

Some global properties of neural networks

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Abstract. Some properties of the global behaviour of a model of neural network are considered.

The geometric concept of “quadrant–degeneration” is studied and it is shown to be independent of the algebraic concept of rank–degeneration. The results obtained are employed to solve some global problems of synthesis (i.e., independent of the initial state of the network) without the use of the theory of linear inequalities.

1 Introduction

The mathematical model of a neural network with which we will be concerned has been introduced by Caianiello (1961). In this model the evolution of a network is described by means of “Neuronic Equations” (N.E.) which describe the instantaneous activity of the net, and “Mnemonic Equations” (M.E.) describing the learning processes. The Adiabatic Learning Hypothesis which asserts that learning processes take place more slowly than the processes described by the neuronic equations, allows to study independently the N.E. and the M.E. within a time interval, whose width is discussed by Caianiello (1961).

We will here be concerned exclusively with the study of neuronic equations in the special form obtained under “self–duality” conditions which characterize the so–called “normal systems” (Caianiello, 1966).

In Section 1, after a brief recall of the formalism, we consider the “individual” problem of synthesis and we show the equivalent, for this case, of a well–known theorem enables us to give some helpful criterion to solve concrete problems.

In Section 2 the “global” problem of synthesis is stated and a result is proved which generalizes a theorem given in Accardi (1971). Moreover, there are some practical considerations of the synthesis of reverberations whose lengths must not cross preassigned upper or lower bounds.

Finally, in Section 3 the solution of some concrete problems of synthesis are presented. An example which shows the independence between “rank–degeneration” and “quadrant–degeneration” is also given.

2 The individual problem of synthesis

We consider the formal neurons as binary elements which can assume the values 1 or -1 . The states of the network are vectors, whose i -th components represent the state of the i -th neuron. Time is quantized in the mode. In the case of “normal systems” the state σ_{t+1} of an N -neuron network at the time $t + 1$ can be expressed by the (vector) equation:

$$\sigma_{t+1} = \text{sgn} (A\sigma_t) \quad (1)$$

where A is an $N \times N$ matrix, and $\text{sgn} [\cdot]$ is the “signum” function, defined componentwise on $R - \{0\}$ by:

$$\text{sgn } x = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \quad (2)$$

The problem of synthesis of neural networks may be expressed in the following manner.

Given a sequence $\sigma, \dots, \sigma_{R+1}$ of states, determine a matrix A such that

$$\sigma_{i+1} = \text{sgn} (A\sigma_i), \quad 1 \leq i \leq R \quad (3)$$

We will refer to this as the “individual” problem of synthesis, that is, the network is required to realize a *single* preassigned sequence of states. If $\sigma_{R+1} = \sigma_1$ the sequence will be called a “reverberation”; if $\sigma_l \neq \sigma_m$ for $l < m$ we will say that the sequence belongs to a transient.

It may be useful to state the equivalent for this model of a well known result about threshold functions and some of its consequences (Elgot, 1960).

Denoting with $\langle \cdot, \cdot \rangle$ the scalar product which maps the vectors $x = (x_i)$; $y = (y_i)$ into the number $\langle x, y \rangle = \sum_{i=1}^N x_i y_i$ and denoting with a_i the i -th row of the matrix A , then Eq. (3), written componentwise, becomes (σ_{ij} being the j -th component of the vector σ_i):

$$\sigma_{i+1,j} = \text{sgn} \langle a_j, \sigma_i \rangle; \quad 1 \leq i \leq R; \quad 1 \leq j \leq N \quad (4)$$

one can thus define the sets

$$\begin{aligned} I_j^+ &= \{\sigma_i | i \leq R; \sigma_{i+1,j} = +1\} \\ I_j^- &= \{\sigma_i | i \leq R; \sigma_{i+1,j} = -1\} \end{aligned}$$

(5)

we then have $I_j^+ \cap I_j^- = \emptyset$ and from $I_j^+ \sqcup I_j^- = \{\sigma_1, \dots, \sigma_R\}$ for every $j \leq N$, Eqs. (3) can be written:

$$\begin{aligned} \langle \sigma_i, a_j \rangle &> 0 && \text{if } \sigma_i \in I_j^+ \\ \langle \sigma_i, a_j \rangle &< 0 && \text{if } \sigma_i \in I_j^-; \quad i \leq j \leq N \end{aligned} \quad (6)$$

