J. Phys. A: Math. Gen. 24 (1991) 3483–3512. Printed in the UK

The low-density limit of quantum systems

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Received 28 September 1990, in final form 27 February 1991

Abstract. The present paper concludes a research programme developed, in the past three years, by various authors in several papers. The basic idea of this programme is to explain the irreversible behaviour of quantum systems as a limiting case (in a sense to be made precise) of usual quantum dynamics. One starts with a system interacting with a reservoir and, in the first attempts to deal with this problem, only limits of observables of the system and deduced master equations were considered. In our approach we study the limits of quantities related to the *whole compound system*. As a corollary we obtain an explanation of the physical origins of the quantum Brownian motion. In the present paper we study the low-density limit of a system coupled to a quasi-free boson reservoir in the equilibrium state at inverse temperature β and fugacity z, through an interaction of the *scattering type*, i.e. one which preserves the total number of particles of the reservoir. We obtain a macroscopic equation, for the limit of the compound system, which is a quantum stochastic differential equation of the Poisson type, in the Frigerio and Maassen sense, whose coefficients are uniquely determined by the one-particle scattering operator of the original Hamiltonian system and whose driving noises are the creation annihilation and number (or gauge) processes living in the space of a Fock quantum Brownian motion over the space $L^2(R, \mathrm{d}t, K_{0,1})$ where $K_{0,1}$ is an Hilbert space depending on the inverse temperature β, the one-particle reservoir dynamics, the free-system dynamics and the interaction.

1. Introduction

The present paper brings to a conclusion a long-term investigation developed through several papers. The main problem of this investigation is to understand the irreversible and dissipative behaviour of quantum systems. Two basic schemes have been developed in order to address this problem. Firstly, where the quantum noise is considered as an intrinsic property of a single system, arising from chaotic properties of the dynamics. An axiomatic approach was proposed in [1] and investigations of particular Hamiltonians which exhibit chaotic behaviour have been carried out by several authors [2, 3]. Another, more traditional approach consists in looking at a system coupled to another system (called, according to the interpretation, reservoir, heat bath, apparatus, noise, ...) and to consider the *reduced evolution* of observables of the system in the following sense: if X is an observable of the system, by the effect of the interaction with the reservoir the Heisenberg evolution up to time t, leads to an observable $X(t)$ which acts on the space of the composite system, i.e. $\mathscr{H}_{\mathrm{syst}} \otimes \mathscr{H}_{\mathrm{res}}$; by taking a partial expectation over the reservoir degrees of freedom, one obtains a new observable $\bar{X}(t)$,

† Supported in part by Ministero della Pubblica and CNR-GNAFA.

‡ Supported in part by CNR-GNAFA, Bando N.211.01.24. On leave of absence from Beijing Normal University.

0305-4470/91/153483+30$03.50 © 1991 IOP Publishing Ltd

acting only on the space $\mathscr{H}_{syst}$ of the system. The map $t \mapsto \bar{X}(t)$ is called the reduced *evolution of the observable X. The equation obeyed by $\bar{X}(t)$ is no longer of the* Heisenberg type, but is usually a complicated integrodifferential equation which gives little insight into the dynamical behaviour of the system. To obtain additional insight one usually considers the reduced dynamics in some limiting, idealized, conditions which should show the different time scales of the system and of the reservoir: it is only in these limiting conditions that the basic physical differences between *system* and *reservoir* arise. Before that, the two play a perfectly symmetric role.

The basic goal of such an idealized limiting procedure is to obtain a description of the limit evolution depending on the reservoir degrees of freedom only through a small set of physically measurable (usually macroscopic) parameters (such as temperature, dumping constants). Several limiting procedures have been investigated up to now. Two of the best known are the so called *weak-coupling* and *low-density* limit. We shall use the abbreviated notation WCL and LDL.

In the WCL, the strength of the system–reservoir interaction is driven by a constant λ and one considers limits, as $\lambda \mapsto 0$, of expectation values in a vacuum or thermal state of time-rescaled observables of the form X_{t/λ^2}. The condition $\lambda \to 0$ means that one considers a weak-coupling situation; the rescaling $t \to t/\lambda^2$ means that one considers long-time cumulative effects of the interactions. The scaling $t \to t/\lambda^2$ has its origins in some results of second-order perturbation theory due to Friedrichs [4]. More precisely: *if $U_t^{(\lambda)}$ denotes the Schrödinger evolution in interaction representation* (cf (2.11) below); u_t^0 denotes the free evolution of the coupled system and φ_S (or φ_R) is the initial state of the system (or reservoir), then in the WCL one studies the limit of

$$\varphi_S \otimes \varphi_R(U_{t/\lambda^2}^{(\lambda)*} u_{t/\lambda^2}^0 (X \otimes 1) U_{t/\lambda^2}^{(\lambda)}). \tag{1.1}$$

In all the cases studied up to now, it has been convenient to include the action of the free Hamiltonian of the system either in the interacting Hamiltonian or in a modified effective Hamiltonian of the reservoir. The effect of this operation is that u_s^0 acts trivially on the system observables, so that the expression (1.1) is equal to

$$\varphi_S \otimes \varphi_R(U_{t/\lambda^2}^{(\lambda)*} (X \otimes 1) U_{t/\lambda^2}^{(\lambda)}). \tag{1.2}$$

In the LDL one also studies limits of the form (1.2) but there are the following important differences:

(i) The parameter λ represents *fugacity, not an interaction constant.*

(ii) In (1.2) the state φ_R *depends on* λ, while the unitary $U_{t/\lambda^2}^{(\lambda)} = U_{t/\lambda^2}$ depends on λ *only through the time rescaling.* This is exactly opposite to the WCL case.

The study of the limits (1.2), both in the WCL and LDL, has been the object of a number of investigations (e.g. [4–8]) whose main result can be formulated as follows: in the limit $\lambda \to 0$ the quantity (1.2) converges, for every observable X of the system, to the limit

$$\varphi_S(P^t(X)) \tag{1.3}$$

where P^t is a *quantum Markovian semigroup* acting on the observables of the system. The whole influence of the reservoir on the dynamics of the system is then concentrated in the generator of the semigroup P^t. The generator of the reduced evolution in the LDL case was found by Dümcke [10] (with techniques completely different from the present ones and based on the BBGKY hierarchy and scattering theory).

The trouble with this approach is that all the information on the reservoir is swept away and only some information on the system is retained. In several situations of physical interest, however, one is also interested in retaining some information on the reservoir and on the coupled evolution system + reservoir. For example, if the reservoir is a field and the system an atom, sometimes one is interested in deducing some information on the atom by means of a measurement of the field. Thus, in the idealized description (WCL or LDL) one would like to also have a mathematical model of the reservoir. In view of this, the following question arises very naturally: *can we claim that in some sense, in the LDL or WCL, some reservoir observables converge to some limit?*

As stated the problem is very vague: Which reservoir observables are expected to converge? Which kind of convergence? What is the limit?

The basic achievement of the series of papers [20–35] has been to give an answer to the above questions which is not only mathematically rigorous and plausible from the point of view of physics, but also very explicit, given in a form which provides an intuitive insight of the physical interactions involved as well as of all the relevant parameters.

In the above-mentioned papers, the full solution of the problem was given only in the WCL case, while in the LDL case, notwithstanding two important preliminary result (cf [36]), the problem remained open.

In the present paper we complete the whole picture by adding the missing steps to the solution of the LDL problem.

The detailed proof is very long and technically involved, therefore, in the present paper, we limit ourselves to outline the basic ideas and techniques involved.

The full solution of the LDL case requires the use of all the techniques developed in [20–35], even those developed for solving problems which seemed to be side variants of the main result in the WCL case (e.g. the fermion case, the case of quadratic interaction, ...).

For this reason, in section 2, we have tried to give a qualitative description of the basic problem, of the physical meaning of our assumptions, and of the connection with our previous results. For a more detailed account of these problems we refer the reader to [36].

2. Statement of the problem

2.1. Notation

We recall from [36] some known facts and notation. Let H_0 and H_1 be Hilbert spaces interpreted respectively as the system and the one-particle reservoir space. Let $W(H_1)$ be the Weyl C^*-algebra on H_1, i.e. the closure of the linear space spanned by the set (of unitaries) $\{W(f):f\in H_1\}$ with commutation relations

$$W(f)\,W(g)=\mathrm{e}^{\mathrm{i}\,\mathrm{Im}\langle f,g\rangle}\,W(g)\,W(f) \tag{2.1}$$

for the unique C^*-norm on it (cf [37]). Let H be a self-adjoint bounded below operator on H_1 and $\beta>0$ and μ be real numbers interpreted as inverse temperature and chemical potential respectively. Let the fugacity z be given by $z=\mathrm{e}^{\beta\mu}$ and define

$$Q_z := (1+z\,\mathrm{e}^{-\beta H})(1-z\,\mathrm{e}^{-\beta H})^{-1}=\coth[\tfrac{1}{2}\beta(H-\mu)] \tag{2.2}$$

and suppose that, for each z in an interval $[0,Z]$, Q_z is a self-adjoint operator on a domain $\mathcal{D}$, independent of z. Denote φ_{Q_z} the mean zero gauge-invariant quasi-free state on $W(H_1)$ with covariance operator Q_z, characterized by the property

$$\varphi_{Q_z}(W(f))=\exp(-\tfrac{1}{2}\langle f,Q_zf\rangle)\qquad \forall f\in H_1 \tag{2.2}$$

and let $\{\mathcal{H}_{Q_z}, \pi_{Q_z}, \Phi_{Q_z}\}$ be the GNS triple of $\{W(H_1), \varphi_{Q_z}\}$, so that

$$\langle \Phi_{Q_z}, \pi_{Q_z}(W(f))\Phi_{Q_z}\rangle = \varphi_{Q_z}(W(f)). \tag{2.3}$$

We shall write W_{Q_z} for $\pi_{Q_z}\circ W$. The *Fock representation* corresponds to the case $Q_z = 1$, which corresponds to the limiting case $\beta = \infty$ (or $z = 0$). In this case the GNS representation will be simply denoted $\{\mathcal{H}, \pi, \Phi\}$. Let S_t^1 be a unitary group on $B(H_1)$ (the one-particle free evolution of the reservoir) and suppose that

$$S_t^1 Q_z = Q_z S_t^1 \qquad \forall t \geq 0 \tag{2.4}$$

where the equality is meant on $\mathcal{D}$. Typically we shall choose $S_t^1 = \exp(\mathrm{i}tH)$ so that (2.4) is obviously satisfied. This implies that the second quantization of S_t^1, denoted $W(S_t^1)$ and characterized by the condition $W(S_t^1)W_{Q_z}(f) = W_{Q_z}(S_t^1 f)$, leaves φ_{Q_z} invariant hence, in the GNS representation, the generator $H_{\mathrm{R}}^{(z)}$ of the one-parameter group $W(S_t^1)$ is called the free Hamiltonian of the reservoir in the representation π_{Q_z}. The *system Hamiltonian* is a self-adjoint operator H_{S} on the system space H_0. The *total free Hamiltonian* is defined as

$$H_0^{(z)} := H_{\mathrm{S}} \otimes 1 + 1 \otimes H_{\mathrm{R}}^{(z)} \tag{2.5}$$

where

$$H_{\mathrm{R}}^{(z)} := \mathrm{d}\Gamma(H) := \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} W(S_t^1). \tag{2.6}$$

The *interaction Hamiltonian* V is

$$V := \mathrm{i} \sum_{\varepsilon \in \{0,1\}} D_\varepsilon \otimes A^+(g_\varepsilon) A(g_{1-\varepsilon}) \tag{2.7}$$

where we use the notation

$$D_0 := D \qquad D_1 := -D^+ \tag{2.8}$$

and $g_0, g_1 \in K \subset H_1$ (K to be defined) and D is a bounded operator on H_0 (we make this assumption for the sake of simplicity, but our techniques apply with minor modifications to a large class of unbounded operators, including creation, annihilation and number operators).

With this notation, the *total Hamiltonian* is

$$H^{(z)} := H_{\mathrm{S}} \otimes 1 + 1 \otimes H_{\mathrm{R}}^{(z)} + V \tag{2.9}$$

and the wave operator at time t is defined by

$$U_t := \exp(\mathrm{i}tH_0^{(z)}) \exp(-\mathrm{i}tH^{(z)}). \tag{2.10}$$

Therefore we have the formal identity

$$\frac{\mathrm{d}}{\mathrm{d}t} U_t = \frac{1}{\mathrm{i}} V(t) U_t \qquad U_0 = \mathbb{1} \tag{2.11}$$

on the weakly linear span of the vectors of the form

$$u \otimes \Phi_{Q_z}(\lambda, S, T; f) \tag{2.12}$$

where $u \in H_0$, $f \in K$, S, $T \in \mathbb{R}$, the $\Phi_{Q_z}(\lambda, S, T; f)$ are the collective coherent vectors defined by

$$W_{Q_z}\left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f \, du\right) \Phi_{Q_z} \tag{2.13}$$

and

$$V(t) := \exp(itH_0^{(z)}) V \exp(-itH_0^{(z)}). \tag{2.14}$$

It is well known that the solution of (2.11) is given by the iterated series

$$U_t = \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n V(t_1) \ldots V(t_n) \tag{2.15}$$

which is convergent on the domain (1.12).

The basic example we have in mind is the *free Bose gas*, where

$$H_1 = L^2(\mathbb{R}^d) \qquad d \geqslant 3 \qquad H = -\tfrac{1}{2}\Delta$$

where Δ is the Laplacian on $\mathbb{R}^d$. In this case one usually starts from a finite volume $V \subseteq \mathbb{R}^d$ and the density n is the limit as the volume $|V| \to +\infty$ of the expectation value of the number operator N_V in finite volume in the state $\phi_{Q_z}^{(V)}$ divided by $|V|$. It becomes asymptotically proportional to the fugacity z in the limit as $z \to 0$; for example, for the free Bose gas in three space dimensions one has

$$n = n(\beta, z) = z \int_{\mathbb{R}^3} \exp(\beta k^2/2 - z)^{-1} \, d^3k.$$

The LDL corresponds to letting the density n in the state φ_{Q_z} tend to zero, time being scaled as $\bar{n}t/n$, $\bar{n}$ being a rescaled density to be held fixed. The asymptotic proportionality of z and n, as $z \to 0$, implies that, choosing $\bar{n} = 1$, the LDL is equivalent to the limit $z \to 0$, time being scaled as t/z.

