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The Ito Algebra of Quantum Gaussian Fields

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The notion of mutual quadratic variation (square bracket) is extended to a quantum probabilistic framework. The mutual quadratic variations of the annihilation, creation, and number fields in a Gaussian representation are calculated, in both the Boson and the Fermion case, in the strong topology on a common invariant domain. It is proved that the corresponding Ito table closes at the second order. The Fock representation is characterized, among the Gaussian ones, by the property that its Ito table closes at the first order. © 1989 Academic Press, Inc.

1. ITO ALGEBRAS

A classical complex valued stochastic process on a probability space $(\Omega, \mathscr{F}, P)$ indexed by an interval $T \subseteq \mathbf{R}$ can be looked at as a map X from T with values in the measurable functions on $(\Omega, \mathscr{F}, P)$. The set $\mathscr{A}$ of these functions is a *-algebra for the pointwise operations and for the involution given by

$$f^*(\omega) = \overline{f(\omega)}; \qquad \omega \in \Omega.$$

To every process X as above we can associate an $\mathscr{A}$-valued finitely additive measure on the sub-intervals $I = (a, b]$ of T defined by

$$X(I) = X_b - X_a$$

called the increment of X. Conversely, if $I \subseteq T \mapsto X(I) \in \mathscr{A}$ is such a

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measure then to every random variable X_{t_0} we can associate the stochastic process

$$X_t = X_{t_0} + X(t_0, t) \qquad \text{if} \quad t_0 \leqslant t$$
$$X_t = X_{t_0} - X(t, t_0) \qquad \text{if} \quad t_0 \geqslant t$$

with the property that the increment measure of the process is the initial one. Thus, up to an "initial value" X_{t_0} there is a one-to-one correspondence between complex valued processes on $(\Omega, \mathscr{F}, P)$ indexed by T and $\mathscr{A}$-valued measures on T.

The mutual quadratic variation [6, 7], also called the brackets, of two classical stochastic processes X, Y is defined in terms of the associated measures (still denoted X, Y) by

$$[\![X, Y]\!](I) = \lim \sum_k X(I_k) \cdot Y(I_k),$$

where the limit is taken over some net (I_k) of partitions of I and is meant for some topology on $\mathscr{A}$ (a.e. convergence, convergence in probability, convergence in L^p, ...). Since the notion of mutual quadratic variation is one of the basic tools in classical stochastic calculus it is natural to try and generalize this notion in two main directions:

(i) The index set T is multidimensional (or even a general measurable space).

(ii) The processes (hence the corresponding measures) take value in a noncommutative *-algebra $\mathscr{A}$.

Quantum fields provide examples of both situations and since all the known examples of quantum fields are built as perturbations of free (Gaussian) fields the first step of our program will be the evaluation of the mutual quadratic variations of quantum Gaussian fields.

In order to have a unified treatment of the classical and the quantum cases, we are led to consider a general set T (called the index set), a family $\mathscr{B}$ of parts of T closed under finite unions, finite intersections, and relative complements and a topological *-algebra with identity $\mathscr{A}$ whose topology is given by a family of seminorms $\{v_{f,g} : f, g \in D\}$ (D-a set). The typical case, corresponding to the classical and quantum processes, will be: $T = \mathbf{R}$ (or a sub-interval thereof); $\mathscr{B}$ the Boolean algebra generated by the intervals; $\mathscr{A} = \mathscr{B}(K)$ for some Hilbert space K, or $\mathscr{A}$ a *-algebra of unbounded operators defined on a common invariant dense domain $D \subseteq K$; and the seminorms $\{v_{f,g}\}$ given by

$$v_{f,g}(a) = |\langle f, ag \rangle|; \qquad a \in \mathscr{A}; f, g \in D. \tag{1.1}$$

Another class of seminorms interesting for the stochastic calculus is (cf. [1])

$$v_f(a) = \|af\|; \qquad a \in \mathscr{A}; f \in D. \tag{1.2}$$

In the following for simplicity we write $I \subseteq T$ for $I \in \mathscr{B}$ and, for such an I, we denote $\mathscr{P}(I)$ the set of all finite partitions (I_k), $k = 1, ..., n$, of I with elements of $\mathscr{B}$. Clearly $\mathscr{P}(I)$ is a directed set for the order relation "is a finer partition than" and therefore it can be used to index nets. If $(x_n)_{\pi \in \mathscr{P}(I)}$ is a family of elements of a topological space, indexed by $\mathscr{P}(I)$, the expression

$$\lim_{\pi \in \mathscr{P}(I)} x_\pi$$

will denote the limit of x_π with respect to the net $\mathscr{P}(I)$.

A finitely additive $\mathscr{A}$-valued measure on $\mathscr{B}$ is a map $X: \mathscr{B} \to \mathscr{A}$ such that

$$X(I \cup J) = X(I) + X(J) \tag{1.3}$$

for any pair of disjoint elements $I, J \in \mathscr{B}$. Unless stated otherwise by an operator valued measure we shall mean a finitely additive measure. The vector space of $\mathscr{A}$-valued measures on $(T, \mathscr{B})$ will be denoted $\mathscr{M}(T, \mathscr{B}; \mathscr{A})$ or simply $\mathscr{M}(T; \mathscr{A})$

DEFINITION (1.1). Let $X, Y: \mathscr{B} \to \mathscr{A}$ be finitely additive measures. The mutual quadratic variation $[\![X, Y]\!]$ of X and Y is the measure defined by

$$[\![X, Y]\!](I) = \lim_{(I_k) \in \mathscr{P}(I)} \sum_k X(I_k)\, Y(I_k) \tag{1.4}$$

is the limit on the right-hand side exists in $\mathscr{A}$ for all $I \in \mathscr{B}$. Otherwise it is not defined.

Remark (1). In classical probability the mutual quadratic variation between X and Y is usually denoted $[X, Y]$. We introduced the double bracket notation to prevent confusion with the commutator.

Remark (2). Instead of "mutual quadratic variation of X and Y" we will also use the shorter expression "brackets of X and Y."

Remark (3). The notion of brackets depends not only on the topology on $\mathscr{A}$, but also on the precise meaning of limit in the expressions like (1.4). We have adopted here the meaning described since it has the nice property that the bracket of two finitely additive measures, if it exists, is again a finitely additive measure. Clearly several other notions of limit in (1.4) could be considered. For example, if T is a bounded interval of $\mathbf{R}$ and $\mathscr{B}$ is the class of finite unions of sub-intervals of T of the form $(s, t]$, then the limit (1.4) could be defined by choosing the limit over the sequence of dyadic partitions of T. All the results from Section 3 will not depend on the choice of the notion of limit in (1.4).

The following examples of brackets are not of the usual semi-martingale type. In the language to be adopted from Section 3 on, one might say that these examples concern the "one particle level" and our goal will consist in the computation of the brackets of different types of "second quantized versions" of these measures.

EXAMPLE (1). Let $X: \mathscr{B} \to \mathscr{A}$ be a projection valued measure, then for any $I \subseteq T$ and for any family of seminorms

$$[\![X, X]\!](I) = X(I),$$

EXAMPLE (2). Let $T = \mathbf{R}$; $\mathscr{B}$-the Borel σ-algebra on $\mathbf{R}$; $\mathscr{A} = \mathscr{B}(L^2(\mathbf{R}))$; and let e_P, e_Q be the projection valued measures of the position $((Qf)(x) = xf(x))$ and momentum $((Pf)(x) = (1/i)(\partial/\partial x) f(x))$ operators, respectively. Then for any $I \subseteq T$

$$[\![e_P, e_Q]\!](I) = [\![e_Q, e_P]\!](I) = 0$$

for the topology induced on $\mathscr{B}(H)$ by the seminorms $v_{f,f}(a) = \langle f, af \rangle$; $a \in \mathscr{B}(L^2(\mathbf{R}))$ with f in the Schwartz space $\mathscr{S}(R)$ of rapidly decreasing functions. This follows from the estimate

$$\|e_Q(I)\, e_P(I) f\|^2 \leqslant \|f\|_1\, |I|^3,$$

where $|I|$ is the Lebesgue measure of I.

EXAMPLE (3). The following is an example of a measure m on $\mathbf{R}$ with values in the bounded operators on $L^2(R)$ which satisfies

$$[\![m, m]\!](I) = m(I) \tag{1.5}$$

in the same topology of Example (2) even if m is not projection valued. In the notations of Example (1), let J be a fixed bounded interval and define, for $I \subseteq R$,

$$m(I) = e_Q(I)\, e_P(J^c) = e_Q(I)[1 - e_P(J)],$$

Then

$$m(I)^2 = m(I) - e_Q(I)\, e_P(J)\, e_Q(I)\, e_P(Jc)$$

and (1.5) follows from the estimate

$$|(e_Q(I)\, e_P(J)\, e_Q(I)\, e_P(J^c) \hat{f})(x)| \leqslant \|f\|_{L^1(\mathbf{R})}\, \chi_I(x)\, |I|$$

for any function $f \in \mathscr{S}(R)$, where $\hat{f}$ denotes its Fourier transform.

We define the positive cone $\mathscr{A}_+$ of an abstract topological *-algebra $\mathscr{A}$ as the closed cone generated by the elements of the form $a^* \cdot a$ with $a \in \mathscr{A}$. Thus, by definition, a positive element of $\mathscr{A}$ is a limit of sums of elements of $\mathscr{A}$ of the form $a^* \cdot a$. With this definition it is clear that the map $(X, Y) \mapsto [\![X, Y]\!]$ is bilinear and positive, in the sense that

$$[\![X^+, X]\!](I)$$

is a positive element of $\mathscr{A}$ for each $I \subseteq T$. We want to interpret it as a multiplication on a suitably restricted space of $\mathscr{A}$-valued finitely additive measures.

Definition (1.). A complex vector space $\mathscr{I}$ of $\mathscr{A}$-valued finitely additive measures is called an Ito algebra if it is closed under the operation of mutual quadratic variation, i.e., if $[\![X, Y]\!]$ exists and belongs to $\mathscr{I}$ for each pair of measures $X, Y \in \mathscr{I}$. If we fix a basis of $\mathscr{I}$ then the brackets of any pair of elements of this basis is a linear combination of elements of the basis. The table which gives this linear combination for any pair of elements of the given basis is called an Ito table.

Notice that the notion of Ito algebra depends both on the topology on $\mathscr{A}$ and on the way in which the limit (1.4) is taken.

In this paper we prove that on the quantum fields, defined by any Gaussian (quasi-free) representation of the canonical commutation (or anticommutation) relations over an arbitrary Hilbert space, there is a natural structure of Ito algebra and we compute explicitly their mutual quadratic variations in both the Boson and the Fermion case. More precisely (cf. Section 5): by second quantizing measures with values in the algebra of bounded operators on some Hilbert space H (e.g., spectral measures) we associate to the usual field, creation, annihilation, and number operators in a given representation some measures with values in the (possibly unbounded) operators on the second quantized space.

If the measures on H we started with form a commutative Ito algebra in the sense of Definitions (1.2) and (6.1), then the set of second quantized measures is also an Ito algebra, whose algebraic structure is constructed explicitly in Theorem (6.3). Moreover we prove that the Fock representation can be characterized, among all the quasi-free representations, by the Ito table of the associated creation, annihilation (or fields), and number (or gauge) measures.

The connection with the Ito tables considered up to now in classical and quantum probability is obtained by choosing $H = L^2(\mathbf{R}_+)$ and by fixing the projection valued measure on $\mathbf{R}_+$ to be

$$e\colon (s, t) \subseteq \mathbf{R}_+ \mapsto e_{(s,t)} = \text{multiplication by } \chi_{(s,t)}.$$

It is clear that the complelx multiples of $e(\cdot)$ form a commutative Ito algebra. There are, however, interesting examples of operator valued measures which are not of this type. For example, the number processes (cf. Section 4) naturally lead to the introduction of measures of the form $T \cdot e(\cdot)$ or $e(\cdot) \cdot T$, where T is a bounded operator on H. The necessity to include these measures, and the fact that the passage from these to more general ones does not complicate in an essential way the techniques of the proofs, motivates our choice to consider an arbitrary Ito algebra of measures at the level of the one particle space.

Even in the case when the Ito algebra is generated by a single projection valued measure, and the index set T is an interval in $\mathbf{R}_+$, our results yield a generalization and a technical improvement of the Ito tables obtained by Hudson and Parthasarathy [9] for the Fock representation of the CCR over the space $L^2(\mathbf{R}_+)$ (we consider only scalar valued test functions since the extension to the vector valued case is trivial), by Hudson and Lindsay [10] for the universal invariant (finite temperature) representation of the CCR over $L^2(\mathbf{R}_+)$, by Applebaum and Hudson [5a] for the Fock and the universal invariant representation of the CAR over $L^2(\mathbf{R}_+)$, and by Applebaum [4] for some quasi-free Gaussian representations of the CCR and the CAR over $L^2(\mathbf{R}_+)$. The generalization consists in the following facts:

(i) Our one particle space is not restricted to be $L^2(\mathbf{R}_+)$ but is arbitrary.

(ii) Being independent on the notion of stochastic integral, our method is representation free within the class of Gaussian representations.

(iii) We can evaluate the mutual quadratic variation of a pair of processes which are based on noncommuting filtrations already at the level of one particle space.

(iv) No commutativity condition between past and future is assumed.

Of these generalizations for the moment, the most relevant is the first one. In fact one can construct examples of Gaussian quantum noises satisfying the "chaoticity conditions" of [2] but which are not reducible to Gaussian quantizations of spaces of the form $L^2(\mathbf{R}_+; K)$ (K-a Hilbert space). For such quantum noises a stochastic integration can be developed on the lines of [1, 3]. Therefore, in order to develop a full stochastic calculus for these noises, including the Ito formula, one needs to know the mutual quadratic variations of all the basic integrators, i.e., the results of this paper.

The improvement consists in the fact that our Ito tables are established in the topology of the strong convergence on the invariant domain D

obtained by application of an arbitrary number of field and Weyl operators to the vacuum vector (cf. Lemma (3.1)), while all the above-mentioned Ito tables have been established in the topology of the weak convergence on the smaller and noninvariant domain of coherent vectors. The note [14] contains the evaluation of all the relevant mutual quadratic variations in the topology of weak convergence on the coherent states. In that framework the computations are considerably easier, but the results are not strong enough to handle even the simplest problems concerning, say, the "continuity of the trajectories" (cf. [3]). In this paper we limited our discussion to mean zero gauge invariant Gaussian fields but, as shown in [13], our techniques are applicable to the general case. Stochastic integrals for some representations of the CCR and CAR over the space $L^2(\mathbf{R}_+)$ have been considered in [5b] but the notion of mutual quadratic variation was not studied in that paper.

As already said, the notion of Ito algebra and the corresponding Ito tables depends neither on the notion of stochastic integral nor on any filtration; in Section 2 we introduce a formal notion of stochastic integration in order to show that, in case of a one dimensional index set and under the additional assumption that the increments of the integrator processes commute with the past fitration, the Ito multiplication $[\![\cdot, \cdot]\!]$ is associative. Associativity turns out to hold, in the Gaussian case, also for multidimensional index sets and without any commutativity assumption (cf. Theorem (7.3) and the remark at the end of Section 8), however, we could prove this result only by an explicit calculation of the brackets up to fourth order since, in the multidimensional case, a general argument of the kind used in Section 2 is still missing.

2. One Dimensional Index set: Associativity of the Ito Multiplication

Let $\mathscr{A}$ be as in Section 1 and let M, N be finitely additive measures on $\mathbf{R}$ with values in $\mathscr{A}$. Denote $M(t) = M(0, t)$, $N(t) = N(0, t)$ and assume that for each bounded interval (s, t) the limits

$$\lim_{\mathscr{P}(s,t)} \sum_k M(t_k, t_{k+1})\, N(0, t_k) =: \int_s^t dM \cdot N \tag{2.1}$$

$$\lim_{\mathscr{P}(s,t)} \sum_k M(0, t_k)\, N(t_k, t_{k+1}) =: \int_s^t M \cdot dN \tag{2.2}$$

exist in $\mathscr{A}$. We call (2.1) the left stochastic integral of N with respect to dM

and (2.2) the right stochastic integral of M with respect to dN. For any fixed $dt > 0$ introduce the notation

$$dF(t) = F(t + dt) - F(t) \tag{2.3}$$

(where F is any $\mathscr{A}$-valued function). Then summing the algebraic identity

$$d(MN) = dM \cdot N + M \cdot dN + dM \cdot dN$$

over the intervals $(t_k, t_{k+1}]$ of a partition $\mathscr{P}$ of $(s, t]$, one finds

$$\begin{aligned} M(t)\, N(t) - M(s)\, N(s) &= \sum_k M(t_k, t_{k+1})\, N(0, t_k) \\ &\quad + \sum_k M(0, t_k)\, N(t_k, t_{k+1}) \\ &\quad + \sum_k M(t_k, t_{k+1})\, N(t_k, t_{k+1}), \end{aligned}$$

where the sums on the right-hand side are extended to all the intervals of the partition P. Since the limits (2.1) and (2.2) are supposed to exist in $\mathscr{A}$, it follows that also the mutual quadratic variation

$$[\![M, N]\!](s, t) = \lim_{\mathscr{P}(s,t)} \sum_k M(t_k, t_{k+1})\, N(t_k, t_{k+1}) \tag{2.4}$$

exists in $\mathscr{A}$. By construction one has

$$M(t)\, N(t) - M(s)\, N(s) = \int_s^t M\, dN + \int_s^t dM\, N + [\![M, N]\!](s, t), \tag{2.5}$$

which we can write symbolically in differential form:

$$d(MN) = dM \cdot N + M \cdot dN + d[\![M, N]\!]. \tag{2.6}$$

Because of this connection between the notion of quadratic variation and that of stochastic integration, we shall sometimes use the symbolic notation

$$d[\![X, Y]\!] = dX \cdot dY$$

for finitely additive $\mathscr{A}$-valued measures X and Y.

Equation (2.6) above expresses the essence of Itô's formula to which it reduces when:

— $\mathscr{A}$ is the real algebra of real valued measurable functions on a probability space $(\Omega, \mathscr{F}, P)$ (considered as a *-algebra with the trivial involution given by the identity).

