PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 22, Fasc. 1 (2002), pp. 141–154

STOCHASTIC EVOLUTIONS DRIVEN BY NON-LINEAR WHITE NOISE

BY

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Abstract. We prove the existence and uniqueness theorem for stochastic differential equations with bounded coefficients driven by the renormalized square of white noise. These equations are interpreted as sesquilinear forms on the linear span of the exponential vectors (of the first order white noise) and the existence theorem is established on the space of these forms.

1991 Mathematics Subject Classification: Primary 81S25; Secondary 81V80.

1. INTRODUCTION

Linear quantum stochastic calculus is associated with the stochastic differentials $dB(t) = b(t)\,dt$, $dB^+(t) = b^+(t)\,dt$, and $dN(t) = b^+(t)\,b(t)\,dt$ corresponding to white noise functionals b and b^+ satisfying the commutation relation $[b(t), b^+(s)] = \gamma\cdot\delta(t-s)$, where $\gamma > 0$ is the variance of the quantum Brownian motion defined by B and B^+, and δ is the delta function (see [2] and [5]). It was developed in the case when $\gamma = 1$ and the annihilation, creation, and number operators B, B^+, and N, respectively, act on Boson Fock space in [9]. A general, representation free, quantum stochastic calculus which includes that of [9] and all other known examples of linear quantum noise was developed in [4] (see also [1] and [2]).

Related to non-linear quantum optics, Accardi and Volovich have recently considered in [7] the quantum stochastic differential equation

$$\begin{aligned} dU(t) &= -i\,[c(t)\,dt + g(t)\,dB_2^+(t) + g(t)\,dB_2(t) + \omega(t)\,dN(t)]\,U(t), \\ U(0) &= 1, \end{aligned} \tag{1.1}$$

where c, g, and ω are complex-valued functions of time t, $dB_2(t) = b(t)^2\,dt$, and $dB_2^+(t) = b^+(t)^2\,dt$.

* The author wishes to thank Professor Luigi Accardi for his support and for the hospitality of the Centro Vito Volterra of the Universitá di Roma TorVergata on several occasions.

This required the extension of quantum stochastic calculus (see [5] and [6]) to include normally ordered nonlinear stochastic differentials of the form $dB_{(m,n)} = b^+(t)^m b(t)^n dt$, where $m, n \in \{0, 1, \ldots\}$ and the noise functionals b^+ and b are defined as follows: Let $L^2_{\mathrm{sym}}(\boldsymbol{R}^n)$ denote the space of square-integrable functions on $\boldsymbol{R}^n$ symmetric under permutation of their arguments, and let $F = \bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}(\boldsymbol{R}^n)$, where: if $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in F$, then $\psi^{(0)} \in \boldsymbol{C}$, $\psi^{(n)} \in L^2_{\mathrm{sym}}(\boldsymbol{R}^n)$ and

$$\|\psi\|^2 = |\psi(0)|^2 + \sum_{n=1}^{\infty} \int_{\boldsymbol{R}^n} |\psi^{(n)}(s_1, \ldots, s_n)|^2 \, ds_1 \ldots ds_n.$$

Denote by $S \subset L^2(\boldsymbol{R}^n)$ the Schwartz space of smooth functions decreasing at infinity faster than any polynomial and let

$$D = \{\psi \in F \,|\, \psi^{(n)} \in S, \ \sum_{n=1}^{\infty} n |\psi^{(n)}|^2 < \infty\}.$$

For each $t \in \boldsymbol{R}$ define the linear operator $b(t)\colon D \to F$ by

$$\big(b(t)\psi\big)^{(n)}(s_1, \ldots, s_n) = \sqrt{n+1}\, \psi^{(n+1)}(t, s_1, \ldots, s_n)$$

and the operator valued distribution $b^+(t)$ by

$$\big(b^+(t)\psi\big)^{(n)}(s_1, \ldots, s_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta(t - s_i)\, \psi^{(n-1)}(s_1, \ldots, \hat{s}_i, \ldots, s_n),$$

where $\hat{}$ denotes omission of the corresponding variable.

Then

$$B(t) = \int_0^t b(s)\,ds, \qquad B^+(t) = \int_0^t b^+(s)\,ds, \qquad \text{and} \qquad N(t) = \int_0^t b^+(s)\, b(s)\, ds$$

are, for each t, operators acting on D. Since $L^2_{\mathrm{sym}}(\boldsymbol{R}^n) = L^2_{\mathrm{sym}}(\boldsymbol{R})^{\otimes n}$, we can identify F with the symmetric (Boson) Fock space over S. In the case when the elements of S are defined on $[0, +\infty)$ we denote the Fock space by $\Gamma(S_+)$. If $\psi = \{(n!)^{-1/2} f^{\otimes n}\}$, we denote ψ by $\psi(f)$. We have

$$\begin{aligned} b(t)\psi(f) &= f(t)\psi(f), \qquad b^2(t)\psi(f) = f(t)^2 \psi(f), \\ \langle \psi(g), b^+(t) b(t) \psi(f) \rangle &= \overline{g(t)}\, f(t) \langle \psi(g), \psi(f) \rangle. \end{aligned} \tag{1.2}$$

For an adapted process $X = \{X(t) \,|\, t \geqslant 0\}$ we define its stochastic differential $dX = \{dX(t) \mid t \geqslant 0\}$ by

$$dX(t) = X(t + dt) - X(t).$$

For two adapted processes X and Y we have

$$d(X \cdot Y)(t) = dX(t) \cdot Y(t) + X(t) \cdot dY(t) + dX(t) \cdot dY(t). \tag{1.3}$$

