Quantum decoherence and the Glauber dynamics from the Stochastic limit

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Abstract

The effects of decoherence for quantum system coupled with a bosonic field are investigated. An application of the stochastic golden rule shows that in the stochastic limit the dynamics of such a system is described by a quantum stochastic differential equation.

The corresponding master equation describes convergence of a system to equilibrium. In particular it predicts exponential damping for off–diagonal matrix elements of the system density matrix, moreover these elements for a generic system will decay at least as $\exp(-tN\frac{kT}{\hbar})$, where N is a number of particles in the system.

As an application of the described technique a derivation from first principles (i.e. starting from a Hamiltonian description) of a quantum extension of the Glauber dynamics for systems of spins is given.

1 Introduction

In the present paper we investigate a general model of quantum system interacting with a bosonic reservoir via an Hamiltonian of the form

$$H = H_0 + \lambda H_I$$

where H_0 is called the free Hamiltonian and H_I the interaction Hamiltonian.

The stochastic golden rules, which arise in the stochastic limit of quantum theory as natural generalizations of Fermi golden rule [1], [2], provide a natural tool to associate a stochastic flow, driven by a white noise equation (stochastic Schrödinger) equation, to any discrete system interacting with a quantum field. This white noise Hamiltonian equation which, when put in normal order becomes equivalent to a quantum stochastic differential equation. The Langevin (stochastic Heisenberg) and master equations are deduced from this white noise equation by means of standard procedures which are described in [1].

We use these equations to investigate the decoherence in quantum systems.

In the work [3], extending previous results obtained with perturbative techniques by [9], it was shown on the example of the spin–boson Hamiltonian that the decoherence in quantum systems

can be controlled by the following constants (cf. section 2 for the definition of the quantities involved)

$$\mathrm{Re}\,(g|g)_\omega^+ = \int dk\,|g(k)|^2 2\pi\delta(\omega(k)-\omega)N(k),$$

For the simplest case of the equalibrium state of the reservoir with the temperature $\beta^{-1}=kT$ this constant will be equal to $\frac{kT}{\hbar}$ (actually this is true for large temperatures and for the dispersion function $\omega(k)=|k|$).

In this paper we extend the approach of [3] from 2–level systems to arbitrary quantum systems with discrete spectrum. Our resuts show that the stocastic limit technique gives us an effective method to control quantum decoherence.

We find that, under the above mentioned interaction, all the off–diagonal matrix elements, of the density matrix of a generic discrete quantum system, will decay exponentially if $\mathrm{Re}\,(g|g)$ are nonzero. In other words we obtain the asymptotic diagonalization of the density matrix.

Moreover, we show that for generic quantum system the off–diagonal elements of the density matrix decay exponentially as $\exp(-N\mathrm{Re}\,(g|g)t)$, with the exponent proportional to the number N of particles in the system. Therefore for generic macroscopic (large N) systems the quantum state will collapse into the classical state very quickly. This effect was built in *by hands* in several phenomenological models of the quantum measurement process. In the stochastic limit approach it is deduced from the Hamiltonian model.

This observation contributes to the clarification of one of the old problems of quantum theory: Why macroscopic systems usually behave classically? i.e. why do we observe classical states although the evolution of the system is a unitary operator described by the Shrödinger equation?

Moreover, this result allows to distinguish between macroscopic systems (that behave classically) and microscopic systems (where quantum effects are important). Quantum effects (or effects of quantum interference) are connected with the off–diagonal elements of the density matrix. Therefore the following notion is natural: the macroscopic system is a system where off–diagonal elements of the density matrix decay quickly (faster than the minimal time of observation). Using that off–diagonal elements decay as $\exp(-N\mathrm{Re}\,(g|g)t)$, we get the following definition of the macroscopic system: $N\mathrm{Re}\,(g|g) >> 1$.

