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A ROLE OF SINGLETONS IN QUANTUM CENTRAL LIMIT THEOREMS

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ABSTRACT. A role of singletons in quantum central limit theorems is studied. A common feature of quantum central limit distributions, the *singleton condition* which guarantees the symmetry of the limit distributions, is revisited in the category of discrete groups and monoids. Introducing a general notion of quantum independence, the *singleton independence* which include the singleton condition as an extremal case, we clarify the role of singletons and investigate the mechanism of arising non-symmetric limit distributions.

Introduction

In recent years the notion of statistical independence has been extensively studied in the context of algebraic probability theory with wide applications and we have caught a glimpse of a rich world spreading beyond the classical probability theory. For example, the free independence due to Voiculescu (see e.g., [20]) gives rise to the Wigner semi-circle law as a central limit distribution and is applied to a study of random matrices, of quantum electrodynamics and so forth. There have appeared so far several different approaches to the notion of independence in order to unify the diversity, see e.g., [5], [7], [9], [10], [15], [18]. Having observed the common feature that the limit distributions obtained in these works are always symmetric, we propose in [2] the idea of *singleton condition* which guarantees the symmetry of the limit distributions. In this paper we shall revisit this condition with some examples arising from discrete

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groups and monoids (semigroups with unit). It is proved that the limit distributions obtained from these algebraic structures lie between the Gaussian and the Wigner semi-circle laws in the sense that the moments are bounded by those of them. This is an extension of the result in [13].

On the other hand, there is an interesting example of a limit distribution obtained by means of a certain rescaled limit of the Haagerup states on the free group. The computation was done explicitly and the Ullman family of distributions is obtained [12]. Thus, in a different limit procedure non-symmetric distributions appear. Motivated by this new phenomenon, we introduced the concept of *singleton independence* in the previous paper [3]. In this paper we clarify the role of singletons and investigate the mechanism of surviving odd moments, or equivalently, of arising a non-symmetric limit distribution. It is noticeable that our singleton independence bears an analytical feature, i.e., is expressed in terms of inequalities. As Bożejko pointed out recently, our results are related to the ψ-independence of Bożejko and Speicher [7] (see also [6]) and a careful study of the relation will create an application to orthogonal polynomials [1]. Thus the notion of statistical independence in algebraic probability is expected to bring a new interaction with harmonic analysis on discrete graphs or on discrete algebraic structures such as monoids, hypergroups, etc. Partial results are found in [14], [17] and a study in this direction is now in progress.

1. Singleton condition

Let $(\mathcal{A}, \varphi)$ be an algebraic probability space, that is, $\mathcal{A}$ is a $*$-algebra with unit 1 and φ is a state, i.e., a positive linear function on $\mathcal{A}$. Here by positivity we mean that

$$\varphi\left(\sum_{i=1}^{n} a_i^* a_i\right) \geq 0$$

for any choice of $n \geq 1$ and $a_1, \cdots, a_n \in \mathcal{A}$. For analytic argument we need the following

DEFINITION 1.1. A family of sequences $\{b^{(j)} = (b_n^{(j)})_{n=1}^{\infty}\,;\, j \in J\}$, where $b_n^{(j)} \in \mathcal{A}$, is said to satisfy the condition of *boundedness of the mixed momenta* if for each $k \geq 1$ there exists a positive constant $\nu_k \geq 0$

such that

$$\left|\varphi\left(b_{n_1}^{(j_1)}\cdots b_{n_k}^{(j_k)}\right)\right| \le \nu_k \tag{1.1}$$

for any choice of $j_1, \cdots, j_k \in J$ and $n_1, \cdots, n_k \in \mathbf{N}$.

Now we come to

DEFINITION 1.2. A family of sequences $\{b^{(j)} = (b_n^{(j)})_{n=1}^{\infty}\, ;\, j \in J\}$, where $b_n^{(j)} \in \mathcal{A}$ with mean zero, is said to satisfy the *singleton condition* (with respect to φ) if

$$\varphi\left(b_{n_1}^{(j_1)}\cdots b_{n_k}^{(j_k)}\right) = 0 \tag{1.2}$$

for any choice of $k \ge 1$, $j_1, \cdots, j_k \in J$, and $n_1, \cdots, n_k \in \mathbf{N}$ with a certain n_s being different from all other ones.

Given a sequence $b = (b_n)_{n=1}^{\infty} \subset \mathcal{A}$ we put

$$S_N(b) = \sum_{n=1}^{N} b_n.$$

By a combinatorial argument modeled after von Waldenfels (see e.g., [10]), we come to the following