The physical reason why the LDL should go with an interaction, which preserves the (generalized) number operator, is that scaling a parameter means that it is a controllable parameter and, if the interaction were not to commute with number operator, then the density could take any value in the course of time.

2.2. *The collective states: statement of the main result*

From the introduction it is clear that we are interested in the asymptotic behaviour, as $\lambda := \sqrt{z} \to 0$, of expectation values (or matrix elements) of quantities of the form

$$U^*_{t/\lambda^2}(X \otimes 1) U_{t/\lambda^2}. \tag{2.16}$$

As a preliminary problem we investigate the asymptotic behaviour of the basic dynamical variable

$$U_{t/\lambda^2} = \sum_{n=0}^{\infty} (-i)^n \int_0^{t/\lambda^2} dt_1 \ldots \int_0^{t_{n-1}} dt_n \, V(t_1) \ldots V(t_n). \tag{2.17}$$

The reason why this is only a preliminary problem is that one can usually determine the asymptotic behaviour of some appropriate matrix elements of the operator (2.16) and, unfortunately, as several examples show (compare for example the main result of [28] with [29]), the limit of the product (2.16) is not obtained in the obvious way from the limit of (2.17).

However, the experience accumulated with the study of several models in the WCL case suggests that the most difficult step consists in controlling limit (2.17) and understanding the equation (usually stochastic) which it satisfies. Once this is done, the passage to the limit (2.16) is by no means trivial, but the difficulties are mainly of a technical nature. (A possible exception is the fact that, in the LDL case, the limit of (2.17) can be controlled for all $t \in \mathbb{R}$ in the Fock case, while the same limit in the finite-temperature case and the limit (2.16) can be controlled only for t in a certain small interval. It is not yet clear if this difficulty is of a fundamental nature or a purely technical one.) In any case, the difference between the limits of the two quantities (2.16) and (2.17) can be described as follows: for finite λ, U_{t/λ^2} satisfies the (time-rescaled) Schrödinger equation in interaction representation (2.11), while $U^*_{t/\lambda^2}(X \otimes 1) U_{t/\lambda^2}$ satisfies the corresponding Heisenberg equation.

The basic conceptual idea, underlying the whole series of [20-36] is that, in the limit $\lambda \to 0$, the Schrödinger equation (2.11) goes into a quantum stochastic differential equation, while the corresponding Heisenberg equation goes into a quantum Langevin equation (more precisely: the quantum Langevin equation canonically associated to the quantum stochastic equation satisfies by the limit of U_{t/λ^2}).

Unfortunately there is no hope that the limit (2.16) (also (2.17)) exists in any usual operator topology: this is well known form the WCL case (and also from the classical case). So the *first* problem to be addressed is: *in which sense do we speak of the limit of the operator* U_{t/λ^2} *as* $\lambda \to 0$?

To answer this question was a difficult problem even in the WCL case. In that case we were able to control the limit of expressions of the form

$$\lim_{\lambda \to 0} \langle u \otimes \Phi_\lambda(f, S, T), U_{t/\lambda^2} v \Phi_\lambda(f', S', T') \rangle \tag{2.18}$$

where the choice of the vectors

$$\Phi_\lambda(f, S, T) := W\left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f \, du\right) \Phi \tag{2.19}$$

(called the *collective coherent vectors*) was motivated by the analogy with the quantum central limit theorem in [38].

More precisely, both in the WCL and LDL Fock cases, for a variety of models, one has the following situation (cf. [2, 3, 9] or section 4) of [36]).

One starts from the Hilbert space $\mathcal{H}$, of a system with a given free evolution and a given interaction V. In terms of these, one defines a two-parameters family $U^{(\lambda)}_{t/\lambda^2}$ of unitary operators of the form (2.17). Finally, one proves the following:

Theorem 2.1. There exists

a set $\mathcal{T}$;

a map Φ: $(\lambda, \alpha) \in \mathcal{T} \mapsto \Phi_\lambda(\alpha) \in \mathcal{H}$;

a Hilbert space $\mathcal{K}$;

a total subset $\Phi(\alpha)(\alpha \in \mathcal{T})$;

a one-parameter family $U(t)$ of unitary operators on $\mathcal{K}$, such that the limit (2.18) exists and is equal to

$$\lim_{\lambda \to 0} \langle \Phi_\lambda(\alpha), U^{(\lambda)}_{t/\lambda^2} \Phi_\lambda(\alpha') \rangle = \langle \Phi(\alpha), U(t) \Phi(\alpha') \rangle. \tag{2.20}$$

In particular, for $t = 0$, this yields

$$\lim_{\lambda \to 0} \langle \Phi_\lambda(\alpha), \Phi_\lambda(\alpha') \rangle = \langle \Phi(\alpha), \Phi(\alpha') \rangle. \tag{2.21}$$

Thus, denoting $\mathcal{H}_\lambda$ the closed subspace spanned by the *collective vectors* $\Phi_\lambda(\alpha)$, (2.21) gives a precise meaning to the statement: *as* $\lambda \to 0$ *the space* $\mathcal{H}_\lambda$ *converges to a limit space* $\mathcal{H}$.

As already mentioned, this scheme worked well for several models [21-23, 28, 29], but the attempt to extend this scheme to wider and wider classes of models led to the discovery of pathological phenomena, e.g. the limit (2.18) might exist and be of the form (2.20) without $U(t)$ being unitary (cf [27]).

The appearance of these pathologies led Frigerio to conjecture that *the correct choice of the collective states should depend on the form of the interaction.*

Consider a system interacting with a reservoir, as in the scheme described at the beginning of this section (but not necessarily with the interaction (2.7)). One should:

(i) Identify, for each λ, a subspace $\mathcal{H}_\lambda$ of the state space of the coupled system—the space of *collective vectors.*

(ii) Identify a *limiting space* $\mathcal{H}$, characterized by the following properties: there exist a dense subspace $\mathcal{D} \subseteq \mathcal{H}$,

for each $\lambda > 0$, a dense subspace $\mathcal{D}_\lambda \subseteq \mathcal{H}_\lambda$,

a one-to-one map $\Phi_\lambda \in \mathcal{D}_\lambda \mapsto \Phi \in \mathcal{D}$ such that, if (Φ_λ), $(\Phi'_\lambda)(\lambda > 0)$ are families in $\mathcal{D}_\lambda$ such that $\Phi_\lambda \to \Phi$, $\Phi'_\lambda \to \Phi'$, with Φ, $\Phi' \in \mathcal{D}$, then

$$\lim_{\lambda \to 0} \langle \Phi_\lambda, \Phi'_\lambda \rangle = \langle \Phi, \Phi' \rangle. \tag{2.22}$$

(iii) Prove the existence of the limit

$$\lim_{\lambda \to 0} \langle \Phi_\lambda, U_{t/\lambda^2} \Phi'_\lambda \rangle = \langle \Phi, U(t) \Phi' \rangle \tag{2.23}$$

for the basic dynamical variable, defined by (2.11) and for (Φ_λ), (Φ'_λ), Φ, Φ' as in (ii) above.

(iv) Deduce an equation (usually a quantum stochastic differential equation in the sense of Hudson and Parthasarathy [15]) satisfied by $U(t)$.

(v) Prove that $U(t)$ is unitary.

(vi) For every observable X of that system prove that, as $\lambda \to 0$,

$$\langle \Phi_\lambda, U_{t/\lambda^2} (X \otimes 1) U^*_{t/\lambda^2} \Phi'_\lambda \rangle \to \langle \Phi, U(t)(X \otimes 1) U^*(t) \Phi' \rangle \tag{2.24}$$

$$\langle \Phi_\lambda, U^*_{t/\lambda^2} (X \otimes 1) U_{t/\lambda^2} \Phi'_\lambda \rangle \to \langle \Phi, U^*(t)(X \otimes 1) U(t) \Phi' \rangle. \tag{2.25}$$

When there is an *a priori* privileged state Φ^0 of the reservoir (vacuum state, thermal state, ...), the corollary of step (vi) given by the choices

$$\Phi_\lambda = u \otimes \Phi^0_\lambda(\alpha) \qquad \Phi'_\lambda = v \otimes \Phi^0_\lambda(\alpha') \qquad u, v \in H_0 \tag{2.26}$$

gives the *reduced evolution* of the system through the identity

$$\langle u \otimes \Psi, U^*(t)(X \otimes 1) U(t) v \otimes \Psi' \rangle = \langle u, P^t(X) v \rangle \tag{2.27}$$

where Ψ corresponds to Φ^0 in the map of point (ii) above and P^t is the Markovian semigroup of (1.3).

2.3. The basic assumptions and their role

The basic assumption, which has been common to *all* the investigations on the WCL and the LDL, concerns the one-particle reservoir and dynamics and is the following:

there exists a non-zero subspace K of H_1 (in all the examples it is a dense subspace) such that

$$\int_{\mathbb{R}} |\langle f, S_t^1 g\rangle| \, dt < \infty \qquad \forall f, g \in K. \tag{2.28}$$

Moreover, we suppose that

$$Q_t K \subseteq K. \tag{2.29}$$

For example, for the free Bose gas, $H_1 = L^2(R^d)$ for some $d \geq 3$, $H = -\Delta/2$, where Δ is the Laplacian on $\mathbb{R}^d$ and K can be chosen to be $L^1 \cap L^\infty(\mathcal{R}^d)$. From lemma 3.2 of [21], we know that the assumption (2.28) implies that the sesquilinear form $(\cdot|\cdot): K \times K \to \mathbb{C}$ defined by

$$(f|g) := \int_{\mathbb{R}} \langle f, S_t^1 g\rangle \, dt \qquad f, g \in K \tag{2.30}$$

defines a pre-scalar product on K. We denote $\{K, (\cdot|\cdot)\}$, or simply K, the completion of the quotient of K by the zero $(\cdot|\cdot)$ norm elements.

We assume, moreover, that the system Hamiltonian H_S is related to the coefficient D, of the interaction, by the identity

$$e^{itH_S} D \, e^{-itH_S} = e^{-it(\omega_0 - \omega_1)} D \tag{2.31}$$

a familiar assumption, satisfies by all the Hamiltonians commonly used in quantum optics. For the sake of definition we shall assume that

$$\omega_j > 0 \qquad (j = 0, 1) \qquad \omega_0 \neq \omega_1. \tag{2.32}$$

We assume moreover that the test functions g_0, g_1 which define the interaction (2.7), have *disjoint energy spectra*, i.e.

$$\langle g_0, S_t^1 g_1\rangle = 0 \qquad \forall t \in \mathbb{R} \tag{2.33}$$

and we fix two mutually orthogonal projections P_0, P_1 commuting with H such that

$$P_\varepsilon g_\varepsilon = g_\varepsilon \qquad \varepsilon = 0, 1. \tag{2.34}$$

This assumption means that, even if the particles of the reservoir have generically a continuous energy spectrum, they behave like a two-level system *as far as their interaction with the system is concerned*: if P_0 and P_1 project onto disjoint intervals (*energy bands*) I_0 and I_1, these energy bands act as the counterpart of the energy levels ω_0, ω_1 of the system.

This assumption has the following effect: when we go in interaction representation with the interaction (2.7) and the free Hamiltonian (2.5), we obtain

$$\begin{aligned} V(t) &= i \sum_{\varepsilon \in \{0,1\}} e^{itH_S} D_\varepsilon \, e^{-itH_S} \otimes A^+(S_t^1 g_\varepsilon) A(S_t^1 g_{1-\varepsilon}) \\ &= i \sum_{\varepsilon \in \{0,1\}} e^{-it(\omega_\varepsilon - \omega_{1-\varepsilon})} D_\varepsilon \otimes A^+(S_t^1 g_\varepsilon) A(S_t^1 g_{1-\varepsilon}). \end{aligned} \tag{2.35}$$

Therefore if we define the group $\{S_t;\ t \in \mathbb{R}\}$ of unitary operator on H_1 by

$$S_t := S_t^1 \, e^{-it(\omega_0 P_0 + \omega_1 P_1)} = \exp[it(H - \omega_0 P_0 - \omega_1 P_1)] \tag{2.36}$$

then

$$S_t g_0 := S_t^1 \, e^{+it\omega_0} g_0 \qquad S_t g_1 := S_t^1 \, e^{-it\omega_1} g_1 \tag{2.37}$$

and therefore

$$V(t) = \mathrm{i} \sum_{\varepsilon \in \{0,1\}} D_\varepsilon \otimes A^+(S_t g_\varepsilon) A(S_t g_{1-\varepsilon}). \tag{2.38}$$

Thus we see that *as far as the interaction V is concerned*, under the assumption (2.33), one can assume that $\omega_0 \equiv \omega_1$ at the cost of replacing the one-particle reservoir dynamics S_t^1 by the *effective* one-particle reservoir dynamics (2.36) (which depends both on the frequencies ω_j of the system and on the interaction). The infinitesimal generator H' of S_t is given by

$$H' = H - \omega_0 P_0 - \omega_1 P_1. \tag{2.39}$$

2.4. Connection with scattering theory

The connection between the stochastic process $U(t)$ and scattering theory is the second crucial step of our programme. It was established in [36] and here we recall the basic results of that paper.

Because of number conservation for interactions of scattering type, the closed subspace of $H_0 \otimes \mathscr{H}_\mathrm{R}$ generated by vectors of the form $u \otimes A^+(f)\Phi$ ($u \in H_0, f \in H_1$), which is naturally isomorphic to $H_0 \otimes H_1$, is globally invariant under the time evolution operator $\exp[\mathrm{i}(H_\mathrm{S} \otimes 1 + 1 \otimes H_\mathrm{R} + V)t]$, and the restriction of the time evolution operator to this subspace corresponds to an evolution operator on $H_0 \otimes H_1$ given by

$$\exp[\mathrm{i}(H_\mathrm{S} \otimes 1 + 1 \otimes H + V_1)t] \tag{2.40}$$

where

$$V_1 = \mathrm{i}(D \otimes |g_0\rangle\langle g_1| - \mathrm{cc}). \tag{2.41}$$

Dümcke's results [10] tell us that the reduced evolution of observables in $\boldsymbol{B}(H_0)$ is completely determined, in the LDL, by the scattering operator for the evolution (2.40) on $H_0 \otimes H_1$ and by the temperature of the reservoir. *This corresponds to the physical intuition that particles of a dilute gas should scatter independently, one at a time, in the system.* The relevant operators are the *one-particle Møller wave operators*

$$\Omega_\pm = s - \lim_{t \to \pm\infty} \exp[\mathrm{i}(H_\mathrm{S} \otimes 1 + 1 \otimes H + V_1)t] \exp[-\mathrm{i}(H_\mathrm{S} \otimes 1 + 1 \otimes H)t] \tag{2.42}$$

the *one-particle T-operator*

$$T = V_1 \Omega_+ \tag{2.43}$$

and the *one-particle S-operator*

$$S = \Omega_-^* \Omega_+. \tag{2.44}$$

Under rather general assumptions, which are satisfied in the present case, S is unitary.