The quantum Markov semigroup we obtain lives invariant the algebra of the spectral projections of the system Hamiltonian and the associated master equation, when restricted to the diagonal part of the density matrix, takes the form of a standard classical kinetic equation, describing the convergence to equilibrium (Gibbs state) of the system, coupled with the given reservoir (quantum field).

Summing up: the convergence to equilibrium is a result of quantum decoherence.

If we can control the interaction so that some of the constants $\mathrm{Re}\,(g|g)$ are zero, then the corresponding matrix elements will not decay in the stochastic approximation, i.e. in a time scale which is extremely long with respect to the *slow clock* of the discrete system. In this sense the stochastic limit approach provides a method for controlling quantum coherence.

The general idea of the stochastic limit (see [1]) is to make the time rescaling $t \to t/\lambda^2$ in the solution of the Schrödinger (or Heisenberg) equation in interaction picture $U_t^{(\lambda)} = e^{itH_0}e^{-itH}$, associated to the Hamiltonian H, i.e.

$$\frac{\partial}{\partial t}U_t^{(\lambda)} = -i\lambda H_I(t)U_t^{(\lambda)}$$

with $H_I(t) = e^{itH_0} H_I e^{-itH_0}$. This gives the rescaled equation

$$\frac{\partial}{\partial t} U_{t/\lambda^2}^{(\lambda)} = -\frac{i}{\lambda} H_I(t/\lambda^2) U_{t/\lambda^2}^{(\lambda)} \tag{1}$$

and one wants to study the limits, in a topology to be specified,

$$\lim_{\lambda \to 0} U_{t/\lambda^2}^{(\lambda)} = U_t \; ; \quad \lim_{\lambda \to 0} \frac{1}{\lambda} H_I\left(\frac{t}{\lambda^2}\right) = H_t \tag{2}$$

The limit $\lambda \to 0$ after the rescaling $t \to t/\lambda^2$ is equivalent to the simultaneous limit $\lambda \to 0$, $t \to \infty$ under the condition that $\lambda^2 t$ tends to a constant (interpreted as a new slow time scale). This limit captures the dominating contributions to the dynamics, in a regime of long times and small coupling, arising from the cumulative effects, on a large time scale, of small interactions ($\lambda \to 0$). The physical idea is that, looked from the slow time scale of the atom, the field looks like a very chaotic object: a *quantum white noise*, i.e. a δ–correlated (in time) quantum field $b^*(t, k)$, $b(t, k)$ also called a *master field*.

The structure of the present paper is as follows.

In section 2 we introduce the model and consider its stochastic limit.

In section 3 we derive the Langevin equation.

In section 4 we derive the master equation for the density matrix and show that for non–zero decoherence the master equation describes the collapse of the density matrix to the classical Gibbs distribution and discuss the connection of this fact with the procedure of quantum measurement.

In section 5, using the characterization of quantum decoherence obtained in section 4 and generalizing arguments of [3], we find that our general model exhibits macroscopic quantum effects (in particular, conservation of quantum coherence). These effects are controllable by the state of the reservoir (that can be controlled by filtering).

In section 6 we apply our general scheme to the model of a quantum system of spins interacting with bosonic field and derive a quantum extension of the Glauber dynamics.

Thus our stochastic limit approach provides a microscopic interpretation, in terms of fundamental Hamiltonian models, to the dynamics of quantum spin systems. Moreover we deduce the full stochastic equation and not only the master equation. This is new even in the case of classical spin systems.

2   The model and it's stochastic limit

In the present paper we consider a general model, describing the interaction of a system S with a reservoir, represented by a bosonic quantum field. Particular cases of this general model were investigated in [3], [4], [5]. The total Hamiltonian is

$$H = H_0 + \lambda H_I = H_S + H_R + \lambda H_I$$

where H_R is the free Hamiltonian of a bosonic reservoir R:

$$H_R = \int \omega(k) a^*(k) a(k) dk$$

acting in the representation space $\mathcal{F}$ corresponding to the state $\langle \cdot \rangle$ of bosonic reservoir generated by the density matrix $\mathbf{N}$ that we take in the algebra of spectral projections of the reservoir

3

Hamiltonian. The reference state $\langle \cdot \rangle$ of the field is a mean zero gauge invariant Gaussian state, characterized by the second order correlation function equal to

$$\langle a(k)a^*(k') \rangle = (N(k) + 1)\delta(k - k')$$

$$\langle a^*(k)a(k') \rangle = N(k)\delta(k - k')$$

where the function $N(k)$ describes the density of bosons with frequency k. One of the examples is the (gaussian) bosonic equilibrium state at temperature β^{-1}.