LEMMA 1.3. [2] *Assume that $\{b^{(j)} = (b_n^{(j)})_{n=1}^{\infty}\, ;\, j \in J\}$, where $b_n^{(j)} \in \mathcal{A}$ with mean $\varphi(b_n^{(j)}) = 0$, satisfies the condition of boundedness of the mixed momenta. Then, for any $\alpha > 0$ it holds that*

$$\begin{aligned}
&\lim_{N\to\infty} \varphi\left(\frac{S_N(b^{(1)})}{N^\alpha}\cdot\frac{S_N(b^{(2)})}{N^\alpha}\cdots\frac{S_N(b^{(k)})}{N^\alpha}\right)\\
&= \lim_{N\to\infty} N^{-\alpha k} \sum_{\alpha k \le p \le k}\ \sum_{\substack{\pi:\{1,\cdots,k\}\to\{1,\cdots,p\}\\ \text{surjective}}}\ \sum_{\substack{\sigma:\{1,\cdots,p\}\to\{1,\cdots,N\}\\ \text{order-preserving}}} \varphi\left(b_{\sigma\circ\pi(1)}^{(1)}\cdots b_{\sigma\circ\pi(k)}^{(k)}\right)
\end{aligned} \tag{1.3}$$

in the sense that one limit exists if and only if the other does and the limits coincide. Moreover, assume that the singleton condition is satisfied. Then

$$\lim_{N\to\infty} \varphi\left(\frac{S_N(b^{(1)})}{N^\alpha}\cdot\frac{S_N(b^{(2)})}{N^\alpha}\cdots\frac{S_N(b^{(k)})}{N^\alpha}\right) = 0 \tag{1.4}$$

takes place if $\alpha > 1/2$ or if $\alpha = 1/2$ and k is odd. If $\alpha = 1/2$ and $k = 2n$ is even,

$$\lim_{N\to\infty} \varphi\left(\frac{S_N(b^{(1)})}{N^{1/2}} \cdot \frac{S_N(b^{(2)})}{N^{1/2}} \cdots \frac{S_N(b^{(2n)})}{N^{1/2}}\right)$$

$$(1.5) \qquad = \lim_{N\to\infty} N^{-n} \sum_{\substack{\pi:\{1,\cdots,2n\}\to\{1,\cdots,n\} \\ 2-1 \text{ map}}} \sum_{\substack{\sigma:\{1,\cdots,n\}\to\{1,\cdots,N\} \\ \text{order-preserving}}} \varphi\left(b^{(1)}_{\sigma\circ\pi(1)} \cdots b^{(2n)}_{\sigma\circ\pi(2n)}\right),$$

and the Gaussian bound takes place:

$$(1.6) \qquad \limsup_{N\to\infty} \left|\varphi\left(\frac{S_N(b^{(1)})}{N^{1/2}} \cdot \frac{S_N(b^{(2)})}{N^{1/2}} \cdots \frac{S_N(b^{(2n)})}{N^{1/2}}\right)\right| \le \frac{(2n)!}{2^n n!} \nu_{2n},$$

where ν_k appears in (1.1).

The existence of the limit such as (1.5) can be discussed in terms of the entangled ergodic theorem [3]; however, little is known about the convergence, in this connection see [16].

2. Minimal generators of a discrete group

A symmetric random walk on a discrete group provides a geometric interpretation of the singleton condition. Let G be a discrete group and $(\pi, \mathcal{H} = \ell^2(G))$ the left regular representation:

$$\pi(g)\xi(h) = \xi(g^{-1}h), \qquad g, h \in G, \quad \xi \in \mathcal{H}.$$

Let $\delta_g = \chi_{\{g\}}$ be the characteristic function of $\{g\}$ and define a state φ on $\mathbf{B}(\mathcal{H})$ by

$$\varphi(a) = \langle a\delta_e, \delta_e\rangle,$$

where e is the unit of G. Let $\mathcal{A}$ be the $*$-algebra generated by $\{\pi(g)\,;\, g \in G\}$ and consider the algebraic probability space $(\mathcal{A}, \varphi)$.

DEFINITION 2.1. Let G be a discrete group and $\Sigma \subset G$ a subset which generates G, i.e., $G = \langle\Sigma\rangle \equiv \{s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}\,;\, s_i \in \Sigma, \epsilon_i \in \{\pm 1\}, n \ge 0\}$. Then Σ is called *minimal* if no proper subset of Σ generates the whole G.

THEOREM 2.2. *Let G be a discrete group equipped with countably infinite generators $\Sigma = \{g_1, g_2, \dots\}$ and consider the sequence of algebraic random variables $a_n = \pi(g_n)$. If Σ is minimal, $\{(a_n)_{n=1}^\infty, (a_n^*)_{n=1}^\infty\}$ satisfies the singleton condition with respect to φ.*

Proof. Consider a product

$$u = a_{i_1}^{\epsilon_1} \cdots a_{i_s}^{\epsilon_s} \cdots a_{i_n}^{\epsilon_n}, \qquad \epsilon_k \in \{+1, *\}$$

and assume that an index i_s appears only once. It is sufficient to prove that $\varphi(u) = 0$. For each k take $g_{i_k} \in \Sigma$ such that $a_{i_k} = \pi(g_{i_k})$. Since π is a unitary representation, we need only to show that $g_{i_1}^{\eta_1} \cdots g_{i_n}^{\eta_n} \neq e$, where $\eta_k = +1$ or $= -1$ according as $\epsilon_k = +1$ or $= *$. If $g_{i_1}^{\eta_1} \cdots g_{i_n}^{\eta_n} = e$ happens, g_{i_s} can be expressed as a product of others and hence $G = \langle \Sigma \backslash \{g_{i_s}\} \rangle$. This is contradiction by the minimality of Σ. □

Notations and assumptions being the same as in Theorem 2.2, we consider real quantum random variables:

$$b_n = a_n + a_n^*.$$

Then $\phi(b_n) = 0$ and $\phi(b_n^2) = 2 + 2\delta_e(g_n^2)$. For simplicity assume that $g_n^2 \neq e$ for any $g_n \in \Sigma$. Then, in connection with central limit theorems we are interested in the asymptotic behavior of

$$\lim_{N\to\infty} \frac{b_1 + b_2 + \cdots + b_N}{\sqrt{2N}}.$$

Here we mention the following result on bounds of the momenta.