The *renormalized Itô table* derived in [6] corresponding to dt, dB, dB^+, dB_2, dB_2^+, and dN is of the form:

	dt	dB	dB^+	dB_2	dB_2^+	dN
dt	0	0	0	0	0	0
dB	0	0	γdt	0	$2\gamma dB^+$	γdB
dB^+	0	0	0	0	0	0
dB_2	0	0	$2\gamma dB$	0	$4\gamma dN$	$2\gamma dB_2$
dB_2^+	0	0	0	0	0	0
dN	0	0	γdB^+	0	$2\gamma dB_2^+$	γdN

We couple $\Gamma(S_+)$ with a system Hilbert space H_0, we define an adapted process $A = \{A(t) \mid t \geqslant 0\}$ to be a family of operators on $H_0 \otimes \Gamma(S_+^{t]})$ such that, for each t, $A(t) = A_t \otimes 1$, where A_t acts on $H_0 \otimes \Gamma(S_+)$ and 1 is the identity operator on $\Gamma(S_+^{(t)})$, where

$$S_+^{t]} = \{f \cdot \chi_{[0,t]} \mid f \in S\} \quad \text{and} \quad S_+^{(t} = \{f \cdot \chi_{(t,+\infty)} \mid f \in S\}.$$

If, for each t, $A(t) = A \otimes 1$, where A is an operator on H_0 and 1 is the identity on $\Gamma(S_+)$, then A is a constant process. If, for each t, $A(t)$ is a bounded operator, then A is a bounded process, etc. In what follows we identify $B(t)$, $B^+(t)$, $B_2(t)$, $B_2^+(t)$, and $N(t)$ with $1 \otimes B(t)$, $1 \otimes B^+(t)$, $1 \otimes B_2(t)$, $1 \otimes B_2^+(t)$, and $1 \otimes N(t)$, respectively, where 1 is the identity on H_0. For a constant adapted process $A = \{A(t) \mid t \geqslant 0\}$ we denote $A(t)$ simply by A.

Stochastic integrals with respect to $dt, dB, \ldots, dN$ are defined in Propositions 1 and 2. Once a quantum stochastic calculus has been constructed, one usually considers the problem of finding conditions under which stochastic differential equations driven by quantum noise admit unitary solutions. To simplify expressions let $N_1(t) = t \cdot 1$, $N_2(t) = B(t)$, $N_3(t) = B^+(t)$, $N_4(t) = B_2(t)$, $N_5(t) = B_2^+(t)$, $N_6(t) = N(t)$, where, with $*$ denoting the dual operator, $N_1^* = N_1$, $N_2^* = N_3$, $N_4^* = N_5$, $N_6^* = N_6$.

Under the assumption of existence of a unique adapted process $U = \{U(t) \mid t \geqslant 0\}$ satisfying

$$\text{(1.4)} \qquad \begin{aligned} dU(t) &= \Big[\sum_{i=1}^{6} A_i \, dN_i(t)\Big] U(t), \\ U(0) &= U_0, \quad 0 \leqslant t \leqslant T < +\infty, \end{aligned}$$

where the coefficients $A_1, A_2, \ldots, A_6$ are bounded, constant adapted processes, it was shown in [3], with the use of the renormalized Itô table and the linear independence of the stochastic differentials $dt, dB, \ldots, dN$, that necessary and sufficient conditions for the unitarity of U (i.e. in order for $U(t)\,U^*(t) =$

$U^*(t)\,U(t) = 1$ for each $t \in [0, T]$) are:

$$
(1.5)\quad
\begin{aligned}
A_1 + A_1^* + A_2 A_2^* \gamma &= 0,\\
A_2 + A_3^* + A_4 A_2^* 2\gamma + A_2 A_2^* 2\gamma + A_2 A_6^* \gamma &= 0,\\
A_4 + A_5^* + A_4 A_6^* 2\gamma &= 0,\\
A_6 + A_6^* + A_4 A_4^* 4\gamma + A_6 A_6^* \gamma &= 0,\\
A_1^* + A_1 + A_3^* A_3 \gamma &= 0,\\
A_3^* + A_2 + A_3^* A_6 \gamma + A_5^* A_3 2\gamma &= 0,\\
A_5^* + A_4 + A_5^* A_6 2\gamma &= 0,\\
A_6^* + A_6 + A_5^* A_5 4\gamma + A_6^* A_6 \gamma &= 0.
\end{aligned}
$$

The same conditions with the same proof are also valid in the case when the coefficients $A_1, \ldots, A_6$ are time dependent. It was also shown that if

$$A_1 = iH - \frac{\gamma}{2} L^* L, \qquad A_2 = -L^* W, \qquad A_3 = L, \qquad A_6 = \frac{W-1}{2\gamma}$$

and

$$A_4 = -\left(\frac{1-\Re W}{8\gamma^2}\right)^{1/2} MW, \qquad A_5 = M^* \left(\frac{1-\Re W}{8\gamma^2}\right)^{1/2},$$

where L, H, W, M are bounded operators with H self-adjoint, and W, M are unitary operators satisfying

$$L^*(1-W) + \sqrt{2}(1-\Re W)^{1/2} ML = 0,$$

where $\Re$ denotes real part, then $A_1, \ldots, A_6$ satisfy (1.5).