From (2.41) it follows that

$$\Omega_\pm = s - \lim_{t \to \pm\infty} U_t^{(1)} \tag{2.45}$$

where $U_t^{(1)}$ is the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} U_t^{(1)} = \left(\sum_{\varepsilon \in \{0,1\}} D_\varepsilon \otimes |S_t g_\varepsilon\rangle\langle S_t g_{1-\varepsilon}| \right) U_t^{(1)} \qquad U_0^{(1)} = 1. \tag{2.46}$$

Let $H_{1,0}$ denote the closed subspace of H_1 spanned by the vectors

$$S_t g_\varepsilon \qquad \varepsilon = 0, 1 \qquad t \in \mathbb{R}. \tag{2.47a}$$

Notice that $U_t^{(1)}$, and therefore also $\Omega_\pm$, T, S, V_1, leave $H_0 \otimes H_{1,0}$ invariant and annihilate on $H_0 \otimes H_{1,0}^\perp$, so we can restrict our attention to $H_{1,0}$, and in the following we assume that

$$H_{1,0} \equiv H_1 . \tag{2.47b}$$

Given this, the spectral subspace corresponding to a fixed real number E, in the spectrum of the total unperturbed one-particle energy operator H' (defined by (2.39)) is two-dimensional. In fact H' commutes with the self-adjoint involution J, characterized by the property $Jg_{1-\varepsilon} = g_\varepsilon$, $\varepsilon = 0, 1$ and, by assumption (2.47b), the function of H' and J, applied to g_0 (or to g_1) span the whole space. Moreover an orthonormal basis of this space is given by the pair

$$g_{0,E}, g_{1,E}$$

where $g_{\varepsilon,E}$ is the component of the vector g_ε in the Eth energy shell. This means that, if we decompose the space $H_0 \otimes H_1$ as a direct integral over the energy spectrum of H', then the space of each energy shell is isomorphic to $H_0 \otimes \mathbb{C}^2$.

If $\mathscr{H}_E \simeq H_0 \otimes \mathbb{C}^2$ is the space corresponding to the energy E, then the scalar product on $\mathscr{H}_E$ is given by

$$(g|f)(E) := \int_{\mathbb{R}} \exp(-\mathrm{i}Et) \langle g, S_t f \rangle \, \mathrm{d}t \qquad (u \otimes g | v \otimes f)(E) := \langle u, v \rangle (g|f)(E) \tag{2.48}$$

where f, $g \in K$, so that the scalar product $(g|f)$ defined by (2.30) corresponds to the energy shell $E = 0$.

Now, from (2.45) and (2.46) it follows that the operator S, given by (2.44), commutes with S_t and therefore with $1 \otimes H'$. This implies that, in the integral decomposition of $H_0 \otimes H_1$ into energy shells, the operator S will leave the space of each shell invariant. So, denoting by $S(E)$ the action of S on the Eth energy shell, one has $S(E) \in \boldsymbol{B}(H_0) \otimes M(2, \mathbb{C})$ given by

$$S(E) = \sum_{\varepsilon, \varepsilon' = 0,1} S_{\varepsilon'\varepsilon}(E) \otimes e_{\varepsilon'\varepsilon} \tag{2.49}$$

$e_{\varepsilon'\varepsilon}$ ($\varepsilon, \varepsilon' = 0, 1$) being matrix units of $M(2, \mathbb{C})$. The unitarity of S requires that (almost) every $S(E)$ is unitary. In order to determine the form of the operators $S_{\varepsilon'\varepsilon}(E) \in B(H_0)$, let us introduce the following notation

$$(g|f)_-(E) := \int_{-\infty}^0 \exp(-\mathrm{i}Et) \langle g, S_t f \rangle \, \mathrm{d}t \tag{2.50a}$$

$$T_\varepsilon(E) := (g_0|g_0)_-(E)(g_1|g_1)_-(E) D_0 D_1 \tag{2.50b}$$

and note that for all real E one has

$$|(g|g)_-(E)| \leq \int_{-\infty}^0 |\langle g, S_t g \rangle| \, \mathrm{d}t = \|g\|_-^2$$

so that, under the assumption

$$16 \|D\|^2 \max(\|g_0\|_-, \|g_1\|_-) < 1 \tag{2.51}$$

we have the convergent geometric series

$$\sum_{n=0}^{\infty} T_\varepsilon(E)^n = (1 - T_\varepsilon(E))^{-1}. \tag{2.52}$$

Define also

$$R_{00}(E) := (g_1|g_1)_-(E)D_0D_1(1-T_0(E))^{-1} \tag{2.53a}$$

$$R_{01}(E) := (1-T_0(E))^{-1}D_0 \tag{2.53b}$$

$$R_{10}(E) := (1-T_1(E))^{-1}D_1 \tag{2.53c}$$

$$R_{11}(E) := (g_0|g_0)_-(E)D_1D_0(1-T_1(E))^{-1}. \tag{2.53d}$$

Under these assumptions and notation in [39] it has been proved that the T-operator has the form

$$T(u\otimes f) = \frac{\mathrm{i}}{2\pi}\sum_{\varepsilon,\varepsilon'=0,1}\int_{-\omega_\varepsilon}^{\infty} R_{\varepsilon'\varepsilon}(E)u\otimes g_{\varepsilon'}(g_\varepsilon|f)(E)\,\mathrm{d}E \qquad \forall u\in H_0, f\in K \tag{2.54}$$

and the matrix elements of the S-operator are given by

$$S_{\varepsilon'\varepsilon}(E) = \delta_{\varepsilon'\varepsilon}1 + [(g_{\varepsilon'}|g_{\varepsilon'})(E)]^{1/2}R_{\varepsilon'\varepsilon}(E)[(g_\varepsilon|g_\varepsilon)(E)]^{1/2}. \tag{2.55}$$

Given all this, the stochastic process $U(t)$, deduced in the Fock case, on the right-hand side of (2.21), is the solution of the stochastic differential equation

$$\mathrm{d}U(t) = \mathrm{d}N_t(S(0)-1;0)U(t) \qquad U(0)=1 \tag{2.56}$$

where

$$\sum_{\varepsilon,\varepsilon'=0,1} R_{\varepsilon'\varepsilon}(0)\otimes N_s(g_{\varepsilon'},g_\varepsilon) := N_s(S(0)-1;0) \tag{2.57}$$

is the quantum Poisson process of zero intensity corresponding to the S-operator on the energy shell of total energy $E=0$. The reformulation (2.56) (as compared with equation (6.1) of [28]) has important consequences. The first of them is that the unitarity of the solution $U(t)$ of (2.56), which has been proved with heavy direct calculations in [28], follows from the unitarity of $S(0)$ and a general theorem of Frigerio and Maassen [40]. The second is that it provides a precious heuristic indication for estimating what should be the full stochastic equation, i.e. involving all energy shells. The basic idea originated in a paper by Alicki and Frigerio [39] and was fully realized in [36]. It can be briefly described as follows: first one writes the Dümcke LDL generator in a form that naturally suggests a unitarity dilation with quantum Poisson noise in the sense of Frigerio and Maassen; then one actually builds such a dilation. As usual there is a wide arbitrarity in the construction of a specific dilation, but one can hope that the general structure of the stochastic equation of the dilation will be at least similar to the true stochastic equation satisfies by the LDL of U_{t/λ^2}.

More precisely, in [36] it has been proved that the Dümcke LDL generator can be written in the form

$$L(X) = \frac{1}{2\pi}\sum_{\varepsilon,\varepsilon'=0,1}\int_{-\omega_\varepsilon}^{\infty}\mathrm{d}E\,\exp[-\beta(E+\omega_\varepsilon)](S_{\varepsilon'\varepsilon}(E)^*XS_{\varepsilon'\varepsilon}(E)-\delta_{\varepsilon'\varepsilon}X) \tag{2.58}$$

($X\in B(H_0)$) and that the semigroup with generator (2.58) admits a unitary dilation in terms of a quantum Poisson process, which can be described as follows. Introduce the Hilbert space

$$\mathcal{K} := L^2((-\omega_0,\infty),\mathrm{d}E)\oplus L^2((-\omega_1,\infty),\mathrm{d}E) \tag{2.59}$$

and the von Neumann subalgebra M of $L^\infty(\mathbb{R}, \mathrm{d}E;\ M(2,\mathbb{C}))$ of functions $E \mapsto \{Y_{\varepsilon'\varepsilon}(E): \varepsilon', \varepsilon = 0, 1\}$ such that $Y_{\varepsilon'\varepsilon}(E) = 0$ for $E < -\omega_\varepsilon$, where M acts on $\mathscr{K}$ in the obvious way. Define a positive linear functional μ on M by

$$\mu(Y) := \frac{1}{2\pi} \sum_{\varepsilon=0,1} \int_{-\omega_\varepsilon}^{\infty} \mathrm{d}E \exp[-\beta(E+\omega_\varepsilon)]\, Y_{\varepsilon\varepsilon}(E) = \langle \xi, Y\xi \rangle_{\mathscr{K}} \tag{2.60}$$

where $\xi \in \mathscr{K}$ is given by

$$\xi(E) := (2\pi)^{-1/2} \left[\chi_{(-\omega_0,\infty)}(E) \exp\left(-\frac{\beta}{2}(E+\omega_0)\right) \oplus \chi_{(-\omega_1,\infty)}(E) \exp\left(-\frac{\beta}{2}(E+\omega_1)\right) \right]. \tag{2.61}$$

The function $E \mapsto S(E)$, $S(E)$ given by (2.49) and (2.55), can be regarded as an element S of $B(H_0) \otimes M$. With this notation, the generator (2.58) can be rewritten in the form of [40] as follows:

$$L(X) = (\mathrm{i}d \otimes \mu)[S^*(X \otimes 1)S - (X \otimes 1)]. \tag{2.62}$$

Then it follows from [40] that, for all $X \in \boldsymbol{B}(H_0)$ and $t \in \mathbb{R}^+$,

$$\exp[Lt](X) = \boldsymbol{E}[U^*(t)(X \otimes 1)U(t)] \tag{2.63}$$

where $U(t)$ is the (unitary) solution of the QSDE:

$$\mathrm{d}U(t) = \mathrm{d}N_t(S-1;\xi)\,U(t) \qquad U(0) = 1. \tag{2.64}$$

Notice that in contrast to the Fock case where only $S(0)$ appears, here we have continuously many values for the total energy E, and an integration over E weighted by the Boltzmann factor $\exp[-\beta(E+\omega_\varepsilon)]$.

In the equilibrium state at strictly positive temperatures $T = 1/\beta$, particles of all energies are present, with numbers which become proportional to $z\exp[-\beta k^2/2]$ in the limit as $z \to 0$. The total energy of a particle with momentum k and of type ε ($\varepsilon \in \{0, 1\}$) is redefined to be $E = \frac{1}{2}k^2\omega_\varepsilon$, to take into account the energies of the energy levels of the system on which the reservoir particles scatter. This allows the description of scattering of a particle on the system by saying that a scattering particle changes its type from ε to $\varepsilon' = 1 - \varepsilon$, while its total energy remains unchanged: $\frac{1}{2}k^2 - \omega_\varepsilon = \frac{1}{2}k'^2 - \omega_{\varepsilon'}$. At the same time, the system performs a transition under the action of the operator $D_{\varepsilon'}$. The particle type $\varepsilon \in \{0, 1\}$ remains a quantum degree of freedom, interacting with the quantum system. On the other hand, the total energy E becomes a *classical* variable (with a continuous spectrum), since the uncertainty relation $\Delta E \Delta(t/z) \geqslant \hbar$ involving energy and rescaled time t/z is no longer a restriction in the limit as $z \to 0$. This gives rise to the peculiar structure of M as an algebra of 2×2 matrices whose entries are functions of E. Each value E of total energy contributes to L with the Boltzmann factor $\exp[-\beta k^2/2] = \exp[-\beta(E+\omega_\varepsilon)]$ for an incoming particle of type ε.

In view of the above considerations one expects that in the general LDL case and with a suitable choice of collective vectors, the matrix elements of $U_{t/z}$ should converge, in the limit as $z \to 0$, to a matrix elements of the solution $U(t)$ of a QSDE of a form similar to (2.64). The main conceptual difficulty in order to verify this conjecture consists in the individuation of the collective vectors with respect to which to form the matrix elements.