The system Hamiltonian has the following spectral decomposition

$$H_S = \sum_r \varepsilon_r P_{\varepsilon_r}$$

where the index r labels the spectral projections of H_S. For example, for a non–degenerate eigenvalue ε_r of H_S the corresponding spectral projection is

$$P_{\varepsilon_r} = |\varepsilon_r\rangle\langle\varepsilon_r|$$

where $|\varepsilon_r\rangle$ is the corresponding eigenvector.

The interaction Hamiltonian H_I (acting in $\mathcal{H}_S \otimes \mathcal{F}$) has the form

$$H_I = \sum_j \left(D_j^* \otimes A(g_j) + D_j \otimes A^*(g_j) \right), \quad A(g) = \int dk \overline{g}(k)a(k) \ ,$$

where $A(g)$ is a smeared quantum field with cutoff function (form factor) $g(k)$. To perform the construction of the stochastic limit one needs to calculate the free evolution of the interaction Hamiltonian: $H_I(t) = e^{itH_0} H_I e^{-itH_0}$.

Using the identity

$$1 = \sum_r P_{\varepsilon_r}$$

we write the interaction Hamiltonian in the form

$$H_I = \sum_j \sum_{rr'} P_{\varepsilon_r} D_j^* P_{\varepsilon_r'} \int dk \overline{g}_j(k)a(k) + h.c. \tag{3}$$

Let us introduce the set of energy differences (Bohr frequencies)

$$F = \{\omega = \varepsilon_r - \varepsilon_{r'} : \varepsilon_r, \varepsilon_{r'} \in \mathrm{Spec}\, H_S\}$$

and the set of all energies of the form

$$F_\omega = \{\varepsilon_r : \exists \varepsilon_{r'} \ (\varepsilon_r, \varepsilon_{r'} \in \mathrm{Spec}\, H_S) \text{ such that } \varepsilon_r - \varepsilon_{r'} = \omega\}$$

With these notations we rewrite the interaction Hamiltonian (3) in the form

$$H_I = \sum_j \sum_{\omega \in F} \sum_{\varepsilon_r \in F_\omega} P_{\varepsilon_r} D_j^* P_{\varepsilon_r - \omega} \int dk \overline{g}_j(k)a(k) + h.c. =$$

$$= \sum_j \sum_{\omega \in F} E_\omega^*(D_j) \int dk \overline{g}_j(k)a(k) + h.c. \tag{4}$$

4

where

$$E_\omega(X) := \sum_{\varepsilon_r \in F_\omega} P_{\varepsilon_r - \omega} X P_{\varepsilon_r} \tag{5}$$

It is easy to see that the free volution of $E_\omega(X)$ is

$$e^{itH_S} E_\omega(X) e^{-itH_S} = e^{-it\omega} E_\omega(X)$$

Using the formula for the free evolution of bosonic fields

$$e^{itH_R} a(k) e^{-itH_R} = e^{-it\omega(k)} a(k)$$

we get for the free evolution of the interaction Hamiltonian:

$$H_I(t) = \sum_j \sum_{\omega \in F} E_\omega^*(D_j) \int dk \overline{g}_j(k) e^{-it(\omega(k)-\omega)} a(k) + h.c. \tag{6}$$

In the stochastic limit the field $H_I(t)$ gives rise to a family of quantum white noises, or master fields. To investigate these noises, let us suppose the following:

1) $\omega(k) \geq 0, \forall k$;

2) The $d - 1$–dimensional Lebesgue measure of the surface $\{k : \omega(k) = 0\}$ is equal to zero (so that $\delta(\omega(k)) = 0$) (for example $\omega(k) = k^2 + m$ with $m \geq 0$).