In the case of the Accardi–Volovich equation (1.1), letting $A_1 = -ic$, $A_4 = -ig$, $A_5 = -ig$, $A_6 = -i\omega$, $A_2 = A_3 = 0$ in (1.5) we infer that the solution of (1.1) is unitary if and only if the functions c, g, ω satisfy

$$\Im c = 0, \qquad 2\Im\omega + |g|^2 4\gamma + |\omega|^2 \gamma = 0, \qquad \Im g + \gamma \bar{g}\omega = 0,$$

where $\Im$ denotes imaginary part.

In this paper we prove that the assumption made on the existence of a unique adapted process U satisfying (1.4) is valid even for time-dependent coefficients $A_1, \ldots, A_6$. We also provide a direct proof of the fact that if $A_1, \ldots, A_6$ are, in general, time-dependent and satisfy (1.5), then U is bounded and in fact unitary. These results extend those obtained in [9] for stochastic evolutions driven by linear noise only. Finally, we show that our results imply the unitarity of the solution of a quantum stochastic differential equation recently considered in [7].

The main result of this paper is:

MAIN THEOREM. *Let the coefficient processes A_i, $i = 1, \ldots, 6$, be adapted and such that*

$$\sup_{0 \leqslant s \leqslant T} \|A_i(s)\| < +\infty .$$

Then the quantum stochastic differential equation

$$dU(t) = \Big[\sum_{i=1}^{6} A_i(t)\, dN_i(t)\Big] U(t),$$

$$U(0) = U_0, \quad 0 \leqslant t \leqslant T < +\infty,$$

or in its equivalent integral form

$$U(t) = U_0 + \sum_{i=1}^{6} \int_0^t A_i(s)\, U(s)\, dN_i(s), \quad 0 \leqslant t \leqslant T < +\infty,$$

where U_0 is a bounded operator on $H_0 \otimes \Gamma(S_+)$, admits a unique adapted strongly continuous solution $U = \{U(t) \mid 0 \leqslant t \leqslant T < +\infty\}$ defined on span $\{u \otimes \psi(f)\}$, where $|f(s)| \leqslant 1$ for all $s \in [0, T]$.

Moreover, if the A_i's satisfy (1.5) *and $U_0 = 1$, then $U = \{U(t) \mid 0 \leqslant t \leqslant T < +\infty\}$ is a unitary process, i.e. $U(t)^* U(t) = U(t) U(t)^* = 1$ for all $t \in [0, T]$.*

The proof of the above theorem is provided in Section 3.

2. THE BASIC ESTIMATES

The following propositions are non-linear noise analogues of those of linear quantum stochastic calculus (see [9]). We assume that the coefficient processes A_i, C_i, $i = 1, 2, \ldots, 6$, are such that the right-hand sides of (2.1)–(2.3) and (2.5) make sense.

PROPOSITION 1. *Let $\Pi(t) = \sum_{i=1}^{6} \int_0^t A_i(s)\, dN_i(s)$, let $u, v \in H_0$ and $f, g \in S_+$. Then*:

$$\text{(2.1)} \quad \langle u \otimes \psi(f), \Pi(t)\, v \otimes \psi(g)\rangle = \sum_{i=1}^{6} \int_0^t \varrho_i(s) \langle u \otimes \psi(f), A_i(s)\, v \otimes \psi(g)\rangle\, ds$$

and

$$\text{(2.2)} \quad \langle \Pi(t)\, u \otimes \psi(f), v \otimes \psi(g)\rangle = \sum_{i=1}^{6} \int_0^t \sigma_i(s) \langle u \otimes \psi(f), A_i^*(s)\, v \otimes \psi(g)\rangle\, ds,$$

where

$$\sigma_i(s) = \begin{cases} 1, & i=1, \\ \overline{f(s)}, & i=2, \\ g(s), & i=3, \\ \overline{f(s)^2}, & i=4, \\ g(s)^2, & i=5, \\ \overline{f(s)}\,g(s), & i=6, \end{cases} \quad \textit{and} \quad \varrho_i(s) = \begin{cases} 1, & i=1, \\ g(s), & i=2, \\ \overline{f(s)}, & i=3, \\ g(s)^2, & i=4, \\ \overline{f(s)^2}, & i=5, \\ \overline{f(s)}\cdot g(s), & i=6. \end{cases}$$

Proof. Proposition 1 follows directly from (1.2). ∎

PROPOSITION 2. *Let*

$$\Pi_1(t) = \sum_{i=1}^{6} \int_0^t A_i(s)\,dN_i(s), \qquad \Pi_2(t) = \sum_{j=1}^{6} \int_0^t C_j(s)\,dN_j(s),$$

and let $u, v \in H_0$, $f, g \in S_+$. *Then*:

$$(2.3) \quad \langle \Pi_1(t)u\otimes\psi(f), \Pi_2(t)v\otimes\psi(g)\rangle$$

$$= \sum_{i,j=1}^{6} \Big[\int_0^t \sigma_i(s)\int_0^s \varrho_j(s')\langle A_i(s)u\otimes\psi(f), C_j(s')v\otimes\psi(g)\rangle\,ds'\,ds$$

$$+\int_0^t \varrho_j(s)\int_0^s \sigma_i(s')\langle A_i(s')u\otimes\psi(f), C_j(s)v\otimes\psi(g)\rangle\,ds'\,ds$$