PROPOSITION 2.3. *Let G be a discrete group with minimal generators $\Sigma = \{g_1, g_2, \cdots\}$. Assume that $g_n^2 \neq e$ for any $g_n \in \Sigma$. Then for the sequence of real quantum random variables $b_n = \pi(g_n) + \pi(g_n)^*$ we have*

$$\limsup_{N\to\infty} \varphi\left(\left(\frac{b_1 + b_2 + \cdots + b_N}{\sqrt{2N}}\right)^{2n}\right) \leq \frac{(2n)!}{2^n n!} \tag{2.1}$$

and

$$\liminf_{N\to\infty} \varphi\left(\left(\frac{b_1 + b_2 + \cdots + b_N}{\sqrt{2N}}\right)^{2n}\right) \geq \frac{(2n)!}{(n+1)!n!}. \tag{2.2}$$

The Gaussian bound (2.1) follows from (1.6) in Lemma 1.3 and is achieved, for example, by the free abelian group with countably infinite generators. For the proof of the Wigner bound (2.2) we rely on the universal property of free groups. By using the canonical homomorphism $p : (F_\infty, \{f_i\}) \to (G, \Sigma)$ with $p(f_i) = g_i$, where F_∞ is the free group generated by $\{f_i\}$, we may estimate the number of the return paths, and hence the momenta, see [13] for details. The right-hand side of (2.2) coincides with the moments of the Wigner semi-circle law:

$$\frac{1}{2\pi}\int_{-2}^{2} t^{2n}\sqrt{4-t^2}\,dt = \frac{(2n)!}{n!(n+1)!},$$

which is obtained from the symmetric random walk on a free group, see e.g., [19].

3. Minimal generators of a discrete monoid

The discussion in the previous section can be extended to a discrete monoid (semigroup with unit). Let G be a monoid with unit e, that is, $ge = eg = g$ for any $g \in G$. Let $\mathcal{H} = \ell^2(G)$ and for $\xi \in \mathcal{H}$ put

$$\pi(g)\xi(h) = \xi(gh), \qquad g, h \in G.$$

For boundedness of $\pi(g)$ we need notation. For $g \in G$ and $S \subset G$ we put

$$R(g \to S) = \{x \in G\,;\, gx \in S\}.$$

If $S = \{h\}$ we write simply $R(g \to h)$ for $R(g \to \{h\})$.

LEMMA 3.1. *$\pi(g) \in \mathbf{B}(\mathcal{H})$ if and only if $\sup_{h\in G} |R(g \to h)| < \infty$, where $|\cdot|$ denotes the cardinality. In that case*

$$\|\pi(g)\|^2 = \sup_{h\in G} |R(g \to h)|. \tag{3.1}$$

Proof. By definition we have

$$\begin{aligned}\|\pi(g)\xi\|^2 &= \sum_{x\in G} |\pi(g)\xi(x)|^2 = \sum_{x\in G} |\xi(gx)|^2 \\ &= \sum_{h\in G} |\{x \in G\,;\, gx = h\}||\xi(h)|^2 = \sum_{h\in G} |R(g \to h)||\xi(h)|^2.\end{aligned}$$

Hence $\|\pi(g)\|^2 \le \sup_{h\in G} |R(g \to h)|$. Equality (3.1) is examined by taking $\xi = \delta_x$, $x \in G$. □

From now on we assume that a monoid G under consideration satisfies the condition:

$$\sup_{h\in G} |R(g \to h)| < \infty \qquad \text{for any } g \in G.$$

Let $\mathcal{A}$ be a $*$-algebra generated by $\{\pi(g)\,;\, g \in G\} \subset \mathbf{B}(\mathcal{H})$ and let φ be the state defined by $\varphi(a) = \langle a\delta_e, \delta_e\rangle$, $a \in \mathcal{A}$.