Now let us investigate the limit of $H_I(t/\lambda^2)$ using one of the basic formulae of the stochastic limit:

$$\lim_{\lambda \to 0} \frac{1}{\lambda^2} \exp\left(\frac{it}{\lambda^2} f(k)\right) = 2\pi \delta(t) \delta(f(k)) \tag{7}$$

which shows that the term $\delta(f(k))$ in (7) is not identically equal to zero only if $f(k) = 0$ for some k in a set of nonzero $d - 1$–dimensional Lebesgue measure. This explains condition (2) above.

The rescaled interaction Hamiltonian is expressed in terms of the rescaled creation and annihilation operators

$$a_{\lambda,\omega}(t, k) = \frac{1}{\lambda} e^{-i\frac{t}{\lambda^2}(\omega(k)-\omega)} a(k), \quad \omega \in F$$

After the stochastic limit every rescaled annihilation operator corresponding to any transition from $\varepsilon_{r'}$ to ε_r with the frequency $\omega = \varepsilon_r - \varepsilon_{r'}$ generates one non–trivial quantum white noise

$$b_\omega(t, k) = \lim_{\lambda \to 0} a_{\lambda,\omega}(t, k) = \lim_{\lambda \to 0} \frac{1}{\lambda} e^{-i\frac{t}{\lambda^2}(\omega(k)-\omega)} a(k)$$

with the relations

$$[b_\omega(t, k), b_\omega^*(t', k')] = \lim_{\lambda \to 0} [a_{\lambda,\omega}(t, k), a_{\lambda,\omega}^*(t', k')] =$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda^2} e^{-i\frac{t-t'}{\lambda^2}(\omega(k)-\omega)} \delta(k - k') = 2\pi \delta(t - t') \delta(\omega(k) - \omega) \delta(k - k') \tag{8}$$

$$[b_\omega(t, k), b_{\omega'}^*(t', k')] = 0$$

(cf. 7). This shows, in particular that quantum white noises, corresponding to different Bohr frequencies, are mutually independent.

The stochastic limit of the interaction Hamiltonian is therefore equal to

$$h(t) = \sum_j \sum_{\omega \in F} E_\omega^*(D_j) \int dk \overline{g}_j(k) b_\omega(t, k) + h.c. \tag{9}$$

5

The state of the master field (white noise) $b_\omega(t, k)$, corresponding to our choice of the initial state of the field, is the mean zero gauge invariant Gaussian state with correlations:

$$\langle b_\omega^*(t, k) b_\omega(t', k')\rangle = 2\pi\delta(t - t')\delta(\omega(k) - \omega)\delta(k - k')N(k)$$

$$\langle b_\omega(t, k) b_\omega^*(t', k')\rangle = 2\pi\delta(t - t')\delta(\omega(k) - \omega)\delta(k - k')(N(k) + 1)$$

and vanishes for noises corresponding to different Bohr frequences.

Now let us investigate the evolution equation in interaction picture for our model. According to the general scheme of the stochastic limit, we get the (singular) white noise equation

$$\frac{d}{dt}U_t = -ih(t)U_t \tag{10}$$

whose normally ordered form is the quantum stochastic differential equation [6]

$$dU_t = (-idH(t) - Gdt)U_t \tag{11}$$

where $h(t)$ is the white noise Hamiltonian (9) given by the stochastic limit of the interaction Hamiltonian and

$$dH(t) = \sum_j \sum_{\omega \in F} \left(E_\omega^*(D_j)\, dB_{j\omega}(t) + E_\omega(D_j)\, dB_{j\omega}^*(t)\right) \tag{12}$$

$$dB_{j\omega}(t) = \int dk\, \overline{g}_j(k) \int_t^{t+dt} b_\omega(\tau, k)d\tau \tag{13}$$