3. The choice of collective vectors

Our starting point for the solution of the LDL problem is an analysis due to Palmer [12]. If we denote H_1^ι the conjugate Hilbert space of H_1 (cf. [23]), then it is known that, up to a unitary ismorphism

$$\Phi_{Q_z} = \Phi_F \otimes \Phi_F^\iota \tag{3.1}$$

and

$$W_{Q_z}(f) = W(Q_+ f) \otimes W(Q^- f) \tag{3.2}$$

$$A_{Q_z}(f) = A(Q_+ f) \otimes 1 + 1 \otimes A^+(Q^- f) \tag{3.3}$$

where

$$Q_+ := \sqrt{\frac{Q_z + 1}{2}} \qquad Q^- := \iota \sqrt{\frac{Q_z - 1}{2}} =: \iota \sqrt{z} Q_- . \tag{3.4}$$

Since

$$Q_z = \frac{1 + z\,e^{-\beta H}}{1 - z\,e^{-\beta H}} \tag{3.5}$$

by expanding in z we obtain

$$Q_+ = 1 + \tfrac{1}{2} z\,e^{-\beta H} + o(z) \qquad Q^- = \iota z^{1/2}\,e^{-BH/2} + o(z) \tag{3.6}$$

so that, in the correspondence (3.3),

$$A_{Q_z}(f) = A(f) \otimes 1 + \frac{z}{2} A(e^{-\beta H} f) \otimes 1 + z^{1/2} 1 \otimes A^+(\iota\,e^{-\beta H/2} f) + o(z) \tag{3.7a}$$

and naturally

$$A^+_{Q_z}(f) = A^+(f) \otimes 1 + \frac{z}{2} A^+(e^{-\beta H} f) \otimes 1 + z^{1/2} 1 \otimes A(\iota\,e^{-\beta H/2} f) + o(z). \tag{3.7b}$$

Thus, in the canonical representation (3.4), the interaction $A^+_{Q_z}(f) A_{Q_z}(g)$ takes the form

$$A^+(f)A(g) \otimes 1 + z^{1/2}(A^+(f) \otimes A^+(\iota\,e^{-\beta H/2} g) + A(g) \otimes A(\iota\,e^{-\beta H/2} f))$$
$$+ \frac{z}{2}(A^+(f)A(e^{-\beta H} g) + A(g)A^+(e^{-\beta H} f)) \otimes 1 + o(z^{3/2}) \tag{3.8}$$

where one recognizes:

(i) the *purely Fock space* term $A^+(f)A(g) \otimes 1$, which is *independent of* z;

(ii) the $z^{1/2}$-terms, which are typical WCL terms;

(iii) the z-terms, which give very simple contribution in the limit;

(iv) the $o(z^{3/2})$ term, which (as has been proved in [31]) tends to zero as $z \to 0$.

The fact that the term $A^+(f)A(g) \otimes 1$ acts only on the Fock space is qualitatively new with respect to the WCL case because this term corresponds to a finite interaction, i.e. not vanishing for $z \to 0$. These considerations led us to study the interaction

$$i(D^+ \otimes A^+(f)A(g) - D \otimes A^+(g)A(f)) \tag{3.9}$$

in the Fock space as a first step towards the understanding of the full low-density interaction (2.7). Motivated by the analogy with the WCL we considered *the same* *collective coherent vectors* as in the WCL, i.e. we study the limit as $\lambda := z^{1/2} \to 0$ of the matrix elements

$$\left\langle u \otimes W\left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f \, \mathrm{d}u\right)\Phi,\, U_{t/\lambda^2} v \otimes W\left(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' \, \mathrm{d}u\right)\Phi\right\rangle. \tag{3.10}$$

Generalizing of the techniques developed for the WCL, in [28] it was proved that the limit (3.10), as $\lambda \to 0$, exists for all $t \in \mathbb{R}$ and satisfies a QSDE driven by a pure number process (cf theorem II of [36]).

Later results (cf [31]) showed, however, that the natural extension of the above-mentioned procedure to the finite-temperature case led to a breakdown of unitarity in the limit evolution.

An analogous pathological result had already appeared in the nonlinear WCL case [27].

Motivated by Frigerio's conjecture (*the form of the collective vectors should depend on the interaction*) we investigated in [35] the WCL of a boson model with a quadratic interaction. In such a case the use of *nonlinear coherent vectors* turned out to be too complicated, but another technique, developed to handle the fermion WCL case and based on *collective number vectors* [23], turned out to be applicable and we were able to solve the problem with the introduction of *nonlinear collective number vectors* defined in terms not of the creation operators but on their squares, e.g.

$$\lambda \int_{S/\lambda^2}^{T/\lambda^2} A(S_u f)^2 \, \mathrm{d}u. \tag{3.11}$$

The decomposition (3.8), together with first-order perturbation theory, suggests the following choice of the collective fields:

$$A^+_{0,t}(\lambda; g_0, g_1) := \lambda \int_0^{t/\lambda^2} \mathrm{d}s (A^+(S_s Q_+ g_0) \otimes A^+(\iota S_s Q_- g_1) + A(S_s Q_+ g_1) \otimes A(\iota S_s Q_- g_0)) \tag{3.12}$$

and

$$A_{0,t}(\lambda; g_0, g_1) := \lambda \int_0^{t/\lambda^2} \mathrm{d}s (A^+(S_s Q_+ g_1) \otimes A^+(\iota S_s Q_- g_0) + A(S_s Q_+ g_0) \otimes A(\iota S_s Q_- g_1)). \tag{3.13}$$

However, a simple computation shows that, as $\lambda \to 0$, the two-point function

$$\langle \Phi_F \otimes \Phi'_F, A_{0,t}(\lambda; g_0, g_1) A^+_{0,t}(\lambda; g'_0, g'_1) \Phi_F \otimes \Phi'_F \rangle \tag{3.14}$$

tends to

$$t \int_{-\infty}^{\infty} \langle g_0, S_t g'_0 \rangle \overline{\langle g_1, S_t \, \mathrm{e}^{-\beta\Delta/2} g'_1 \rangle} \, \mathrm{d}t. \tag{3.15}$$

Therefore, if we replace (3.12), (3.13) by the simpler expressions

$$A^+_{0,t}(\lambda; g_0, g_1) := \lambda \int_0^{t/\lambda^2} \mathrm{d}s \, A^+(S_s Q_+ g_0) \otimes A^+(\iota S_s Q_- g_1) \tag{3.16}$$

and its adjoint, then the corresponding two-point function tends to the same limit. The definition of the collective fields is suggested by (3.16) or, more generally, by

$$A^+_{S,T}(\lambda; f_0, f_1) := \lambda \int_{S/\lambda^2}^{T/\lambda^2} \mathrm{d}s \, A^+(S_s Q_+ f_0) \otimes A^+(\iota S_s Q_- f_1) \tag{3.17}$$

and

$$A_{S,T}(\lambda; f_0, f_1) := (A^+_{S,T}(\lambda; f_0, f_1))^* = \lambda \int_{S/\lambda^2}^{T/\lambda^2} \mathrm{d}s\, A(S_s Q_+ f_0) \otimes A(\iota S_s Q_- f_1) \tag{3.18}$$

where f_0, f_1 are arbitrary elements of K not necessarily satisfying condition (2.33).

Given this choice of the *collective fields* we define the *collective nonlinear number vectors* by

$$A^+_{S_1,T_1}(\lambda; f_{0,1}, f_{1,1}) A^+_{S_2,T_2}(\lambda; f_{0,2}, f_{1,2}) \ldots A^+_{S_N,T_N}(\lambda; f_{0,N}, f_{1,N}) \Phi_F \otimes \Phi^\iota_F$$
$$=: \prod_{h=1}^{N} A^+_{S_h,T_h}(\lambda; f_{0,h}, f_{1,h}) \Phi_F \otimes \Phi^\iota_F. \tag{3.19}$$

Of course the operators (3.17), (3.18) are not true creation and annihilation operators nor are the vectors (3.19) true number vectors. However, a simple corollary of theorem 3.1 below, whose precise statement will not be spelled out here, implies that *in the limit* they do behave respectively as creation and annihilation operators and number vectors. Starting from these collective number vectors and with the same argument as lemma 2.2 of [35], one can prove the following theorem which shows that the collective number vectors, defined with the operators $A^\pm(\lambda, f_0, f_2)$, converge to the corresponding number vectors of a quantum Brownian motion. This quantum Brownian motion takes values in a Hilbert space $K_{0,1}$ (see below) which is of high physical interest because *it contains all the information on the microscopic model, which is preserved under passage to the limit.* As shown by the form (3.23), of its scalar product, the space $K_{0,1}$ is defined in a highly non-trivial way by the physics of the problem, in particular by:

(i) the one-particle reservoir dynamics;
(ii) the characteristic frequencies of the system;
(iii) the inverse temperature β;
(iv) the interaction (via the choice of the coherent vectors).

Finally, notice that, even if we start from an equilibrium state at inverse temperature β, in the limit we obtain a *Fock quantum Brownian motion* and not a finite-temperature quantum Brownian motion (as was the case in the WCL). Also, this is physically reasonable because we are considering the LDL.

The following theorem determines the quantum Brownian motion of the LDL corresponding to the physical characteristics (i)–(iv) listed above. Its proof follows from the arguments of the proof of lemma 2.2 in [35].

Theorem 3.1. For any $\varepsilon = 0, 1, \{S_h, T_h\}_{h=1}^{N}, \{S'_h, T'_h\}_{h=1}^{N'} \subset \mathbb{R}$ and

$$\{f_{\varepsilon,h}\}_{h=1}^{N} \subset K, \{f'_{\varepsilon,h}\}_{h=1}^{N'} \subset K \tag{3.20}$$

as $\lambda \to 0$, the limit of the scalar product

$$\left\langle \prod_{h=1}^{N} A^+_{S_h,T_h}(\lambda; f_{0,h}, f_{1,h}) \Phi_F \otimes \Phi^\iota_F, \prod_{h=1}^{N'} A^+_{S'_h,T'_h}(\lambda; f'_{0,h}, f'_{1,h}) \Phi_F \otimes \Phi^\iota_F \right\rangle \tag{3.21}$$

exists and is equal to

$$\left\langle \prod_{h=1}^{N} A^+(\chi_{[S_h,T_h]} \otimes (f_{0,h}, f_{1,h}))\Psi, \prod_{h=1}^{N'} A^+(\chi_{[S'_h,T'_h]} \otimes (f'_{0,h}, f'_{1,h}))\Psi \right\rangle \tag{3.22}$$

on $\Gamma(L^2(\mathbb{R})\otimes K_{0,1})$, Ψ is the vacuum of $\Gamma\,(\Gamma^2(\mathbb{R})\otimes K_{0,1})$ and $K_{0,1}$ is the Hilbert space defined as the completion of

$$K^{\circ}_{0,1} := \{(f_0, f_1) : f_\varepsilon \in K,\ \varepsilon = 0, 1\}$$

with the scalar product

$$\begin{aligned}
&((f_0,f_1)|(f_0',f_1'))\\
&\quad:= \int_{-\infty}^{\infty} \langle f_0, S_t f_0'\rangle \overline{\langle f_1, S_t\,\mathrm{e}^{-\beta\Delta/2} f_1'\rangle}\,\mathrm{d}t\\
&\quad= \lim_{\lambda\to 0}\Big\langle \Phi_{\mathrm{F}}\otimes\Phi_{\mathrm{F}}^{\iota}, \lambda\int_0^{1/\lambda^2} \mathrm{d}s\, A(S_s Q_+ f_0)\otimes A(\iota S_s Q_- f_1)\\
&\qquad\times\lambda\int_0^{1/\lambda^2} \mathrm{d}t\, A^+(S_t Q_+ f_0')\otimes A^+(\iota S_t Q_- f_1')\Phi_{\mathrm{F}}\otimes\Phi_{\mathrm{F}}^{\iota}\Big\rangle.
\end{aligned} \tag{3.23}$$

Remark. $K_{0,1}$ can be rewritten as $K\otimes_\beta K$, which denotes the algebraic tensor product $K\otimes K$ completed under the scalar product (3.23). By this notation, (3.22) can be rewritten as

$$\Big\langle \prod_{h=1}^{N} A^+(\chi_{[S_h,T_h]}\otimes f_{0,h}\otimes_\beta f_{1,h})\Psi, \prod_{h=1}^{N'} A^+(\chi_{[S_h',T_h']}\otimes f_{0,h}'\otimes_\beta f_{1,h}')\Psi\Big\rangle. \tag{3.24}$$

4. The low-density limit

Now we expand the basic dynamical variable U_{t/λ^2} in the iterated series and consider its matrix elements with respect to two collective nonlinear number vectors. The nth term of this expansion will have the form

$$\begin{aligned}
\Delta_n(\lambda, t) := \Big\langle u\otimes\prod_{h=1}^{N} A^+_{S_h,T_h}(\lambda; f_{0,h}, f_{1,h})\Phi_{\mathrm{F}}\otimes\Phi_{\mathrm{F}}^{\iota}, (-\mathrm{i})^n\int_0^{t/\lambda^2}\mathrm{d}t_1\ldots\int_0^{t_{n-1}}\mathrm{d}t_n\\
\times V(t_1)\ldots V(t_n)v\otimes\prod_{h=1}^{N'} A^+_{S_h',T_h'}(\lambda; f_{0,h}', f_{1,h}')\Phi_{\mathrm{F}}\otimes\Phi_{\mathrm{F}}^{\iota}\Big\rangle.
\end{aligned} \tag{4.1}$$

Let us first consider the case of $n = 1$ in order to illustrate the strategy of the proof. In the notation (4.1),

$$\begin{aligned}
\Delta_1(\lambda, t) = \Big\langle u\otimes\prod_{h=1}^{N} A^+_{S_h,T_h}(\lambda; f_{0,h}, f_{1,h})\Phi_{\mathrm{F}}\otimes\Phi_{\mathrm{F}}^{\iota}, \int_0^{t/\lambda^2}\mathrm{d}t_1\\
&\times\sum_{\varepsilon\in\{0,1\}} D_\varepsilon\otimes[A^+(S_{t_1}Q + g_\varepsilon)A(S_{t_1}Q_+ g_{1-\varepsilon})\otimes 1\\
&+\lambda(A^+(S_{t_1}Q_+ g_\varepsilon)\otimes A^+(\iota S_{t_1}Q_- g_{1-\varepsilon}) + A(S_{t_1}Q_+ g_{1-\varepsilon})\otimes A(\iota S_{t_1}Q_- g_\varepsilon))\\
&+\lambda^2 1\otimes A^+(\iota S_{t_1}Q_- g_{1-\varepsilon})A(\iota S_{t_1}Q_- g_\varepsilon)]\\
&\times v\otimes\prod_{h=1}^{N'} A^+_{S_h',T_h'}(\lambda; f_{0,h}', f_{1,h}')\Phi_{\mathrm{F}}\otimes\Phi_{\mathrm{F}}^{\iota}\Big\rangle\\
:= \Delta_1(1,\lambda,t) + \Delta_1(2,\lambda,t) + \Delta_1(3,\lambda,t) + \Delta_1(4,\lambda,t).
\end{aligned} \tag{4.2}$$