LEMMA 3.2. *For $S \subset G$ it holds that*

$$\pi(g)\chi_S = \chi_{R(g\to S)}, \tag{3.2}$$

$$\pi(g)^*\chi_S(x) = |S \cap R(g \to x)|. \tag{3.3}$$

In particular,

$$\pi(g)^*\delta_h = \delta_{gh}. \tag{3.4}$$

Proof. In view of

$$\pi(g)\chi_S(x) = \chi_S(gx) = \begin{cases} 1, & gx \in S, \\ 0, & \text{otherwise,} \end{cases}$$

we come to (3.2). Next, by definition we have

$$\begin{aligned} \pi(g)^*\chi_S(x) &= \langle \pi(g)^*\chi_S, \delta_x\rangle = \langle \chi_S, \pi(g)\delta_x\rangle \\ &= \sum_{h\in S} \pi(g)\delta_x(h) = \sum_{h\in S} \delta_x(gh) = |S \cap R(g \to x)|, \end{aligned}$$

which proves (3.3). Setting $S = \{h\}$ in (3.3), we obtain

$$\pi(g)^*\delta_h(x) = |\{h\} \cap R(g \to x)| = \begin{cases} 1, & gh = x, \\ 0, & \text{otherwise,} \end{cases}$$

from which (3.4) follows immediately. □

For a subset $S \subset G$ we denote by $\langle S\rangle$ the smallest submonoid of G which contains S and e. For a monoid G we consider the condition:

(A) $R(a \to b) \subset \langle a, b\rangle$ for any $a, b \in G$.

This is equivalent to the following

(A′) $R(a \to S) \subset \langle a, S\rangle$ for any $a \in G$ and $S \subset G$.

Under condition (A), for any $a, b \in G$ the solutions to the equation $ax = b$ belong to $\langle a, b\rangle$ whenever they exist. Note that, in general, an element of a monoid may have more than one inverse.

LEMMA 3.3. *Let G be a monoid satisfying (A). Then, for any $g_1, \cdots, g_n \in G$ and $\epsilon_1, \cdots, \epsilon_n \in \{+1, *\}$ we have*

$$\operatorname{supp} \pi(g_1)^{\epsilon_1} \cdots \pi(g_n)^{\epsilon_n}\delta_e \subset \langle g_1, g_2, \cdots, g_n\rangle. \tag{3.5}$$

Proof. We prove the assertion by induction on n. For $n = 1$ the statement is rather obvious. In fact, by (3.2) we see that $\pi(g_1)\delta_e = \chi_{R(g_1 \to e)}$, and hence by condition (A) we have

$$\operatorname{supp} \pi(g_1)\delta_e = R(g_1 \to e) \subset \langle g_1 \rangle.$$

On the other hand, by (3.4) we have $\pi(g_1)^*\delta_e = \delta_{g_1}$ and

$$\operatorname{supp} \pi(g_1)^*\delta_e = \{g_1\} \subset \langle g_1 \rangle.$$

Thus (3.5) is valid for $n = 1$.

Suppose next that the statement is valid up to $n - 1$. Write

$$\pi(g_1)^{\epsilon_1}\pi(g_2)^{\epsilon_2} \cdots \pi(g_n)^{\epsilon_n}\delta_e = \pi(g_1)^{\epsilon_1}\left(\pi(g_2)^{\epsilon_2} \cdots \pi(g_n)^{\epsilon_n}\delta_e\right) \equiv \pi(g_1)^{\epsilon_1}\Psi.$$

By the assumption of induction $W \equiv \operatorname{supp} \Psi \subset \langle g_2, \cdots, g_n \rangle$. If $W = \emptyset$, $\Psi = 0$ and (3.5) is obvious. Suppose that $W \neq \emptyset$. Then

$$\Psi = \sum_{h \in W} c_h \delta_h.$$

Now consider $\pi(g_1)\Psi$. In view of (3.2) and condition (A) we see that

$$\operatorname{supp} \pi(g_1)\Psi \subset \bigcup_{h \in W} R(g_1 \to h) \subset \bigcup_{h \in W} \langle g_1, h \rangle \subset \langle g_1, W \rangle \subset \langle g_1, g_2, \cdots, g_n \rangle.$$

As for $\pi(g_1)^*\Psi$, by (3.4) we obtain

$$\pi(g_1)^*\Psi = \sum_{h \in W} c_h \delta_{g_1 h}$$

and hence

$$\operatorname{supp} \pi(g_1)^*\Psi \subset g_1 W \subset \langle g_1, g_2, \cdots, g_n \rangle.$$

Thus (3.5) is also valid for n. □

During the above proof we have seen that

$$\pi(g_1)^{\epsilon_1} \cdots \pi(g_n)^{\epsilon_n}\delta_e = 0 \quad \text{or} \quad \sum_{w \in W} c_w \delta_w,$$

where $\emptyset \neq W \subset \langle g_1, g_2, \cdots, g_n \rangle$ and $c_w \in \mathbf{N}$.

DEFINITION 3.4. Let G be a monoid. We say that a subset $\Sigma \subset G$ is a *set of generators* if $\langle \Sigma \rangle = G$. A set of generators Σ is called *minimal* if $B\langle A \rangle \cap \langle A \rangle = \emptyset$ for any pair of non-empty subsets $A, B \subset \Sigma$ with $A \cap B = \emptyset$.

A set of generators Σ is minimal if and only if for any $s \in \Sigma$ and $b \in \langle \Sigma \backslash \{s\} \rangle$, the equation $sx = b$ has no solution in $\langle \Sigma \backslash \{s\} \rangle$.