Defining the sets, for $1 \leq j \leq N$

$$-I_j^- = \{\sigma \mid -\sigma \in I_j^-\}; \quad I'_j = I_j^+ \sqcup (-I_j^-) \quad (7)$$

Eqs. (6) can be written in compact form:

$$\langle \sigma_i, a_j \rangle > 0; \quad \sigma_i \in I'_j; \quad 1 \leq j \leq N \quad (8)$$

As a consequence, a matrix A satisfying our problem exists if and only if the N systems of linear inequalities

$$\langle \sigma_i, x \rangle > 0; \quad \sigma_i \in I'_j; \quad 1 \leq j \leq N \quad (9)$$

are consistent. A well known theorem of convex analysis (Rockafellar, 1969) asserts that this is the case if and only if the null vector does *not* belong to the convex hull of the set I'_j for $1 \leq j \leq N$, that is, if there exist no R non-negative numbers t_σ whose sum equals 1, such that, for some j ,

$$\sum_{\sigma \in I'_j} t_\sigma \cdot \sigma = 0 \quad (10)$$

This is equivalent to

$$\sum_{\sigma \in I'_j} t_\sigma \cdot u_\sigma = \frac{1}{2} \mathbf{1} \quad (11)$$

where $u_\sigma = \frac{1}{2}(\sigma + \mathbf{1})$, i.e., u_σ is the ‘‘boolean state’’ associated to σ . Introducing the matrix $u_j = [u_1, \dots, u_R]$, $\sigma_i = 2u_i - \mathbf{1} \in I'_j$, Eq. (11) is equivalent to:

$$u_j t = \frac{1}{2} \mathbf{1} \quad (12)$$

Eqs. (11), (12) provide, through classical theorems of linear algebra, some simple sufficient criteria of realizability. For example, if $(u_i)_{1 \leq i \leq R}$ is a generating system for the set $2I'_j - \mathbf{1}$, i.e., a maximum number of independent

vectors, a sufficient condition for realizability is that the vector $\mathbf{1}$ does not belong to the space generated by the u_i . Suppose now that the sequence $(\sigma)_{1 \leq i \leq R+1}$ is realizable, i.e., there exists a network which contains this sequence in its evolution. If σ is an arbitrary vector of Q^N , then, from Eq. (10) it follows that the sequence $(\sigma_i)_{1 \leq i \leq R+2}$; $\sigma_{R+2} = \sigma$, will be realizable if and only if no vectors λ , with negative components exist, such that for some $j = 1, 2, \dots, N$, the equation

$$S_j \lambda = \tau \quad (13)$$

where the R columns of the matrix S_j are the vectors of I'_j in the preceding notations, and $\tau = \pm \sigma_{R+1}$ according to the sign of the j -th component of σ_{R+2} .

The geometric meaning of condition (13) can be expressed in this way: denote $P_j \sigma$ the j -th component of the state $\sigma \in Q^N$; then the sequence $(\sigma_i)_{1 \leq i \leq R+2}$ can be realized if and only if for every $j = 1, \dots, N$ the vector $-(P_j \sigma) \sigma_{R+1} = \tau_j$ does not belong to the convex cone generated by the set I'_j (see (7)); this means that for no positive numbers $(\lambda_{\sigma'})_{\sigma' \in I'_j}$ one has:

$$\sum_{\sigma' \in I'_j} \lambda_{\sigma'} \cdot \sigma' = \tau_j \quad (14)$$

taking scalar products for τ_j on both sides, one finds

$$\sum_{\sigma' \in I'_j} \lambda_{\sigma'} \langle \sigma', \tau_j \rangle = N \quad (15)$$

so that a sufficient condition for the realizability of the sequence $(\sigma_i)_{1 \leq i \leq R+2}$ is that for every $j = 1, 2, \dots$, one has:

$$\langle \sigma', \tau_j \rangle \leq 0; \quad \forall \sigma' \in I'_j \quad (16)$$

Suppose now that the sequence $(\sigma_i)_{1 \leq i \leq R}$ is N -net realizable. Then defining in the above notations the sets $\Gamma(I'_j)$; $1 \leq j \leq N$, as the closed convex hull of I'_j , the following proposition answers the question.