The same argument as in theorem 4.3 of [31] shows that

$$\Delta_1(4, \lambda, t) \to 0. \tag{4.3}$$

A simple adaptation of the arguments of theorem 3.4 of [28], required by the fact that here we use the collective number vectors rather than the collective coherent vectors, shows that the quantity

$$\Delta_1(1,\lambda,t)=\sum_{\varepsilon\in\{0,1\}}\Big\langle u\otimes\prod_{h=1}^{N}A^{+}_{S_h,T_h}(\lambda;f_{0,h},f_{1,h})\Phi_{\mathrm F}\otimes\Phi^{\iota}_{\mathrm F},$$

$$\times\int_0^{t/\lambda^2}\mathrm dt_1 1\otimes A^{+}(S_{t_1}Q_+g_\varepsilon)A(S_{t_1}Q_+g_{1-\varepsilon})\otimes 1$$

$$\times D_\varepsilon v\otimes\prod_{h=1}^{N'}A^{+}_{S'_h,T'_h}(\lambda;f'_{0,h},f'_{1,h})\Phi_{\mathrm F}\otimes\Phi^{\iota}_{\mathrm F}\Big\rangle$$

$$=\sum_{\varepsilon\in\{0,1\}}\sum_{l_1=1}^{N}\sum_{l_2=1}^{N'}\Big\langle u\otimes\prod_{1\le h\le N,h\ne l_1}A^{+}_{S_h,T_h}(\lambda;f_{0,h},f_{1,h})\Phi_{\mathrm F}\otimes\Phi^{\iota}_{\mathrm F},$$

$$\times D_\varepsilon v\otimes\prod_{1\le h\le N',h\ne l_2}A^{+}_{S'_h,T'_h}(\lambda;f'_{0,h},f'_{1,h})\Phi_{\mathrm F}\otimes\Phi^{\iota}_{\mathrm F}\Big\rangle$$

$$\times\lambda^2\int_{S_{l_1}/\lambda^2}^{T_{l_1}/\lambda^2}\mathrm du_{l_1}\int_{S'_{l_2}/\lambda^2}^{T'_{l_2}/\lambda^2}\mathrm dv_{l_2}\int_0^{t/\lambda^2}\mathrm dt_1\langle S_{u_{l_1}}Q_+f_{0,l_1},S_{t_1}Q_+g_\varepsilon\rangle$$

$$\times\langle S_{t_1}Q_+g_{1-\varepsilon},S_{v_{l_2}}Q_+f'_{0,l_2}\rangle\overline{\langle S_{u_{l_1}}Q_-f_{1,l_1}S_{v_{l_2}}Q_-f'_{1,l_2}\rangle}$$

converges, as $\lambda\to0$, to

$$\sum_{\varepsilon\in\{0,1\}}\sum_{l_1=1}^{N}\sum_{l_2=1}^{N'}\Big\langle u\otimes\prod_{1\le h\le N,h\ne l_1}A^{+}(\chi_{[S_h,T_h]}\otimes f_{0,h}\otimes_\beta f_{1,h})\Psi,\ D_\varepsilon v$$

$$\otimes\prod_{1\le h\le N',h\ne l_2}A^{+}(\chi_{[S'_h,T'_h]}\otimes f'_{0,h}\otimes_\beta f'_{1,h})\Psi\Big\rangle\int_0^t\mathrm dt_1\,\chi_{[S_{l_1},T_{l_1}]}(t_1)\chi_{[S'_{l_2},T'_{l_2}]}(t_1)$$

$$\times\int_{-\infty}^{\infty}\mathrm du_{l_1}\int_{-\infty}^{\infty}\mathrm dv_{l_2}\langle S_{u_{l_1}}f_{0,l_1},g_\varepsilon\rangle\langle g_{1-\varepsilon},S_{v_{l_2}}f'_{0,l_2}\rangle\overline{\langle S_{u_{l_1}}f_{1,l_1},S_{v_{l_2}}\mathrm e^{+\beta\Delta/2}f'_{1,l_2}\rangle}.\tag{4.4}$$

If we define

$$T_{g,1}(\varepsilon)(f_0\otimes_\beta f_1):=\int_{-\infty}^{\infty}\mathrm du\langle g_{1-\varepsilon},S_uf_0\rangle g_\varepsilon\otimes_\beta S_uf_1\tag{4.5}$$

then the right-hand side of (4.4) can be written, in terms of the limiting Fock space of theorem 3.1, as

$$\sum_{\varepsilon\in\{0,1\}}\Big\langle u\otimes\prod_{1\le h\le N}A^{+}(\chi_{[S_h,T_h]}\otimes f_{0,h}\otimes_\beta f_{1,h})\Psi,\int_0^t D_\varepsilon\otimes\mathrm dN(\chi_{[0,t_1]}\otimes T_{g,1}(\varepsilon))$$

$$\times v\otimes\prod_{1\le h\le N'}A^{+}(\chi_{[S'_h,T'_h]}\otimes f'_{0,h}\otimes_\beta f'_{1,h})\Psi\Big\rangle\tag{4.6}$$

i.e. as a matrix element of a stochastic integral with respect to the number process on that Fock space.

It follows from the arguments of section 4 of [35] ('cross-terms' give only zero contribution) and from theorem 5.2 of [35] that

$$\Delta_1(2,\lambda,t)=\sum_{\varepsilon\in\{0,1\}}\left\langle u\otimes\prod_{h=1}^{N}A^{+}_{S_h,T_h}(\lambda;f_{0,h},f_{1,h})\Phi_F\otimes\Phi_F^{\iota},\lambda\int_0^{t/\lambda^2}dt_1\,A^{+}(S_{t_1}Q_{+}g_{\varepsilon})\right.$$
$$\left.\otimes A^{+}(\iota S_{t_1}Q_{-}g_{1-\varepsilon})D_{\varepsilon}v\otimes\prod_{h=1}^{N'}A^{+}_{S'_h,T'_h}(\lambda;f'_{0,h},f'_{1,h})\Phi_F\otimes\Phi_F^{\iota}\right\rangle$$

converges to

$$\sum_{\varepsilon\in\{0,1\}}\left\langle u\otimes\prod_{h=1}^{N}A^{+}(\chi_{[S_h,T_h]}\otimes f_{0,h}\otimes_{\beta}f_{1,h})\Psi,\int_0^{t}D_{\varepsilon}\otimes dA^{+}(\chi_{[0,t_1]}\otimes g_{\varepsilon}\otimes_{\beta}g_{1-\varepsilon})\right.$$
$$\left.\times v\otimes\prod_{h=1}^{N'}A^{+}(\chi_{[S'_h,T'_h]}\otimes f'_{0,h}\otimes_{\beta}f'_{1,h})\Psi\right\rangle \tag{4.7}$$

which is a matrix element of a stochastic integral with respect to the creation operator in the given Fock space. Similarly the quantity

$$\Delta_1(3,\lambda,t)=\sum_{\varepsilon\in\{0,1\}}\left\langle u\otimes\prod_{h=1}^{N}A^{+}_{S_h,T_h}(\lambda;f_{0,h},f_{1,h})\Phi_F\otimes\Phi_F^{\iota},\lambda\int_0^{t/\lambda^2}dt_1\,A(S_{t_1}Q_{+}g_{\varepsilon})\right.$$
$$\left.\otimes A(\iota S_{t_1}Q_{-}g_{1-\varepsilon})D_{1-\varepsilon}v\otimes\prod_{h=1}^{N'}A^{+}_{S'_h,T'_h}(\lambda;f'_{0,h},f'_{1,h})\Phi_F\otimes\Phi_F^{\iota}\right\rangle$$

converges to the matrix element of a stochastic integral with respect to a number operator, namely

$$\sum_{\varepsilon\in\{0,1\}}\left\langle u\otimes\prod_{h=1}^{N}A^{+}(\chi_{[S_h,T_h]}\otimes f_{0,h}\otimes_{\beta}f_{1,h})\Psi,\int_0^{t}dA(\chi_{[0,t_1]}\otimes g_{\varepsilon}\otimes_{\beta}g_{1-\varepsilon})\right.$$
$$\left.\times D_{1-\varepsilon}v\prod_{h=1}^{N'}A^{+}(\chi_{[S'_h,T'_h]}\otimes f'_{0,h}\otimes_{\beta}f'_{1,h})\Psi\right\rangle. \tag{4.8}$$

Notice that all the matrix elements are with respect to *the same pair* of number vectors in the Fock space of theorem 3.1. This gives an idea of how in the LDL the creation, annihilation and number processes arise.

In the general case, in order to study the limit of the expressions (4.1), we consider the following three types of terms, since the other terms are negligible in the limit: *pure Fock* LDL, *the* WCL *terms and the interacting terms.*

The basic idea is that the pure Fock LDL terms give rise to the number process; the WCL terms to the creation and annihilation processes and the interacting terms also to the creation and annihilation processes: this comes from the Ito product of the number and creation (annihilation) processes which gives the creation (annihilation) process. Moreover, the limits of the general terms, arising from products of terms of different groups, will give rise to the products of the corresponding stochastic differentials which are dealt with by the Ito table (cf [15]). More precisely, the same arguments as in section 4) of [31] show that one needs only to consider the following types of terms:

The dN terms (pure Fock LDL)

$$\left\langle u\otimes\prod_{h=1}^{N}A^{+}_{S_h,T_h}(\lambda;f_{0,h},f_{1,h})\Phi_F\otimes\Phi_F^{\iota},\int_0^{t/\lambda^2}dt_1\right.$$
$$\times\int_0^{t_1}dt_2\ldots\int_0^{t_{n-1}}dt_n\,A^{+}(S_{t_1}Q_{+}g_{\varepsilon(1)})A(S_{t_1}Q_{+}g_{1-\varepsilon(1)})\otimes 1$$

$$\times A^{+}(S_{t_2}Q_{+}g_{\varepsilon(2)})A(S_{t_2}Q_{+}g_{1-\varepsilon(2)})\otimes 1\ldots$$
$$\times A^{+}(S_{t_n}Q_{+}g_{\varepsilon(n)})A(S_{t_n}Q_{+}g_{1-\varepsilon(n)})\otimes 1$$
$$\times D_{\varepsilon(1)}\ldots D_{\varepsilon(n)}v\otimes\prod_{h=1}^{N'}A^{+}_{S'_h,T'_h}(\lambda;f'_{0,h},f'_{1,h})\Phi_{\mathrm{F}}\otimes\Phi^{\iota}_{\mathrm{F}}\Big\rangle. \qquad (4.9)$$

The $\mathrm{d}A^{\pm}$ terms (pure weak coupling)

$$\Big\langle u\otimes\prod_{h=1}^{N}A^{+}_{S_h,T_h}(\lambda;f_{0,h},f_{1,h})\Phi_{\mathrm{F}}\otimes\Phi^{\iota}_{\mathrm{F}},\lambda^{n}\int_0^{t/\lambda^2}\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\ldots\int_0^{t_{n-1}}\mathrm{d}t_n\,A^{\varepsilon(1)}(S_{t_1}Q_{+}g_{\sigma(1)})$$
$$\otimes A^{\varepsilon(1)}(\iota S_{t_1}Q_{-}g_{1-\sigma(1)})\ldots A^{\varepsilon(n)}(S_{t_n}Q_{+}g_{\sigma(n)})\otimes A^{\varepsilon(n)}(\iota S_{t_n}Q_{-}g_{1-\sigma(n)})$$
$$\times D_{\sigma_\varepsilon(1)}\ldots D_{\sigma_\varepsilon(n)}v\otimes\prod_{h=1}^{N'}A^{+}_{S'_h,T'_h}(\lambda;f'_{0,h},f'_{1,h})\Phi_{\mathrm{F}}\otimes\Phi^{\iota}_{\mathrm{F}}\Big\rangle \qquad (4.10)$$

where

$$\sigma_\varepsilon(h):=\begin{cases}\sigma(h) & \text{if } \varepsilon(h)=1\\ 1-\sigma(h) & \text{if } \varepsilon(h)=0.\end{cases} \qquad (4.11)$$

The $\mathrm{d}t$ terms (weak coupling-Fock LFL-weak coupling)

$$\Big\langle u\otimes\prod_{h=1}^{N}A^{+}_{S_h,T_h}(\lambda;f_{0,h},f_{1,h})\Phi_{\mathrm{F}}\otimes\Phi^{\iota}_{\mathrm{F}},\lambda^{2}$$
$$\times\int_0^{t/\lambda^2}\mathrm{d}t_0\int_0^{t_0}\mathrm{d}t_1\ldots\int_0^{t_{2n-1}}\mathrm{d}t_{2n}\int_0^{t_{2n}}\mathrm{d}t_{2n+1}\,A(S_{t_0}Q_{+}g_{\varepsilon})\otimes A(\iota S_{t_0}Q_{-}g_{1-\varepsilon})$$
$$\times A^{+}(S_{t_1}Q_{+}g_{\varepsilon})A(S_{t_1}Q_{+}g_{1-\varepsilon})A^{+}(S_{t_2}Q_{+}g_{1-\varepsilon})A(S_{t_2}Q_{+}g_{\varepsilon})\otimes 1\ldots$$
$$\times A^{+}(S_{t_{2n-1}}Q_{+}g_{\varepsilon})A(S_{t_{2n-1}}Q_{+}g_{1-\varepsilon})A^{+}(S_{t_{2n}}Q_{+}g_{1-\varepsilon})A(S_{t_{2n}}Q_{+}g_{\varepsilon})\otimes 1$$
$$\times A^{+}(S_{t_{2n+1}}Q_{+}g_{\varepsilon})\otimes A^{+}(\iota S_{t_{2n+1}}Q_{-}g_{1-\varepsilon})$$
$$\times(D_{1-\varepsilon}D_{\varepsilon})^{n+1}v\otimes\prod_{h=1}^{N'}A^{+}_{S'_h,T'_h}(\lambda;f'_{0,h},f'_{1,h})\Phi_{\mathrm{F}}\otimes\Phi^{\iota}_{\mathrm{F}}\Big\rangle. \qquad (4.12)$$