— The topology on $\mathscr{A}$ is the topology of convergence in probability.

— The finitely additive $\mathscr{A}$-valued measures M are defined by $M(s, t) = X_t - X_s$, where $X = (X_t)$ is a real valued stochastic process on $(\Omega, \mathscr{F}, P)$.

— In (2.6), M and N are powers of a continuous martingale X, say $M = X^m$; $N = X^n$. In this case, in the notation (2.3) above one has

$$d(MN) = d(X^{m+n}) = (m+n)\, X^{m+n-1}\, dX \\ + (m+n-1)\, X^{m+n-2}\, dX \cdot dX$$

from which one easily deduces the Ito formula for polynomials in X:

$$df(X) = f'(X)\, dX + \tfrac{1}{2} f''(X)\, dX \cdot dX \\ = f'(X)\, dX + \tfrac{1}{2} f''(X)\, d[\![X, X]\!]. \tag{2.7}$$

Given a family of classical or quantum processes $\{X_\alpha(t) : \alpha \in T;\, t \in \mathbf{R}_+\}$ one of the basic problems in the quantum stochastic calculus consists in the explicit evaluation of the mutual quadratic variations $[\![X_\alpha, X_\beta]\!]$ of all the pairs of processes in the given family. The notion of the Ito table introduced in Definition (1.2) can be extended by requiring that the mutual quadratic variation of any two processes in the family is a sum of finitely many stochastic integrals with respect to processes in the slame family—in symbols,

$$[\![X_\alpha, X_\beta]\!](t, t+dt) = dX_\alpha(t) \cdot dX_\beta(t) \\ = \int_t^{t+dt} \sum_\gamma c^\gamma_{\alpha,\beta}(s)\, dX_\gamma(s)$$

for some adapted processes $c^\gamma_{\alpha,\beta}(s)$. The processes $c^\gamma_{\alpha,\beta}(s)$ are called, in analogy with the Lie algebra case, the structure processes of the family $\{X_\alpha(t)\}$. If the family of stochastic processes $\{X_\alpha(t) : \alpha \in T;\, t \in \mathbf{R}_+\}$ admits a closed Ito table with structure processes $c^\gamma_{\alpha,\beta}(s) = c^\gamma_{\alpha,\beta}$ which are complex constants, then the linear combinations of the measures $\{dX_\alpha(t)\}$ are an Ito algebra in the sense of Definition (1.2). In this paper we shall limit ourselves to this case because the above-mentioned extension depends on the notion of stochastic integral.

The following theorem shows that, if the space of integrable processes is rich enough to separate the integrators, then, under the additional commutativity condition (2.8), the Ito multiplication is necessarily associative.

THEOREM (2.1). *In the notations above, let $\mathscr{A}$ be an associative algebra* *and let* X_j $(j=1, 2, 3)$ *be finitely additive* $\mathscr{A}$*-valued measures on* $\mathbf{R}$ *and let* $F_j\colon \mathbf{R} \to \mathscr{A}$ $(j=1, 2, 3)$ *be step functions such that the integrals*

$$I_j(t) := \int_0^t F_j \, dX_j; \qquad \int_0^t I_k F_j \, dX_j;$$

$$\int_0^t I_k \, dI_j; \qquad j, k = 1, 2, 3$$

exist for every finite t *in the sense of* (2.1), (2.2) *and satisfy*

$$\int_0^t I_k \, dI_j = \int_0^t I_k F_j \, dX_j.$$

Assume moreover that for each $t < u$ *and for any three (adapted) processes* F_j *one has*

$$F_j(t)[X_k(u) - X_k(t)] = [X_k(u) - X_k(t)] \, F_j(t). \tag{2.8}$$

Then

$$\int_0^t F_1 F_2 F_3 \, d[\![[\![X_1, X_2]\!], X_3]\!] = \int_0^t F_1 F_2 F_3 \, d[\![X_1, [\![X_2, X_3]\!]]\!]. \tag{2.9}$$

Proof. Notice that, because of (2.8), it is not necessary to distinguish between right and left stochastic integrals. Therefore the argument at the beginning of this section and our assumptions imply that the quadratic variation $[\![X_i, X_j]\!]$ exists in $\mathscr{A}$ and the identity

$$\left(\int_0^t F_i \, dX_i\right)\left(\int_0^t F_j \, dX_j\right) = \int_0^t F_i I_j \, dX_i + \int_0^t I_i F_j \, dX_j + \int_0^t F_i F_j \, d[\![X_i, X_j]\!] \tag{2.10}$$

(Ito formula) holds. Because of our assumptions

$$\int F_i \, dX_i \in \mathscr{A}, \qquad i = 1, 2, 3.$$

This, together with the associativity of the multiplication in $\mathscr{A}$ and Ito formula (2.10), will imply the associativity of Ito's multiplication. To prove this we shall use the shorthand notation

$(G)_j$ for the function $t \mapsto \int G_{\chi[0,t]} \, dX_j$, $\qquad j = 1, 2, 3$

$(G)_{i,j}$ for the function $t \mapsto G_{\chi[0,t]} \, d[\![X_i, X_j]\!]$, $\qquad i, j = 1, 2, 3$

$$A := \left[\left(\int F_1\, dX_1\right)\cdot\left(\int F_2\, dX_2\right)\right]\cdot\left(\int F_3\, dX_3\right)$$

$$B := \left(\int F_1\, dX_1\right)\cdot\left[\left(\int F_2\, dX_2\right)\cdot\left(\int F_3\, dX_3\right)\right],$$

where throughout the paper χ_I will denote the characteristic function of the set I. Using (2.5) we find

$$\begin{aligned} A &= \left[\int F_1 (F_2)_2\, dX_1 + \int (F_1)_1 F_2\, dX_2 \right.\\ &\quad \left. + \int F_1 F_2\, d[\![X_1, X_2]\!]\right]\cdot\left(\int F_3\, dX_3\right)\\ &= \int F_1 (F_2)_2 (F_3)_3\, dX_1 + \int (F_1 (F_2)_2)_1\, dX_3\\ &\quad + \int F_1 (F_2)_2 F_3\, d[\![X_1, X_3]\!] + \int (F_1)_1 F_2 (F_3)_3\, dX_2\\ &\quad + \int (F_1)_1 (F_2)_2 F_3\, dX_2 + \int (F_1)_1 F_2 F_3\, d[\![X_2, X_3]\!]\\ &\quad + \int (F_1)_1 F_2 (F_3)_3\, d[\![X_1, X_2]\!] + \int (F_1 F_2)_{12} F_3\, dX_3\\ &\quad + \int F_1 F_2 F_3\, d[\![[\![X_1, X_2]\!], X_3]\!]. \end{aligned}$$

On the other hand, again using (2.5), we have that

$$\begin{aligned} B &= \left(\int F_1\, dX_1\right)\left[\int F_2 (F_3)_3\, dX_2 + \right.\\ &\quad \left. + \int (F_2)_2 F_3\, dX_3 + \int F_2 F_3\, d[\![X_2, X_3]\!]\right]\\ &= \int F_1 (F_2 (F_3)_3)_2\, dX_1 + \int (F_1)_1 (F_2 (F_3)_3\, dX_2\\ &\quad + \int F_1 F_2 (F_3)_3\, d[\![X_1, X_2]\!] + \int F_1 ((F_2)_2 F_3)_3\, dX_1\\ &\quad + \int (F_1)_1 (F_2)_2 F_3\, dX_3 + \int F_1 (F_2)_2 F_3\, d[\![X_1, X_3]\!]\\ &\quad + \int (F_1)_1 F_2 F_3\, d[\![X_2, X_3]\!]\\ &\quad + \int F_1 (F_2 F_3)_{23}\, dX_1 \int F_1 F_2 F_3\, d[\![X_1, [\![X_2, X_3]\!]]\!]. \end{aligned}$$

By the associativity of the multiplication in $\mathscr{A}$ we have that $A = B$. Comparing the terms in A and B it is clear that the terms with integrators of the form $[\![X_i, X_j]\!]$ cancel. Consider now the terms which are integrated with respect to dX_1 in B. These can be written as

$$\int F_1 \cdot [(F_2(F_3)_3)_2 + ((F_2)_2 F_3)_3 + (F_2 F_3)_{23}] \cdot dX_1$$

but due to Ito's formula (2.10) the term in square brackets is equal to $(F_2)_2(F_3)_3$. Hence the dX_1-integrals cancel and similarly for the dX_2- and dX_3-integrals. So we are left with (2.9) and this ends the proof.

Remark (1). The fact that an Ito algebra should be associative as a consequence of general arguments was pointed out to the authors by M. Schürmann.

Remark (2). In the above theorem we have specifiedneither the class of integrands (adapted processes) nor the class of integrators (semi-martigales). However, all the properties required by the theorem are verified in the rigorous theory of stochastic integration developed in [3] (cf. [1] for a preliminary exposition).

3. Boson Gaussian (Quasi-free) States

In this section we recall known facts on Boson Gaussian (quasi-free) states and we prove the technical Lemma (3.2) which will be frequently used in the remainder of the paper.

By a pre-Hilbert space we mean a vector space H over $\mathbf{C}$ with a non-degenerate scalar product denoted $\langle \cdot, \cdot \rangle$ which is real bilinear and satisfies

$$\langle f, g \rangle = \overline{\langle g, f \rangle}; \qquad f, g \in H.$$

We will use the notation

$$\sigma(f, g) = \operatorname{Im}\langle f, g \rangle; \qquad f, g \in H.$$

Given a pre-Hilbert space there exists a unique C^*-algebra $W(H)$ with the following properties:

(i) There exists a map $W: H \to W(H)$ such that the linear space generated by the $W(f)$ $(f \in H)$ is dense in $W(H)$.

(ii) $W(f)^* = W(-f); \qquad f \in H.$

(iii) $W(f)\, W(g) = e^{-i\sigma(f, g)}\, W(f + g); \qquad f, g \in H.$

$W(H)$ is called the CCR (or the Weyl) algebra over H. In our notations the scalar product is conjugate linear in the first variable. A gauge invariant regular mean zero quasi-free (or Gaussian) state with covariance Q is the state φ_Q on $W(H)$ satisfying

$$\varphi_Q(W(f)) = e^{-(1/2)\langle f, Qf\rangle}; \qquad f \in H, \tag{3.1}$$

where Q is an operator on H satisfying

$$\langle Qf, f\rangle \geqslant \|f\|^2; \qquad \forall f \in H. \tag{3.2}$$

For the definition of a Gaussian state only the real linearity of Q is required. In this paper we restrict ourselves to complex linear Q, but our techniques are applicable to the case of a real linear Q as well. For a general mean zero Gaussian state the term gauge invariant means that Q is complex linear and not only real linear. Any state on $W(H)$ is uniquely determined by the condition (3.1) and conversely any operator Q satisfying (3.2) defines a unique Gaussian state on $W(H)$. The Gaussian state corresponding to the choice

$$Q = 1 \tag{3.3}$$

in (3.1) is called the Fock state on $W(H)$ and denoted φ_1 or φ_F. Thus, be definition, the Fock state on $W(H)$ is characterized by

$$\varphi_F(W(f)) = e^{-(1/2)\|f\|^2}; \qquad f \in H. \tag{3.4}$$

If $\{\mathscr{H}, \pi, \Phi\}$ is the GNS triple [15] associated to the pair $\{W(H), \varphi_F\}$ then π is called the Fock representation of the CCR over H. The Gaussian state corresponding to the choice

$$Q = \lambda \cdot 1, \qquad \lambda \in \mathbf{R}_+, \lambda > 1 \tag{3.5}$$

is called the universal invariant state with parameter λ and its GNS representation is called the universal invariant representation (of the CCR over H) with parameter λ.

When no confusion might arise the Gaussian state φ_Q will be denoted φ. In the remainder of this section we fix a Gaussian state $\varphi = \varphi_Q$ with covariance Q and denote $\{\mathscr{H}, \pi, \Phi\}$ the associated GNS triple. It is known that the GNS triple of the pair $\{W(H), \varphi_Q\}$ can be easily described in terms of the Fock representation of the CCR over H and over its conjugate space $\bar{H}$ which is defined as follows:

(i) As a set $\bar{H}$ coincides with H.

(ii) The identity map $\iota: f \in H \to \iota(f) \in \bar{H}$ is conjugate linear (additive and $\iota(\lambda f) = \bar{\lambda}\iota(f)$).

(iii) For each $f, g \in H$

$$\langle \iota(f), \iota(g)\rangle_{\bar{H}} = \langle g, f\rangle_H. \tag{3.6}$$

Notice that in this case the map $A \in B(H) \to \bar{A} \in \mathscr{B}(\bar{H})$ defined by

$$\bar{A}\iota(f) = \iota(Af), \qquad f \in H \tag{3.7}$$

is a conjugate linear *-isomorphism of $\mathscr{B}(H)$ onto $\mathscr{B}(\bar{H})$.

Denoting φ_F and $\bar{\varphi}_F$ the Fock states on $W(H)$ and $W(\bar{H})$, respectively, and $\{\mathscr{H}_F, \pi_F, \Phi_F\}$; $\{\bar{\mathscr{H}}_F, \bar{\pi}_F, \bar{\Phi}_F\}$ the corresponding GNS triples, and denoting

$$C = \sqrt{\frac{Q+1}{2}}; \qquad S = \sqrt{\frac{Q-1}{2}}, \tag{3.8}$$

it follows that, if $\{\mathscr{H}, \pi, \Phi\}$ is the GNS triple of $\{W(H), \varphi_Q\}$ then the map

$$\pi(W(f)) \mapsto \pi_F(W(Cf)) \otimes \bar{\pi}_F(W(\bar{S}\iota(f))) \tag{3.9a}$$

$$\Phi \mapsto \Phi_F \otimes \bar{\Phi}_F \tag{3.9b}$$

extends to a unitary isomorphism of the triple $\{\mathscr{H}, \pi, \Phi\}$ with the triple $\{\mathscr{H}_F \otimes \bar{\mathscr{H}}_F \otimes \bar{\pi}_F, \Phi_F \otimes \bar{\Phi}_F\}$. Thus in principle every quasi-free calculation can be reduced to a Fock calculation. However, since this reduction does not simplify the calculations in a significant way, we prefer to deal with the general case directly.

LEMMA (3.1). *For any $f \in H$, the family $\{\pi(W(tf))\}$ $(t \in R)$ is a strongly continuous unitary group and its generator $B(f)$, called the field operator in the representation π, is such that for all $n \in N$, and all $g_1, ..., g_n \in H$, one has*

$$B(g_1) \cdot \cdots \cdot B(g_n)\, \pi(W(f))\, \Phi \in D(B(g)). \tag{3.10}$$

Proof. This is standard result which follows by explicit computation from the CCR and the fact that the function $t \in \mathbf{R} \to \varphi(W(f+tg))$ is analytic for all f and g.

LEMMA (3.2). *There exists a unique real polynomials P_n in the variables $s_1, ..., s_n, t_{1,2}, ..., t_{n-1,n}$, homogeneous of degree n if* deg $(s_j) = 1$, $\deg(t_{i,j}) = 2$ *and independent on the Gaussian state φ_Q such that for all $n \in N$, and $g_1, ..., g_n, f, g \in H$, one has*

$$\begin{aligned} &\langle \pi(W(h))\Phi, B(g_1) \cdot \cdots \cdot B(g_n)\, \pi(W(f))\Phi \rangle \\ &\quad = \langle \pi(W(h))\Phi, \pi(W(f))\Phi > P_n(s_1, ..., s_n, t_{1,2}, ... t_{n-1,n}) \end{aligned} \tag{3.11}$$

with

$$\begin{aligned} s_j &= i\,\mathrm{Re}\langle Q(f-h), g_j \rangle + \sigma(f+h, g_j); \\ t_{i,j} &= \mathrm{Re}\langle Qg_i, g_j \rangle + i\sigma(g_i, g_j). \end{aligned} \tag{3.12}$$

The polynomials P_n satisfy the recursion relation

$$P_{n1}(s_1, ..., s_n, s_{n+1}, t_{1,2}, ..., t_{n,n+1})$$
$$= s_{n+1} P_n(s_1, ..., s_n, t_{1,2}, ..., t_{n-1,n})$$
$$+ \sum_{j=1}^{n} \frac{\partial}{\partial s_j} P_n(s_1, ..., s_n, t_{1,2}, ..., t_{n-1,n}) \cdot t_{j,n+1}. \tag{3.13}$$

Moreover the coefficients of the monomials in P_n are 1 *if all the indices occur in the monomial,* 0 *otherwise.*

Remark. We write down the first five polynomials P_n, some of which will be used in the following.

$$P_0 = 1; \qquad P_1(s_1) = s_1; \qquad P_2(s_1, s_2, t_{1,2}) = s_1 s_2 + t_{1,2} \tag{3.14}$$

$$P_3(s_1, s_2, s_3, t_{1,2}, t_{1,3}, t_{2,3}) = s_1 s_2 s_3 + s_1 t_{2,3} + s_2 t_{1,3} + s_3 t_{1,2} \tag{3.15}$$

$$P_4(s_1, s_2, s_3, s_4, t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,4})$$
$$= s_1 s_2 s_3 s_4 + s_1 s_4 t_{2,3} + s_2 s_4 t_{1,3} + s_3 s_4 t_{1,2} + s_2 s_3 t_{1,4}$$
$$+ s_1 s_3 t_{2,4} + s_1 s_2 t_{3,4} + t_{2,3}, t_{1,4} + t_{2,4} t_{1,3} + t_{3,4} t_{1,2} \tag{3.16}$$

$$P_5(\{s_j\}_{j=1,...,5}, \{t_{j,k}\}_{j,k=1,...,5})$$
$$= s_1 s_2 s_3 s_4 s_5 + s_1 s_4 s_5 t_{2,3} + s_2 s_4 s_5 t_{1,3} + s_3 s_4 s_5 t_{1,2}$$
$$+ s_2 s_3 s_5 t_{1,4} + s_s t_{2,3} t_{1,4} + s_5 t_{2,4} t_{1,3} + s_5 t_{3,4} t_{1,2}$$
$$+ s_2 s_3 s_4 t_{1,5} + s_4 t_{2,3} t_{1,5} + s_1 s_2 s_3 t_{4,5} + s_1 s_3 s_4 t_{2,5}$$
$$+ s_4 t_{1,3} t_{2,5} + s_3 t_{1,4} t_{2,5} + s_1 s_2 s_4 t_{3,5}$$
$$+ s_1 t_{2,3} t_{4,5} + s_2 t_{1,3} t_{4,5} + s_3 t_{1,2} t_{4,5}$$
$$+ s_3 t_{1,5} t_{2,4} + s_1 t_{3,5} t_{2,4} + s_2 t_{1,5} t_{3,4} + s_1 t_{2,5} t_{3,4}. \tag{3.17}$$

The general rule for P_n is the following: from the set $\{1, ..., n\}$ one picks k pairs (i_α, j_α) with

$$i_\alpha < j_\alpha, \qquad \alpha = 1, ..., k; k = 1, ..., [n/2] = \text{integral part of } n/2$$

then one forms the products

$$\left(\prod_{h \in \{1,...,n\} - \{i_1, j_1, ..., i_k, j_k\}} s_h \right) \cdot \left(\prod_{\alpha=1}^{k} t_{i_\alpha, j_\alpha} \right)$$

and one sums all these products over all the k and all the choices of the (i_α, j_α); this gives P_n.