Proposition 1 *If the sequence $(\sigma_i)_{1 \leq i \leq R}$ is N -net realizable, and ν is the number of indices $j = 1, \dots, N$, such that the line $y = \lambda \in \mathbb{R}$, does not intersect the set $\Gamma(I'_j)$, then there exist 2^ν vectors $\sigma \in Q^N$ such that the sequence $\{(\sigma_i), \sigma\}$ is N -net realizable.*

In fact, from Eq. (10) one can see that the sequence $\{(\sigma_i), \sigma\}$ is N -net realizable if and only if for every $j = 1, \dots, N$ vector $\lambda(P_j\sigma)\sigma_R$ does not belong to $\Gamma(I'_j)$ where $\lambda < 0$ and $P_j\sigma$ is the j -th component of σ . So, if the straight line $y = \lambda\sigma_R$ does not intersect $\Gamma(I'_j)$, then the j -th component of σ can be arbitrary. If $\lambda\sigma_R$ belongs to $\Gamma(I'_j)$ for some λ , then $-\lambda\sigma_R$ cannot belong to $T(I'_j)$ since $(\sigma_i)_{1 \leq i \leq R}$ is realizable.

Thus our hypothesis is equivalent to saying that ν components of σ are arbitrary.

3 The “Global” Problem of Synthesis

The formulation of the problem of synthesis given in the preceding paragraph depends on the initial state: i.e., if a matrix A is found which realizes a preassigned sequence of states $(\sigma_i)_{1 \leq i \leq R+1}$, nothing in general is known about the evolution of the network when an initial state τ is chosen different from any of the σ_i in the sequence. There are, nevertheless, some problems in which one is mainly interested in the *final stable configuration* of the network resulting from an arbitrary initial excitation. One can refer, for example, to Caianiello *et al.* (1967) for a theoretical statement of the problem, and to Burattini and Liesis (1970) for an analysis of this kind of situation from an experimental point of view. This problem can clearly be reduced to the preceding one by the consideration of a sufficiently large number of systems of linear equalities, but this technique soon becomes very involved, even for small N (number of neurons) and R (length of reverberations). To handle this problem from an intrinsic standpoint, we want to develop a geometric formulation of it. We denote Q^N the set of states of the network, identified, as usual, with the set of vertices of the unit symmetric cube in \mathbb{R}^N , and define the mapping

$$T_A : \sigma \in Q^N \rightarrow \text{sgn}(A\sigma) \in Q^N \quad (17)$$

the evolution equation of the network is then written

$$T_A(\sigma_i) = \sigma_{i+1} \quad (18)$$

and a generic state σ belongs to a reverberation of length R if and only if R is the smallest positive integer such that:

$$T_A^R(\sigma) = \sigma ; \quad \text{where} \quad T_A^0(\sigma) = \sigma ; \quad T_A^{n+1} = T_A(T_A^n) \quad (19)$$

If, on the contrary, for every natural integer R one has $T_A^R(\sigma) \neq \sigma$ the state σ is said to belong to a transient. The function T_A has the following two properties:

- (i) T_A is symmetric, i.e., $T_A(-\sigma) = -T_A(\sigma)$ for every σ in Q^N .
- (ii) T_A maps Q into itself: i.e., $T_A Q^N \subseteq Q^N$.

Since Q^N is a finite set, and, for every integer n one has:

$$T_A^{n+1}(Q^N) \subset T_A^n(Q^N) \tag{20}$$

the inclusion can be proper only for a finite number of steps. Let m be the least integer such that equality holds in (20) and set

$$T_A^m(Q^N) = V \tag{21}$$

the subset QV of Q^N thus defined, is the largest among the subsets of Q^N on which the function T_A acts as a permutation. This will be called the “core” of the network whose matrix is A and is characterized by the following property: *a state belongs to a reverberation of the network if and only if it belongs to V .* In analogy with Q^N , the points of V may be considered as the vertices of a “polyhedron” (in general not a cube!). Like Q^N , the polyhedron V is symmetric because such is the map T_A . We can sum up the above consideration in this way: to every self-dual network A corresponds a symmetric subset V of Q^N such that T_A acts as a permutation on the elements of V , and V is the largest subset of Q with this property.