The $\mathrm{d}A\,\mathrm{d}N$ terms (weak coupling-Fock LDL)

$$\Big\langle u\otimes\prod_{h=1}^{N}A^{+}_{S_h,T_h}(\lambda;f_{0,h},f_{1,h})\Phi_{\mathrm{F}}\otimes\Phi^{\iota}_{\mathrm{F}},\lambda\int_0^{t/\lambda^2}\mathrm{d}t_0\int_0^{t_0}\mathrm{d}t_1\ldots$$
$$\times\int_0^{t_{n-1}}\mathrm{d}t_n\,A(S_{t_0}Q_{+}g_{\varepsilon})\otimes A(\iota S_{t_0}Q_{-}g_{1-\varepsilon})$$
$$\times A^{+}(S_{t_1}Q_{+}g_{\varepsilon(1)})A(S_{t_1}Q_{+}g_{1-\varepsilon(1)})$$
$$\times A^{+}(S_{t_2}Q_{+}g_{\varepsilon(2)})A(S_{t_2}Q+g_{1-\varepsilon(2)})\ldots A^{+}(S_{t_{n-1}}Q_{+}g_{\varepsilon(n-1)})$$
$$\times A(S_{t_{n-1}}Q_{+}g_{1-\varepsilon(n-1)})$$
$$\times A^{+}(S_{t_n}Q_{+}g_{\varepsilon(n)})A(S_{t_n}Q_{+}g_{1-\varepsilon(n)})\otimes 1$$
$$\times D_{1-\varepsilon}D_{\varepsilon(1)}\ldots D_{\varepsilon(n)}v\otimes\prod_{h=1}^{N'}A^{+}_{S'_h,T'_h}(\lambda;f'_{0,h},f'_{1,h})\Phi_{\mathrm{F}}\otimes\Phi^{\iota}_{\mathrm{F}}\Big\rangle. \qquad (4.13)$$

The $\mathrm{d}N\,\mathrm{d}A^+$ terms (Fock LDL-weak coupling)

$$\left\langle u\otimes\prod_{h=1}^{N}A^+_{S_h,T_h}(\lambda;f_{0,h},f_{1,h})\Phi_F\otimes\Phi_F^{\iota},\lambda\int_0^{t/\lambda^2}\mathrm{d}t_1\ldots\int_0^{t_{n-1}}\mathrm{d}t_n\int_0^{t_n}\mathrm{d}s\,A^+(S_{t_1}Q_+g_{\varepsilon(1)})\right.$$
$$\times A(S_{t_1}Q_+g_{1-\varepsilon(1)})A^+(S_{t_2}Q_+g_{\varepsilon(2)})A(S_{t_2}Q_+g_{1+\varepsilon(n)})\otimes 1\ldots$$
$$\times A^+(S_{t_{n-1}}Q_+g_{\varepsilon(n-1)})A(S_{t_{n-1}}Q_+g_{1-\varepsilon(n+1)})A^+(S_{t_n}Q_+g_{\varepsilon(n)})$$
$$\times A(S_{t_n}Q_+g_{1-\varepsilon(n)})\otimes 1$$
$$\times A^+(S_sQ_+g_\varepsilon)\otimes A^+(S_sQ_-g_{1-\varepsilon})D_{\varepsilon(1)}\ldots$$
$$\left.\times D_{\varepsilon(n)}D_\varepsilon v\otimes\prod_{h=1}^{N'}A^+_{S'_h,T'_h}(\lambda;f'_{0,h},f'_{1,h})\Phi_F\otimes\Phi_F^{\iota}\right\rangle. \tag{4.14}$$

Moreover, combining the uniform estimate arguments in [27, 28, 31, 36], we have the following result:

Theorem 4.1. If the test functions g_0, g_1 and the operator D satisfy the condition (2.51) then there exists a $t_0>0$ such that for each $t\in[0,t_0)$ we can take the limit (2.23) term by term, where U_t is expanded in the form (2.15).

5. The stochastic differential equation

Having understood which terms of the iterated series contribute, in the limit, to define which type of stochastic integrals, the next step is to find the explicit form of the quantum stochastic differential equation.

Now the situation is not the same as in the WCL. Recall that in that case, the $\mathrm{d}A$ and $\mathrm{d}A^+$ terms come exactly from $V(t_1)$, while from $V(t_1)\ldots V(t_n)(n\geqslant 2)$ we can only obtain the products of $\mathrm{d}t_j$, $\mathrm{d}A(t_k)$ and $\mathrm{d}A^+(t_h)$ (of course corresponding to different time intervals $\mathrm{d}t_j$, $\mathrm{d}t_k$ and $\mathrm{d}t_h$) and, moreover, the $\mathrm{d}t$ term comes only from $V(t_1)V(t_2)$. The present situation is similar to the situation of the LDL Fock case: for each $n\geqslant 1$, from $V(t_1)\ldots V(t_n)$ we can get the $\mathrm{d}A$, $\mathrm{d}A^+$ and $\mathrm{d}N$ terms, and for each $n=2,4,6,\ldots$ we can get a $\mathrm{d}t$ term. More precisely, let us illustrate this idea with some computation on (4.10), (4.12), (4.13) and (4.14).

Terms of type (4.10), i.e. the $\mathrm{d}N$ terms, are controlled as in [28] (LDL Fock case). Bringing the product of annihilation and creation operators in (4.12) to the normally ordered form and by the same argument as in [35], we find that, in the limit $\lambda\to 0$, (4.12) tends to

$$\left\langle u\otimes\prod_{1\leqslant h\leqslant N}A^+(\chi_{[S_h,T_h]}\otimes f_{0,h}\otimes_\beta f_{1,h})\Psi,\int_0^t\mathrm{d}t_0\int_{-\infty}^0\mathrm{d}t_1\int_{-\infty}^0\mathrm{d}t_2\ldots\int_{-\infty}^0\mathrm{d}t_{2n}\int_{-\infty}^0\mathrm{d}t_{2n+1}\right.$$
$$\times\langle g_\varepsilon,S_{t_1}g_\varepsilon\rangle\langle g_{1-\varepsilon},S_{t_2}g_{1-\varepsilon}\rangle\langle g_\varepsilon,S_{t_3}g_\varepsilon\rangle\ldots\langle g_{1-\varepsilon},S_{t_{2n}}g_{1-\varepsilon}\rangle\langle g_\varepsilon,S_{t_{2n+1}}g_\varepsilon\rangle$$
$$\times\overline{\langle g_{1-\varepsilon},S_{t_1+t_2+\ldots+t_{2n}+t_{2n+1}}\,\mathrm{e}^{-\beta\Delta/2}g_{1-\varepsilon}\rangle}$$
$$\left.\times(D_{1-\varepsilon}D_\varepsilon)^{n+1}v\otimes\prod_{1\leqslant h\leqslant N'}A^+(\chi_{[S'_h,T'_h]}\otimes f'_{0,h}\otimes_\beta f'_{1,h})\Psi\right\rangle. \tag{5.1}$$

Bringing the product of annihilation and creation operators in (4.13) to the normally ordered form, we obtain may terms, among which only one can produce a limit of the form of $\int_0^t dA(\ldots)$, which is

$$\sum_{l_2=1}^{N'} \Big\langle u \otimes \prod_{h=1}^{N} A^+_{S_h,T_h}(\lambda; f_{0,h}, f_{1,h})\Phi_F \otimes \Phi_F^{\iota}, \lambda^2 \int_0^{t/\lambda^2} dt_0 \int_0^{t_0} dt_1 \ldots \int_0^{t_{n-1}} dt_n \langle S_{t_0} Q_+ g_\varepsilon, S_{t_1} Q_+ g_{\varepsilon(1)}\rangle$$
$$\times \langle S_{t_1} Q_+ g_{1-\varepsilon(1)}, S_{t_2} Q_+ g_{\varepsilon(2)}\rangle \ldots \langle S_{t_{n-1}} Q_+ g_{1-\varepsilon(n-1)}, S_{t_n} Q_+ g_{\varepsilon(n)}\rangle$$
$$\times \int_{S'_{l_2}/\lambda^2}^{T'_{l_2}/\lambda^2} dv_{l_2} \langle S_{t_n} Q + g_{1-\varepsilon(n)}, S_{v_{l_2}} Q_+ f'_{0,l_2}\rangle$$
$$\times \langle S_{t_0} Q_- g_{1-\varepsilon}, S_{v_{l_2}} Q_- f'_{1,l_2}\rangle_\iota$$
$$\times D_{1-\varepsilon} D_{\varepsilon(1)} \ldots D_{\varepsilon(n)} v \otimes \prod_{1\le h\le N', h\ne l_2} A^+_{S'_h,T'_h}(\lambda; f'_{0,h}, f'_{1,h})\Phi_F \otimes \Phi_F^{\iota}\Big\rangle. \quad (5.2)$$

The same arguments as [21] (or [28], [31] or [35]) imply that (5.2) tends to

$$\sum_{l_2=1}^{N'} \Big\langle u \otimes \prod_{h=1}^{N} A^+(\chi_{[S_h,T_h]} \otimes f_{0,h} \otimes_\beta f_{1,h})\Psi, \int_0^t dt_0\, \chi_{[S'_l,T'_{l_2}]}(t_0) \int_{-\infty}^0 dt_1 \ldots \int_{-\infty}^0 dt_n$$
$$\times \langle g_\varepsilon, S_{t_1} g_{\varepsilon(1)}\rangle \langle g_{1-\varepsilon(1)}, S_{t_2} g_{\varepsilon(2)}\rangle \ldots \langle g_{1-\varepsilon(n-1)}, S_{t_n} g_{\varepsilon(n)}\rangle$$
$$\times \int_{-\infty}^{\infty} dv_{l_2} \langle g_{1-\varepsilon(n)}, S_{v_{l_2}} f'_{0,l_2}\rangle \overline{\langle g_{1-\varepsilon}, S_{v_{l_2}+t_n+\ldots+t_1} e^{-\beta\Delta/2} f'_{1,l_2}\rangle}$$
$$\times D_{1-\varepsilon} D_{\varepsilon(1)} \ldots D_{\varepsilon(n)} v \otimes \prod_{1\le h\le N', h\ne l_2} A^+(\chi_{[S'_h,T'_h]}(f'_{0,h} \otimes_\beta f'_{1,h}))\Big\rangle$$
$$= \Big\langle u \otimes \prod_{h=1}^{N} A^+(\chi_{[S_h,T_h]} \otimes f_{0,h} \otimes_\beta f_{1,h})\Psi, \int_0^t dA_s(\Pi^-_{n,\varepsilon,g})$$
$$\times (D_{1-\varepsilon} D_\varepsilon)^{(n+1)/2} (\chi_{2\mathbb{N}-1}(n) + \chi_{2\mathbb{N}}(n) D_{1-\varepsilon}) v$$
$$\otimes \prod_{1\le h\le N'} A^+(\chi_{[S'_h,T'_h]} \otimes f'_{0,h} \otimes_\beta f'_{1,h})\Psi\Big\rangle \quad (5.3)$$

where $\chi_{2\mathbb{N}}$ (or $\chi_{2\mathbb{N}-1}$) is the characteristic function on the even (or odd) natural integers and $\Pi^-_{n,\varepsilon,g}$ is a element of $K_{0,1}$

$$\Pi^-_{n,\varepsilon,g} := \int_{-\infty}^0 dt_1 \ldots \int_{-\infty}^0 dt_n \overline{\langle g_\varepsilon, S_{t_1} g_\varepsilon\rangle \langle g_{1-\varepsilon}, S_{t_2} g_{1-\varepsilon}\rangle} \ldots \overline{\langle g_{\varepsilon_n}, S_{t_n} g_{\varepsilon_n}\rangle}$$
$$\times g_{1-\varepsilon_n} \otimes_\beta S_{-(t_n+\ldots+t_1)} g_{1-\varepsilon} \quad (5.4)$$

and

$$\varepsilon_n := \begin{cases} 1-\varepsilon & \text{if } n \text{ is even} \\ \varepsilon & \text{if } n \text{ is odd.} \end{cases} \quad (5.4)$$

Similarly, from (4.14) one can only obtain one term in the form of $\int_0^t dA^+(\ldots)$, whose limit is

$$\Big\langle u\otimes\prod_{h=1}^{N} A^+(\chi_{[S_h,T_h]}\otimes f_{0,h}\otimes_\beta f_{1,h})\Psi,$$
$$\times\int_0^t dA_s^+(\Pi^+_{n,\varepsilon,g})(\chi_{2N-1}(n)+\chi_{2N}(n)D_\varepsilon)(D_{1-\varepsilon}d_\varepsilon)^{(n+1)/2}v$$
$$\otimes\prod_{1\leq h\leq N'} A^+[\chi_{[S_h',T_h']}\otimes f'_{0,h}\otimes_\beta f'_{1,h})\Psi\Big\rangle \tag{5.6}$$

where

$$\Pi^+_{n,\varepsilon,g}=\int_{-\infty}^0 dt_1\ldots\int_{-\infty}^0 dt_n\langle g_{\varepsilon_n}, S_{t_1}g_{\varepsilon_n}\rangle\langle g_{1-\varepsilon_n}, S_{t_2}g_{1-\varepsilon_n}\rangle$$
$$\ldots\langle g_\varepsilon, S_{t_n}g_\varepsilon\rangle g_{1-\varepsilon_n}\otimes_\beta S_{t_n+\ldots+t_1}g_{1-\varepsilon}. \tag{5.7}$$

The above discussions suggest the following:

Theorem 5.1. Let, in (2.23), Φ_λ, Φ'_λ, denote any pair of collective nonlinear number vectors of the form (3.19), and Φ, Φ' the corresponding vectors in the quantum Brownian motion space. Then, as $\lambda\to 0$, the limit of (2.23) exists and the operator $U(t)$, on the right-hand side of (2.23), is the unique unitary solution of quantum stochastic differential equation

$$U(t)=1+\sum_{\varepsilon\in\{0,1\}}\int_0^t\Big(\sum_{n=1}^\infty (D_{1-\varepsilon}D_\varepsilon)^n\otimes dA_s^+(\Pi^+_{2n-1,\varepsilon,g})$$
$$+\sum_{n=0}^\infty D_\varepsilon(D_{1-\varepsilon}D_\varepsilon)^n\otimes dA_s^+(\Pi^+_{2n,\varepsilon,g})$$
$$+\sum_{n=1}^\infty (D_{1-\varepsilon}D_\varepsilon)^n\otimes dA_s(\Pi^-_{2n-1,\varepsilon,g})+\sum_{n=0}^\infty (D_{1-\varepsilon}D_\varepsilon)^n D_{1-\varepsilon}\otimes dA_s(\Pi^-_{2n,\varepsilon,g})$$
$$+\sum_{n=1}^\infty (D_\varepsilon D_{1-\varepsilon})^n\otimes dN_s(T_{g,2n}(\varepsilon))$$
$$+\sum_{n=1}^\infty (D_\varepsilon D_{1-\varepsilon})^{n-1}D_\varepsilon\otimes dN_s(T_{g,2n-1}(\varepsilon))$$
$$+\sum_{n=0}^\infty ((g_\varepsilon, g_{1-\varepsilon})|\Pi^+_{2n,\varepsilon,g})_-(D_{1-\varepsilon}D_\varepsilon)^{n+1}\otimes 1\, ds\Big)U(s) \tag{5.8}$$

where by definition

$$T_{g,n}(\varepsilon)(f_0\otimes_\beta f_1)$$
$$:=\int_{-\infty}^\infty du\int_{-\infty}^0 dt_2\ldots\int_{-\infty}^0 dt_n\langle g_{1-\varepsilon}, S_{t_2}g_{1-\varepsilon}\rangle\langle g_\varepsilon, S_{t_3}g_\varepsilon\rangle\ldots\langle g_{\varepsilon_n}, S_{t_n}g_{\varepsilon_n}\rangle$$
$$\times\langle S_{t_2+\ldots+t_n}g_{1-\varepsilon_n}, S_u f_0\rangle g_\varepsilon\otimes_\beta S_u f_1 \tag{5.9}$$

$$(f_0\otimes_\beta f_1|f'_0\otimes_\beta f'_1)_-:=\int_{-\infty}^0 dt\langle f_0, S_t f_1\rangle\overline{\langle f'_0, S_t\, e^{-\beta\Delta/2}f'_1\rangle} \tag{5.10}$$

$$(f_0\otimes_\beta f_1|f'_0\otimes_\beta f'_1)_+:=\int_0^\infty dt\langle f_0, S_t f_1\rangle\overline{\langle f'_0, S_t\, e^{-\beta\Delta/2}f'_1\rangle} \tag{5.11}$$

and the vectors $\Pi^\pm$ are defined by (5.4) and (5.5).