THEOREM 3.5. *Let G be a monoid satisfying condition (A), and let $\Sigma \subset G$ be a countable infinite set of generators $\Sigma = \{g_1, g_2, \cdots\}$. Define a sequence of quantum random variables by $a_j = \pi(g_j)$. If Σ is minimal, $\{(a_j)_{j=1}^{\infty}, (a_j^*)_{j=1}^{\infty}\}$ satisfies the singleton condition.*

Proof. Consider the product

$$x = a_{i_1}^{\epsilon_1} \cdots a_{i_l}^{\epsilon_l} a_s^{\epsilon} a_{j_1}^{\eta_1} \cdots a_{j_m}^{\eta_m}, \qquad \epsilon_k, \epsilon, \eta_k \in \{+1, *\},$$

and assume that $s \neq i_1, \cdots, i_l, j_1, \cdots, j_m$. Since the argument is similar, we may assume $\epsilon = *$. Then

$$\varphi(x) = \langle a_s^* a_{j_1}^{\eta_1} \cdots a_{j_m}^{\eta_m} \delta_e, a_{i_l}^{\epsilon_l *} \cdots a_{i_1}^{\epsilon_1 *} \delta_e \rangle. \tag{3.6}$$

From Lemma 3.3 it follows that both supp $(a_{j_1}^{\eta_1} \cdots a_{j_m}^{\eta_m} \delta_e)$ and supp $(a_{i_l}^{\epsilon_l *} \cdots a_{i_1}^{\epsilon_1 *} \delta_e)$ are contained in $\langle g_{i_1}, \cdots, g_{i_l}, g_{j_1}, \cdots, g_{j_m} \rangle \subset \langle \Sigma \backslash \{g_s\} \rangle$. Hence

$$\text{supp}\,(a_{i_l}^{\epsilon_l *} \cdots a_{i_1}^{\epsilon_1 *} \delta_e) \subset \langle \Sigma \backslash \{g_s\} \rangle,$$

and

$$\text{supp}\,(a_s^* a_{j_1}^{\eta_1} \cdots a_{j_m}^{\eta_m} \delta_e) \subset g_s \langle \Sigma \backslash \{g_s\} \rangle.$$

Then, by the minimality of Σ we see that $\varphi(x) = 0$ for (3.6) is the inner product of two functions with disjoint supports. □

Notations and assumptions being the same as in Theorem 3.5, we consider real quantum random variables:

$$b_n = \pi(g_n) + \pi(g_n)^* = a_n + a_n^*.$$

Then, $\varphi(b_n) = 0$ and $\varphi(b_n^2) = 1 + 2\delta_e(g_n^2) + |R(g_n \to e)|$. For simplicity we assume that $g_n^2 \neq e$ and $R(g_n \to e) = \emptyset$ for all $n \geq 1$. We are then interested in

$$\lim_{N \to \infty} \frac{b_1 + b_2 + \cdots + b_N}{\sqrt{N}}.$$

The following result is compared with Proposition 2.3.

PROPOSITION 3.6. *Notations and assumptions being the same as above, we have*

$$\limsup_{N \to \infty} \varphi \left(\left(\frac{b_1 + b_2 + \cdots + b_N}{\sqrt{N}} \right)^{2n} \right) \leq \frac{(2n)!}{2^n n!} \tag{3.7}$$

and

$$\liminf_{N \to \infty} \varphi \left(\left(\frac{b_1 + b_2 + \cdots + b_N}{\sqrt{N}} \right)^{2n} \right) \geq \frac{(2n)!}{(n+1)!n!}. \tag{3.8}$$

Proof. We need only to show the Wigner bound (3.8). To this end it is sufficient to show that $\varphi(x) \geq 1$ for any $x = a_{i_1}^{\epsilon_1} \cdots a_{i_{2n}}^{\epsilon_{2n}}$ such that $(i_1, \cdots, i_{2n})$ forms a non-crossing pair-partition with $\epsilon_p = +1$ and $\epsilon_q = *$ for each pair (i_p, i_q) with $p < q$. In fact, one has

$$\pi(g)\pi(g)^* \delta_z = \pi(g)\delta_{gz} = \chi_{R(g \to gz)} = \delta_z + \sum_{w \in R(g \to gz) \setminus \{z\}} \delta_w,$$

whence $\varphi(x) \geq 1$ by induction. □

4. Singleton independence

We need some combinatorial notion. For $\alpha = (j, \epsilon) \in \mathbf{N} \times \{+1, *\}$ we put

$$\alpha^* = \begin{cases} (j, *), & \text{if } \epsilon = +1, \\ (j, +1), & \text{if } \epsilon = *. \end{cases}$$

DEFINITION 4.1. Consider a finite sequence $\alpha_1 \cdots \alpha_m$, where $\alpha_p \in \mathbf{N} \times \{+1, *\}$. Then α_s is called a *singleton* if $\alpha_s \neq \alpha_k^*$ for any $k \neq s$. A singleton α_s is called *outer* if $\alpha_p \neq \alpha_q^*$ for any $p < s < q$, and is called *inner* if $\alpha_p = \alpha_q^*$ for some $p < s < q$.

For example, consider the product $\alpha_1 \alpha_2 \alpha_1^* \alpha_3 \alpha_2$. The second α_2 is an inner singleton and the forth α_3 and the last α_2 are outer singletons. Notice that both α_2's are singletons though appearing twice.