We can express the global problem of synthesis as follows: given a symmetric subset V of Q^N and a permutation P on the points of V , does a self-dual network A exist, such that V is the “core” of A and the permutation induced by T_A on V coincides with P ? The “core” of a network is its final stable configuration since starting from an arbitrary state after a finite number of steps ($\leq m$) the network decays into a state belonging to the “core” which is “stable” with respect to the evolution of the net, in the sense that if the initial state belongs to the “core”, all the subsequent states of the network also belong to it.

Thus the problem of vector realizability of networks can be generalized as follows: given a set S and an action on S of a subgroup G of the group of permutations on elements of S , we look for the minimum integer N such that there exists a one to one map φ of S into a polyhedron P of K^N (K

an arbitrary field) and an homomorphism ϱ of G into the group of transformations (not necessarily linear) of P into itself such that if $g \in G$ one has $\varphi \circ g = \varrho(g) \circ \varpi$. This approach is by no means restricted to the case of autonomous threshold nets, but extend to the general problem of vector realizations of automata. This formulation allows a mathematical approach to the problem of realizability of nets (i.e., systems of boolean functions) which is different from the usual ones. As long as we are concerned with the reverberation behavior of the net we see that the central concept involved is that of *action of the permutation group on the vertices of a polyhedron*. This action is usually expressed by means of the Heaviside function composed with an affine mapping, denoting with φ the transformation which maps the boolean cube onto the unit symmetric one, with H the action of the Heaviside function, with S the action of the Heaviside function, with S the action of the “signum” function one obtains the equality:

$$\varphi H = S\varphi \tag{22}$$

The transformation φ has two peculiar geometric features: 1) it preserves dimension; 2) it maps a cube onto a cube. The meaning of these properties is simple: the transition from one model to the other preserves both the number of neurons and the code. It is apparent that in equality (22) every trace of these properties is lost, so that this equation can be assumed (φ being one to one) as the *definition* of equivalence of two vector-models of the same neural net. More precisely: if V is an arbitrary polyhedron of a space K^N and S is an action of a subgroup of a permutation group, we say that the couple (V, S) is an equivalent vector-model of a couple (Q, H) if there exists a map $\varphi : Q^N \rightarrow V$ such that $\varphi H = S\varphi$. As an example of global property of a network we consider the following proposition which extends a recent result of one of the authors.

Proposition 2 *Let A be an N -network of rank K , then, if we denote with $(R_i)_{1 \leq i \leq m}$ the lengths of all its reverberations, we have*

$$\sum_{i=1}^m R_i \leq 2^N - 2^{N-K+1} + 2 \tag{23}$$

Proof. Since A has rank K , it projects all the points of Q^N (states of the network) on a K -dimensional linear manifold. It has been shown (Accardi,

1971) that such a manifold can intersect at most $2^N - 2^{N-K+1} + 2$ “quadrants” so that this will be “a fortiori” an upper bound for the number of points belonging to the core (which must belong to one of these “quadrants”). This number, because of the characteristic property of the core stated previously, is more than the sum of the lengths of all the reverberations of the network.

The limitation expressed in (??) is the best possible one, depending only on the rank of the matrix. We refer, for this and other considerations, to the above-mentioned paper in which an example is given where equality holds in (??).

Thus the rank of a network provides an upper bound for the number of states in its core. It can be shown (see Section 3) that this is not the case for a lower bound of this number; therefore it may be of interest to know some *type* of sufficient conditions which supply such a lower bound. This will ensure, as in the following example, that the network is sufficiently rich in reverberations, either in number *or* in length.

Let, for simplicity, $N = 2^r$. In such a case one can choose from Q an orthogonal basis $(\sigma_i)_{1 \leq i \leq N}$ (this is trivially verified by induction on the tensor product of q copies of \mathbb{R}^2). Put $K = 2^q$ with $p < q$ and $S = [\sigma_1, \dots, \sigma_N]$; $S_1 = [-\sigma_1, \dots, \sigma_K; s_{K+1}, \dots, s_N]$ where s_j is an arbitrary linear combination of the $(\sigma_i)_{1 \leq i \leq K}$ for $j = K + 1, \dots, N$. Now define A by

$$AS = S_1 ; \quad A = \frac{1}{N} S_1 S^T \quad (24)$$

Then clearly A has the rank K ; its core contains the ν vertices of Q^N which are linear combinations of the $(\sigma_i)_{1 \leq i \leq K}$ and the network A acts on each of these vertices as a non-linear oscillator. Obviously a similar argument may hold for any matrix of the type AB where B is a matrix which leaves invariant the polyhedron $Q(K)$ generated by the $(\sigma_i)_{1 \leq i \leq K}$. For any such matrix, denoted by (R_i) the set of the lengths of *all* the reverberations, one has:

$$\nu \leq \sum_i R_i \leq 2^N - 2^{N-K+1} + 2 \quad (25)$$

(for example if $K = N - 1$ then $\nu = \binom{N}{N/2}$).