$$+\int_0^t \omega_{ij}^{\gamma}(s)\langle A_i(s)u\otimes\psi(f), C_j(s)v\otimes\psi(g)\rangle\,ds\Big],$$

where σ_i, ϱ_j *are determined as in Proposition* 1 *and*

$$\omega_{i,j}^{\gamma}(t) = \begin{cases} \gamma, & i=j=3, \\ 2\gamma\overline{f(t)}, & i=3, j=5, \\ \gamma g(t), & i=3, j=6, \\ 2\gamma g(t), & i=5, j=3, \\ 4\gamma\overline{f(t)}\,g(t), & i=j=5, \\ 2\gamma g(t)^2, & i=5, j=6, \\ \gamma f(t), & i=6, j=3, \\ 2\gamma\overline{f(t)^2}, & i=6, j=5, \\ \gamma\overline{f(t)}\,g(t), & i=j=6, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. By (1.3),

$$d\langle \Pi_1(t)u\otimes\psi(f), \Pi_2(t)v\otimes\psi(g)\rangle = \langle d\Pi_1(t)u\otimes\psi(f), \Pi_2(t)v\otimes\psi(g)\rangle \tag{2.4}$$

$$+\langle \Pi_1(t)u\otimes\psi(f), d\Pi_2(t)v\otimes\psi(g)\rangle + \langle d\Pi_1(t)u\otimes\psi(f), d\Pi_2(t)v\otimes\psi(g)\rangle$$

$$= \sum_{i,j=1}^{6} [\langle A_i(t)dN_i(t)u\otimes\psi(f), \int_0^t C_j(s')dN_j(s')v\otimes\psi(g)\rangle$$

$$+\langle \int_0^t A_i(s)dN_i(s)u\otimes\psi(f), C_j(t)dN_j(t)v\otimes\psi(g)\rangle$$

$$+\langle A_i(t)dN_i(t)u\otimes\psi(f), C_j(t)dN_j(t)v\otimes\psi(g)\rangle].$$

When, for example, $i = 3$ and $j = 6$, then by (1.2) and the fact that, by the renormalized Itô table, $dN_6^*\cdot dN_3 = dN_6\cdot dN_3 = \gamma dN_3$ the expression in brackets equals

$$g(t)\int_0^t \overline{f(s')}\cdot g(s')\langle A_3(t)u\otimes\psi(f), C_6(s')v\otimes\psi(g)\rangle\, ds'\, dt$$

$$+\int_0^t g(s)\langle A_3(s)u\otimes\psi(f), C_6(t)v\otimes\psi(g)\rangle\, ds\overline{f(t)}\, g(t)\, dt$$

$$+\gamma\cdot g(t)\langle A_3(t)u\otimes\psi(f), C_6(t)v\otimes\psi(g)\rangle\, dt.$$

The rest of the terms are computed similarly, and the result follows by integrating (2.4) from 0 to z. ■

PROPOSITION 3. *Let $\Pi_1, \Pi_2, u, v, f, g, \sigma_i, \varrho_j$ and ω_{ij}^{γ} be determined as in Proposition 1. Then:*

$$\langle \Pi_1(t)u\otimes\psi(f), \Pi_2(t)v\otimes\psi(g)\rangle \tag{2.5}$$

$$= \int_0^t [\sum_{i=1}^{6} \sigma_i(s)\langle A_i(s)u\otimes\psi(f), \Pi_2(s)v\otimes\psi(g)\rangle$$

$$+\sum_{j=1}^{6} \varrho_j(s)\langle \Pi_1(s)u\otimes\psi(f), C_j(s)v\otimes\psi(g)\rangle$$

$$+\sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(s)\langle A_i(s)u\otimes\psi(f), C_j(s)v\otimes\psi(g)\rangle]\, ds.$$

Proof. As in the proof of Proposition 2, by (1.3), (1.2) and the renormalized Itô table, we have

$$d\langle \Pi_1(t)u\otimes\psi(f), \Pi_2(t)v\otimes\psi(g)\rangle = \langle d\Pi_1(t)u\otimes\psi(f), \Pi_2(t)v\otimes\psi(g)\rangle$$

$$+\langle \Pi_1(t)u\otimes\psi(f), d\Pi_2(t)v\otimes\psi(g)\rangle + \langle d\Pi_1(t)u\otimes\psi(f), d\Pi_2(t)v\otimes\psi(g)\rangle$$

$$= \sum_{i=1}^{6} \sigma_i(t)\langle A_i(t)u\otimes\psi(f), \Pi_2(t)v\otimes\psi(g)\rangle\, dt$$

$$+\sum_{j=1}^{6}\varrho_j(t)\langle\Pi_1(t)u\otimes\psi(f),\ C_j(t)v\otimes\psi(g)\rangle\,dt$$
$$+\sum_{i,j=1}^{6}\omega_{ij}^{\gamma}(t)\langle A_i(t)u\otimes\psi(f),\ C_j(t)v\otimes\psi(g)\rangle]\,dt,$$

and the result follows by integrating from 0 to z using $\Pi_1(0)=\Pi_2(0)=0$. ∎

COROLLARY 1. *In the notation of Proposition* 2 *we have*

$$\|\Pi_1(t)u\otimes\psi(f)\|^2=\sum_{i,j=1}^{6}\Big[\int_0^t\sigma_i(s)\int_0^s\bar\sigma_j(s')\langle A_i(s)u\otimes\psi(f),\ A_j(s')u\otimes\psi(f)\rangle\,ds'\,ds$$
$$+\int_0^t\bar\sigma_j(s)\int_0^s\sigma_i(s')\langle A_i(s')u\otimes\psi(f),\ A_j(s)u\otimes\psi(f)\rangle\,ds'\,ds$$
$$+\int_0^t\omega_{ij}^{\gamma}(s)\langle A_i(s)u\otimes\psi(f),\ A_j(s)u\otimes\psi(f)\rangle\,ds\Big].$$