According to the stochastic golden rule (11) the limit dynamical equation is obtained as follows: the first term in (11) is just the limit of the iterated series solution for (1)

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \int_t^{t+dt} H_I\left(\frac{\tau}{\lambda^2}\right) d\tau$$

The second term Gdt, called the drift, is equal to the limit of the expectation value in the reservoir state of the second term in the iterated series solution for (1)

$$\lim_{\lambda \to 0} \frac{1}{\lambda^2} \int_t^{t+dt} dt_1 \int_t^{t_1} dt_2 \langle H_I\left(\frac{t_1}{\lambda^2}\right) H_I\left(\frac{t_2}{\lambda^2}\right)\rangle$$

Making in this formula the change of variables $\tau = t_2 - t_1$ we get

$$\lim_{\lambda \to 0} \frac{1}{\lambda^2} \int_t^{t+dt} dt_1 \int_{t-t_1}^0 d\tau \langle H_I\left(\frac{t_1}{\lambda^2}\right) H_I\left(\frac{t_1}{\lambda^2} + \frac{\tau}{\lambda^2}\right)\rangle \tag{14}$$

Computing the expectation value and using the fact that the limits of oscillating factors of the form $\lim_{\lambda \to 0} e^{\frac{ict_1}{\lambda^2}}$ vanish unless the constant c is equal to zero, we see that we can have non–zero limit only when all oscillating factors of a kind $e^{\frac{ict_1}{\lambda^2}}$ (with t_1) in (14) cancel. In conclusion we get

$$G = \sum_{ij} \sum_{\omega \in F} \int_{-\infty}^0 d\tau \left(\int dk\, \overline{g_i(k)} g_j(k) e^{i\tau(\omega(k)-\omega)}(N(k) + 1)E_\omega^*(D_i) E_\omega(D_j) + \right.$$

$$\left. + \int dk\, g_i(k)\overline{g_j(k)} e^{-i\tau(\omega(k)-\omega)} N(k) E_\omega(D_i) E_\omega^*(D_j) \right)$$

and therefore, from the formula

$$\int_{-\infty}^{0} e^{it\omega} dt = \frac{-i}{\omega - i0} = \pi\delta(\omega) - i\,\mathrm{P.P.}\,\frac{1}{\omega} \tag{15}$$

we get the following expression for the drift G:

$$\sum_{ij}\sum_{\omega\in F}\left(\int dk\,\overline{g_i(k)}g_j(k)\frac{-i(N(k)+1)}{\omega(k)-\omega-i0}E_\omega^*(D_i)\,E_\omega(D_j)+\right.$$

$$\left.+\int dk\,g_i(k)\overline{g_j(k)}\frac{iN(k)}{\omega(k)-\omega+i0}E_\omega(D_i)\,E_\omega^*(D_j)\right) =$$

$$= \sum_{ij}\sum_{\omega\in F}\left((g_i|g_j)_\omega^- E_\omega^*(D_i)\,E_\omega(D_j) + \overline{(g_i|g_j)}_\omega^+ E_\omega(D_i)\,E_\omega^*(D_j)\right) \tag{16}$$

Let us note that for (16) we have the following Cheshire Cat effect found in [3]: even if the frequency ω is negative and therefore does not generate a quantum white noise the corresponding values $(g|g)_\omega^\pm$ in (16) will be non–zero. In other terms: negative Bohr frequencies contribute to an energy shift in the system, but not to its damping.

Remark If F is any subset of $\mathrm{Spec}\,H_S$ and X_r are arbitrary bounded operators on $\mathcal{H}_S$ then for any $t \in R$

$$e^{itH_S}\sum_{\varepsilon_r\in F}P_{\varepsilon_r}X_rP_{\varepsilon_r} = \sum_{\varepsilon_r\in F}e^{it\varepsilon_r}P_{\varepsilon_r}X_rP_{\varepsilon_r} = \sum_{\varepsilon_r\in F}P_{\varepsilon_r}X_rP_{\varepsilon_r}e^{itH_S}$$

In other words: $\sum_{\varepsilon_r\in F}P_{\varepsilon_r}X_rP_{\varepsilon_r}$ belongs to the commutant $L^\infty(H_S)'$ of the abelian algebra $L^\infty(H_S)$, generated by the spectral projections of H_S.