The upper bound given in Proposition 2 for the number of states of the core of a network of rank K can be improved with some additional hypothesis on the matrix A of the network. We consider here one of these cases as

an illustration of the method. Suppose $x^T U = 0$ is the equation of the K -dimensional subspace defined by A . The matrix U (which is of order $N \times (N - K)$ and rank $N - K$) can be decomposed into: $U = \left[\frac{B}{\sqrt{B}} \right]$ where B is of rank $N - K$ and the above equation is equivalent to the following:

$$x^T = [-y^T V | y^T] \quad (26)$$

where y is an arbitrary vector with K components.

We want to look for the number of signs obtainable with vectors x of the form (??) when the matrix V is such that in each of its columns there may occur some zero and, if $v^{(j)}$ denotes its j -th column, for every couple of indices h and i there exist at least two components, say the α -th and the β -th, non null both in $v^{(h)}$ and $v^{(i)}$ and such that:

$$\begin{aligned} \operatorname{sgn} v_{\alpha}^{(i)} &= -\operatorname{sgn} v_{\alpha}^{(h)} \\ \operatorname{sgn} v_{\beta}^{(i)} &= \operatorname{sgn} v_{\beta}^{(h)} \end{aligned} \quad (27)$$

(this is surely possible if, for instance, $2^{K-1} \geq N - K$).

Then, if ν_h is the number of zeros in the column $v^{(h)}$, the set M_h of all the $\sigma \in Q^K$ such that the sign of the product $\langle x, v^{(h)} \rangle$ does not depend on the $|x_i|$'s whenever $\operatorname{sgn} x = \sigma$, contains exactly 2^{ν_h+1} elements. Furthermore $M_h \cap M_i = \emptyset$ if $h \neq i$, in fact, because of our hypothesis there exist two components α and β such that if

$$\operatorname{sgn}(x_{\alpha}) \operatorname{sgn}(v_{\alpha}^{(h)}) = \operatorname{sgn}(x_{\beta}) \operatorname{sgn}(v_{\beta}^{(h)}) \quad (28)$$

then, necessarily:

$$\operatorname{sgn}(x_{\alpha}) \operatorname{sgn}(v_{\alpha}^{(i)}) \neq \operatorname{sgn}(x_{\beta}) \operatorname{sgn}(v_{\beta}^{(i)}) \quad (29)$$

which means that if $\operatorname{sgn} x = \sigma \in M_h$, then the product $\langle x, v^{(i)} \rangle$ depends on the $|x_i|$'s.

Now for every $\sigma \in Q^K$, let h_{σ} be the number of columns $v^{(j)}$ in V such that the product $\langle x, v^{(j)} \rangle$ does not depend on the $|x_i|$'s whenever $\operatorname{sgn} x = \sigma$. Our hypothesis implies that $h_{\sigma} = 1$ or $h_{\sigma} = 0$. Then the vector $x^T V$ can give rise to at most $2^{N-K-h_{\sigma}}$ different signs, so that, when σ ranges in Q^K the number

$$\sum_{\sigma \in Q^K} 2^{N-K-h_{\sigma}} \quad (30)$$

is an upper bound for the number of vertices in the core of the network. We have, then,

$$\sum_{\sigma \in Q^K} 2^{-h_\sigma} = \sum_{j=1}^{N-K} \sum_{\sigma \in M_j} 2^{-h_\sigma} + \sum_{\tau \in L} 2^{-h_\tau} \quad (31)$$

where L denotes the set of the $\sigma \in Q^K$ not belonging to any of the M_j , i.e., such that if $\text{sgn } x = \sigma$ the product $\langle x, v^{(i)} \rangle$ depends on the $|x_j|$'s for every $i = 1, 2, \dots, N - K$. Since the M_j are disjoint and each of them contains 2^{ν_j+1} elements, then