Proof. The corollary follows from Proposition 2 and the fact that $f=g\Rightarrow\varrho_j=\bar\sigma_j$. ∎

COROLLARY 2. *In the notation of Proposition* 2 *we have*

$$\|\Pi_1(t)u\otimes\psi(t)\|^2\leqslant 2\Big[\sum_{i=1}^{6}\|\sigma_i\|_{2,t}\|A_i\|_{2,t}^{u,f}\Big]^2+\sum_{i,j=1}^{6}\|\omega_{ij}^{\gamma}\|_{2,t}\|A_i\|_{4,t}^{u,f}\|A_j\|_{4,t}^{u,f},$$

where

$$\|\sigma_i\|_{2,t}=\Big[\int_0^t|\sigma_i(s)|^2\,ds\Big]^{1/2},\qquad \|\omega_{ij}^{\gamma}\|_{2,t}=\Big[\int_0^t|\omega_{ij}^{\gamma}(s)|^2\,ds\Big]^2,$$

and for $p=1,2,\ldots$

$$\|A_i\|_{p,t}^{u,f}=\Big[\int_0^t\|A_i(s)u\otimes\psi(f)\|^p\,ds\Big]^{1/p}.$$

Proof. By Corollary 1 we obtain

$$\|\Pi_1(t)u\otimes\psi(f)\|^2 \tag{2.6}$$
$$\leqslant\sum_{i,j=1}^{6}\Big[\int_0^t|\sigma_i(s)|\,\|A_i(s)u\otimes\psi(f)\|\int_0^s|\bar\sigma_j(s')|\,\|A_j(s')u\otimes\psi(f)\|\,ds'\,ds$$
$$+\int_0^t|\bar\sigma_j(s)|\,\|A_j(s)u\otimes\psi(f)\|\int_0^s|\sigma_j(s')|\,\|A_i(s')u\otimes\psi(f)\|\,ds'\,ds$$
$$+\int_0^t|\omega_{ij}^{\gamma}(s)|\,\|A_i(s)u\otimes\psi(f)\|\,\|A_j(s)u\otimes\psi(f)\|\,ds\Big]$$
$$\leqslant\sum_{i,j=1}^{6}\Big[2\int_0^t|\sigma_i(s)|\,\|A_i(s)u\otimes\psi(f)\|\,ds\int_0^t|\sigma_j(s)|\,\|A_j(s)u\otimes\psi(f)\|\,ds$$
$$+\int_0^t|\omega_{ij}^{\gamma}(s)|\,\|A_i(s)u\otimes\psi(f)\|\,\|A_j(s)u\otimes\psi(f)\|\,ds\Big],$$

which by the Cauchy–Schwartz inequality is less than or equal to

$$\sum_{i,j=1}^{6}\Big[2\Big(\int_0^t|\sigma_i(s)|^2\,ds\Big)^{1/2}\Big(\int_0^t\|A_i(s)\,u\otimes\psi(f)\|^2\,ds\Big)^{1/2}$$
$$\times\Big(\int_0^t|\sigma_j(s)|^2\,ds\Big)^{1/2}\Big(\int_0^t\|A_j(s)\,u\otimes\psi(f)\|^2\,ds\Big)^{1/2}$$
$$+\Big(\int_0^t|\omega_{ij}^{\gamma}(s)|^2\,ds\Big)^{1/2}\Big(\int_0^t\|A_i(s)\,u\otimes\psi(f)\|^2\,\|A_j(s)\,u\otimes\psi(f)\|^2\,ds\Big)^{1/2}\Big]$$

from which the result follows by applying the Cauchy–Schwartz inequality to the last parentheses. ■

COROLLARY 3. *In the notation of Proposition* 2 *suppose that* $A_i(s) = \alpha_i(s)\,\beta(s)$, *where, for each i,* α_i *is a bounded adapted process on* $H_0\otimes\Gamma(S_+)$ *such that, for all* $t\geqslant 0$, $\sup_{0\leqslant s\leqslant t}\|\alpha_i(s)\|<\infty$, *and* β *is a strongly continuous adapted process, i.e.* $s\in[0,t]\to\beta(s)\,u\otimes\psi(f)$ *is continuous for all* $u\in H_0, f\in S_+$. *Then for* $0\leqslant t\leqslant T<+\infty$

$$\|\Pi_1(t)\,u\otimes\psi(f)\|^2\leqslant K_{\gamma,T}\cdot T\sup_{0\leqslant s\leqslant T}\|\beta(s)\,u\otimes\psi(f)\|^2,$$

where

$$K_{\gamma,T}=\Big[2\sum_{i=1}^{6}\|\sigma_i\|_{2,T}\sup_{0\leqslant s\leqslant T}\|\alpha_i(s)\|^2+\sum_{i,j=1}^{6}\|\omega_{ij}^{\gamma}\|_{2,T}\sup_{0\leqslant s\leqslant T}\|\alpha_i(s)\|\sup_{0\leqslant s\leqslant T}\|\alpha_j(s)\|\Big].$$