Combining the techniques of [29] and the above arguments, we find

Theorem 5.2. In the same notation as in theorem 5.1, as $\lambda \to 0$, for each $X \in B(H_0)$, the limits (2.24), (2.25) exist and the unitary operators $U(t)$, on the right-hand sides of (2.24), (2.25), are precisely those defined by theorem 5.1.

6. Further discussion

We now rewrite (5.8) in a form giving physical insight and establish unitarity.

First of all notice that our limiting space $K_{0,1} = K \otimes_\beta K$ consist of elements $f_0 \otimes_\beta f_1$, and $B(K_{0,1})$ acts naturally on $K_{0,1}$ by

$$(A \otimes_\beta B)(f_0 \otimes_\beta f_1) = Af_0 \otimes_\beta Bf_1 \qquad \forall A, B \in B(K).$$

Proceeding as in [17] let us introduce the *energy* representation

$$S_E := \int_{-\infty}^{\infty} S_t \, e^{-iEt} \, dt \tag{6.1}$$

so that S_E is a linear map from K to K (more precisely, it is an operator-valued distribution) with the properties

$$S_E^* = S_E \tag{6.2}$$

$$S_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_E \, e^{-itE} \, dE. \tag{6.3}$$

Denote then

$$(f|g)(E) := \int_{-\infty}^{\infty} \langle f, S_t g\rangle \, e^{-itE} \, dt = \langle f, S_E g\rangle \tag{6.4a}$$

$$(f|g)_-(E) := \int_{-\infty}^{0} \langle f, S_t g\rangle \, e^{-itE} \, dt \tag{6.4b}$$

$$(f|g)_+(E) := \int_{0}^{\infty} \langle f, S_t g\rangle \, e^{-itE} \, dt. \tag{6.4c}$$

These definitions imply that

$$\overline{(f|g)(E)} = (g|f)(E) \qquad \overline{(f|g)_\pm(E)} = (g|f)_\mp(E). \tag{6.5}$$

Moreover

$$\begin{aligned} S_E S_{E'} &= \int_{-\infty}^{\infty} S_t \, e^{-itE} \, dt \int_{-\infty}^{\infty} S_s \, e^{-isE'} \, ds \\ &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds \, S_{t+s} \, e^{-i(t+s)E} \, e^{-is(E'-E)} \\ &= \int_{-\infty}^{\infty} d\tau \, S_\tau \, e^{-i\tau E} \int_{-\infty}^{\infty} ds \, e^{-is(E'-E)} = 2\pi \, \delta(E - E') S_E. \end{aligned} \tag{6.6}$$

Using the energy representation, the operators $T_{g,n}(\varepsilon)$ defined by (5.9) can be rewritten as

$$T_{g,2n}(\varepsilon)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}E(g_{1-\varepsilon}|g_{1-\varepsilon})_{-}(E)^{n}(g_{\varepsilon}|g_{\varepsilon})_{-}(E)^{n-1}|g_{\varepsilon}\rangle\langle g_{\varepsilon}|S_E\otimes_{\beta}S_E$$

$$:=\int_{-\infty}^{\infty}\mathrm{d}E\,T_{g,2n}(\varepsilon,E) \tag{6.7}$$

$$T_{g,2n-1}(\varepsilon)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}E(g_{1-\varepsilon}|g_{1-\varepsilon})_{-}(E)^{n-1}(g_{\varepsilon}|g_{\varepsilon})_{-}(E)^{n-1}|g_{\varepsilon}\rangle\langle g_{1-\varepsilon}|S_E\otimes_{\beta}S_E$$

$$:=\int_{-\infty}^{\infty}\mathrm{d}E\,T_{g,2n-1}(\varepsilon,N). \tag{6.8}$$

Their adjoints (notice that in general $(A\otimes_{\beta}B)^{*}=A^{*}\otimes_{\beta}B^{*}$ is *not* true) have the following forms:

$$T^{*}_{g,2n}(\varepsilon)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}E\overline{(g_{1-\varepsilon}|g_{1-\varepsilon})_{-}(E)^{n}(g_{\varepsilon}|g_{\varepsilon})_{-}(E)^{n-1}}|g_{\varepsilon}\rangle\langle g_{\varepsilon}|S_E\otimes_{\beta}S_E$$

$$:=\int_{-\infty}^{\infty}\mathrm{d}E\,T^{*}_{g,2n}(\varepsilon,E) \tag{6.9}$$

$$T^{*}_{g,2n-1}(\varepsilon)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}E\overline{(g_{1-\varepsilon}|g_{1-\varepsilon})_{-}(E)^{n-1}(g_{\varepsilon}|g_{\varepsilon})_{-}(E)^{n-1}}|g_{1-\varepsilon}\rangle\langle g_{\varepsilon}|S_E\otimes_{\beta}S_E$$

$$:=\int_{-\infty}^{\infty}\mathrm{d}E\,T^{*}_{g,2n-1}(\varepsilon,E). \tag{6.10}$$

Applying the energy representation to (5.4) and (5.7), we obtain

$$\Pi^{+}_{2n,\varepsilon,g}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}E(g_{1-\varepsilon}|g_{1-\varepsilon})_{-}(E)^{n}(g_{\varepsilon}|g_{\varepsilon})_{-}(E)^{n}g_{\varepsilon}\otimes_{\beta}S_E g_{1-\varepsilon}$$

$$:=\int_{-\infty}^{\infty}\mathrm{d}E\,\Pi^{+}_{2n,\varepsilon,g}(E) \tag{6.11}$$

$$\Pi^{+}_{2n-1,\varepsilon,g}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}E(g_{1-\varepsilon}|g_{1-\varepsilon})_{-}(E)^{n-1}(g_{\varepsilon}|g_{\varepsilon})_{-}(E)^{n}g_{1-\varepsilon}\otimes_{\beta}S_E g_{1-\varepsilon}$$

$$:=\int_{-\infty}^{\infty}\mathrm{d}E\,\Pi^{+}_{2n-1,\varepsilon,g}(E) \tag{6.12}$$

$$\Pi^{-}_{2n,\varepsilon,g}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}E\overline{(g_{1-\varepsilon}|g_{1-\varepsilon})_{-}(E)^{n}(g_{\varepsilon}|g_{\varepsilon})_{-}(E)^{n}}g_{\varepsilon}\otimes_{\beta}S_E g_{1-\varepsilon}$$

$$:=\int_{-\infty}^{\infty}\mathrm{d}E\,\Pi^{-}_{2n,\varepsilon,g}(E) \tag{6.13}$$

and

$$\Pi^{-}_{2n-1,\varepsilon,g}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}E\overline{(g_{1-\varepsilon}|g_{1-\varepsilon})_{-}(E)^{n-1}(g_{\varepsilon}|g_{\varepsilon})_{-}(E)^{n}}g_{1-\varepsilon}\otimes_{\beta}S_E g_{1-\varepsilon}$$

$$:=\int_{-\infty}^{\infty}\mathrm{d}E\,\Pi^{-}_{2n-1,\varepsilon,g}(E). \tag{6.14}$$

Now for each $g \in K$, denote g^- the orthogonal projection of g into the negative-energy spectral subspace of S_t. Then g^- is characterized by the property

$$(g^-|f)(E) = \int_{-\infty}^{0} dE \langle g, S_t f\rangle e^{-itE} = (g|f)_-(E) \qquad \forall E \in \mathbb{R}, f \in K. \tag{6.15}$$

Clearly if $\langle g, S_t f\rangle = 0, \forall t \in \mathbb{R}$, i.e. if f, g have disjoint energy supports, then $(g^-|f)(E) = (f|g^-)(E) = 0$. Finally let $\xi: \mathbb{R} \to K \otimes_\beta K$ denote the function

$$\xi(E) := \sum_{\varepsilon \in \{0,1\}} g_\varepsilon^- \otimes_\beta g_\varepsilon \frac{1}{(g_\varepsilon | g_\varepsilon)_-(E)}. \tag{6.16}$$

The role of the function ξ is explained by the following:

Theorem 6.1. For each $n \in \mathbb{N}$,

$$T_{g,2n}(\varepsilon, E)\xi(E) = \Pi^+_{2n-1,1-\varepsilon,g}(E) \tag{6.17a}$$

$$T_{g,2n-1}(\varepsilon, E)\xi(E) = \Pi^+_{2(n-1),\varepsilon,g}(E) \tag{6.17b}$$

$$T^*_{g,2n}(\varepsilon, E)\xi(E) = \Pi^+_{2n-1,1-\varepsilon,g}(E) \tag{6.17c}$$

$$T^*_{g,2n-1}(\varepsilon, E)\xi(E) = \Pi^+_{2(n-1),\varepsilon,g}(E) \tag{6.17d}$$

$$(\xi(E)| T_{g,2n-1}(\varepsilon, E)\xi(E)) = 0 \tag{6.18a}$$

$$(\xi(E)| T_{g,2n}(1-\varepsilon, E)\xi(E)) = (g_{1-\varepsilon}|g_{1-\varepsilon})_-(E)^n (g_\varepsilon|g_\varepsilon)_-(E)^n \tag{6.18b}$$

Proof. To prove (6.17a), by (6.7) and (6.16),

$$T_{g,2n}(\varepsilon, E)\xi(E) = \frac{1}{2\pi}(g_{1-\varepsilon}|g_{1-\varepsilon})_-(E)^n (g_\varepsilon|g_\varepsilon)_-(E)^{n-1}$$

$$|g_\varepsilon\rangle\langle g_\varepsilon| S_E \otimes_\beta S_E \sum_{\sigma\in\{0,1\}} g_\sigma^- \otimes_\beta g_\sigma \frac{1}{(g_\sigma|g_\sigma)_-(E)}$$

$$= \sum_{\sigma\in\{0,1\}} \frac{1}{2\pi}(g_{1-\varepsilon}|g_{1-\varepsilon})_-(E)^n (g_\varepsilon|g_\varepsilon)_-(E)^{n-1} |g_\varepsilon\rangle\langle g_\varepsilon|$$

$$\times S_E g_\sigma^- \otimes_\beta S_E g_\sigma \frac{1}{(g_\sigma|g_\sigma)_-(E)}$$

$$= \frac{1}{2\pi}(g_{1-\varepsilon}|g_{1-\varepsilon})_-(E)^n (g_\varepsilon|g_\varepsilon)_-(E)^{n-1} g_\varepsilon \otimes_\beta S_E g_\varepsilon (g_\varepsilon|S_E g_\varepsilon^-) \frac{1}{(g_\varepsilon|g_\varepsilon)_-(E)}$$

$$= \frac{1}{2\pi}(g_{1-\varepsilon}|g_{1-\varepsilon})_-(E)^n (g_\varepsilon|g_\varepsilon)_-(E)^{n-1} g_\varepsilon \otimes_\beta S_E g_\varepsilon. \tag{6.19a}$$

Comparing the right-hand side of (6.19) with (6.12), (6.17a) follows. Similarly, one obtains (6.17b, c, d).

To prove (6.18a) it is enough to notice that for each E in vector $\xi(E)$, defined by (6.16), the vector has either the subscript $(0,0)$ or $(1,1)$ and, by (6.8), the operator $T_{g,2n-1}(\varepsilon, E)$ is a scalar multiple of the operator $|g_\varepsilon\rangle\langle g_{1-\varepsilon}|S_E\otimes_\beta S_E$. Therefore, the vector $T_{g,2n-1}(\varepsilon, E)\xi(E)$ has either the subscript $(0,1)$ or $(1,0)$, and these facts imply (6.17a) because g_0 and g_1 have disjoint energy support.

Using similar arguments as in the proof of (6.17a) and (6.18a), one can easily get (6.18b).