Proof. The result is proved by induction on n. The lemma is obviously true for $n=0, 1$. Suppose now that it holds for $n=0, 1, ..., k$. Then we have

$$\begin{aligned}
&\langle \pi(W(h))\Phi, B(g_1)\cdot \cdots \cdot B(g_k)\, B(g_{k+1})\, \pi(W(f))\Phi\rangle \\
&= \frac{\partial}{i\partial\lambda}[\langle \pi(W(h))\Phi, B(g_1)\cdot \cdots \\
&\quad \cdot B(g_k)\, \pi(W(\lambda g_{k+1}+f)\Phi\rangle\, e^{-i\lambda\sigma(g_{k+1},f)}]_{\lambda=0} \\
&= \frac{\partial}{i\partial\lambda}[\langle \pi(W(h))\Phi, \pi(W(\lambda g_{k+1}+f)\Phi\rangle \\
&\quad \cdot e^{-i\lambda\sigma(g_{k+1},f)}\, P_k(s_1(\lambda), ..., s_k(\lambda), t_{1,2}, ..., t_{k-1,k})]_{\lambda=0},
\end{aligned} \tag{3.18}$$

where $t_{i,j}$ is as in (3.12) and

$$s_j(\lambda) = i\,\mathrm{Re}\langle Q(\lambda g_{k+1}+f-h), g_j\rangle + \sigma(\lambda g_{k+1}+f+h, g_j) \tag{3.19a}$$

so that

$$\frac{d}{id\lambda} s_j(\lambda) = t_{j,k+1}. \tag{3.19b}$$

Moreover

$$\begin{aligned}
&\frac{\partial}{i\partial\lambda}[\langle \pi(W(h))\Phi, \pi(W(\lambda g_{k+1}+f)\Phi\rangle\, e^{-i\lambda\sigma(g_{k+1},f)}]_{\lambda=0} \\
&= \frac{\partial}{i\partial\lambda}[\langle \Phi, \pi(W(\lambda g_{k+1}+f-h)\Phi\rangle\, e^{-i\lambda\sigma(g_{k+1},f)-i\lambda\sigma(g_{k+1}+f,h)}]_{\lambda=0} \\
&= \frac{\partial}{i\partial\lambda}[e^{-(1/2)\langle Q(\lambda g_{k+1}+f-h), \lambda g_{k+1}+f-h\rangle + i\lambda\sigma(f+h, g_{k+1}) + i\lambda\sigma(h,f)}]_{\lambda=0} \\
&= e^{-(1/2)\langle Q(f-h), f-h\rangle + i\lambda\sigma(h,f)}\cdot[\mathrm{Re}\langle Q(f-h), g_{k+1}\rangle + \sigma(f+h, g^{k+1})] \\
&= \langle \pi(W(h))\Phi, \pi(W(f))\Phi\rangle\, s_{k+1}.
\end{aligned} \tag{3.20}$$

Using (3.19) and (3.20) to compute the derivative in (3.18) we find the result.

If no confusion can arise from now on we shall suppress the notation π for the representation and in the sequel we shall use the notation

$$D := \text{linear span of } \{B(g_1)\cdot \cdots \cdot B(g_n)\, W(f)\Phi : n\in\mathbf{N}, g_1, ..., g_n, f\in H\}. \tag{3.21}$$

By Lemma (3.1), D is an invariant domain for all the field operators $B(f)$. Moreover from the CCR one easily derives the relations

$$B(f+g)=B(f)+B(g), \qquad B(\lambda f)=\lambda B(f), \qquad \lambda\in\mathbf{R} \tag{3.22}$$

$$[B(f), B(g)]\subseteq 2i\sigma(f,g) \tag{3.23}$$

$$W(f)^* B(g)\, W(f)=2\sigma(f,g)+B(g), \tag{3.24}$$

all the relations being meant on D. From (3.24) it follows that D is also invariant under the action of the Weyl operators $W(f)$.

COROLLARY (3.3). *If $T_\alpha^{(1)}, ..., T_\alpha^{(n+1)}$ are $n+1$ nets of operators on H converging strongly to $T^{(1)}, ..., T^{(n+1)}$ on H then for all $g_1, ..., g_n, f\in H$ one has*

$$\lim_\alpha B(T_\alpha^{(1)} g_1)\cdot \cdots \cdot B(T_\alpha^{(n)} g_n)\cdot W(T_\alpha^{(n+1)} f)\Phi$$
$$=B(T^{(1)} g_1)\cdot \cdots \cdot B(T^{(n)} g_n)\cdot W(T^{(n+1)} f)\Phi.$$

Proof. This result follows in a straightforward way from Lemma (3.2).

COROLLARY (3.4). *If $g_1, ..., g_n, f\in H$, then*

$$\lim_{\lambda\to 0} B(g_1)\cdots B(g_n)\frac{W(\lambda f)-1}{i\lambda}=B(g_1)\cdots B(g_n)\, B(f)$$

strongly on D.

Proof. This result follows from Lemma (3.1) and from the relation (3.24).

4. NUMBER FIELDS

Let H be a pre-Hilbert space. By $\mathscr{B}_i(H)$ we denote the set of all bounded linear operators T on H such that the unique continuous extension $\bar{T}$ of T to the completion of H has the properties

$$T_1 H\subseteq H; \qquad T_2 H\subseteq H$$
$$e^{itT_1}H\subseteq H; \qquad e^{itT_2}H\subseteq H, \qquad \forall t\in\mathbf{R},$$

where

$$T_1=\frac{1}{2}(\bar{T}+\bar{T}^*); \qquad T_2=\frac{1}{2i}(\bar{T}-\bar{T}^*)$$

are respectively the self-adjoint and anti-self-adjoint parts of $\bar{T}$.

Remark that if $T \in \mathscr{B}_i(H)$ then $\bar{T}^* H \subseteq H$. We will denote the restriction of $\bar{T}^*$ to H by T^*. Clearly $T^* \in \mathscr{B}_i(H)$. If $T \in \mathscr{B}_i(H)$ is such that $\bar{T}$ is self-adjoint we will denote the restriction of $e^{it\bar{T}}$ to H by e^{itT}.

Clearly, if H is a Hilbert space then $\mathscr{B}_i(H) = \mathscr{B}(H)$. However, in some physical examples the covariance Q of the Gaussian state is unbounded, hence its domain is only a pre-Hilbert space and therefore, as will be evident in the following, the notion of $\mathscr{B}_i(H)$ will be needed to define the number fields.

Consider now the Weyl C^*-algebra $W(H)$ over a pre-Hilbert space and the Gaussian state φ as described in Section 3. Take $T \in \mathscr{B}_i(H)$ such that $\bar{T}$ is self-adjoint. Then it is known that the map $W(f) \in W(H) \mapsto W(e^{itT} f) \in W(H)$ defines a one parameter automorphism group of $W(H)$ which we denote by α_t^T. If

$$[Q, e^{itT}] = 0$$

for all $t \in \mathbf{R}$ or equivalently if

$$\varphi \circ \alpha_t^T = \varphi$$

for all $t \in \mathbf{R}$ then α_t^T can be implemented unitarily in the GNS representation, i.e., for each $t \in \mathbf{R}$ there exists a unique unitary operator U_t^T on $\mathscr{H}$ such that

$$\pi \circ \alpha_t^T = U_t^T \cdot \pi(\cdot) \cdot U_t^{T*}; \qquad U_t^T \Phi = \Phi$$

or equivalently

$$U_t^T \pi(W(f)) \Phi = \pi(W(e^{itT} f)) \Phi$$

for all $f \in H$.

PROPOSITION (4.1). *The one parameter family U_t^T is a strongly continuous unitary group. Denoting by N_T its generator and by D the dense subspace of $\mathscr{H}$ defined in* (3.21) *we have that*

$$D \subseteq D(N_T); \qquad N_T(D) \subseteq D.$$

Moreover, for each $n \in N$, $g_1, ..., g_n, f \in H$ we have that

$$\begin{aligned} N_T \cdot B(g_1) \cdot \cdots \cdot B(g_n)\, W(f) \Phi \\ = \Big(-i \sum_{j=1}^{n} B(g_1) \cdot \cdots \cdot B(iTg_j) \cdot \cdots \cdot B(g_n) + B(g_1) \cdot \cdots \\ \cdot B(g_n) \cdot B(iTf) - B(g_1) \cdot \cdots \cdot B(g_n) \cdot \langle f, Tf \rangle \Big)\, W(f) \Phi. \quad (4.1) \end{aligned}$$

Proof. Using Corollary (3.4) it is easy to check that

$$U_t^T B(g_1) \cdot \cdots \cdot B(g_n)\, W(f)\Phi = B(e^{itT}g_1) \cdot \cdots \cdot B(e^{itT}g_n)\, W(e^{itT}f)\Phi.$$

Hence,

$$\begin{aligned} &\frac{U_t^T - 1}{it} B(g_1) \cdot \cdots \cdot B(g_n)\, W(f)\Phi \\ &\quad = -i \sum_{j=1}^{n} B(g_1) \cdot \cdots \cdot \frac{B(e^{itT}g_j - g_j)}{it} \cdot \cdots \cdot B(g_n)\, W(f)\Phi \\ &\qquad + B(g_1) \cdot \cdots \cdot B(g_n) \cdot \frac{W(e^{itT}f) - W(f)}{it} \cdot \Phi. \end{aligned} \tag{4.2}$$

By Corollary (3.4), as $t \to 0$, the sum in the right-hand side of (4.2) converges to

$$-i \sum_{j=1}^{n} B(g_1) \cdot \cdots \cdot B(iTg_j) \cdot \cdots \cdot B(g_n)\, W(f)\Phi. \tag{4.3}$$

So we have to show that

$$\begin{aligned} \lim_{t \to 0} \Bigg\| B(g_1) \cdot \cdots \cdot B(g_n) \cdot \Bigg[&\frac{W(e^{itT}f) - W(f)}{it} \\ &- (B(iTf) + \langle f, Tf \rangle)\, W(f) \Bigg] \Phi \Bigg\| = 0. \end{aligned} \tag{4.4}$$

Now

$$\begin{aligned} &\Bigg\| B(g_1) \cdot \cdots \cdot B(g_n) \cdot \Bigg[\frac{W(e^{itT}f) - W(f)}{it} \\ &\qquad - (B(iTf) + \langle f, Tf \rangle) \Bigg] \Phi \Bigg\| \\ &= \Bigg\| B(g_1) \cdot \cdots \cdot B(g_n) \cdot \Bigg[\frac{W(e^{itT}f - f)\, e^{i\sigma(e^{itT}f, f)} - 1}{it} \\ &\qquad - (B(iTf) + \langle f, Tf \rangle) \Bigg] W(f)\Phi \Bigg\| \end{aligned}$$

is surely majorized by the sum of the four quantities

$$\|B(g_1) \cdot \cdots \cdot B(g_n) \cdot [W(e^{itT}f - f) - 1]\, W(f)\Phi\| \frac{|e^{i\sigma(e^{itT}f, f)} - 1|}{|t|} \tag{i}$$

$$\frac{1}{|t|}\|B(g_1)\cdot\cdots\cdot B(g_n)\cdot[W(e^{itT}f-f)-W(itTf)]\,W(f)\Phi\| \tag{ii}$$

$$\left\|B(g_1)\cdot\cdots\cdot B(g_n)\cdot\left[\frac{W(itTf)-1}{it}-B(iTf)\right]W(f)\Phi\right\| \tag{iii}$$

$$\|B(g_1)\cdot\cdots\cdot B(g_n)\,W(f)\Phi\|\cdot\left|\frac{e^{i\sigma(e^{itT}f,f)}-1}{it}+\langle f,Tf\rangle\right|. \tag{iv}$$

It is clear that the quantities (i) and (iv) tend to 0 as $t\to 0$ and, because of Corollary (3.4), also the term (iii) tends to zero. So it remains to show that the term (ii) tends to 0 as $t\to 0$. To prove this, notice that the term (ii) is majorized by the sum of the two expressions

$$\frac{1}{|t|}\|B(g_1)\cdot\cdots\cdot B(g_n)\cdot[W(e^{itT}f-f)-W(itTf+f)]\Phi\| \tag{iia}$$

$$\|B(g_1)\cdot\cdots\cdot B(g_n)\cdot W(itTf+f)\Phi\|\,\frac{1}{|t|}\,|e^{i\sigma(e^{itT}f,f)}-e^{it\langle Tf,f\rangle}|. \tag{iib}$$

Since (iib) tends clearly to 0 as $t\to 0$, it remains to estimate (iia). To this goal notice that, because of Lemma (3.2), the square of (iia) is equal to

$$\begin{aligned}
&\frac{1}{t^2}[P_{2n}(s_1^{(1)},\ldots,s_{2n}^{(1)},t_{1,2},\ldots,t_{2n-1,2n})\\
&\quad-P_{2n}(s_1^{(2)},\ldots,s_{2n}^{(2)},t_{1,2},\ldots,t_{2n-1,2n})\\
&\quad-P_{2n}(s_1^{(3)},\ldots,s_{2n}^{(3)},t_{1,2},\ldots,t_{2n-1,2n})\\
&\quad+P_{2n}(s_1^{(4)},\ldots,s_{2n}^{(4)},t_{1,2},\ldots,t_{2n-1,2n})]\\
&\quad+\frac{1}{t^2}[1-\langle W(e^{itT}f)\Phi,W(e^{itT}f+f)\Phi\rangle]\\
&\quad\times P_{2n}(s_1^{(2)},\ldots,s_{2n}^{(2)},t_{1,2},\ldots,t_{2n-1,2n})\\
&\quad+\frac{1}{t^2}[1-\langle W(e^{itT}f+f)\Phi,W(e^{itT}f)\Phi\rangle]\\
&\quad\times P_{2n}(s_1^{(3)},\ldots,s_{2n}^{(3)},t_{1,2},\ldots,t_{2n-1,2n}),
\end{aligned} \tag{4.5}$$

where we have used the notations

$$\rho(j)=n+1-j\quad\text{for } 1\leqslant j\leqslant n;\qquad \rho(j)=j-n\quad\text{for } n+1\leqslant j\leqslant 2n$$

$$s_j^{(1)}=2\sigma(e^{itT}f,g_{\rho(j)})$$

$$s_j^{(2)} = i\,\mathrm{Re}\langle Q(itTf + f - e^{itT}f),\ g_{\rho(j)}\rangle + \sigma((itTf + f + e^{itT}f),\ g_{\rho(j)})$$

$$s_j^{(3)} = \bar{s}_j^{(2)}$$

$$s_j^{(4)} = 2\sigma(itTf + f,\ g_{\rho(j)})$$

$$t_{i,j} = \mathrm{Re}\langle Qg_{\rho(i)},\ g_{\rho(j)}\rangle + i\sigma(g_{\rho(i)},\ g_{\rho(j)}).$$

It is easily seen that the last two terms of (4.5) vanish at $t \to 0$ and, since the polynomials P_{2n} are homogeneous of degree $2n$ (with $\deg s_j = 1$, $\deg t_{ij} = 2$) and the $t_{i,j}$ independent on t, our statement is equivalent to proving that

$$\lim_{t \to 0} \frac{1}{t^2} [s_{k_1}^{(1)} s_{k_2}^{(1)} \cdots s_{k_{2l}}^{(1)} - s_{k_1}^{(2)} s_{k_2}^{(2)} \cdots s_{k_{2l}}^{(2)}$$

$$- s_{k_1}^{(3)} s_{k_2}^{(3)} \cdots s_{k_{2l}}^{(3)} + s_{k_1}^{(4)} s_{k_2}^{(4)} \cdots s_{k_{2l}}^{(4)}] = 0 \tag{4.6}$$

for $l \geqslant 1$ and $k_1, k_2, ..., k_{2l} \in \{1, ..., n\}$. But (4.6) follows in a straightforward way from the fact that for all $j \in \{1, ..., n\}$ and for all $h, k = 1, 2, 3, 4$, one has

$$s_j^{(h)}\,(t=0) = s_j^{(k)}\,(t=0)$$

$$\frac{d}{dt} s_j^{(h)}(t)|_{t=0} = \frac{d}{dt} s_j^{(k)}(t)|_{t=0}$$

$$\frac{d^2}{dt^2} (s_j^{(1)}(t) - s_j^{(2)}(t) - s_j^{(3)}(t) + s_j^{(4)}(t))|_{t=0} = 0.$$

Now let $T \in \mathscr{B}$ such that for all $t \in \mathbf{R}$

$$[Q, e^{itT_1}] = [Q, e^{itT_2}] = 0, \tag{4.7}$$

where T_1, T_2 are respectively the restrictions of the self-adjoint and anti-self-adjoint parts of $\bar{T}$ and H. Then both N_{T_1} and N_{T_2} are well defined on D by the preceding discussion and we denote

$$N_T = N_{T_1} + iN_{T_2}. \tag{4.8}$$

Notice that D is a invariant domain for N_T. In the following we shall denote $\mathscr{L}(D)$ the vector space of all linear operators with invariant domain D and by $\mathscr{B}_Q(H)$, a *-algebra, for the usual operations, of operators $X \in \mathscr{B}_i(H)$, satisfying (4.2) and closed in $\mathscr{B}(H)$ for the fopology of strong convergence on H.