Proof. In the notation of Corollary 2 we have

$$\|A_i\|_{2,t}^{u,f}\leqslant\|A_i\|_{2,T}^{u,f}\leqslant T^{1/2}\sup_{0\leqslant s\leqslant T}\|\alpha_i(s)\|\sup_{0\leqslant s\leqslant T}\|\beta(s)\,u\otimes\psi(f)\|$$

and

$$\|A_i\|_{4,t}^{u,f}\leqslant\|A_i\|_{4,T}^{u,f}\leqslant T^{1/4}\sup_{0\leqslant s\leqslant T}\|\alpha_i(s)\|\sup_{0\leqslant s\leqslant T}\|\beta(s)\,u\otimes\psi(f)\|$$

and the result follows from Corollary 2. ■

COROLLARY 4. *In the notation of Proposition* 2, *if* $|f(s)|\leqslant 1$ *for all* $s\in[0,T]$ (*as in* [8]) *and the* A_i*'s are as in Corollary* 3, *then*

$$\|\Pi_1(t)\,u\otimes\psi(f)\|^2\leqslant L_{\gamma,T}\cdot T\sup_{0\leqslant s\leqslant T}\|\beta(s)\,u\otimes\psi(f)\|^2,$$

where

$$L_{\gamma,T}=(2+4\gamma)\,T\sum_{i=1}^{6}\sup_{0\leqslant s\leqslant T}\|\alpha_i(s)\|^2.$$

Proof. The corollary follows from Corollary 3 and the definition of σ_i and ω_{ij}^{γ}. ■

COROLLARY 5. *In the notation of Proposition* 2, *if* $|f(s)| \leqslant 1$ *for all* $s \in [0, T]$ *and the* A_i*'s are as in Corollary* 3, *then*

$$\|\Pi_i(t)\, u \otimes \psi(f)\|^2 \leqslant M_{\gamma,T} \int_0^t \|\beta(s)\, u \otimes \psi(f)\|^2\, ds,$$

where

$$M_{\gamma,T} = (2T+4\gamma)\Big[\sum_{i=0}^{6} \sup_{0 \leqslant s \leqslant T} \|\alpha_i(s)\|\Big]^2.$$

Proof. As in the proof of Corollary 2, (2.6) implies

$$\begin{aligned}
\|\Pi_1(t)\, u \otimes \psi(f)\|^2 &\leqslant \sum_{i,j=1}^{6} \Big[2\int_0^t \|A_i(s)\, u \otimes \psi(f)\|\, ds \int_0^t \|A_j(s)\, u \otimes \psi(f)\|\, ds \\
&\qquad + 4\gamma \int_0^t \|A_i(s)\, u \otimes \psi(f)\|\, \|A_j(s)\, u \otimes \psi(f)\|\, ds\Big] \\
&\leqslant \sum_{i,j=1}^{6} \Big[2\int_0^t \|\alpha_i(s)\, \beta(s)\, u \otimes \psi(f)\|\, ds \int_0^t \|\alpha_j(s)\, \beta(s)\, u \otimes \psi(f)\|\, ds \\
&\qquad + 4\gamma \int_0^t \|\alpha_i(s)\|\, \|\alpha_j(s)\|\, \|\beta(s)\, u \otimes \psi(f)\|^2\, ds\Big] \\
&\leqslant \sum_{i,j=1}^{6} \Big[2 \sup_{0 \leqslant s \leqslant T} \|\alpha_i(s)\| \sup_{0 \leqslant s \leqslant T} \|\alpha_i(s)\| \int_0^t \|\beta(s)\, u \otimes \psi(f)\|\, ds \\
&\qquad + 4\gamma \sup_{0 \leqslant s \leqslant T} \|\alpha_i(s)\| \sup_{0 \leqslant s \leqslant T} \|\alpha_j(s)\| \int_0^t \|\beta(s)\, u \otimes \psi(f)\|^2\, ds\Big],
\end{aligned}$$

which by the Cauchy–Schwartz inequality is less than or equal to

$$\begin{aligned}
&\sum_{i,j=1}^{6} \Big[2T \sup_{0 \leqslant s \leqslant T} \|\alpha_i(s)\| \sup_{0 \leqslant s \leqslant T} \|\alpha_j(s)\|^{1/2} \Big(\int_0^t \|\beta(s)\, u \otimes \psi(f)\|^2\, ds\Big)^{1/2} \\
&\qquad + 4\gamma \sup_{0 \leqslant s \leqslant T} \|\alpha_i(s)\| \sup_{0 \leqslant s \leqslant T} \|\alpha_j(s)\| \int_0^t \|\beta(s)\, u \otimes \psi(f)\|^2\, ds\Big] \\
&\leqslant (2T+4\gamma) \sum_{i,j=0}^{6} \sup_{0 \leqslant s \leqslant T} \|\alpha_i(s)\| \sup_{0 \leqslant s \leqslant T} \|\alpha_j(s)\| \int_0^t \|\beta(s)\, u \otimes \psi(f)\|^2\, ds \\
&= M_{\gamma,T} \int_0^t \|\beta(s)\, u \otimes \psi(f)\|^2\, ds. \quad \blacksquare
\end{aligned}$$