Given a sequence $\{g_j\}_{j=1}^{\infty}$ of elements of $*$-algebra $\mathcal{A}$, for $\alpha \in \mathbf{N} \times \{+1, *\}$ we put

$$g_\alpha = \begin{cases} g_j & \text{if } \alpha = (j, +1) \\ g_j^* & \text{if } \alpha = (j, *). \end{cases}$$

DEFINITION 4.2. Let $\mathcal{A}$ be a $*$-algebra and let $\{\varphi_\gamma \,;\, 0 \leq \gamma \leq \overline{\gamma}\}$ be a family of states on $\mathcal{A}$, $\overline{\gamma} > 0$. Let $\{g_j\}_{j=1}^{\infty}$ be a sequence of elements of $\mathcal{A}$ such that $\varphi_\gamma(g_\alpha) = \gamma$ for all $\alpha \in \mathbf{N} \times \{+1, *\}$. Then the sequence $\{g_j\}$ is called *singleton independent* with respect to φ_γ if for any $k \geq 1$ there exists $M_k \geq 0$ such that

$$(4.1) \qquad |\varphi_\gamma(g_{\alpha_1} \cdots g_{\alpha_k})| \leq \gamma M_k |\varphi_\gamma(g_{\alpha_1} \cdots \hat{g}_{\alpha_s} \cdots g_{\alpha_k})|,$$

for any choice of $\alpha_1, \cdots, \alpha_k$ with some α_s being a singleton for $\alpha_1 \cdots \alpha_k$. Here $\hat{g}_{\alpha_s}$ stands for the omission. (The case of $\gamma = 0$ is related to the singleton condition, see Definition 1.2.)

By repeated application of (4.1) we come to

$$|\varphi_\gamma(g_{\alpha_1} \cdots g_{\alpha_k})| \leq \gamma^s \tilde{M}_k |\varphi_\gamma(g_{\beta_1} \cdots g_{\beta_{k-s}})|, \tag{4.2}$$

for any choice of $g_{\alpha_1}, \cdots, g_{\alpha_k}$ with s singletons, where $\tilde{M}_k = M_k M_{k-1} \cdots M_1$ and $\beta_1 \cdots \beta_{k-s}$ is obtained from $\alpha_1 \cdots \alpha_k$ by removing the s singletons.

In the following we assume that the boundedness condition (1.1) is fulfilled for any φ_γ uniformly in γ. Namely, for each $k \geq 1$ there exists $C_k \geq 0$ such that

$$|\varphi_\gamma(g_{\alpha_1} \cdots g_{\alpha_k})| \leq C_k \tag{4.3}$$

for any choice of $g_{\alpha_1}, \cdots, g_{\alpha_k}$ and $0 \leq \gamma \leq \overline{\gamma}$.

When φ_γ is fixed we write $\tilde{g}_\alpha = g_\alpha - \gamma$ so that $\varphi_\gamma(\tilde{g}_\alpha) = 0$. Put

$$S_N^\epsilon = \sum_{j=1}^N \tilde{g}_j^\epsilon, \qquad \epsilon \in \{+1, *\}.$$

Throughout we fix $k \geq 1$ and $\epsilon_1, \cdots, \epsilon_k \in \{+1, *\}$ and consider the product

$$S_N^{\epsilon_1} \cdots S_N^{\epsilon_k} = \sum_{j_1, \cdots, j_k = 1}^N \tilde{g}_{j_1}^{\epsilon_1} \cdots \tilde{g}_{j_k}^{\epsilon_k} = \sum_{\alpha \in A_N} \tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}, \tag{4.4}$$

where $A_N = A_N(\epsilon_1, \cdots, \epsilon_k)$ is the set of maps $\alpha : \{1, \cdots, k\} \to \{1, \cdots, N\} \times \{+1, *\}$ such that the second component of α_l coincides with the given ϵ_l, $1 \leq l \leq k$. Let $\pi : \{1, \cdots, N\} \times \{+1, *\} \to \{1, \cdots, N\}$ be the projection defined by $\pi(j, \epsilon) = j$ and put $\overline{\alpha} = \pi \circ \alpha$.

Each $\alpha \in A_N$ determines a partition of $\{1, \cdots, k\}$ by the inverse image of $\overline{\alpha}$. Thereby the sum (4.4) over A_N is divided according to the cardinality of the inverse image of $\overline{\alpha}$. Let $\mathcal{P}_{k,p}$ be the collection of partitions of $\{1, \cdots, k\}$ into a disjoint union of p non-empty subsets. For $(S_1, \cdots, S_p) \in \mathcal{P}_{k,p}$ we denote by $[S_1, \cdots, S_p]$ the set of $\alpha \in A_N$ such that $\overline{\alpha}$ is constant on each S_j and takes different values on different S_j's.

With these notations we have

$$\begin{aligned}
&\varphi_{\lambda/\sqrt{N}}\left(\frac{S_N^{\epsilon_1}}{\sqrt{N}}\cdots\frac{S_N^{\epsilon_k}}{\sqrt{N}}\right) \\
&= N^{-k/2}\sum_{\alpha\in A_N}\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}) \qquad (4.5)\\
&= N^{-k/2}\sum_{p=1}^{k}\sum_{(S_1,\cdots,S_p)\in\mathcal{P}_{k,p}}\sum_{\alpha\in[S_1,\cdots,S_p]}\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k})
\end{aligned}$$

and the large N asymptotics is in question.