PROPOSITION (4.2). *The map* $N\colon T \in \mathscr{B}_Q(H) \to N_T = N(T) \in \mathscr{L}(D)$ *is complex linear in the sense that*

$$N(aT_1 + bT_2) = aN(T_1) + bN(T_2). \tag{4.9}$$

Moreover

$$N(T^*) = N(T)^*, \tag{4.10}$$

where both equalities hold on D.

Proof. The statement (4.4) follows from (4.3) and the explicit action of N_T, for $T \in \mathscr{B}_Q(H)$, given in Proposition (4.1). Using the same proposition one easily sees that

$$\langle N_{T^*}\xi, \eta \rangle = \langle \xi, N_T \eta \rangle$$

for all $\xi, \eta \in D$, which implies (4.10).

5. Mutual Quadratic Variations of Quantum Fields Measures with Respect to Gaussian States

Keeping the notations of the preceding sections, from now on we fix a complex pre-Hilbert space H, a mean zero gauge invariant Gaussian state φ on $W(H)$ with covariance Q and GNS triple $\{\mathscr{H}, \pi, \Phi\}$, a measurable space $(T, \mathscr{B})$, and a *-algebra $\mathscr{B}_Q(H)$ as defined before Proposition (4.2). We shall denote $\mathscr{M}(T; \mathscr{B}_Q(H))$ the vector space of all $\mathscr{B}_Q(H)$-valued finitely additive measures on $\mathscr{B}$ and write $I \subseteq T$ to mean that $I \subseteq T$ and $I \in \mathscr{B}$. The field operators $B(f)$ $(f \in \mathscr{H})$ and the number fields N_T $(T \in \mathscr{B}(H))$ are defined as in Sections 3 and 4; the domain D is defined by (3.21) and $\mathscr{L}(D)$ is defined before Proposition (4.2).

Using the real linearity of the mappings $X \in \mathscr{B}_Q(H) \mapsto B(Xf) \in \mathscr{L}(D)$ and $X \in \mathscr{B}_Q(H) \mapsto N_X \in \mathscr{L}(D)$ one can lift finitely additive measures on $(T, \mathscr{B})$ with values in $\mathscr{B}_Q(H)$ to (second quantized) measures $A^f_X, A^{f+}_X, B^f_X, N_X$ taking values in $\mathscr{L}(D)$.

Definition (5.1). For $X \in \mathscr{M}(T; \mathscr{B}_Q(H))$ and $f \in H$ define the measures

$$B^f_X(I) = B(X(I)f) \tag{5.1}$$

$$A^f_X = \tfrac{1}{2}\,[B^{if}_X - iB^f_X], \qquad (A^f_X)^+ = A^{f+}_X = \tfrac{1}{2}\,[B^{if}_X + iB^f_X] \tag{5.2}$$

$$N_X(I) = N_{X(I)} \tag{5.3}$$

for all $I \subseteq T$ and $X \in \mathscr{M}(T; \mathscr{B}_Q(H))$.

The measures $B^f_X, A^f_X, A^{f+}_X, N_X$ together with the scalar measures

$$\langle f, X(\cdot)\, g \rangle, \qquad f, g \in H, \qquad X \in \mathscr{M}(T; \mathscr{B}_Q(H))$$

will be called the basic integrators. In this section we compute their mutual quadratic variations. In order to achieve this goal we introduce the following regularity conditions which roughly express the nonatomicity of the measures involved.

DEFINITION (5.2). A $\mathscr{B}_Q(H)$-valued measure X on $(T, \mathscr{B})$ will be called *regular* if for each $I \subseteq T$ one has

$$\lim_{(I_k) \in \mathscr{P}(I)} \sum_k |\langle f, X(I_k) g \rangle|^2 = 0 \qquad \forall f, g \in H \tag{5.4}$$

$$\lim_{(I_k) \in \mathscr{P}(I)} \sum_{k,l} |\langle X(I_l) f, X(I_k) g \rangle|^2 = 0 \qquad \forall f, g \in H \tag{5.5}$$

and if for some constant M_I, every f in D, and every partition $(I_k) \in \mathscr{P}(I)$

$$\sum_k \|X(I_k) f\|^2 \leqslant M_I \|f\|^2. \tag{5.6}$$

Two $\mathscr{B}_Q(H)$-valued measures X, Y are called jointly regular if both X and Y are regular and for every f, g in H

$$\lim_{(I_k) \in \mathscr{P}(I)} \sum_{k,l} |\langle X(I_l) f, Y(I_k) g \rangle|^2 = 0 \qquad \forall f, g \in H. \tag{5.7}$$

In agreement with the notation introduced before Definition (1.1), we shall denote

$$\mathscr{M}(T; \mathscr{B}_Q(H)) \quad \text{resp.} \quad \mathscr{M}(T; \mathscr{L}(D)) \tag{5.8}$$

the vector space of all $\mathscr{B}_Q(H)$-valued (resp. $\mathscr{L}(D)$-valued) regular finitely aditive measures on $\mathscr{B}$.

LEMMA (5.3). *If $X, Y \in \mathscr{M}(T; \mathscr{B}_Q(H))$ are regular measures on T then*:

(a) *for all $f, g \in H$ and $I \subseteq T$ one has*

$$\lim_{(I_k) \in \mathscr{P}(I)} \sum_k |\langle f, X(I_k) g \rangle| \; Y(I_k) = 0 \tag{5.9}$$

in the strong operator topology on H;

(b) *for all $g_1, ..., g_n, g, f, f' \in H$ one has*

$$\lim_{(I_k) \in \mathscr{P}(I)} B(g_1) \cdot \cdots \cdot B(g_n) \cdot \sum_k B(X_k g) \langle f, Y_k f' \rangle = 0 \tag{5.10}$$

strongly on D.

Proof. Part (a) follows from the regularity conditions since

$$\left\| \sum_k |\langle f, X(I_k)\, g\rangle| \; Y(I_k) h \right\| \leqslant \left(\sum_k |\langle f, X(I_k)\, g\rangle|^2 \right)^{1/2} \cdot \left(\sum_k \|Y(I_k) h\|^2 \right)^{1/2}.$$

Part (b) follows from Lemma (3.3), (a) and the real linearity of the fields, since

$$\sum_k B(X_k\, g) \cdot \langle f, Y_k f' \rangle = \sum_k B(\mathrm{Re}\langle f, Y_k f' \rangle \cdot X_k\, g) + i \sum_k B(\mathrm{Im}\langle f, Y_k f' \rangle \cdot X_k\, g).$$

LEMMA (5.4). *Let $X, Y \in \mathcal{M}_Q(T; \mathcal{B}(H))$ be two jointly regular self-adjoint measures such that $[\![X, Y]\!]$ exists strongly on H, i.e., for all $I \subseteq T$, $f \in H$ one has*

$$\lim_{(I_k) \in \mathcal{P}(I)} \sum_k X(I_k)\, Y(I_k) f = [\![X, Y]\!](I) f. \tag{5.11}$$

Then for each $f, g \in H$ the bracket $[\![B_Y^g, B_X^f]\!]$ exists strongly on D and, on this domain, the following identity holds:

$$[\![B_Y^g, B_X^f]\!] = \mathrm{Re}\langle [\![X, Y]\!]\, g, Qf \rangle + i\, \mathrm{Im}\langle [\![X, Y]\!]\, g, f \rangle. \tag{5.12}$$

Proof. Using the shorthand notation X_k, Y_k instead of $X(I_k)$, $Y(I_k)$, we have to prove that

$$\sum_k B(Y_k g)\, B(X_k f) \to \mathrm{Re}\langle Qg, [\![Y, X]\!]\, f \rangle + i\, \mathrm{Im}\langle g, [\![Y, X]\!]\, f \rangle \tag{5.13}$$

strongly on D. To this goal denote

$$z = \mathrm{Re}\langle Qg, [\![Y, X]\!]\, f \rangle + i\, \mathrm{Im}\langle g, [\![Y, X]\!]\, f \rangle \tag{5.14}$$

$$Z_p = \sum_k B(Y_k\, g)\, B(X_k\, f) - z. \tag{5.15}$$

Then we have

$$\begin{aligned} &\|Z_p B(g_1) \cdot \cdots \cdot B(g_n)\, W(h) \Phi\| \\ &\quad \leqslant \|(Z_p B(g_1) - B(g_1) Z_p)\, B(g_2) \cdot \cdots \cdot B(g_n)\, W(h) \Phi\| \\ &\qquad + \|B(g_1) \cdot Z_p \cdot B(g_2) \cdot \cdots \cdot B(g_n)\, W(h) \Phi\|. \end{aligned} \tag{5.16}$$

The first term of the right-hand side of (5.16) equals

$$\left\| \left(\sum_k 2i \operatorname{Im}\langle X_k f, g_1 \rangle B(Y_k g) + \sum_k 2i \operatorname{Im}\langle Y_k g, g_1 \rangle B(X_k f) \right) \cdot B(g_2) \cdot \cdots \cdot B(g_n)\, W(h)\Phi \right\|$$

and by Lemma (5.3) and Corollary (3.3) this tends to zero in the limit of finer partitions. So we have to show that

$$\|B(g_1) \cdot Z_p \cdot B(g_2) \cdot \cdots \cdot B(g_n)\, W(h)\Phi\| \to 0.$$

By repeating the commutation argument above, we see it is sufficient to prove that

$$\|B(g_1) \cdot B(g_2) \cdot \cdots \cdot B(g_n) Z_p \cdot W(h)\Phi\| \to 0. \tag{5.17}$$

First we check that

$$\|Z_p \cdot W(h)\Phi\| \to 0. \tag{5.18}$$

Denote

$$x_k = 2\langle h, X_k f \rangle; \qquad y_k = 2\langle h, Y_k f \rangle \tag{5.19a}$$

$$r_{i,j} = \operatorname{Re}\langle QX_i f, Y_j g \rangle + i \operatorname{Im}\langle X_i f, Y_j g \rangle \tag{5.19b}$$

$$s_{i,j} = \operatorname{Re}\langle QX_i f, X_j f \rangle + i \operatorname{Im}\langle X_i f, X_j f \rangle \tag{5.19c}$$

$$t_{i,j} = \operatorname{Re}\langle QY_i f, Y_j f \rangle + i \operatorname{Im}\langle Y_i f, Y_j f \rangle \tag{5.19d}$$

then, using Lemma (3.2) we find

$$\begin{aligned}
\|Z_p \cdot W(h)\Phi\|^2 &= \left\langle W(h)\Phi, \sum_l B(X_l f) \cdot B(Y_l g) \cdot \sum_k B(Y_k g)\, B(X_k f)\, W(h)\Phi \right\rangle \\
&\quad - 2 \operatorname{Re}\left(\bar{z} \left\langle W(h)\Phi, \sum_k B(Y_k g)\, B(X_k f)\, W(h)\Phi \right\rangle \right) + |z|^2 \\
&= \sum_{k,l} P_4(x_l, y_l, y_k, x_k, r_{l,l}, r_{l,k}, s_{l,k}, t_{l,k}, \bar{r}_{k,l}, \bar{r}_{k,k}) \\
&\quad - 2 \operatorname{Re}\left(\bar{z} \sum_k P_2(y_k, x_k; \bar{r}_{k,k}) \right) + |z|^2 \\
&= \sum_{k,l} (x_l y_l y_k x_k + x_l y_l \bar{r}_{k,k} + x_l y_k \bar{r}_{k,l} + x_l x_k t_{l,k} + y_l y_k s_{l,k} \\
&\quad + y_l x_k r_{l,k} + y_k x_k r_{l,l} + r_{l,l} \bar{r}_{k,k} + r_{l,k} \bar{r}_{k,l} + s_{l,k} t_{l,k}) \\
&\quad - 2 \operatorname{Re}\left(\bar{z} \sum_k (y_k x_k + \bar{r}_{k,k}) \right) + |z|^2
\end{aligned}$$

and this tends to zero by the arguments

$$\left|\sum_k x_k y_k\right| \leqslant 4 \sum_k |\langle h, X_k g\rangle| \cdot |\langle h, Y_k g\rangle|$$

$$\leqslant 4 \left(\sum_k |\langle h, X_k g\rangle|^2\right)^{1/2} \cdot \left(\sum_k |\langle h, Y_k g\rangle|^2\right)^{1/2}$$

and the right-hand side tends to zero by the regularity of X and Y. Furthermore

$$\left|\sum_{k,l} x_l y_k \bar{r}_{k,l}\right| \leqslant 4 \left|\operatorname{Re}\left\langle\left(\sum_k y_k X_k Qf\right), \sum_l x_l Y_l g\right\rangle\right|$$

$$+ 4 \left|\operatorname{Im}\left\langle Q\left(\sum_k y_k X_k Qf\right), \sum_l x_l Y_l g\right\rangle\right|,$$

which tends to zero by Lemma (5.3)(a). For the same reason

$$\left|\sum_{l,k} x_l x_k t_{l,k}\right|, \qquad \left|\sum_{l,k} y_l y_k s_{l,k}\right|, \qquad \left|\sum_{l,k} y_l y_k r_{l,k}\right|$$

tend to zero. Moreover

$$\left|\sum_{l,k} r_{l,k} \bar{r}_{l,k}\right| \leqslant 2 \sum_{l,k} |\langle X_l Qf, Y_k g\rangle|^2 + 2 \sum_{l,k} |\langle X_l f, Y_k g\rangle|^2,$$

which tends to zero by the joint regularity of X and Y. One has also

$$\left|\sum_{l,k} s_{l,k} t_{l,k}\right| \leqslant 2 \left(\sum_{l,k} |\langle X_l(Q+1) f, X_k g\rangle|^2\right)^{1/2}$$

$$\cdot 2 \left(\sum_{l,k} |\langle Y_l(Q+1) g, Y_k f\rangle|^2\right)^{1/2}$$

and again this tends to zero by the regularity of X and Y. Finally, since by assumption $[\![X, Y]\!]$ exists in the strong operator topology, then

$$\sum_k r_{k,k} = \sum_k \langle Qf, X_k Y_k g\rangle + i\langle f, X_k Y_k g\rangle \to \bar{z}.$$

In conclusion

$$\lim \|Z_p \cdot W(h) \Phi\| = \lim \left(|z|^2 - 2 \operatorname{Re}\left(z \sum_k r_{k,k}\right)\right) = 0$$

and this proves (5.18). With exactly the same arguments one can check that

$$\lim \|Z_p^* \cdot W(h)\Phi\| = 0.$$

In order to prove (5.17) we use again the commutation argument (5.16) to reduce the problem to prove that

$$\langle W(h)\Phi, Z_p^* Z_p B(h_1) \cdot \cdots \cdot B(h_m)\, W(h)\Phi \rangle \to 0. \tag{5.20}$$

This can be done by an induction argument. We checked already that (5.20) holds for $m = 0$. Suppose now it also holds for $m = 1, ..., n-1$. Then, by Lemma (3.2) and the recursion formula (3.13) we have

$$\begin{aligned}
&\langle W(h)\Phi, Z_p^* Z_p B(h_1) \cdot \cdots \cdot B(h_n)\, W(h)\Phi \rangle \\
&\quad = z_n \langle W(h)\Phi, Z_p^* Z_p B(h_1) \cdot \cdots \cdot B(h_{n-1})\, W(h)\Phi \rangle && \text{(i)} \\
&\qquad + \sum_{j=1}^{n-1} \langle W(h)\Phi, Z_p^* Z_p B(h_1) \cdot \cdots \cdot \hat{B}(h_j) \cdot \cdots \cdot B(h_n)\, W(h)\Phi \rangle\, t_{j,n} && \text{(ii)} \\
&\qquad + \sum_k \langle W(h)\Phi, B(Y_k g) \cdot Z_p B(h_1) \cdot \cdots \cdot B(h_n)\, W(h)\Phi \rangle\, u_{k,n} && \text{(iii)} \\
&\qquad + \sum_k \langle W(h)\Phi, B(X_k f) \cdot Z_p B(h_1) \cdot \cdots \cdot B(h_n)\, W(h)\Phi \rangle\, v_{k,n} && \text{(iv)} \\
&\qquad + \sum_k \langle W(h)\Phi, Z_p^* \cdot B(X_k f)\, B(h_1) \cdot \cdots \cdot B(h_n)\, W(h)\Phi \rangle\, u_{k,n} && \text{(v)} \\
&\qquad + \sum_k \langle W(h)\Phi, Z_p^* \cdot B(Y_k f)\, B(h_1) \cdot \cdots \cdot B(h_n)\, W(h)\Phi \rangle\, v_{k,n}, && \text{(vi)}
\end{aligned}$$

where

$$\begin{aligned}
z_n = 2\langle h, h_n \rangle, \qquad t_{j,n} &= \mathrm{Re}\langle Qh_j, h_n \rangle + i\,\mathrm{Im}\langle h_j, h_n \rangle \\
u_{j,n} &= \mathrm{Re}\langle QX_j f, h_n \rangle + i\,\mathrm{Im}\langle X_j f, h_n \rangle; \\
v_{j,n} &= \mathrm{Re}\langle QY_j g, h_n \rangle + i\,\mathrm{Im}\langle X_j g, h_n \rangle.
\end{aligned}$$

By the induction hypothesis both (i) and (ii) tend to zero. Also

$$\begin{aligned}
|(v)| &= \left| \sum_k \langle Z_p \cdot W(h)\Phi, B(X_k f)\, B(h_1) \cdot \cdots \cdot B(h_n)\, W(h)\Phi \rangle\, u_{j,n} \right| \\
&\leqslant \|Z_p \cdot W(h)\Phi\| \\
&\quad \cdot \left(\left\| \sum_k B(\mathrm{Re}\langle QY_k g, h_n \rangle X_k f)\, B(h_1) \cdot \cdots \cdot B(h_n)\, W(h)\Phi \right\| \right. \\
&\quad \left. + \left\| \sum_k B((\langle Y_k g, h_n \rangle)\, X_k f)\, B(h_1) \cdot \cdots \cdot B(h_n)\, W(h)\Phi \right\| \right),
\end{aligned}$$

which tends to zero by (5.18), Lemma (3.3), and Corollary (3.3). Analogously (vi) $\to 0$. Finally,

$$\begin{aligned}
|(\text{iii})| &= \left| \sum_k \langle W(h)\Phi, B(Y_k g)\cdot Z_p B(h_1)\cdot \cdots \cdot B(h_n)\, W(h)\Phi\rangle\, u_{k,n} \right| \\
&\leqslant \Big| \sum_k \langle Z_p^* \cdot W(h)\Phi, [B(\operatorname{Re}\langle QY_k f, h_n\rangle\, Y_k g) \\
&\quad + iB(\langle X_k f, h_n\rangle\, Y_k g)] \cdot \cdots \cdot B(h_n)\, W(h)\Phi\rangle \Big| \\
&\quad + 2\Big| \sum_{l,k} \langle W(h)\Phi, [B(\langle Y_j g, X_k f\rangle, Y_l g) \\
&\quad + B(\langle Y_l g, Y_l g\rangle\, X_k f)]\cdot B(h_1)\cdot \cdots \cdot B(h_n)\, W(h)\Phi\rangle\, u_{l,n} \Big| \qquad (5.21)
\end{aligned}$$

and the first term in the right-hand side of (5.21) tends to zero for the same reason as for the term (v). The second term tends to zero by Corollary (3.3), in fact

$$\begin{aligned}
&\left\| \sum_{l,k} |u_{k,n}|\, (\langle Y_k g, X_l f\rangle, Y_l g) \right\| \\
&\qquad \leqslant \sum_{l,k} |u_{k,n}| \cdot \|Y_k g\| \cdot \|X_l f\| \cdot \|Y_l g\| \\
&\qquad \leqslant \left(\sum_k |u_{k,n}|^2 \right)^{1/2} \cdot \left(\sum_k \|Y_k g\|^2 \right) \cdot \left(\sum_l \|X_l f\|^2 \right)^{1/2}
\end{aligned}$$

and the same upper bound holds for

$$\left\| \sum_{l,k} |u_{k,n}|\, (\langle Y_k g, Y_l f\rangle, X_l g) \right\|.$$

From this our statement follows since this upper bound tends to zero by the regularity of X and Y.