$$\sum_{\tau \in L} 2^{-h_\tau} = 2^K - \sum_{j=1}^{N-K} 2^{\nu_j+1} \quad (32)$$

but if $\sigma \in M_j$ then $h_\sigma = 1$ and therefore

$$\sum_{j=1}^{N-K} \sum_{\sigma \in M_j} 2^{-h_\sigma} = \sum_{j=1}^{N-K} 2^{\nu_j} \quad (33)$$

which yields

$$\sum_{\sigma \in Q^K} 2^{N-K-h_\sigma} = 2^N - \frac{3}{2} 2^{N-K} \left(\sum_{j=1}^{N-K} 2^{\nu_j} \right) \quad (34)$$

It has already been observed (see Accardi, 1971) that besides the rank-degeneration which has been investigated in Proposition 2 there may arise another kind of degeneration, called “quadrant-degeneration” which takes place when different vertices of Q^N are mapped into a single quadrant of \mathbb{R}^N so that the “signum” function, when applied to them, maps them all into the same vertex of Q^N . The following example shows that these two kinds of “degeneration” are independent within the limits of Proposition 2 and that the essential feature of the equivalence of nets is topological-stability of the evolution within some “tolerance cone” - rather than algebraic - rank invariance.

Moreover, this example solves, without the introduction of any controlling element, the problem, examined in Caianiello *et al.* (1967), of synthesizing a net which has the same immediate reaction regardless of the impinging excitation.

We are thus looking for a matrix A such that for a preassigned $\tau \in Q^N$

$$\text{sgn } A\sigma = \pm\tau ; \quad \forall \sigma \in Q^N \quad (35)$$

i.e., A must map the whole symmetric cube into the quadrants defined by τ and $-\tau$. And a sufficient condition for this is that A maps Q^N into the cone $C(\tau)$ of the axis the direction of τ and tangent to the “walls” delimiting the quadrant identified by τ itself.

Denoting $\theta_j(\tau)$ the angle between τ and its projection on the j -th wall of the τ -quadrant, then for $j = 1, \dots, N$,

$$\cos \theta_j(\tau) = \frac{\langle \tau, \tau \rangle - (\langle \tau, e_j \rangle)^2}{\sqrt{N(N-1)}} = \frac{N-1}{\sqrt{N(N-1)}} = \sqrt{1 - \frac{1}{N}} \quad (36)$$

Therefore a vector x belongs to the cone $C(\tau)$ if $|\cos(\widehat{x\tau})| \geq \sqrt{1 - \frac{1}{N}}$. Thus, choosing A such that τ is an eigenvector of A^T corresponding to the positive eigenvalue λN one has,

$$\langle A\sigma, \tau \rangle = \langle \sigma, A^T \tau \rangle = \lambda N \langle \sigma, \tau \rangle \quad (37)$$

and if N is *odd*

$$|\cos(A\widehat{\sigma\tau})| = \frac{|\langle A\sigma, \tau \rangle|}{\|A\sigma\| \cdot \|\tau\|} = \lambda N \frac{|\langle \sigma, \tau \rangle|}{\|A\sigma\| \cdot \|\tau\|} \geq \frac{\lambda N}{\|A\| \cdot N} = \frac{\lambda}{\|A\|}$$

where $\|A\|$ is a norm for the matrix A satisfying the inequality $\|Ax\| \leq \|A\| \cdot \|x\|$ see, for example, Gantmacher (1964) and the required conditions on A are (N being odd):

$$\begin{aligned} A^T \tau &= \lambda N \cdot \tau \\ \lambda \|A\| &\geq \sqrt{1 - \frac{1}{N}} \end{aligned}$$

It is instructive to give a geometrical interpretation of this problem. Suppose first, as is always possible, that the matrix A in (??) is positive definite. Then it is well known (see Gantmacher, 1964) that its action on the vectors of \mathbb{R}^N consists of a “stretching” of their components along N orthogonal directions (the “eigen-axis” of the matrix). Condition (??) can therefore be interpreted as imposing that the “stretching” along the axes of τ must be much bigger than along other axes of A .

In particular, we can choose A in (??) as any positive (not necessarily definite) matrix of *arbitrary rank* K ($1 \leq K \leq N$) which shows that the essential feature for the evolution of a network A is the geometrical action of A on the points of Q and not the rank of A .

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