3. PROOF OF THE MAIN THEOREM

Define a sequence of adapted processes $\{U_n\}_{n=0}^{\infty}$ by $U_0(t) = U_0$ and, for $n \geqslant 1$,

$$U_n(t) = U_0 + \sum_{i=1}^{6} \int_0^t A_i(s)\, U_{n-1}(s)\, dN_i(s).$$

Then for $0 \leqslant t' \leqslant t \leqslant T < +\infty$ we have

$$\|[U_n(t) - U_n(t')]\, u\otimes\psi(f)\|^2 = \Big\| \sum_{i=1}^{6} \int_{t'}^{t} A_i(s)\, U_{n-1}(s)\, dN_i(s)\, u\otimes\psi(f)\Big\|^2$$

$$\leqslant L_{\gamma,T}(t-t') \sup_{0\leqslant s\leqslant T} \|U_{n-1}(s)\, u\otimes\psi(f)\|^2 \qquad \text{(by Corollary 4)}$$

from which we conclude by induction that $\{U_n\}_{n=0}^{\infty}$ is a sequence of strongly continuous adapted processes. Moreover,

$$\|[U_n(t) - U_{n-1}(t)]\, u\otimes\psi(f)\|^2$$

$$= \Big\| \sum_{i=1}^{6} \int_0^t A_i [U_{n-1}(t_1) - U_{n-2}(t_1)]\, dN_i(t_1)\, u\otimes\psi(f)\Big\|^2$$

$$\leqslant M_{\gamma,T} \int_0^t \|[U_{n-1}(t_1) - U_{n-2}(t_1)]\, u\otimes\psi(f)\|^2\, dt_1 \qquad \text{(by Corollary 5)}$$

$$\leqslant (M_{\gamma,T})^2 \int_0^t \int_0^{t_1} \|[U_{n-2}(t_2) - U_{n-3}(t_2)]\, u\otimes\psi(f)\|^2\, dt_2\, dt_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\leqslant (M_{\gamma,T})^{n-1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \Big\| \sum_{i=1}^{6} \int_0^{t_{n-1}} A_i(s) U_0\, dN_i(s)\, u\otimes\psi(f)\Big\|^2 dt_{n-1} \ldots dt_1$$

$$\leqslant (M_{\gamma,T})^n \|U_0\|^2 \|u\otimes\psi(f)\|^2 \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} ds\, dt_{n-1} \ldots dt_1$$

$$\leqslant (M_{\gamma,T})^n \|U_0\|^2 \|u\otimes\psi(f)\|^2 \frac{t^n}{n!},$$

and so

$$\|[U_n(t) - U_{n-1}(t)]\, u\otimes\psi(f)\| \leqslant (M_{\gamma,T})^{n/2} \|U_0\| \cdot \|u\otimes\psi(f)\| \frac{t^{n/2}}{\sqrt{n!}}$$

from which we conclude that

$$\sum_{n=1}^{\infty} \|[U_n(t) - U_{n-1}(t)]\, u\otimes\psi(f)\| < \infty.$$

By the completeness of $H_0 \otimes \Gamma(S_+)$ the above series also converges nonabsolutely, and so we may define

$$U(t)\, u \otimes \psi(f) \overset{\text{def}}{=} \lim_n U_n(t)\, u \otimes \psi(f),$$

where the limit is uniform for $t \in [0, T]$. As a strong limit of adapted processes U is also adapted. By the uniformity of convergence, U is strongly continuous. Moreover,

$$\begin{aligned}
&\Big\|\Big[U(t) - U_0 - \sum_{i=1}^{6} \int_0^t A_i(s)\, U(s)\, dN_i(s)\Big] u \otimes \psi(f)\Big\| \\
&\quad = \Big\|\Big[U(t) - U_{n+1}(t) + U_{n+1}(t) - U_0 - \sum_{i=1}^{6} \int_0^t A_i(s)\, U_n(s)\, dN_i(s) \\
&\qquad + \sum_{i=1}^{6} \int_0^t A_i(s)\, U_n(s)\, dN_i(s) - \sum_{i=1}^{6} \int_0^t A_i(s)\, U(s)\, dN_i(s)\Big] u \otimes \psi(f)\Big\| \\
&\quad \leqslant \|[U(t) - U_{n+1}(t)]\, u \otimes \psi(f)\| \\
&\qquad + \Big\|\Big[U_{n+1} - U_0 - \sum_{i=1}^{6} \int_0^t A_i(s)\, U_n(s)\, dN_i(s)\Big] u \otimes \psi(f)\Big\| \\
&\qquad + \Big\|\sum_{i=1}^{6} \int_0^t A_i(s)\, [U_n(s) - U(s)]\, dN_i(s)\, u \otimes \psi(f)\Big\|.
\end{aligned}$$

The first term of the above goes to zero as $n \to \infty$ by the definition of U. The second term is equal to zero by the definition of U_n. The third term is, by Corollary 4, less than or equal to

$$\Big[L_{\gamma,T}\, T \sup_{0 \leqslant s \leqslant T} \|[U_n(s) - U(s)]\, u \otimes \psi(f)\|^2\Big]^{1/2},$$

which goes to zero as $n \to \infty$ by the uniformity of the convergence in the definition of U. Thus

$$U(t) = U_0 + \sum_{i=1}^{6} \int_0^t A_i(s)\, U(s)\, dN_i(s),$$

i.e. U is a solution of (1.4).