COROLLARY (5.5). *Let X, Y be as in Lemma (5.5). Then for all $g_1, ..., g_n, g, f_1, ..., f_n, f \in H$*

$$\begin{aligned}
&\lim_{(I_k)\in\mathscr{P}(I)} \sum_k B(g_1)\cdot \cdots \cdot B(g_n)\, B(Y(I_k) g)\, B(f_1)\cdot \cdots \cdot B(f_n)\, B(X(I_k) f) \\
&\quad = B(g_1)\cdot \cdots \cdot B(g_n)\, B(f_1)\cdot \cdots \cdot B(f_n)[\operatorname{Re}\langle [\![X, \tilde{Y}]\!]\, g, Qf\rangle \\
&\qquad + i\operatorname{Im}\langle [\![X, Y]\!]\, g, f\rangle] \qquad (5.22)
\end{aligned}$$

strongly on D.

Proof. Since

$$[B(Y_k g), B(g_j)] \subseteq 2i \operatorname{Im}\langle Y_k g, g_j\rangle$$

in (5.22) we can, because of (b) in Lemma (5.3), commute $B(Y_k g)$ and $B(X_k f)$ to the left. Therefore the thesis follows from Lemma (5.5).

COROLLARY (5.6). *Let X, Y be as in Lemma* (5.5) *and A_X^f, A_Y^{g+} as defined by* (5.2). *Then*

$$[\![A_X^f, A_Y^{g+}]\!] = \left\langle [\![X, Y]\!] f, \frac{Q+1}{2} g\right\rangle \tag{5.23}$$

$$[\![A_X^{f+}, A_Y^g]\!] = \left\langle \frac{Q-1}{2} f, [\![X, Y]\!] g\right\rangle \tag{5.24}$$

$$[\![A_X^f, A_Y^g]\!] = [\![A_X^{f+}, A_Y^{g+}]\!] = 0, \tag{5.25}$$

where as usual all the identities are meant on D.

Proof. The identities (5.23), (5.24), (5.25) follow from (5.12) using (5.7) and the complex bilinearity of the brackets.

THEOREM (5.7). *Let $X, Y \in \mathcal{M}(T; \mathcal{B}_Q(H))$ be self-adjoint jointly regular measures such that $[\![X, Y]\!]$ exists strongly on H. Then the brackets*

$$[\![N_X, N_Y]\!], [\![N_X, A_Y^f]\!], [\![A_X^f, N_Y]\!], [\![N_X, A_Y^{f+}]\!], [\![A_X^{f+}, N_Y]\!]$$

exist in the topology of strong convergence on D. Moreover the following relations hold on D (the hat on a symbol means that it is missing):

$$\begin{aligned}
&[\![N_X, N_Y]\!]\, B(g_1)\cdot \cdots \cdot B(g_n)\, W(h)\Phi \\
&\quad = \Big\{ - \sum_{1 \leqslant j_1 < j_2 \leqslant n} B(g_1)\cdot \cdots \cdot \hat{B}(g_{j_1})\cdot \cdots \cdot \hat{B}(g_{j_2})\cdot B(g_n) \\
&\qquad \cdot [\operatorname{Re}\langle Q g_{j_1}, [\![X, Y]\!] g_{j_2}\rangle + i \operatorname{Im}\langle g_{j_1}, [\![X, Y]\!] g_{j_2}\rangle \\
&\qquad + \operatorname{Re}\langle [\![X, Y]\!] g_{j_1}, Q g_{j_2}\rangle + i \operatorname{Im}\langle [\![X, Y]\!] g_{j_1}, g_{j_2}\rangle] \\
&\qquad - i \sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n) \\
&\qquad \cdot [\operatorname{Re}\langle [\![X, Y]\!] g_j, Q h\rangle + i \operatorname{Im}\langle [\![X, Y]\!] g_j, h\rangle \\
&\qquad + \operatorname{Re}\langle Q g_j, [\![X, Y]\!] h\rangle + i \operatorname{Im}\langle g_j, [\![X, Y]\!] h\rangle] \\
&\qquad + \sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B([\![X, Y]\!] g_j)\cdot \cdots \cdot B(g_n) \\
&\qquad + B(g_1)\cdot \cdots \cdot B(g_n)\cdot B([\![X, Y]\!] h) + i B(g_1)\cdot \cdots \cdot B(g_n) \\
&\qquad \cdot [\operatorname{Re}\langle [\![X, Y]\!] h, Q h\rangle + i \operatorname{Im}\langle [\![X, Y]\!] f, f\rangle] \Big\}\, W(h)\Phi
\end{aligned} \tag{5.26}$$

$$\llbracket N_X, A_Y^f \rrbracket(I)\, B(g_1) \cdot \cdots \cdot B(g_n)\, W(h)\Phi$$
$$= \left[-A^f_{\llbracket X, Y\rrbracket}(I) + \left\langle \llbracket X, Y\rrbracket(I) f, \frac{Q+1}{2} h \right\rangle \right] B(g_1) \cdot \cdots \cdot B(g_n)\, W(h)\Phi$$
$$- i \sum_{j=1}^{n} \left\langle \llbracket X, Y\rrbracket(I) f, \frac{Q+1}{2} g_j \right\rangle$$
$$\cdot B(g_1) \cdot \cdots \cdot \hat{B}(g_j) \cdot \cdots \cdot B(g_n)\, W(h)\Phi \tag{5.27}$$

$$\llbracket N_X, A_Y^{f+} \rrbracket(I)\, B(g_1) \cdot \cdots \cdot B(g_n)\, W(h)\Phi$$
$$= \left[A^{f+}_{\llbracket X, Y\rrbracket}(I) + \left\langle \frac{Q-1}{2} h, \llbracket X, Y\rrbracket(I) f \right\rangle \right] B(g_1) \cdot \cdots \cdot B(g_n)\, W(h)\Phi$$
$$- i \sum_{j=1}^{n} \left\langle \frac{Q-1}{2} g_j, \llbracket X, Y\rrbracket(I) f \right\rangle$$
$$\cdot B(g_1) \cdot \cdots \cdot \hat{B}(g_j) \cdot \cdots \cdot B(g_n)\, W(h)\Phi \tag{5.28}$$

$$\llbracket A_X^f, N_Y \rrbracket(I)\, B(g_1) \cdot \cdots \cdot B(g_n)\, W(h)\Phi$$
$$= \left[\left\langle \frac{Q+1}{2} f, \llbracket X, Y\rrbracket(I) h \right\rangle \right] B(g_1) \cdot \cdots \cdot B(g_n)\, W(h)\Phi$$
$$- i \sum_{j=1}^{n} \left\langle \frac{Q+1}{2} f, \llbracket X, Y\rrbracket(I) g_j \right\rangle$$
$$\cdot B(g_1) \cdot \cdots \cdot \hat{B}(g_j) \cdot \cdots \cdot B(g_n)\, W(h)\Phi \tag{5.29}$$

$$\llbracket A_X^{f+}, N_Y \rrbracket(I)\, B(g_1) \cdot \cdots \cdot B(g_n)\, W(h)\Phi$$
$$= \left[\left\langle \llbracket X, Y\rrbracket(I) h, \frac{Q-1}{2} f \right\rangle \right] B(g_1) \cdot \cdots \cdot B(g_n)\, W(h)\Phi$$
$$- i \sum_{j=1}^{n} \left\langle \llbracket X, Y\rrbracket(I) g_j, \frac{Q-1}{2} f \right\rangle$$
$$\cdot B(g_1) \cdot \cdots \cdot \hat{B}(g_j) \cdot \cdots \cdot B(g_n)\, W(h)\Phi. \tag{5.30}$$

Proof. Let $X, Y \in M(T; \mathscr{B}_Q(H))$ be as stated in the theorem. Then for every $I \subseteq T$ and for every partition $(I_k) \in \mathscr{P}(I)$ we have, using Proposition (4.1),

$$N_{X_k} N_{Y_k} B(g_1) \cdot \cdots \cdot B(g_n)\, W(f)\Phi$$
$$= -i \sum_{j=1}^{n} N_{X_k} B(g_1) \cdot \cdots \cdot B(iY_k g_j) \cdot \cdots \cdot B(g_n)\, W(f)\Phi$$
$$+ N_{X_k} \cdot B(g_1) \cdot \cdots \cdot B(g_n) \cdot B(iY_k h)\, W(h)\Phi$$
$$- N_{X_k} B(g_1) \cdot \cdots \cdot B(g_n) \langle h, Y_k h\rangle\, W(h)\Phi. \tag{5.31}$$

And, again by Proposition (4.1), this can be written as the sum of the terms

$$-\sum_{1\leqslant j_1<j_2\leqslant n} B(g_1)\cdot B(iX_k g_{j_1})\cdot \cdots \cdot B(iY_k g_{j_2})\cdot \cdots \cdot B(g_n)\, W(h)\Phi \quad (1)$$

$$-\sum_{1\leqslant j_1<j_2\leqslant n} B(g_1)\cdot \cdots \cdot B(iY_k g_{j_1})\cdot \cdots \cdot B(iX_k g_{j_2})\cdot \cdots \cdot B(g_n)\, W(f)\Phi \quad (2)$$

$$+\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(X_k Y_k g_j)\cdot \cdots \cdot B(g_n)\, W(h)\Phi \quad (3)$$

$$-i\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(iY_k g_j)\cdot \cdots \cdot B(g_n)\, B(iX_k h)\cdot W(h)\Phi \quad (4)$$

$$-i\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(iY_k g_j)\cdot \cdots \cdot B(g_n)\langle h, X_k h\rangle \cdot W(h)\Phi \quad (5)$$

$$-i\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(iX_k g_j)\cdot \cdots \cdot B(g_n)\, B(iY_k h)\cdot W(h)\Phi \quad (6)$$

$$+iB(g_1)\cdot \cdots \cdot B(g_n)\, B(X_k Y_k h)\, W(h)\Phi \quad (7)$$

$$+B(g_1)\cdot \cdots \cdot B(g_n)\, B(Y_k h)\, B(X_k h)\, W(h)\Phi \quad (8)$$

$$-B(g_1)\cdot \cdots \cdot B(g_n)\, B(iY_k h)\langle h, X_k h\rangle\, W(h)\Phi \quad (9)$$

$$+i\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(X_k g_j)\cdot \cdots \cdot B(g_n)\langle h, Y_k h\rangle\, W(h)\Phi \quad (10)$$

$$+B(g_1)\cdot \cdots \cdot B(g_n)\, B(iX_k h)\langle h, X_k h\rangle\, W(h)\Phi \quad (11)$$

$$-B(g_1)\cdot \cdots \cdot B(g_n)\langle h, Y_k h\rangle\langle h, X_k h\rangle\, W(h)\Phi. \quad (12)$$

By point (a) of Lemma (5.3) we see that, after summing over k, the term (12) will tend to zero in the limit of infinitely fine partitions of I. By point (b) of the same lemma the terms (5), (9), (10), and (11) will also give zero contributions to $[\![N_X, N_Y]\!]$. By Corollary (3.3) and the real linearity of $f\mapsto B(f)$, the contributions of (3) and (7) will be respectively

$$\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B([\![X, Y]\!]\, g_j)\cdot \cdots \cdot B(g_n)\, W(h)\Phi \quad (3a)$$

$$iB(g_1)\cdot \cdots \cdot B(g_n)\cdot B([\![X, Y]\!]\, h)\cdot W(h)\Phi. \quad (7a)$$

Finally, by Corollary (5.5) the contributions of (1), (2), (4), (6), and (8) can be respectively written as

$$-\sum_{1\leqslant j_1<j_2\leqslant n} B(g_1)\cdot \cdots \cdot \hat{B}(g_{j_1})\cdot \cdots \cdot \hat{B}(g_{j_2})\cdot \cdots \cdot B(g_n)\, W(h)\Phi$$

$$\cdot\,[\mathrm{Re}\langle Qg_{j_1}, [\![X, Y]\!]\, g_{j_1}\rangle + i\,\mathrm{Im}\langle g_{j_1}, [\![X, Y]\!]\, g_{j_2}\rangle] \quad (1a)$$

$$- \sum_{1 \leqslant j_1 < j_2 \leqslant n} B(g_1) \cdot \cdots \cdot \hat{B}(g_{j_1}) \cdot \cdots \cdot \hat{B}(g_{j_2}) \cdot \cdots \cdot B(g_n)\, W(h)\Phi$$

$$\cdot [\mathrm{Re}\langle [\![X, Y]\!]\, g_{j_1}, Qg_{j_2}\rangle + i\, \mathrm{Im}\langle [\![X, Y]\!]\, g_{j_1}, g_{j_2}\rangle] \tag{2a}$$

$$-i \sum_{j=1}^{n} B(g_1) \cdot \cdots \cdot \hat{B}(g_j) \cdot \cdots \cdot B(g_n)\, W(h)\Phi$$

$$\cdot [\mathrm{Re}\langle [\![X, Y]\!]\, g_j, Qh\rangle + i\, \mathrm{Im}\langle [\![X, Y]\!]\, g_j, Qh\rangle] \tag{4a}$$

$$-i \sum_{j=1}^{n} B(g_1) \cdot \cdots \cdot \hat{B}(g_j) \cdot \cdots \cdot B(g_n)\, W(h)\Phi$$

$$\cdot [\mathrm{Re}\langle [\![X, Y]\!]\, h, Qg_j\rangle + i\, \mathrm{Im}\langle [\![X, Y]\!]\, h, g_j\rangle] \tag{6a}$$

$$+B(g_1) \cdot \cdots \cdot B(g_n)\, W(f)\Phi\, [\mathrm{Re}\langle [\![X, Y]\!]\, h, Qh\rangle + i\, \mathrm{Im}\langle [\![X, Y]\!]\, h, Qh\rangle]. \tag{8a}$$

Thus we conclude that

$$\begin{aligned} \lim_{(I_k) \in \mathscr{P}} \sum_k N_{X_k} N_{Y_k} B(g_1) \cdot \cdots \cdot B(g_n)\, W(h)\Phi \\ &= [\![N_X, N_Y]\!](I)\, B(g_1) \cdot \cdots \cdot B(g_n)\, W(h)\Phi \\ &= (1a) + (2a) + (3a) + (4a) + (6a) + (7a) + (8a) \end{aligned}$$

and this proves (5.26).

The verification of the identities (5.27), ..., (5.30) is based on exactly the same techniques as above. We conclude the proof of the theorem by a description of the strategy through which all these identities are proved. We believe that, by doing so, we shall provide the reader greater insight than by proving one by one all the identities.

Let X, Y be two $\mathscr{B}_Q(H)$-valued measures and let M_X, L_Y be two of the associated $\mathscr{L}(D)$-valued measures (e.g., B_X^f, N_Y, ...). Writing down explicitly the action of $M_X(I_k) \cdot L_Y(I_k)$ on a vector in the domain D, only the following four kinds of terms will occur:

(a) terms containing the products of matrix elements of $X(I_k)$ and of $Y(I_k)$ (cf. the term (12) in the above computation);

(b) terms containing the product of a matrix element of $X(I_l)$ with a field $B(Y(I_k)f)$ (cf. the term (5) in the above computation);

(c) terms containing a field evaluated at the product of the two measures $B(X(I_l) \cdot Y(I_k)g)$ (cf. the term (3) in the above computation);

(d) terms containing the product of two fields $B(X(I_l)g_1)$ and $B(Y(I_k)g_2)$ possibly sleparated by a product of field operators (cf. the term (6) in the above computation).

The terms of type (a) or (b) will not contribute to the brackets by Lemma (5.3) (a) and (c).

By the real linearity of $f \mapsto B(f)$ and Corollary (3.3) the terms which are of type (c) will give a contribution where the factor $B(X(I_l) \cdot Y(I_l)h)$ is replaced by $B([\![X, Y]\!] h)$.

Finally, by Corollary (5.5), the terms that belong to the class (d) will give rise to a contribution where the factors $B(X(I_l)h)$ and $B(Y(I_l)h)$ are replaced by a scalar factor.