If $\alpha\in[S_1,\cdots,S_p]$ and $S_1=\{l\}$, then $\alpha_l\equiv(j_l,\epsilon_l)$ is a singleton in the sequence $\alpha_1\cdots\alpha_k$. For the index j_l appears only once in $j_1,\cdots,j_k$.

LEMMA 4.3. *For $0\le s\le k$ let $\mathcal{P}_{k,p}^s$ denote the set of partitions $(S_1,\cdots,S_p)\in\mathcal{P}_{k,p}$ with s singletons, i.e., $|\{i\,;\,|S_i|=1\}|=s$. Then it holds that $p\le(k+s)/2$. Moreover, if $p<(k+s)/2$ then*

$$\lim_{N\to\infty}N^{-k/2}\sum_{(S_1,\cdots,S_p)\in\mathcal{P}_{k,p}^s}\sum_{\alpha\in[S_1,\cdots,S_p]}\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k})=0. \qquad (4.6)$$

Proof. For $(S_1,\cdots,S_p)\in\mathcal{P}_{k,p}^s$ we have

$$k=\sum_{j=1}^{p}|S_j|=\sum_{|S_j|\ge2}|S_j|+s\ge2(p-s)+s=2p-s.$$

Then, in view of (4.2) and (4.3), we see that

$$\begin{aligned}
&N^{-k/2}\sum_{(S_1,\cdots,S_p)\in\mathcal{P}_{k,p}^s}\sum_{\alpha\in[S_1,\cdots,S_p]}\left|\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k})\right|\\
&\le N^{-k/2}|\mathcal{P}_{k,p}^s|\frac{N^p}{p!}\tilde{M}_kC_{k-s}\left(\frac{\lambda}{\sqrt{N}}\right)^s=\tilde{M}_kC_{k-s}|\mathcal{P}_{k,p}^s|\frac{\lambda^s}{p!}N^{p-(k+s)/2}.
\end{aligned}$$

The last quantity goes to 0 as $N\to\infty$ if $p<(k+s)/2$, thereby (4.6) follows. □

It follows from Lemma 4.3 that the non-trivial contribution to the limit of (4.5) comes from those partitions $(S_1,\cdots,S_p)\in\mathcal{P}_{k,p}^s$ satisfying $p=(k+s)/2$, that is, $k=2p-s$. In that case, $1\le|S_j|\le2$ for all j. In fact, if $|S_1|\ge3$, we have

$$k\ge3+\sum_{j\ge2,|S_j|\ge2}|S_j|+s\ge3+2(p-s-1)+s=2p-s+1,$$

which is incompatible with $k = 2p - s$. Taking this into account, let A'_N denote the set of $\alpha \in A_N$ which determines a pair-partition with singletons, i.e., the corresponding partition $(S_1, \cdots, S_s, T_1, \cdots, T_t)$ of $\{1, 2, \cdots, k\}$ satisfies:

$$|S_j| = 1, \quad |T_j| = 2, \quad s + 2t = k, \quad s \geq 0, \quad t \geq 0. \tag{4.7}$$

Then

$$\lim_{N\to\infty} \varphi_{\lambda/\sqrt{N}} \left(\frac{S_N^{\epsilon_1}}{\sqrt{N}} \cdots \frac{S_N^{\epsilon_k}}{\sqrt{N}} \right) = \lim_{N\to\infty} N^{-k/2} \sum_{\alpha \in A'_N} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}). \tag{4.8}$$

Let $\alpha \in A'_N$ and $(S_1, \cdots, S_s, T_1, \cdots, T_t)$ the corresponding partition as in (4.7). Then $\alpha \in [S_1, \cdots, S_s, T_1, \cdots, T_t]$. Going back to (4.2), we have

$$\left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}) \right| \leq \tilde{M}_k \left(\frac{\lambda}{\sqrt{N}} \right)^s \left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\beta_1} \cdots \tilde{g}_{\beta_{2t}}) \right|, \tag{4.9}$$

where $(\beta_1, \cdots, \beta_{2t})$ is obtained from $(\alpha_1, \cdots, \alpha_k)$ by removing the singletons. Then

$$\begin{aligned} & N^{-k/2} \sum_{\alpha \in [S_1, \cdots, S_s, T_1, \cdots T_t]} \left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}) \right| \\ \leq \; & N^{-k/2} \frac{N^s}{s!} \frac{N^t}{t!} \tilde{M}_k \left(\frac{\lambda}{\sqrt{N}} \right)^s \left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\beta_1} \cdots \tilde{g}_{\beta_{2t}}) \right| \\ = \; & \frac{\lambda^s}{s!t!} \tilde{M}_k \left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\beta_1} \cdots \tilde{g}_{\beta_{2t}}) \right|. \end{aligned} \tag{4.10}$$