A corollary of this remark is that, for each $\omega \in F$, for any bounded operator $X \in L^\infty(H_S)'$ and for each pair of indices (i,j) the operators

$$E_\omega(D_i)\,X\,E_\omega^*(D_j),\qquad E_\omega^*(D_i)\,X\,E_\omega(D_j) \tag{17}$$

belong to the commutant $L^\infty(H_S)'$ of $L^\infty(H_S)$. In particular, if H_S has non–degenerate spectrum so that $L^\infty(H_S)$ is a maximal abelian subalgebra of $B(\mathcal{H}_S)$, the operators (17) also belong to $L^\infty(H_S)$.

3 The Langevin equation

Now we will find the Langevin equation, which is the limit of the Heisenberg evolution, in interaction representation. Let X be an observable. The Langevin equation is the equation satisfied by the stochastic flow j_t, defined by:

$$j_t(X) = U_t^* X U_t$$

where U_t satisfies equation (11) in the previous section, i.e.

$$dU_t = (-idH(t) - Gdt)U_t \tag{18}$$

To derive the Langevin equation we consider

$$dj_t(X) = j_{t+dt}(X) - j_t(X) = dU_t^* X U_t + U_t^* X dU_t + dU_t^* X dU_t \qquad (19)$$

The only nonvanishing products in the quantum stochastic differentials are

$$dB_{i\omega}(t)dB_{j\omega}^*(t) = 2\text{Re}\,(g_i|g_j)_\omega^- dt, \quad dB_{i\omega}^*(t)dB_{j\omega}(t) = 2\text{Re}\,(g_i|g_j)_\omega^+ dt \qquad (20)$$

Combining the terms in (19) and using (18), (12), (16) and (20) we get the Langevin equation

$$dj_t(X) = \sum_\alpha j_t \circ \theta_\alpha(X) dM^\alpha(t) = \sum_{n=-1,1;j\omega} j_t \circ \theta_{nj\omega}(X) dM^{nj\omega}(t) + j_t \circ \theta_0(X) dt \qquad (21)$$

where

$$dM^{-1,j\omega}(t) = dB_{j\omega}(t), \quad \theta_{-1,j\omega}(X) = -i[X, E_\omega^*(D_j)] \qquad (22)$$

$$dM^{1,j\omega}(t) = dB_{j\omega}^*(t), \quad \theta_{1,j\omega}(X) = -i[X, E_\omega(D_j)] \qquad (23)$$

and

$$\theta_0(X) = \sum_{ij} \sum_{\omega \in F} \left(-i\,\text{Im}\,(g_i|g_j)_\omega^-[X, E_\omega^*(D_i) E_\omega(D_j)] + i\,\text{Im}\,(g_i|g_j)_\omega^+[X, E_\omega(D_i) E_\omega^*(D_j)] + \right.$$

$$+2\text{Re}\,(g_i|g_j)_\omega^- \left(E_\omega^*(D_i) X E_\omega(D_j) - \frac{1}{2}\{X, E_\omega^*(D_i) E_\omega(D_j)\} \right) +$$

$$\left. +2\text{Re}\,(g_i|g_j)_\omega^+ \left(E_\omega(D_i) X E_\omega^*(D_j) - \frac{1}{2}\{X, E_\omega(D_i) E_\omega^*(D_j)\} \right) \right) \qquad (24)$$

is a quantum Markovian generator. The structure map $\theta_0(X)$ has the standard form of the generator of a master equation [7]