6. The Gaussian Ito Table

The basic integrators

$$B_X^f,\ A_X^f,\ A_X^{f+},\ N_X,\ \langle f, X(\cdot)\, g\rangle,$$
$$f, g \in H,\ X, Y \in \mathscr{M}(T; \mathscr{B}(H)) \tag{6.1}$$

can be defined for any representation $\{\mathscr{H}, \pi\}$ of the CCR over a Hilbert space H as elements of $\mathscr{M}(T; \mathscr{L}(D; \mathscr{H}))$. Taking Ito products (i.e., quadratic variations) of the basic integrators we can form new elements of $\mathscr{M}(T; \mathscr{L}(D; \mathscr{H}))$ called 1st-order integrators. By induction we can define an nth-order integrator as the Ito product of a basic integrator with an $(n-1)$st-order integrator and a (p, q)th-order integrator as the Ito product of a pth-order integrator with a qth-order one. A pleasant feature of Gaussian representations is that for them this hierarchy closes at the second order, i.e., the Ito product of two first-order integrators as well as any second-order integrator can be expressed as a linear combination of first-order integrators (provided the underlying measures satisfy the commutativity condition of Definition (6.1)). In Section 7 we shall prove that, even without this commutativity condition, the closure of the hierarchy at the first order is a characterizing property of the Fock representation.

Definition (6.1). We say that two regular measures $X, Y \in \mathscr{M}(T; \mathscr{B}_Q(H))$ Ito-commute if both $[\![X, Y]\!]$ and $[\![Y, X]\!]$ exist and $[\![X, Y]\!] = [\![Y, X]\!]$.

Definition (6.2). Let $\mathscr{I}$ be an Ito algebra in $\mathscr{M}(T, \mathscr{B}_Q(H))$. We say that $\mathscr{I}$ is regular if all $X, Y \in \mathscr{I}$ are jointly regular in the sense of Definition (5.2). $\mathscr{I}$ is called self-adjoint if for all $X \in \mathscr{I}$ and all $I \subseteq T$, the measure X^* defined by

$$X^*(I) = X(I)^*$$

belongs to $\mathscr{I}$.

THEOREM (6.3). *Let $\mathscr{I}$ be an associative, Ito-commutative, regular, self-adjoint Ito algebra in $\mathscr{M}(T, \mathscr{B}_Q(H))$. Then, for any operator C in $\mathscr{L}(D)$, denoting $C^{\#}$ either C itself or its adjoint, the following statements are true*:

(i) *The linear span of*

$$\{\langle f, X(\cdot)g\rangle, A_X^{f\#}, N_X, [\![A_X^{f\#}, N_X]\!], [\![N_X, A_X^{f\#}]\!], [\![N_X, N_Y]\!]; \quad f, g \in H, X, Y \in \mathscr{I}\}$$

is an associative self-adjoint Ito algebra $\mathscr{K}$ in $\mathscr{L}(D)$ for the multiplication $[\![\cdot, \cdot]\!]$.

(ii) (*Associativity of the Ito Multiplication*) *For all $C_1, C_2 \in \mathscr{K}$ we have that*

$$[\![C_1, C_2]\!]^* = [\![C_2^*, C_1^*]\!] \tag{6.2}$$

and for all self-adjoint $X, Y, Z \in \mathscr{I}$ one has

$$[\![M_{[\![X,Y]\!]}, M'_Z]\!] = [\![M_X, M'_{[\![Y,Z]\!]}]\!], \tag{6.3}$$

where $M_{\#}$ and $M'_{\#}$ stand for $A_{\#}^{f\#}$ or $N_{\#}$.

(iii) *Using* (6.2) *and* (6.3) *and the associativity, the algebraic structure of $\mathscr{K}$ is completely determined by the relations* (5.23), (5.24), *and* (5.25) *and by the following relations which hold on the domain $\mathscr{D}$ for all $f, g \in H$ and all self-adjoint $X, Y, Z \in \mathscr{I}$*:

$$[\![N_X, A_Y^f]\!] = -A_{[\![X,Y]\!]}^f + [\![A_Y^{f\#}, N_X]\!] \tag{6.4}$$

$$[\![A_X^{f\#}, [\![A_Y^{g\#}, N_Z]\!]]\!] = 0 \tag{6.5}$$

$$[\![A_X^f, [\![N_Y, N_Z]\!]]\!] = [\![A_X^f, N_{[\![Y,Z]\!]}]\!] \tag{6.6}$$

$$[\![N_X, [\![N_Y, N_Z]\!]]\!] = N_{[\![X,[\![Y,Z]\!]]\!]}. \tag{6.7}$$

Remark (1). From (6.2) it follows that all brackets in the Ito algebra $\mathscr{K}$ exist not only in the topology of strong convergence on $\mathscr{D}$ but even on the topology of strong-* convergence on $\mathscr{D}$.

Proof of Theorem (6.3). The validity of (6.3) and (6.4) follows immediately from Theorem (5.7), the associativity, and the Ito-commutativity of $\mathscr{I}$. Therefore the theorem will be proved if we show the validity of the relations (6.5), (6.6), (6.7) and of the following list of relations (for all $f, g \in H$ and all self-adjoint $X, Y, Z, V \in \mathscr{I}$):

$$[\![N_X, A_Y^{f+}]\!] = A_{[\![X,Y]\!]}^{f+} + [\![A_Y^{f+}, N_X]\!] \tag{6.8}$$

$$[\![[\![A_X^f, N_Y]\!], N_Z]\!] = [\![A_X^f, N_{[\![Y,Z]\!]}]\!] \tag{6.9}$$

$$[\![[\![A_X^{f+}, N_Y]\!], N_Z]\!] = -[\![A_X^{f+}, N_{[\![Y,Z]\!]}]\!] \tag{6.10}$$

$$[\![[\![A_X^{f+}, N_Y]\!], N_Z]\!] = -[\![A_{[\![X,Y]\!]}^{f+}, N_Z]\!] \tag{6.11}$$

$$[\![N_X, [\![A_Y^{f\#}, N_Z]\!]]\!] = [\![[\![N_X, A_Y^{f\#}]\!], N_Z]\!] = 0 \tag{6.12}$$

$$[\![[\![N_X, N_Y]\!], N_Z]\!] = N_{[\![[\![X,Y]\!],Z]\!]} \tag{6.13}$$

$$[\![[\![A_X^f, N_Y]\!], A_Z^{g+}]\!] = \left\langle \frac{Q+1}{2} f, [\![[\![X, Y]\!], Z]\!]\, g \right\rangle \tag{6.14}$$

$$[\![[\![A_X^{f+}, N_Y]\!], A_Z^g]\!] = \left\langle [\![[\![X, Y]\!], Z]\!]\, g, \frac{Q-1}{2} f \right\rangle \tag{6.15}$$

$$[\![[\![A_X^f, N_Y]\!], A_Z^g]\!] = [\![[\![A_X^{f+}, N_Y]\!], A_Z^{g+}]\!] = 0 \tag{6.16}$$

$$[\![[\![A_X^{f\#}, N_Y]\!], [\![A_Z^{g\#}, N_V]\!]]\!] = 0 \tag{6.17}$$

$$[\![[\![A_X^f, N_Y]\!], [\![N_Z, N_V]\!]]\!] = [\![A_{[\![,Y]\!]}^f, N_{[\![Z,V]\!]}]\!] \tag{6.18}$$

$$[\![[\![A_X^{f+}, N_Y]\!], [\![N_Z, N_V]\!]]\!] = -[\![A_{[\![X,Y]\!]}^{f+}, N_{[\![Z,V]\!]}]\!] \tag{6.19}$$

$$[\![[\![N_X, N_Y]\!], A_Z^f]\!] = -[\![N_{[\![X,Y]\!]}, A_Z^f]\!] \tag{6.20}$$

$$[\![[\![N_X, N_Y]\!], A_Z^{f+}]\!] = [\![N_{[\![X,Y]\!]}, A_Z^{f+}]\!] \tag{6.21}$$

$$[\![[\![N_X, N_Y]\!], [\![A_Z^{f\#}, N_V]\!]]\!] = 0 \tag{6.22}$$

$$[\![[\![N_X, N_Z]\!], [\![N_Z, N_V]\!]]\!] = [\![N_{[\![X,Y]\!]}, N_{[\![Z,V]\!]}]\!]. \tag{6.23}$$

The verification of (6.3), ..., (6.23) is done with the same techniques we already discussed in Theorem (5.7). We shall omit the details. However, in order to illustrate the role of the condition of Ito-commutativity in the proof, we shall prove that (6.7) holds on the vectors of the form $W(h)\Phi$. To prove this we compute, for all $I \subseteq T$ and $h \in H$,

$$\lim_{(I_k) \in \mathscr{P}(I)} \sum_k N_X(I_k) \cdot [\![N_Y, N_Z]\!](I_k) \cdot W(h)\Phi.$$

Using the shorthand notation X_k and $[\![Y, Z]\!]_k$ instead of $X(I_k)$ and $[\![Y, Z]\!](I_k)$, Proposition (5.3) and Proposition (5.6) imply that

$$\begin{aligned} &N_X(I_k) \cdot [\![N_Y, N_Z]\!](I_k)\, W(h)\Phi \\ &\quad = N_X(I_k) \cdot iB([\![Y, Z]\!]_k h)\, W(h)\Phi + [\mathrm{Re}\langle [\![Y, Z]\!]_k h, Qh\rangle \\ &\qquad + i\,\mathrm{Im}\langle [\![Y, Z]\!]_k h, h\rangle]\, N_X(I_k)\, W(h)\Phi \\ &\quad = B(iX_k [\![Y, Z]\!]_k h)\, W(h)\Phi + iB([\![Y, Z]\!]_k h) \cdot B(iX_k)\, W(h)\Phi \\ &\qquad + [\mathrm{Re}\langle [\![Y, Z]\!]_k h, Qh\rangle + i\,\mathrm{Im}\langle [\![Y, Z]\!]_k h, h\rangle] \\ &\qquad \cdot [B(iX_k) - \langle h, X_k h\rangle]\, W(h)\Phi. \end{aligned} \tag{6.24}$$

By Lemma (5.3) the third and the fourth terms vanish in the limit when they are summed over k. By Lemma (3.3) the first term in the limit gives the contribution

$$B(i[\![X, [\![Y, Z]\!]]\!](I)h)\, W(h)\Phi \tag{6.25}$$

and by Corollary (5.5)(c) the contribution of the second term is

$$i[\mathrm{Re}\langle [\![X, [\![Y, Z]\!]]\!](I)\, h, iQh\rangle + i\,\mathrm{Im}\langle [\![X, [\![Y, Z]\!]]\!](I)\, h, ih\rangle]\; W(h)\Phi,$$

which, due to the self-adjointness and the Ito-commutativity of X and Y, is equal to

$$-\langle h, [\![X, [\![Y, Z]\!]]\!](I)h\rangle\, W(h)\Phi.$$

Thus in the limit and when summed over k the right-hand side of (6.6) converges to

$$B(i[\![X, [\![Y, Z]\!]]\!](I)h)\, W(h)\Phi - \langle h, [\![X, [\![Y, Z]\!]]\!](I)h\rangle\, W(h)\Phi,$$

which (cf. Proposition (4.1)) is equal to the left-hand side of (6.7).

The only kind of Ito algebras considered up to now in the literature are those corresponding to the choice of a single spectral measure $e: I \subseteq T \to e(I) \in \mathrm{Proj}(H)$. In this case $[\![e, e]\!] = e$ hence the complex multiples of e form indeed a commutative Ito algebra which is full in the sense of Definition (7.1) since

$$e(T) = 1.$$

The Gaussian Ito table corresponding to this Ito algebra assumes a particularly simple form. In order to describe it in the most economical way, it is convenient to introduce the following notations: A^f, A^{f+}, N will denote respectively A_e^f, A_e^{f+}, N_e; and the $\mathscr{L}(D)$-valued measures $S_\pm^f$ are defined by

$$S_+^f = [\![A^{f+}, N]\!] = -A^{f+} + [\![N, A^{f+}]\!] \tag{6.26}$$

$$S_-^f = [\![A^f, N]\!] = A^f + [\![N, A^f]\!]. \tag{6.27}$$

Moreover if we agree to use exchangeably the symbols X and dX dor a generic measure X and if for any $\mathscr{L}(D)$-valued measures M, L we use the notation

$$dM \cdot dL = d[\![M, L]\!]$$

then the Gaussian Ito table for the measures A^f, A^{f+}, N takes the form

$\nearrow$	dA^{g+}	dN	dA^g	$dN\cdot dN$	dS^g_+	dS^g_-
dA^f	$\langle de\cdot f, \frac{1}{2}(Q+1)g\rangle$	dS^f_-	0	dS^g_-	0	0
dN	$dA^{g+}+dS^g_+$	$dN\cdot dN$	$-dA^g+dS^g_-$	dN	0	0
dA^{f+}	0	dS^f_+	$\langle de\cdot g, \frac{1}{2}(Q-1)f\rangle$	$-dS^f_+$	0	0
$dN\cdot dN$	$dA^{g+}+dS^g_+$	dN	0	$dN\cdot dN$	0	0
dS^f_-	$\langle de\cdot f, \frac{1}{2}(Q+1)g\rangle$	dS^f_-	0	dS^f_-	0	0
dS^f_+	0	$-dS^f_+$	$\langle de\cdot g, \frac{1}{2}(Q-1)f\rangle$	$-dS^f_+$	0	0

where the brackets with the scalar measures have not been included since they are all zero.

Notice that, in the Fock case ($Q=1$), $S^f_-=A^f$ and $S^f_+=0$. Therefore in this case the Ito table closes in the first three rows and columns. A corollary of Theorem (7.2), below, is that this property characterizes the Fock representation.

7. Ito Algebra Characterization of the Fock State

DEFINITION (7.1). We say that a subset $\mathcal{M}_0$ of $\mathcal{M}(T;\mathcal{B}_Q(H))$ is full if the set

$$\{[\![X, Y]\!](I)f : f\in H, I\subseteq T, X, Y\in\mathcal{M}(T;\mathcal{B}_Q(H)),$$
$$X, Y \text{ are jointly regular and } [\![X, Y]\!] \text{ exists}\}$$

is dense in H.

Notice that a spectral measure in $\mathcal{M}(T;\mathcal{B}_Q(H))$ is certainly full.

THEOREM (7.2). *For all $f\in H$ and all jointly regular self-adjoint $X, Y\in\mathcal{M}(T;\mathcal{B}_Q(H))$ such that the brackets $[\![X, Y]\!]$ exist consider the following statements (where all equalities are meant to hold on D):*

(i) *φ_Q is the Fock state (i.e., $Q=1$),*
(ii) $[\![A^f_X, N_Y]\!]=A^f_{[\![X,Y]\!]^*}$,
(iii) $[\![A^{f*}_X, N_Y]\!]=0$,
(iv) $[\![N_Y, A^f_X]\!]=0$,
(v) $[\![N_Y, A^{f+}_X]\!]=A^{f+}_{[\![Y,X]\!]}$,
(vi) *N is a $[\![\cdot,\cdot]\!]$-homomorphism, i.e.,*

$$[\![N_X, N_Y]\!]=N_{[\![X,Y]\!]}.$$

Then (i) *implies each of the other statements and, if $\mathcal{M}(T;\mathcal{B}_Q(H))$ is full, then all the statements are equivalent.*

Proof. The fact that (i) implies the statements (ii), ..., (v) follows immediately from Proposition (5.4) and from the validity, for $Q=1$, of the identity

$$\begin{aligned}&A_X^f(I)\,B(g_1)\cdot\cdots\cdot B(g_n)\,W(h)\Phi\\&\quad=-i\sum_{j=1}^{n}B(g_1)\cdot\cdots\cdot\hat{B}(g_j)\cdot\cdots\cdot B(g_n)\langle X(I)\,f,g_j\rangle\,W(h)\Phi\\&\qquad+\langle X(I)\,f,h\rangle\,B(g_1)\cdot\cdots\cdot B(g_n)\,W(h)\Phi,\end{aligned}\tag{7.1}$$

Now we prove that (i)$\Rightarrow$(vi). Using Proposition (5.3) and with the notations

$$Z=[\![X,Y]\!];\qquad Z_s=\frac{1}{2}(Z+Z^*);\qquad Z_a=\frac{1}{2i}(Z-Z^*)\tag{7.2}$$

this amounts to proving that

$$\begin{aligned}&\sum_{1\leqslant j_1<j_2\leqslant n}B(g_1)\cdot\cdots\cdot\hat{B}(g_{j_1})\cdot\cdots\cdot\hat{B}(g_{j_2})\cdot\cdots\cdot B(g_n)\,W(f)\Phi\\&\qquad\cdot[\langle g_{j_1},Zg_{j_2}\rangle+\langle g_{j_2},Zg_{j_1}\rangle]\\&\qquad+\sum_{j=1}^{n}B(g_1)\cdot\cdots\cdot\hat{B}(g_j)\cdot\cdots\cdot B(g_n)\,W(f)\Phi\\&\qquad\cdot[\langle Zg,f\rangle+\langle g,Zf\rangle]\\&\qquad+\sum_{j=1}^{n}B(g_1)\cdot\cdots\cdot B(Zg_j)\cdot\cdots\cdot B(g_n)\,W(f)\Phi\\&\qquad+\sum_{j=1}^{n}B(g_1)\cdot\cdots\cdot B(g_n)\cdot B(Zf)\cdot W(f)\Phi\\&\qquad-B(g_1)\cdot\cdots\cdot B(g_n)\,W(f)\Phi\langle Zf,f\rangle\\&\quad=-i\sum_{j=1}^{n}B(g_1)\cdot\cdots\cdot B(iZ_s\,j)\cdot\cdots\cdot B(g_n)\,W(f)\Phi\\&\qquad+\sum_{j=1}^{n}B(g_1)\cdot\cdots\cdot B(iZ_a\,j)\cdot\cdots\cdot B(g_n)\,W(f)\Phi\\&\qquad+B(g_1)\cdot\cdots\cdot B(g_n)\,B(iZ_s\,f)\,W(f)\Phi+iB(g_1)\\&\qquad\cdot\cdots\cdot B(g_n)\,B(iZ_a\,f)\,W(f)\Phi\\&\qquad-B(g_1)\cdot\cdots\cdot B(g_n)\,W(f)\Phi\langle Zf,f\rangle.\end{aligned}\tag{7.3}$$