In general, a pair-partition $(T_1, \cdots, T_t)$ of $\{1, 2, \cdots, 2t\}$ is called *negligible* if there exists $C \geq 0$ such that

$$|\varphi_\gamma(\tilde{g}_{\beta_1} \cdots \tilde{g}_{\beta_{2t}})| \leq C\gamma \tag{4.11}$$

holds for any $\beta : \{1, 2, \cdots, 2t\} \to \mathbf{N} \times \{+1, *\}$ such that $\beta \in [T_1, \cdots, T_t]$. We say that $\alpha \in A'_N$ is negligible if the pair-partition β determined by α as above is negligible. In that case (4.10) becomes

$$N^{-k/2} \sum_{\alpha \in [S_1, \cdots, S_s, T_1, \cdots T_t]} \left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}) \right| \leq \frac{\lambda^s}{s!t!} \tilde{M}_k C \frac{\lambda}{\sqrt{N}},$$

which goes to 0 as $N \to \infty$.

In conclusion, we state the following

THEOREM 4.4. *Let $\mathcal{A}$ be a $*$-algebra with a family of states $\{\varphi_\gamma\,;\,0\le\gamma\le\overline{\gamma}\}$, $\overline{\gamma}>0$. Let $\{g_j\}_{j=1}^\infty$ be a sequence of elements of $\mathcal{A}$ satisfying the singleton independence* (4.1) *and the uniform boundedness* (4.3). *Then for any $k\ge 1$ and $\epsilon_1,\cdots,\epsilon_k\in\{+1,*\}$ we have*

(4.12)

$$\lim_{N\to\infty}\varphi_{\lambda/\sqrt{N}}\left(\frac{S_N^{\epsilon_1}}{\sqrt{N}}\cdots\frac{S_N^{\epsilon_k}}{\sqrt{N}}\right)=\lim_{N\to\infty}N^{-k/2}\sum_{\alpha\in A_N''}\varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}),$$

where A_N'' denotes the set of $\alpha\in A_N$ determining a non-negligible pair-partition with singletons.

5. Haagerup states on the free group

Let F_∞ be the free group generated by $\{g_1,g_2,\cdots\}$. Each $x\in F_\infty$, $x\ne e$, admits a unique expression of the form:

$$x=g_{\alpha_1}\cdots g_{\alpha_n},\qquad \alpha_i\ne\alpha_{i+1}^*,\quad 1\le i\le n-1,$$

where $g_\alpha=g_j^\epsilon$ for $\alpha=(j,\epsilon)\in\mathbf{N}\times\{\pm1\}$. In that case n is called the *length* of x and we write $|x|=n$. By definition $|e|=0$. For a general theory of length functions see e.g., [4], [8]. Let $\mathcal{A}$ be the group $*$-algebra associated with F_∞, where $g_j^*=g_j^{-1}$. For each $0\le\gamma\le1$ there exists a state φ_γ on $\mathcal{A}$ uniquely determined by

$$\varphi_\gamma(x)=\gamma^{|x|},\qquad x\in F_\infty.$$

This φ_γ is called the *Haagerup state*, see [11].

The two sequences $\{(g_j)_{j=1}^\infty,(g_j^{-1})_{j=1}^\infty\}$ satisfy the singleton condition (cf. Section 1) with respect to the Haagerup state φ_γ only when $\gamma=0$; while the singleton independence and the uniform boundedness (cf. Section 4) hold. In fact, the idea of the singleton independence was motivated by the Haagerup states. In this concrete case (4.12) is computed explicitly, where a pair-partition with singletons is negligible if there appears a crossing pair.

DEFINITION 5.1. Assume that a product $\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}$ contains $s\ge0$ inner singletons and no outer singletons. Let $\alpha_{j_1},\cdots,\alpha_{j_s}$ be the suffices which correspond the singletons and denote the rest by $\beta_1,\cdots,\beta_{k-s}$ in order. We say that the product satisfies the condition (NCI) if $g_{\beta_1}\cdots g_{\beta_{k-s}}=e$. For any $k\ge1$ and $\epsilon_1,\cdots,\epsilon_k\in\{+1,*\}$ let

$\mathrm{NCI}_k(s;\epsilon_1,\cdots,\epsilon_k)$ be the set of equivalence classes of products $\tilde{g}_{\alpha_1}\cdots\tilde{g}_{\alpha_k}$ which consist of $(k-s)/2$ non-crossing pairs and of s inner singletons.

THEOREM 5.2. *Let $k \geq 1$ and $\epsilon_1,\cdots,\epsilon_k \in \{+1,*\}$. For the Haagerup states $\{\varphi_\gamma\}$ on the free group F_∞ it holds that*

$$(5.1)\qquad \lim_{N\to\infty}\varphi_{\lambda/\sqrt{N}}\left(\frac{S_N^{\epsilon_1}}{\sqrt{N}}\cdots\frac{S_N^{\epsilon_k}}{\sqrt{N}}\right)=\sum_{s=0}^{k-2}(-\lambda)^s\cdot|\mathrm{NCI}_k(s;\epsilon_1,\cdots,\epsilon_k)|.$$

For further study concerning the above result see [3]. Another examples of the singleton independence are known from the unitary representations of free groups.

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