\langle A_s \xi, \xi \rangle} \nu(ds), \quad \forall \eta \in N$$

Then the function  $g$  is a solution of the Lévy heat equation (3.4).

**Proof.** For any  $\eta \in N$ , we have

$$\begin{aligned}\partial_\eta^2 g(t, \xi) &= \int_S e^{t\alpha \text{Tr}_C(A_s)} \partial_\eta^2 e^{\langle A_s \xi, \xi \rangle} \nu(ds) \\ &= \int_S e^{t\alpha \text{Tr}_C(A_s)} [(\langle A_s \xi, \eta \rangle + \langle \eta, A_s^* \xi \rangle)^2 + 2\langle A_s \eta, \eta \rangle] e^{\langle A_s \xi, \xi \rangle} \nu(ds)\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{n} \sum_{j=1}^n g''(t, \xi)(e_j, e_j) \\ = \int_S e^{t\alpha \text{Tr}_C(A_s)} e^{\langle A_s \xi, \xi \rangle} \left[ \frac{1}{n} \sum_{j=1}^n (\langle A_s \xi, e_j \rangle + \langle e_j, A_s^* \xi \rangle)^2 + \frac{2}{n} \sum_{j=1}^n \langle A_s e_j, e_j \rangle \right] \nu(ds)\end{aligned}$$

In the limit  $n \rightarrow \infty$ , this is equals to

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_S e^{t\alpha \text{Tr}_C(A_s)} e^{\langle A_s \xi, \xi \rangle} \frac{2}{n} \sum_{j=1}^n \langle A_s e_j, e_j \rangle \nu(ds) = \\ = \int_S e^{t\alpha \text{Tr}_C(A_s)} e^{\langle A_s \xi, \xi \rangle} 2 \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \langle A_s e_j, e_j \rangle \nu(ds) \\ = \int_S 2\text{Tr}_C(A_s) e^{t\alpha \text{Tr}_C(A_s)} e^{\langle A_s \xi, \xi \rangle} \nu(ds)\end{aligned}$$

On the other hand,

$$\partial_t g(t, \xi) = \frac{\alpha}{2} \int_S 2\text{Tr}_C(A_s) e^{t\alpha \text{Tr}_C(A_s)} e^{\langle A_s \xi, \xi \rangle} \nu(ds)$$

and from this the statement follows.  $\square$

**Example.** An important example illustrating the situation of Theorem 15 is given by the choice

$$H = L^2(0, 2\pi); \quad A_s = M_{\chi_{(0,s)}}$$

i.e.,  $A_s$  is the multiplication by the indicator function of the interval  $(0, s)$ . In this case, if  $(e_j)$  is an uniformly bounded and equally dense orthonormal basis of  $L^2(0, 2\pi)$  (it is well known that such basis exist, cf. e. g. [7]), one has

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum \langle e_j, A_s e_j \rangle = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \int_0^s e_j^2(u) du = s, \quad \text{for any } s \geq 0$$



and all the conditions of Theorem 15 are satisfied if the measure  $\nu$  has support in a bounded interval of  $\mathbb{R}$ . It follows that the function

$$g(t, \xi) = \int_{\mathbb{R}} e^{\alpha st} e^{\langle \chi_{(0,s)} \xi, \xi \rangle} \nu(ds)$$

is a solution of the heat equation (3.4). The paper [21] established a relation between this function and the square of white noise.

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