If $V = \{V(t) \mid 0 \leqslant t \leqslant T\}$ is another process such that

$$V(t) = U_0 + \sum_{i=1}^{6} \int_0^t A_i(s)\, V(s)\, dN_i(s),$$

then by Corollary 5 we have

$$\|[U(t) - V(t)]\, u \otimes \psi(f)\|^2 \leqslant M_{\gamma,T} \int_0^t \|[U(s) - V(s)]\, u \otimes \psi(f)\|^2\, ds,$$

from which we conclude by Gronwall's inequality that

$$\|[U(t)-V(t)]\,u\otimes\psi(f)\| = 0.$$

Hence $U = V$ on the exponential domain, thus proving uniqueness. Turning to unitarity we notice that for $u \in H_0$, and $f \in S_+$ with $|f(s)| \leqslant 1$ on $[0, T]$, by the integral form of U,

$$\begin{aligned}
&\langle U(t)\,u\otimes\psi(f),\ U(t)\,v\otimes\psi(g)\rangle - \langle u\otimes\psi(f),\ v\otimes\psi(g)\rangle \\
&= \Big\langle u\otimes\psi(f),\ \sum_{i=1}^{6}\int_0^t A_i(s)\,U(s)\,dN_i(s)\,v\otimes\psi(g)\Big\rangle \\
&\quad + \Big\langle \sum_{i=1}^{6}\int_0^t A_i(s)\,U(s)\,dN_i(s)\,u\otimes\psi(f),\ v\otimes\psi(g)\Big\rangle \\
&\quad + \Big\langle \sum_{i=1}^{6}\int_0^t A_i(s)\,U(s)\,dN_i(s)\,u\otimes\psi(f),\ \sum_{j=1}^{6}\int_0^t A_j(s)\,U(s)\,dN_j(s)\,v\otimes\psi(g)\Big\rangle,
\end{aligned}$$

which by Propositions 1 and 3 is equal to

$$\begin{aligned}
&\int_0^t \sum_{i=1}^{6} \varrho_i(s)\,\langle u\otimes\psi(f),\ A_i(s)\,U(s)\,v\otimes\psi(g)\rangle\, ds \\
&+\int_0^t \sum_{i=1}^{6} \sigma_i(s)\,\langle u\otimes\psi(f),\ A_i^*(s)\,U(s)\,v\otimes\psi(g)\rangle\, ds \\
&+\int_0^t \Big[\sum_{i=1}^{6} \sigma_i(s)\,\big\langle A_i(s)\,U(s)\,u\otimes\psi(f),\ (U(s)-1)\,v\otimes\psi(g)\big\rangle \\
&+\sum_{j=1}^{6} \varrho_j(s)\,\big\langle (U(s)-1)\,u\otimes\psi(f),\ A_j(s)\,U(s)\,v\otimes\psi(g)\big\rangle \\
&+\sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(s)\,\langle A_i(s)\,U(s)\otimes\psi(f),\ A_j(s)\,U(s)\,v\otimes\psi(g)\rangle\Big]\, ds \\
&=\int_0^t \big\langle U(s)\,u\otimes\psi(f), \\
&\Big[\sum_{i=1}^{6} \sigma_i(s)\,A_i^*(s)+\sum_{j=1}^{6} \varrho_j(s)\,A_j(s)+\sum_{i,j=1}^{6} \omega_{ij}^{\gamma}(s)\,A_i^*(s)\,A_j(s)\Big]\,U(s)\,v\otimes\psi(g)\big\rangle\, ds.
\end{aligned}$$

Let $K = \big[\sum_{i=1}^{6}\sigma_i(s)\,A_i^*(s)+\sum_{j=1}^{6}\varrho_j(s)\,A_j(s)+\sum_{i,j=1}^{6}\omega_{ij}^{\gamma}(s)\,A_i^*(s)\,A_j(s)\big]$. We will show that $K = 0$. Using the definition of σ_i, ϱ_j and ω_{ij}^{γ} and collecting terms we have, by (1.5),

$$\begin{aligned}
K &= [A_1^*+A_1+\gamma A_3^*\,A_3]+\bar{f}\,[A_2^*+A_3+2\gamma\,A_3^*\,A_5+\gamma A_6^*\,A_3] \\
&\quad +g\,[A_3^*+A_2+\gamma A_3^* A_6+2\gamma A_5^*\,A_3]+\bar{f}^2\,[A_4^*+A_5+2\gamma A_6^*\,A_5] \\
&\quad +g^2\,[A_5^*+A_4+2\gamma A_5^*\,A_6]+\bar{f}g\,[A_6^*+A_6+4\gamma A_5^*\,A_5+\gamma A_6^*\,A_6] = 0.
\end{aligned}$$

Thus $\langle U(t)u\otimes\psi(f), U(t)v\otimes\psi(g)\rangle = \langle u\otimes\psi(f), v\otimes\psi(g)\rangle$ and by the totality of $\{u\otimes\psi(f)\}$ it follows by Proposition 7.2 of [10] that $U(t)$ extends to a unique linear isometry on $H_0\otimes\Gamma(S_+)$. Similarly, by considering the dual equation

$$U^*(t) = 1 + \sum_{i=1}^{6} \int_0^t U^*(s) A_i^*(s) dN_i^*(s)$$

we conclude that $U(t)$ extends to a coisometry on $H_0\otimes\Gamma(S_+)$, thus proving unitarity.

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Received on 16.4.2001;
revised version on 27.11.2001