Define

$$T_3(E):=\sum_{\varepsilon\in\{0,1\}}\sum_{n=1}^{\infty}((D_\varepsilon D_{1-\varepsilon})^n\otimes T_{g,2n}(\varepsilon,E)+(D_\varepsilon D_{1-\varepsilon})^{n-1}D_\varepsilon\otimes T_{g,2n-1}(\varepsilon,E)) \tag{6.19b}$$

$$T_3:=\int_{-\infty}^{\infty}\mathrm{d}E\,T_3(E)\qquad T_3^*:=\int_{-\infty}^{\infty}\mathrm{d}E\,T_3^*(E) \tag{6.19c}$$

where the reason of the introduction of the subscript 3 is explained in section 7). One important property of T_3, which is a consequence of (6.6), is

Lemma 6.2.

$$T_3T_3^*=T_3^*T_3=T_3+T_3^* \tag{6.20}$$

$$T_3\int_{-\infty}^{\infty}\mathrm{d}E\,T_3^*(E)\xi(E)=T_3^*\int_{-\infty}^{\infty}\mathrm{d}E\,T_3(E)\xi(E)=\int_{-\infty}^{\infty}\mathrm{d}E(T_3(E)+T_3^*(E))\xi(E). \tag{6.21}$$

Proof. We shall only prove $T_3T_3^*=T_3+T_3^*$ and other equalities follow from the same arguments.

The idea of the proof is similar to that used in theorem 7.1 (the unitarity) in [28]. let us introduce the notation

$$T_\varepsilon(E):=D_\varepsilon D_{1-\varepsilon}(g_\varepsilon|g_\varepsilon)_-(E)(g_{1-\varepsilon}|g_{1-\varepsilon})_-(E) \tag{6.22}$$

then, (6.7)-(6.10) imply that

$$T_3(E)=\frac{1}{2\pi}\sum_{\varepsilon\in\{0,1\}}\left(\frac{T_\varepsilon(E)}{(g_\varepsilon|g_\varepsilon)_-(E)(1-T_\varepsilon(E))}\otimes|g_\varepsilon\rangle\langle g_\varepsilon|S_E\otimes_\beta S_E\right.$$
$$\left.+\frac{1}{1-T_\varepsilon(E)}D_\varepsilon\otimes|g_\varepsilon\rangle\langle g_{1-\varepsilon}|S_E\otimes_\beta S_E\right) \tag{6.23a}$$

and

$$T_3^*(E)=\frac{1}{2\pi}\sum_{\varepsilon\in\{0,1\}}\left(\frac{T_\varepsilon^*(E)}{\overline{(g_\varepsilon|g_\varepsilon)_-(E)}(1-T_\varepsilon^*(E))}\otimes|g_\varepsilon\rangle\langle g_\varepsilon|S_E\otimes_\beta S_E\right.$$
$$\left.-D_{1-\varepsilon}\cdot\frac{1}{1-T_\varepsilon^*(E)}\otimes|g_{1-\varepsilon}\rangle\langle g_\varepsilon|S_E\otimes_\beta S_E\right) \tag{6.23b}$$

(notice again that in general $(A\otimes_\beta B)^*\neq A^*\otimes_\beta B^*$). As in [28] the crucial remark is that $T_\varepsilon(E)$ is a *normal operator*, i.e. $T_\varepsilon(E)T_\varepsilon^*(E)=T_\varepsilon^*(E)T_\varepsilon(E)$. This implies in particular that

$$T_3T_3^* = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{d}E \int_{-\infty}^{\infty} \mathrm{d}E' \sum_{\varepsilon\in\{0,1\}} \Bigg((g_\varepsilon|g_\varepsilon)(E)$$

$$\times \frac{T_\varepsilon(E)T_\varepsilon^*(E')}{(g_\varepsilon|g_\varepsilon)_-(E)\overline{(g_\varepsilon|g_\varepsilon)_-(E')}(1-T_\varepsilon(E))(1-T_2^*(E'))} \otimes |g_\varepsilon\rangle\langle g_\varepsilon|S_{E'}\otimes_\beta S_{E'}S_E$$

$$+\frac{1}{1-T_\varepsilon(E)} D_\varepsilon \frac{T_{1-\varepsilon}^*(E')}{\overline{(g_{1-\varepsilon}|g_{1-\varepsilon})(E')}(1-T_{1-\varepsilon}^*(E'))}$$

$$\otimes (g_{1-\varepsilon}|g_{1-\varepsilon})(E)|g_\varepsilon\rangle\langle g_{1-\varepsilon}|S_{E'}\otimes_\beta S_{E'}S_E)$$

$$-\frac{T_\varepsilon(E)}{(g_\varepsilon|g_\varepsilon)_-(E)\cdot(1-T_\varepsilon(E))}$$

$$\times D_\varepsilon \frac{1}{1-T_{1-\varepsilon}^*(E')} \otimes (g_\varepsilon|g_\varepsilon)(E)|g_\varepsilon\rangle\langle g_{1-\varepsilon}|S_{E'}\otimes_\beta S_{E'}S_E$$

$$-\frac{1}{1-T_\varepsilon(E)} D_\varepsilon D_{1-\varepsilon} \frac{1}{1-T_\varepsilon^*(E')}$$

$$\otimes (g_{1-\varepsilon}|g_{1-\varepsilon})(E)(|g_\varepsilon\rangle\langle g_\varepsilon|S_{E'}\otimes_\beta S_{E'}S_E). \tag{6.24}$$

Applying (6.6) to the right-hand side of (6.24) one finds that

$$T_3T_3^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}E \sum_{\varepsilon\in\{0,1\}} \Bigg((g_\varepsilon|g_\varepsilon)(E) \frac{|T_\varepsilon^*(E)|^2}{|(g_\varepsilon|g_\varepsilon)_-(E)|^2|1-T_\varepsilon^*(E)|^2}$$

$$\otimes |g_\varepsilon\rangle\langle g_\varepsilon|S_E\otimes_\beta S_E + \frac{1}{1-T_\varepsilon(E)} D_\varepsilon \frac{T_{1-\varepsilon}^*(E)}{\overline{(g_{1-\varepsilon}|g_{1-\varepsilon})(E)}(1-T_{1-\varepsilon}^*(E))}$$

$$\otimes (g_{1-\varepsilon}|g_{1-\varepsilon})(E)|g_\varepsilon\rangle\langle g_{1-\varepsilon}|S_E$$

$$\otimes_\beta S_E - \frac{T_\varepsilon(E)}{(g_\varepsilon|g_\varepsilon)_-(E)(1-T_\varepsilon(E))} D_\varepsilon \frac{1}{1-T_{1-\varepsilon}^*(E)}$$

$$\otimes (g_\varepsilon|g_\varepsilon)(E)|g_\varepsilon\rangle\langle g_{1-\varepsilon}|S_E\otimes_\beta S_E) - \frac{D_\varepsilon D_{1-\varepsilon}}{|1-T_\varepsilon^*(E)|^2}$$

$$\times \otimes (g_{1-\varepsilon}|g_{1-\varepsilon})(E)|g_\varepsilon\rangle\langle g_\varepsilon|S_E\otimes_\beta S_E)\Bigg). \tag{6.25}$$

Notice that the sum of the first terms (of the system part) of the right-hand side of (6.23a) and (6.23b) is equal to

$$\frac{T_\varepsilon(E)(g_\varepsilon|g_\varepsilon)+(E)(1-T_\varepsilon^*(E))+T_\varepsilon^*(E)(g_\varepsilon|g_\varepsilon)_-(E)(1-T_\varepsilon(E))}{|(g_\varepsilon|g_\varepsilon)_-(E)|^2|1-T_\varepsilon(E)|^2}$$

$$= \frac{-|T_\varepsilon(E)|^2(g_\varepsilon|g_\varepsilon)(E)}{|(g_\varepsilon|g_\varepsilon)_-(E)|^2|1-T_\varepsilon(E)|^2} + \frac{T_\varepsilon(E)(g_\varepsilon|g_\varepsilon)_+(E)+T_\varepsilon^*(E)(g_\varepsilon|g_\varepsilon)_-(E)}{|(g_\varepsilon|g_\varepsilon)_-(E)|^2|1-T_\varepsilon(E)|^2}$$

$$= \frac{-|T_\varepsilon(E)|^2(g_\varepsilon|g_\varepsilon)(E)}{|(g_\varepsilon|g_\varepsilon)_-(E)|^2|1-T_\varepsilon(E)|^2} + \frac{D_\varepsilon D_{1-\varepsilon}}{|1-T_\varepsilon^*(E)|^2}(g_{1-\varepsilon}|g_{1-\varepsilon})(E). \tag{6.26}$$

This shows that the sum of the first terms on the right-hand sides of (6.23a) and (6.23b) is equal to the sum of the first term and the fourth term in the integral of the right-hand side of (6.25). The same arguments imply that the sum of the second terms on the right-hand sides of (6.23a) and (6.23b) is equal to the sum of the second term and the third term in the integral of the right-hand side of (6.25) and this fact ends the proof. □

Theorem 6.1 implies that we can write the quantum stochastic differential equation (5.9) in the Frigerio–Maassen form [40]. To this goal, recall that, for any pair of Hilbert spaces χ_0, χ_1, if N, A denote the number and annihilation processes on the Fock space $\Gamma(\chi_1)$, then for $X_0 \in B(\chi_0)$, $X_1 \in B(\chi_1)$, $x \in \chi_1$, Frigerio and Maassen ([40]) introduced the notation

$$N(X_0 \otimes X_1) := X_0 \otimes N(X_1) \tag{6.27a}$$

$$A(X_0 \otimes X_1 x) := X_0 \otimes A(X_1 x) \tag{6.27b}$$

$$\langle x, X_0 \otimes X_1 x\rangle := X_0 \otimes 1\langle x, X_1 x\rangle. \tag{6.27c}$$

With this notation, angle (5.8) takes the form

$$U(t) = 1 + \int_0^t \int_{-\infty}^{\infty} \mathrm{d}E(\mathrm{d}N_s(T_3(E)) + \mathrm{d}A_s^+(T_3(E)\xi(E)) + \mathrm{d}A_s(T_3^*(E)\xi(E)) + \langle \xi(E), t_3(E)\xi(E)\rangle \,\mathrm{d}s)U(s). \tag{6.28}$$

Now consider the direct integral of Hilbert spaces

$$\mathscr{H} := \int_{\mathbb{R}}^{\oplus} (\mathscr{K} \otimes_\beta \mathscr{K})_E \,\mathrm{d}E \tag{6.29}$$

where, $(\mathscr{K} \otimes_\beta \mathscr{K})_E$ is the Hilbert space $K \otimes_\beta K$ with the scalar product $(\cdot|\cdot)_E$

$$((f \otimes_\beta g)|(f' \otimes_\beta g'))_E := (f|f')(E)\overline{(g|\mathrm{e}^{-\beta H} g')(E)}. \tag{6.30a}$$

In this space we consider the operator X and the vector ξ, defined by

$$\int_{\mathbb{R}}^{\oplus} X(E)\,\mathrm{d}E \qquad \int_{\mathbb{R}}^{\oplus} \xi(E)\,\mathrm{d}E = \xi \tag{6.30b}$$

and keeping into account the fact that (6.20) is equivalent to the statement that T_3 has the form

$$T_3 = S - 1 \tag{6.31}$$

where $S: \mathscr{H} \to \mathscr{H}$ is a unitary operator, (6.28) becomes

$$U(t) = 1 + \int_0^t (\mathrm{d}N_s(S-1) + \mathrm{d}A_s^+((S-1)\xi) + \mathrm{d}A_s((S-1)^*\xi) + \langle \xi, (S-1)\xi\rangle \,\mathrm{d}s)U(s) \tag{6.32}$$

which is exactly of the Frigerio–Maassen type. As a corollary of this fact we obtain:

Theorem 6.3. The solution of QSDE (6.32) (and thus (5.9)) is unitary.

Proof. This follows from theorem 3.4 of [40].

Remark. Notice that *formally* one has

$$T_3^*(E)T_3(E) = T_3(E)T_3^*(E) = T_3^*(E) + T_3(E) \qquad \forall E \in \mathbb{R} \tag{6.33}$$

which means that the unitarity condition holds on every energy shell. Unfortunately, (6.33) involves products of the form $T_3^*(E)T_3(E)$ and $T_3(E)T_3^*(E)$, which make no rigorous sense because of (6.6).

7. Connection with scattering theory

The last step of our programme is to show that the operator S in (6.32) is simply related (*although different*) to the scattering operator described in section 2) and introduced in [39] and [36]. To this goal notice that our limiting reservoir space is $K_{0,1}$ and the system space is the same as the original system space. The elements of $H_0 \otimes K_{0,1} =: \mathscr{H}$ can be written as

$$u \otimes f \otimes_\beta g \qquad u \in H_0, (f, g) \in K_{0,1}$$

with the scalar product

$$\langle u \otimes f \otimes_\beta g, u' \otimes f' \otimes_\beta g' \rangle := \langle u, u' \rangle (f \otimes_\beta g | f' \otimes_\beta g') \tag{7.1}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in H_0 and $(\cdot | \cdot)$ is given by (3.23).

For any operators $D \in B(H_0)$, $X_1, X_2) \in B(K)$, the action of the $D \otimes X_1 \otimes_\beta X_2$ on $\mathscr{H}$ is the following:

$$D \otimes X_1 \otimes_\beta X_2 (u \otimes f \otimes_\beta g) := Du \otimes (X_1 \otimes_\beta X_2)(f \otimes_\beta g) = Du \otimes X_1 f \otimes_\beta X_2 g. \tag{7.2}$$

With this notation, if we denote the triples operator as $(D \otimes X_1) \otimes_\beta X_2$, then the QSDE (6.28) can be rewritten as

$$U(t) = 1 + \int_0^t \int_{-\infty}^{\infty} \mathrm{d}E (\mathrm{d}N_s((T(E) \otimes_\beta S_E)) + \mathrm{d}A_s^+((T(E) \otimes_\beta S_E)\xi(E))$$
$$+ \mathrm{d}A_s((T(E) \otimes_\beta S_E)^* \xi(E)) + \langle \xi(E), (T(E) \otimes_\beta S_E)\xi(E) \rangle \, \mathrm{d}s) U(s). \tag{7.3}$$

The meaning of (7.3) is simple: (6.28) is the equation for U in terms of the operators $T_3(E)$ *which act on triples* (this is the reason for the subscript 3). In (7.3) this action is separated into the action of $T(E)$ *on the space of pair* $H_0 \otimes K$ and the action of S_E *on the single space* K. We claim that the operator

$$T = \int_{-\infty}^{\infty} \mathrm{d}E \, T(E) \tag{7.4}$$

is exactly the operator T defined by (2.43) (see also lemma 5.1 of [36]). This is simply checked by comparing the expansion (6.24) of [36] with (6.19*a*), (6.7) and (6.8).

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