One easily checks that for the Fock representation the following holds for all $f, g \in H$ and $Z \in \mathscr{B}(H)$:

$$(iB(Zg) + \langle Zg, f\rangle)\, W(f)\Phi$$
$$= (B(iZ_s g) + iB(Z_a g) - \langle g, Zf\rangle\, W(f)\Phi. \tag{7.4}$$

Furthermore, using the commutation relations of the B and (7.4) we find

$$\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(iZ_s g_j)\cdot \cdots \cdot B(g_n)\, W(f)\Phi$$
$$+\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(iZ_a g_j)\cdot \cdots \cdot B(g_n)\, W(f)\Phi$$
$$= -i \sum_{1\leqslant j_1 < j_2 \leqslant n} B(g_1)\cdot \cdots \cdot \hat{B}(g_{j_1})\cdot \cdots \cdot \hat{B}(g_{j_2})$$
$$\cdot \cdots \cdot B(g_n)\, W(f)\, \Phi[2i\,\mathrm{Im}\langle Z_s g_{j_1}, g_{j_2}\rangle]$$
$$+ \sum_{1\leqslant j_1 < j_2 \leqslant n} B(g_1)\cdot \cdots \cdot \hat{B}(g_{j_1})\cdot \cdots \cdot \hat{B}(g_{j_2})$$
$$\cdot \cdots \cdot B(g_n)\, W(f)\, \Phi[2i\,\mathrm{Im}\langle Z_a g_{j_1}, g_{j_2}\rangle]$$
$$-i\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n)\, B(iZ_s g_j)\, W(f)\Phi$$
$$+\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n)\, B(iZ_a g_j)\, W(f)\Phi$$
$$= -i \sum_{1\leqslant j_1 < j_2 \leqslant n} B(g_1)\cdot \cdots \cdot \hat{B}(g_{j_1})\cdot \cdots \cdot \hat{B}(g_{j_2})$$
$$\cdot \cdots \cdot B(g_n)\, W(f)\, \Phi[\langle Zg_{j_1}, g_{j_2}\rangle + \langle g_{j_1}, Zg_{j_2}\rangle$$
$$-2i\,\mathrm{Im}\langle Zg_{j_1}, Zg_{j_2}\rangle] + \sum_{j=1}^{n} B(g_1)$$
$$\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n)\, B(Zg_j)\, W(f)\Phi$$
$$-i\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n)\, W(f)\, \Phi[\langle Zg, f\rangle + \langle g, Zf\rangle]$$
$$= \sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(Zg_j)\cdot \cdots \cdot B(g_n)\, W(f)\Phi - i\sum_{j=1}^{n} B(g_1)$$
$$\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n)\, W(f)\, \Phi[\langle Zg, f\rangle + \langle g, Zf\rangle]. \tag{7.5}$$

Combining (7.4) and (7.5) we finally get (7.3). Suppose now that $\mathscr{M}(T; \mathscr{B}_Q(H))$ is full. We prove then that (ii) $\to$ (i). From Proposition (5.4) we know that

$$[\![A_X^f, N_Y]\!]\, W(h)\Phi = \left\langle \frac{Q+1}{2} f, Zh \right\rangle W(h)\Phi$$

with $Z = [\![X, Y]\!]$. Hence by (ii) we have that

$$\langle \Phi, W(h)\Phi \rangle \left\langle \frac{Q+1}{2} f, Zh \right\rangle = \langle \Phi, A^{f}_{Z^*} W(h)\Phi$$

$$= \langle \Phi, W(h)\Phi \rangle \langle f, Zh \rangle$$

so $\langle (1-Q)/2 \cdot f, Zh \rangle = 0$ for all $f, h \in H$ and $Z = [\![X, Y]\!]$, thus $Q = 1$. The other implications are proven in the same way.

THEOREM (7.3). *Let $\mathscr{I}$ be as above. Let $\mathscr{I}$ be a self-adjoint regular Ito algebra in $\mathscr{M}(T; \mathscr{B}_Q(H))$ and denote by $\mathscr{K}$ the linear span in $\mathscr{M}(T; \mathscr{B}_Q(H))$ of the measures*

$$\{\langle f, X(\cdot)\, g\rangle, A^f_X, A^{f+}_X, N_X : X \in \mathscr{I}; f, g \in H\}.$$

Consider the following statements:

(i) φ_Q *is the Fock state* ($Q = 1$),

(ii) $\mathscr{K}$ *is an Ito algebra.*

Then (i) *implies* (ii). *Moreover, if $\mathscr{I}$ is full,* (i) *and* (ii) *are equivalent.*

Proof. The fact that (i) implies (ii) was already contained in Theorem (7.2). Suppose now that $\mathscr{I}$ is full and that $\mathscr{K}$ is an Ito algebra. Fix now any $f \in H$ and $X, Y \in \mathscr{I}$ such that $X^* = X$ and $Y^* = Y$. Since $\mathscr{K}$ is an Ito algebra we know that there exist $f_1, g_1, f_2, g_2 \in H$ and $Z_1, Z_2, Z_3, Z_4, Z_5 \in \mathscr{I}$ such that for all $I \subseteq T$,

$$[\![A^f_X, N_Y]\!](I) = \langle f_1, Z_1(I)\, g_1 \rangle + A^{f_2}_{Z_2}(I) + A^{f_3}_{Z_3}(I) + N_{Z_4}(I).$$

Taking the scalar product of this operator identity between the vectors $W(h)\Phi$ and $W(g)\Phi$ and using the explicit action of the involved operators on these vectors (cf. Theorem (5.7), Proposition (4.1), and Lemma (3.2)) we find that for all $h, g \in H$ and all $I \subseteq T$,

$$\begin{aligned}\langle \tfrac{1}{2}(Q+1) f, [\![X, Y]\!](I)\, g \rangle = {} & \langle f_1, Z_1(I)\, g_1 \rangle \\ & + \langle Z_2(I) f_2, \tfrac{1}{2}(Q+1)\, g \rangle \\ & + \langle Z_2(I) f_2, \tfrac{1}{2}(1-Q) h \rangle \\ & + \langle \tfrac{1}{2}(1+Q) h, Z_3(I) f_3 \rangle \\ & + \langle \tfrac{1}{2}(1-Q)\, g, Z_3(I) f_3 \rangle \\ & + \langle \tfrac{1}{2}(1+Q) h, Z_4(I) f_4 \rangle \\ & + \langle \tfrac{1}{2}(1-Q)\, g, Z_4(I) h \rangle.\end{aligned}$$

Hence letting $h = g = 0$, we have that, for all $I \subseteq T$,

$$\langle f_1, Z_1(I) g_1 \rangle = 0.$$

Similarly, letting $h = 0$ and using the arbitrariness of g, we obtain

$$\tfrac{1}{2}(Q+1)[\![X, Y]\!]^*(I) f = \tfrac{1}{2}(Q+1) Z_2(I) f_2;$$
$$\tfrac{1}{2}(1-Q) Z_3(I) f_3 = 0$$

and by a similar argument

$$\tfrac{1}{2}(1-Q) Z_2(I) f_2 = \tfrac{1}{2}(1+Q) Z_3(I) f_3$$
$$= \tfrac{1}{2}(1+Q) Z_4(I) = \tfrac{1}{2}(1-Q) Z_4(I) = 0$$

and, since $Q \geqslant 1$, this implies that

$$Z_1 = Z_4 = 0; \qquad Z_3(I) f_3 = 0; \qquad Z_1(I) f_1 = [\![X, Y]\!]^*(I) f$$

thus we know that for all $f \in H$ and all self-adjoint $X, Y \in \mathscr{I}$,

$$[\![A_X^f, N_Y]\!] = A^f_{[\![X, Y]\!]^*}.$$

Since $\mathscr{I}$ was supposed to be full this implies by Theorem (7.2) that $Q = 1$.

8. Fermion Ito Algebras

Let H be a complex Hilbert space whose scalar product we shall denote $\langle \cdot, \cdot \rangle$ and let $s(\cdot, \cdot) = \mathrm{Re}\langle \cdot, \cdot \rangle$. The CAR C^*-algebra $A(H)$ over (H, s) is the unique C^*-algebra such that there exists a real linear map $B: f \in H \to B(f) \in A(H)$ satisfying

$$\{B(f), B(g)\} = 2s(f, g) = B(f) \cdot B(g) + B(g) \cdot B(f) \tag{8.1}$$

$$B(f)^* = B(f) \tag{8.2}$$

and the set $\{B(g_1), ..., B(g_n); f_j \in H, j = 1, ..., n\}$ is dense in $A(H)$. The conditions (8.1) and (8.2) imply that each $B(f)$ is a bounded operator with norm $\|f\|$.

Definition (8.1). A gauge invariant mean zero Gaussian state φ on $A(H)$ with covariance Q is determined by the conditions

$$\varphi(B(g_1) \cdot \cdots \cdot B(g_n)) = 0 \tag{8.3}$$

if n is odd, and

$$\delta(B(f)\cdot B(g))=\mathrm{Re}\langle f, g\rangle + i\,\mathrm{Im}\langle f, Qg\rangle \tag{8.4}$$

$$\varphi(B(g_1)\cdot\cdots\cdot B(g_{2n}))=\sum_{j_1,\ldots,j_n}\mathrm{sgn}(j_1,\ldots,j_n)\cdot\varphi(B(g_{j_1})\cdot B(g_{j_2}))\cdot\cdots\cdot\varphi(B(g_{j_{2n-1}})\cdot B(g_{j_{2n}})), \tag{8.5}$$

where the sum is taken over all $(j_1,\ldots,j_n)$ from 1 to $2n$ such that $j_1<j_2,\ldots,j_{2n-1}<j_{2n}$ and $j_1<j_3<\cdots<j_{2n-1}$ and Q is a bounded self-adjoint operator on H satisfying

$$\|Q\|\leqslant 1. \tag{8.6}$$

The Fock state is defined by the condition $Q=1$ and the anti-Fock state by the condition $Q=-1$. We denote $\{\mathscr{H}, \pi, \Phi\}$ the GNS triple of φ and

$$D=\text{linear span of }\{B(g_1)\cdot\cdots\cdot B(g_n)\Phi\colon g_j\in H, j=1,\ldots,n\}, \tag{8.7}$$

where we have omitted the representation π in the notation. For $(T, \mathscr{B})$ as in Section 1 define

$$\mathscr{B}_Q(H)=\{X\in\mathscr{B}(H)\colon [X, Q]=0\}$$

and let $\mathscr{M}(T;\mathscr{B}_Q(H))$, $\mathscr{M}(T;\mathscr{L}(D))$ be $\mathscr{L}(D))$-valued measures on $(T,\mathscr{B})$ as in the beginning of Section 5 with D defined by (8.7). As usual, for $I\subseteq T$, we shall denote $\mathscr{P}(I)$ the set of all finite partitions $(I_1,\ldots,I_n)$ of I.

Let $X, Y\in\mathscr{M}(T;\mathscr{B}_Q(H))$ be such that the brackets $[\![X, Y]\!]$ exist in the strong operator topology on H. Then by the uniform boundedness principle it follows that for each $I\subseteq T$ the operator $[\![X, Y]\!](I)\in\mathscr{B}(H)$. Moreover, since Q is bounded,

$$[[\![X, Y]\!](I), Q]=0 \tag{8.8}$$

therefore $[\![X, Y]\!]\in\mathscr{M}(T;\mathscr{B}_Q(H))$.

DEFINITION (8.2). Let φ be a quasi-free state on the CAR C^*-algebra $A(H)$ with GNS triple $\{\mathscr{H}, \pi, \Phi\}$. Let $X\in\mathscr{M}(T;\mathscr{B}_Q(H))$ and let $f\in H$. Define for any $I\subseteq T$

$$B_X^f(I)=\pi(B(X(I)f)) \tag{8.9}$$

and notice that $B_X^f\in\mathscr{M}(T;\mathscr{B}(\mathscr{H}))$. We call B_X^f the Fermion field measures associated with the test function f and the measure X.

LEMMA (8.3). *If $X, Y \in \mathcal{M}(T; \mathcal{B}(H))$ are regular, then for all $I \subseteq T$ and $f, g, h \in H$ one has*

$$\lim_{(I_k) \in \mathcal{P}(I)} \sum_{k=1}^{n} B(X(I_k) f) \langle g, Y(I_k) h \rangle = 0, \tag{8.10}$$

where the limit is meant in the norm topology of $A(H)$.

Proof. Since for any partition $(I_k) \in \mathcal{P}(I)$ we have that

$$\begin{aligned} &\left\| \sum_{k=1}^{n} B(X(I_k) f) \langle g, Y(I_k) h \rangle \right\| \\ &\qquad \leqslant \sum_{k=1}^{n} \|B(X(I_k) f)\| \cdot |\langle g, Y(I_k) h \rangle| \\ &\qquad \leqslant \left(\sum_{k=1}^{n} \|B(X(I_k) f)\|^2 \right)^{1/2} \cdot \left(\sum_{k=1}^{n} |\langle g, Y(I_k) h \rangle|^2 \right)^{1/2} \end{aligned}$$

the result follows directly from the regularity of X and Y.

PROPOSITION (8.4). *Let $X, Y \in \mathcal{M}(T; \mathcal{B}_Q(H))$ be jointly regular measures and assume that $[\![X^*, Y]\!]$ exists strongly. Then for every $f, g \in H$, $[\![B_X^f, B_Y^g]\!]$ exists in the strong operator topology on $\mathcal{B}(\mathcal{H})$. Moreover*

$$[\![B_X^f, B_Y^g]\!](I) = \operatorname{Re}\langle f, [\![X^*, Y]\!](I)\, g \rangle + i \operatorname{Im}\langle Qf, [\![X^*, Y]\!](I)\, g \rangle. \tag{8.11}$$

Proof. Since

$$\begin{aligned} &\left\| \sum_{k=1}^{n} B(X(I_k) f)\, B(Y(I_k) g) \right\| \\ &\qquad \leqslant \left(\sum_{k=1}^{n} \|B(X(I_k) f)\|^2 \right)^{1/2} \cdot \left(\sum_{k=1}^{n} \|B(Y(I_k) g)\|^2 \right)^{1/2} \\ &\qquad \leqslant M \cdot \|f\| \cdot \|g\| \end{aligned}$$

for all $(I_k) \in \mathcal{P}(I)$ by linearity and continuity it is sufficient to prove strong convergence on vectors of the form $B(g_1) \cdot \cdots \cdot B(g_n) \Phi$. Moreover

$$\begin{aligned} &\sum_{k=1}^{n} B(X(I_k) f)\, B(Y(I_k) g)\, B(g_1) \cdot \cdots \cdot B(g_n) \Phi \\ &\quad = \sum_{k=1}^{n} B(X(I_k) f)\, 2 \operatorname{Re}\langle Y(I_k) g, g_1 \rangle\, B(g_2) \cdot \cdots \cdot B(g_n) \Phi \\ &\qquad - \sum_{k=1}^{n} B(Y(I_k) g)\, 2 \operatorname{Re}\langle X(I_k) f, g_1 \rangle\, B(g_2) \cdot \cdots \cdot B(g_n) \Phi \\ &\qquad + B(g_1) \cdot \sum_{k=1}^{n} B(X(I_k) f)\, B(Y(I_k) g)\, B(g_2) \cdot \cdots \cdot B(g_n) \Phi. \end{aligned}$$

Hence, by Lemma (8.3) and repetition of the argument above, it is sufficient to prove that

$$\left\| \sum_{k=1}^{n} (B(X(I_k) f) B(Y(I_k) g) - c)\Phi \right\|^2 \to 0, \tag{8.12}$$

where

$$c = \mathrm{Re}\langle f, [\![X^*, Y]\!](I) g\rangle + i \,\mathrm{Im}\langle Qf, [\![X^*, Y]\!](I) g\rangle. \tag{8.13}$$

But

$$\left\langle \Phi, \sum_{k=1}^{n} B(X(I_k) f) B(Y(I_k) g)\Phi \right\rangle$$

$$= \sum_{k=1}^{n} \mathrm{Re}\langle X(I_k) f, Y(I_k) g\rangle + i \,\mathrm{Im}\langle QX(I_k) f, Y(I_k) g\rangle \to 0,$$

and

$$\varphi\left(\sum_{k,l=1}^{n} B(Y(I_l) g) B(X(I_l) f) B(X(I_k) f) B(Y(I_k) g)\right)$$

$$= \varphi\left(\sum_{l=1}^{n} B(Y(I_l) g) B(X(I_l) f)\right) \cdot \varphi\left(\sum_{k=1}^{n} B(X(I_k) f) B(Y(I_k) g)\right) \quad \text{(a)}$$

$$- \sum_{k,l=1}^{n} \varphi(B(Y(I_l) g) B(X(I_k) f)) \cdot \varphi(B(X(I_l) f) B(Y(I_k) g)) \quad \text{(b)}$$

$$+ \sum_{k,l=1}^{n} \varphi(B(Y(I_l) g) B(Y(I_k) f)) \cdot \varphi(B(X(I_l) f) B(X(I_k) g)). \quad \text{(c)}$$

Clearly (a) tends to $|c|^2$ whereas (b) and (c) tend to zero because of the assumed regularity of X and Y. This proves (8.12).

Let $X \in \mathscr{B}_Q(H)$ be a self-adjoint operator. Consider the one parameter quasi-free automorphism group of $A(H)$ given by

$$\alpha_\theta^X(B(f)) = B(e^{i\theta X} f), \qquad \theta \in \mathbf{R}. \tag{8.14}$$

Our assumptions imply that φ is α_θ^X-invariant hence, in the GNS representation of φ it is implemented by a unitary operator U_θ^X, i.e.,

$$\pi(\alpha_\theta^X(x)) = U_\theta^X \pi(x) U_\theta^{X*}; \qquad U_\theta^X \Phi = \Phi. \tag{8.15}$$

PROPOSITION (8.5). *The* 1 *parameter unitary group* U_θ^X $(\theta \in \mathbf{R})$ *is strongly continuous and its generator* N_X *has* D *in its domain. Moreover*

$$N_X \pi(B(g_1) \cdot \cdots \cdot B(g_n))\Phi$$

$$= -i \sum_{l=1}^{n} \pi(B(g_1) \cdot \cdots \cdot B(iXg_j) \cdot \cdots \cdot B(g_n))\Phi. \tag{8.16}$$

Proof. Straightforward.

DEFINITION (8.6). For $X \in \mathcal{M}(T; \mathcal{B}_Q(H))$ we define the measure $N_X \in \mathcal{M}(T; \mathcal{L}(\mathcal{D}))$ by

$$N_X(I) = N_{(X(I)+X(I)/2)f} + iN_{(X(I)-X(I)^+/2i)f}, \qquad I \in \mathcal{B}. \tag{8.17}$$

THEOREM (8.7). *Let $X, Y \in \mathcal{M}_Q(T; \mathcal{B}(H))$ be jointly regular self-adjoint measures. Assume that $[\![X, Y]\!]$ exists strongly on H. Then the quadratic variations*

$$[\![N_X, N_Y]\!],\ [\![N_X, B_Y^f]\!],\ [\![B_X^f, N_Y]\!] \tag{8.18}$$

exist strongly on D. Moreover, if the creation and annihilation operators are defined as usual by

$$A_X^f = \tfrac{1}{2}(B_X^{if} - iB_X^f), \qquad A_X^{f+} = \tfrac{1}{2}(B_X^{if} + iB_X^f) \tag{8.19}$$

then we have for self-adjoint X and Y

$$\begin{aligned}
&[\![N_X, N_Y]\!]\, B(g_1) \cdot \cdots \cdot B(g_n)\Phi \\
&\quad = \sum_{j=1}^{n} B(g_1) \cdot \cdots \cdot B([\![X, Y]\!]\, g_j) \cdot \cdots \cdot B(g_n)\Phi \\
&\quad\quad + \sum_{1 \leqslant j < k \leqslant n} (-1)^{j+k} B(g_1) \cdot \cdots \cdot \hat{B}(g_j) \cdot \cdots \cdot \hat{B}(g_k) \cdot \cdots \cdot B(g_n)\Phi \\
&\quad\quad \cdot [\operatorname{Re}\langle g_j, [\![X, Y]\!]\, g_k\rangle + i \operatorname{Im}\langle Qg_j, [\![X, Y]\!]\, g_k\rangle \\
&\quad\quad + \operatorname{Re}\langle [\![X, Y]\!]\, g_j, g_k\rangle + i \operatorname{Im}\langle Q[\![X, Y]\!]\, g_j, g_k\rangle]
\end{aligned} \tag{8.20}$$

$$\begin{aligned}
&[\![N_X, A_Y^f]\!]\, B(g_1) \cdot \cdots \cdot B(g_n)\Phi \\
&\quad = -A^f_{[\![X,Y]\!]} B(g_1) \cdot \cdots \cdot B(g_n)\Phi \\
&\quad = i \sum_{j=1}^{n} (-1)^j B(g_1) \cdot \cdots \cdot \hat{B}(g_j) \cdot \cdots \cdot B(g_n)\Phi \left\langle [\![X, Y]\!]\, f, \frac{Q+1}{2}\, g_j \right\rangle
\end{aligned} \tag{8.21}$$

$$\begin{aligned}
&[\![N_Y, A_X^{f+}]\!]\, B(g_1) \cdot \cdots \cdot B(g_n)\Phi \\
&\quad = A^{f+}_{[\![X,Y]\!]} B(g_1) \cdot \cdots \cdot B(g_n)\Phi \\
&\quad\quad + i \sum_{j=1}^{n} (-1)^j B(g_1) \cdot \cdots \cdot \hat{B}(g_j) \cdot \cdots \cdot B(g_n) \left\langle \frac{1-Q}{2}\, g_j, [\![X, Y]\!]\, f \right\rangle
\end{aligned} \tag{8.22}$$

$$[\![A_X^f, N_Y]\!]\, B(g_1)\cdot \cdots \cdot B(g_n)\Phi$$
$$= -i \sum_{j=1}^{n} (-1)^j B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n) \left\langle \frac{Q+1}{2} f, [\![X, Y]\!]\, g_j \right\rangle \tag{8.23}$$

$$[\![A_X^{f+}, N_Y]\!]\, B(g_1)\cdot \cdots \cdot B(g_n)\Phi$$
$$= i \sum_{j=1}^{n} (-1)^j B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n) \left\langle [\![X, Y]\!]\, g_j, \frac{1-Q}{2} f \right\rangle. \tag{8.24}$$

Proof. To prove (8.20) notice that

$$\sum_k N_X(I_k)\, N_Y(I_k)\, B(g_1)\cdot \cdots \cdot B(g_n)\Phi$$
$$= -i \sum_k N_X(I_k) \sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(iY(I_k)\, g_j)\cdot \cdots \cdot B(g_n)\Phi$$
$$= \sum_k \sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(X(I_k)\, Y(I_k)\, g_j)\cdot \cdots \cdot B(g_n)\Phi \tag{i}$$
$$-\sum_k \sum_{h<j} B(g_1)\cdot \cdots \cdot B(iX(I_k)\, g_h)\cdot \cdots \cdot B(iY(I_k)\, g_j)\cdot \cdots \cdot B(g_n)\Phi \tag{ii}$$
$$-\sum_k \sum_{h>j} B(g_1)\cdot \cdots \cdot B(iY(I_k)\, g_h)\cdot \cdots \cdot B(iX(I_k)\, g_j)\cdot \cdots \cdot B(g_n)\Phi. \tag{iii}$$

Clearly (i) tends to

$$\sum_k B(g_1)\cdot \cdots \cdot B([\![X, Y]\!](I)\, g_j)\cdot \cdots \cdot B(g_n)\Phi.$$

Consider now (ii). Because of Lemma (8.3) we can anticommute the factors $B(iX(I_k)\, g_h)$ and $B(iY(I_k)\, g_j)$ to the right. By Proposition (8.4) we then get that (ii) converges to

$$-\sum_{h<j} (-1)^{n-1-h} B(g_1)\cdot \cdots \cdot \hat{B}(g_h)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n)\cdot [\![B_X^{ig_h}, B_Y^{ig_j}]\!]\,\Phi$$
$$= \sum_{h<j} (-1)^{j+h} B(g_1)\cdot \cdots \cdot \hat{B}(g_n)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n)\cdot [\![B_X^{ig_h}, B_Y^{ig_j}]\!]\,\Phi$$
$$= \sum_{h<j} (-1)^{j+h} B(g_1)\cdot \cdots \cdot \hat{B}(g_h)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n)$$
$$\cdot [\mathrm{Re}\langle g_j, [\![X, Y]\!]\, g_k\rangle + i\, \mathrm{Im}\langle Qg_j, [\![X, Y]\!]\, g_k\rangle] \cdot \Phi. \tag{8.25}$$

The term (iii) is dealt with in the same way. To prove (8.6) notice that

$$\sum_k N_X(I_k)\, A_Y^f(I_k)\, B(g_1)\cdot \cdots \cdot B(g_n)\Phi$$

$$= \sum_k N_X(I_k)\, \tfrac{1}{2}(B(iY(I_k)\, f) - iB(Y(I_k)\, f))\, B(g_1)\cdot \cdots \cdot B(g_n)\Phi$$

$$= \frac{1}{2}\sum_k (iB(X(I_k)\; Y(I_k)\, f) - B(iX(I_k)\; Y(I_k)\, f))\, B(g_1)\cdot \cdots \cdot B(g_n)\Phi$$

$$+ \frac{1}{2}\sum_k (B(iY(I_k)\, f) - iB(Y(I_k)\, f))$$

$$\cdot\left(-i\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(X(I_k)\, g_j)\cdot \cdots \cdot B(g_n)\Phi\right). \tag{8.26}$$

Clearly the first term in the right-hand side of (8.26) tends to

$$A^f_{[\![X,Y]\!]}(I)\, B(g_1)\cdot \cdots \cdot B(g_n)\Phi.$$

Applying the anticommutation relations, Lemma (8.3), and Proposition (8.4), we find that the second term in (8.26) has the same limit as

$$\frac{-i}{2}\sum_{j=1}^{n} (-1)^{n-j}\, B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n)$$

$$\cdot\sum_k (B(iY(I_k)\, f) - iB(Y(I_k)\, f))\, B(iY(I_k)\, g_j)\Phi$$

and this limit is equal to

$$\frac{i}{2}\sum_{j=1}^{n} (-1)^{j}\, B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n)$$

$$\cdot\, [\mathrm{Re}\langle [\![X,\, Y]\!]\, f,\, g_j\rangle + i\,\mathrm{Im}\langle [\![X,\, Y]\!]\, f,\, Qg_j\rangle$$

$$- i\,\mathrm{Re}\langle [\![X,\, Y]\!]\, f,\, g_j\rangle\, \mathrm{Im}\langle [\![X,\, Y]\!]\, f,\, iQg_j\rangle]$$

$$= \frac{i}{2}\sum_{j=1}^{n} (-1)^{j}\, B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot B(g_n)\cdot\left[\left\langle [\![X,\, Y]\!]\, f,\, \frac{Q+1}{2}\, g_j\right\rangle\right].$$

For the identity (8.22) we notice that

$$\sum_k N_X(I_k)\, A_Y^{f+}(I_k)\, B(g_1)\cdot \cdots \cdot B(g_n)\Phi$$

$$= \frac{1}{2}\sum_k (iB(X(I_k)\; Y(I_k)\, f) + B(iX(I_k)\; Y(I_k)\, f))\, B(g_1)\cdot \cdots \cdot B(g_n)\Phi$$

$$+ \frac{1}{2}\sum_k (B(iY(I_k)\, f) + iB(Y(I_k)\, f))$$

$$\cdot\left(-i\sum_{j=1}^{n} B(g_1)\cdot \cdots \cdot B(X(I_k)\, g_j)\cdot \cdots \cdot B(g_n)\Phi\right).$$

Clearly the first term in the above sum tends to

$$A^{f+}_{[\![X,Y]\!]}(I)\,B(g_1)\cdot\cdots\cdot B(g_n)\Phi$$

and as in the previous case we find that the second term tends to

$$\frac{i}{2}\sum_{j=1}^{n}(-1)^j B(g_1)\cdot\cdots\cdot\hat{B}(g_j)\cdot\cdots\cdot B(g_n)\cdot\left[\left\langle\frac{1-Q}{2}g_j, [\![X,Y]\!]f\right\rangle\right].$$

Finally, for the identities (8.23) and (8.24) we have

$$\sum_k A^f_X(I)\,N_Y(I_k)\,B(g_1)\cdot\cdots\cdot B(g_n)\Phi$$

$$=\frac{-i}{2}\sum_k (B(iX(I_k)f)-iB(X(I_k)f))\,B(g_1)$$

$$\cdot\cdots\cdot\hat{B}(iY(I_k)g_j)\cdot\cdots\cdot B(g_n)\Phi,$$

which converges to

$$\frac{i}{2}\sum_{j=1}^{n}(-1)^j B(g_1)\cdot\cdots\cdot\hat{B}(g_j)\cdot\cdots\cdot B(g_n)\cdot\left[\left\langle\frac{Q+1}{2}, [\![X,Y]\!]g_j\right\rangle\right]$$

and

$$\sum_k A^{f+}_X(I_k)\,N_Y(I_k)\,B(g_1)\cdot\cdots\cdot B(g_n)\Phi$$

$$=\frac{-i}{2}\sum_k (B(iX(I_k)f)+iB(X(I_k)f))\,B(g_1)$$

$$\cdot\cdots\cdot\hat{B}(iY(I_k)g_j)\cdot\cdots\cdot B(g_n)\Phi,$$

which converges to

$$\cdot\ \frac{i}{2}\sum_{j=1}^{n}(-1)^j B(g_1)\cdot\cdots\cdot\hat{B}(g_j)\cdot\cdots\cdot B(g_n)\cdot\left[\left\langle[\![X,Y]\!]g_j, \frac{1-Q}{2}f\right\rangle\right].$$

THEOREM (8.8). *For all f in H and all jointly regular self-adjoint $X, Y\in\mathcal{M}(T;\mathcal{B}_Q(H))$ such that $[\![X,Y]\!]$ exists consider the following statements (where all the equalities hold on D):*

$$\varphi \text{ is the Fock state} \tag{8.27}$$

$$Q=1 \tag{8.28}$$

$$[\![A^f_X, N_Y]\!]=A^f_{[\![X,Y]\!]} \tag{8.29}$$

$$[\![A_X^{f+}, N_Y]\!] = 0 \tag{8.30}$$

$$[\![N_X A_Y^f]\!] = 0 \tag{8.31}$$

$$[\![N_X A_Y^{f+}]\!] = A_{[\![X,Y]\!]}^{f+} \tag{8.32}$$

$$[\![N_X, N_Y]\!] = N_{[\![X,Y]\!]}, \tag{8.33}$$

i.e., N is a $[\![\cdot,\cdot]\!]$-homomorphism.

Then (i) *implies all the other statements. If $\mathscr{M}(T; \mathscr{B}_Q(H))$ is full then all statements are equivalent.*

Proof. The fact that (i) implies (ii), ..., (vi) follows immediately from Theorem (8.7) and the validity, for $Q=1$ of the identity

$$A_X^f(I)\cdot B(g_1)\cdot \cdots \cdot B(g_n)\Phi$$
$$= i\sum_{j=1}^{n} (-1)^j B(g_1)\cdot \cdots \cdot B(g_n)\Phi\cdot\langle X(I) f, g_j\rangle.$$

Now we prove that (i) $\Rightarrow$ (iv). Using Proposition (8.5) and the notation

$$Z = [\![X, Y]\!]; \qquad Z_s = \frac{1}{2}(Z+Z^*); \qquad Z_a = \frac{1}{2i}(Z-Z^*)$$

we have that

$$N_Z\cdot B(g_1)\cdot \cdots \cdot B(g_n)\Phi$$
$$= \sum_{j=1}^{n} (-1)^j B(g_1)\cdot \cdots \cdot [B(iZ_a g_j) - iB(iZ_s g_j)]\cdot \cdots \cdot B(g_n)\Phi. \tag{8.34}$$

Using the anticommutation relation, the left-hand side of (8.30) can be rewritten as

$$\sum_{1\leqslant j<k\leqslant n} (-1)^{k-j} B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot \hat{B}(g_k)\cdot \cdots \cdot B(g_n)\Phi$$
$$\cdot [2i\,\mathrm{Im}\langle Z_s g_j, g_k\rangle - 2\,\mathrm{Re}\langle iZ_a g_j, g_k\rangle]$$
$$+ \sum_{j=1}^{n} (-1)^{n-j} B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot [B(iZ_a g_j)$$
$$- iB(iZ_s g_j)]\cdot \cdots \cdot B(g_n)\cdot [B(iZ_a g_j) - iB(iZ_s g_j)]\Phi. \tag{8.35}$$

But since $Q=1$ we have that $A(h)\Phi = 0$ for all h and using this it is easily checked that

$$[B(iZ_a g_j) - iB(iZ_s g_j)]\Phi = B(Zg_j)\Phi.$$

Now, again using the anticommutation relations, the second term of (8.31) can be rewritten as

$$\sum_{1\leqslant j<k\leqslant n} (-1)^{n-k} B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot \hat{B}(g_k)$$
$$\cdot \cdots \cdot B(g_n)\Phi \cdot 2\,\mathrm{Re}\langle Zg_j, g_k\rangle$$
$$+\sum_{1\leqslant j\leqslant n} B(g_1)\cdot \cdots \cdot B(Zg_j)\cdot \cdots \cdot B(g_n)\Phi.$$

After substituting this in (8.31), we find that

$$N_Z\cdot B(g_1)\cdot \cdots \cdot B(g_n)\Phi$$
$$=\sum_{1\leqslant j\leqslant n} B(g_1)\cdot \cdots \cdot B(Zg_j)\cdot \cdots \cdot B(g_n)\Phi$$
$$+\sum_{1\leqslant j<k\leqslant n} (-1)^{k-j} B(g_1)\cdot \cdots \cdot \hat{B}(g_j)\cdot \cdots \cdot \hat{B}(g_k)$$
$$\cdot \cdots \cdot B(g_n)\Phi \cdot [\langle Zg_j, g_k\rangle + \langle g_j, Zg_k\rangle]$$

and comparing this with $[\![N_X, N_Y]\!]\cdot B(g_1)\cdot \cdots \cdot B(g_n)\Phi$ for $Q=1$ as computed in Theorem (8.7) we find that

$$[\![N_X, N_Y]\!] = N_Z,$$

Conversely, assume that $\mathscr{M}(T; \mathscr{B}_Q(H))$ is full. We prove then that (vi) $\Rightarrow$ (ii). The other implications are dealt with in the same way. Suppose that $[\![N_X, N_Y]\!] = N_{[\![X,Y]\!]}$ for all self-adjoint regular X and Y in $\mathscr{M}(T; \mathscr{B}_Q(H))$ whose brackets exist. Taking matrix elements of this operator identity between any vectors of the form $B(f)\Phi$ and $B(g)\Phi$ we find, using Theorem (8.7) and Proposition (8.5), that for all $f, g \in H$,

$$\langle f, Z_s g\rangle = \langle f, QZ_s g\rangle,$$

where Z_s is defined as above. Since $\mathscr{M}(T; \mathscr{B}_Q(H))$ is supposed to be full, it follows that $Q=1$.

Theorem (7.3) can also be reformulated for the Fermi case. Hence also in this case we have the characterization of the Fock state in terms of Ito algebras. If $Q \neq 1$, then the measures

$$[\![N_X, A_Y^{f\#}]\!]; \qquad [\![A_X^{f\#}, N_Y]\!]; \qquad [\![N_X, N_Y]\!]$$

are new objects, i.e., they are measures which cannot be expressed as linear combinations of the scalar and the Fermi field measures. However, the same result as for Boson holds (cf. Theorem (6.3)), i.e., if

$\mathcal{I} \subseteq \mathcal{M}(T; \mathcal{B}_Q(H))$ is an associative Ito-commutative Ito algebra then the linear span $\mathcal{K}$ of the measures

$$N_X; \qquad A_Y^{f\#}; \qquad [\![A_X^{f\#}, N_Y]\!]; \qquad [\![N_X, N_Y]\!]; \qquad \langle f, X(\cdot)\, g\rangle$$

with $f, g \in H$ and $X, Y \in \mathcal{I}$ is an associative Ito algebra whose algebraic structure is completely determined by exactly the same relations as those listed in Theorem (6.3).

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