$$\theta_0(X) = \Psi(X) - \frac{1}{2}\{\Psi(1), X\} + i[H, X]$$

where Ψ is a completely positive map and H is selfadjoint. In our case $\Psi(X)$ is a linear combination of terms of the type

$$E_\omega^*(D_i) X E_\omega(D_j)$$

Remark   A corollary of the remark at the end of section 2 is that the Markovian generator θ_0 maps $L^\infty(H_S)'$ into itself. Moreover, if X in (24) belongs to the $L^\infty(H_S)$ then the Hamiltonian part of $\theta_0(X)$ vanishes and only the dissipative part remains. In particular, if H_S has non–degenerate spectrum then $\theta_0(X)$ maps $L^\infty(H_S)$ and has the form

$$\theta_0(X) = \sum_{ij} \sum_{\omega \in F} \left(2\text{Re}\,(g_i|g_j)_\omega^- (E_\omega^*(D_i) X E_\omega(D_j) - X E_\omega^*(D_i) E_\omega(D_j)) + \right.$$

$$\left. +2\text{Re}\,(g_i|g_j)_\omega^+ (E_\omega(D_i) X E_\omega^*(D_j) - X E_\omega(D_i) E_\omega^*(D_j)) \right)$$

for any $X \in L^\infty(H_S)$.

The structure maps θ_α in (21) satisfy the following stochastic Leibnitz rule, see the paper [8].

Theorem. *For any pair of operators in the system algebra X, Y, the structure maps in the Langevin equation (21) satisfy the equation*

$$\theta_\alpha(XY) = \theta_\alpha(X)Y + X\theta_\alpha(Y) + \sum_{\beta,\gamma} c_\alpha^{\beta\gamma}\theta_\beta(X)\theta_\gamma(Y)$$

where the structure constants $c_\alpha^{\beta\gamma}$ is given by the Ito table

$$dM^\beta(t)dM^\gamma(t) = \sum_\alpha c_\alpha^{\beta\gamma}dM^\alpha(t)$$

The conjugation rules of $dM^\alpha(t)$ and θ_α are connected in such a way that formula (21) defines a $*$–flow ($* \circ j_t = j_t \circ *$).

3.1 Evolution for the density matrix

Let us now investigate the master equation for the density matrix ρ.

We will show that if the reservoir is in the equilibrium state at temperature β^{-1} then for the generic system with decoherence the solution of the master equation $\rho(t)$ with $t \to \infty$ tends to the classical Gibbs state with the same temperature β^{-1}. This phenomenon realizes the quantum measurement procedure — the quantum state (density matrix) collapses into the classical state.

To show this we use the control of quantum decoherence that was found in the stochastic approximation of quantum theory, see [3] and discussion below.

Let us consider the evolution of the state (positive normed linear functional on system observables) given by the density matrix ρ, $\rho(X) = \operatorname{tr} \hat\rho X$. The evolution of the state is defined as follows

$$\rho_t = j_t^*(\rho) = \rho \circ j_t$$

Therefore from (21) we get the evolution equation

$$d\rho_t(X) = \rho \circ dj_t(X) = \rho \circ \sum_\alpha j_t \circ \theta_\alpha(X)dM^\alpha(t) = \sum_\alpha \rho_t\left(\theta_\alpha(X)dM^\alpha(t)\right)$$

Only the stochastic differential dt in this formula will survive and we get the master equation

$$\frac{d}{dt}\rho_t(X) = \rho_t \circ \theta_0(X) \equiv \theta_0^*(\rho_t)(X) \tag{25}$$

Let us consider the density matrix $\hat\rho = \hat\rho_S \otimes \hat\rho_R$,

$$\hat\rho_{S,t} = \sum_{\mu,\nu} \rho(\mu, \nu, t)|\mu\rangle\langle\nu|$$

where $|\mu\rangle$, $|\nu\rangle$ are eigenvectors of the system Hamiltonian H_S.

Using the form (24) of θ_0 and the identities

$$\operatorname{tr} Y[X, A] = -\operatorname{tr} [Y, A]X$$

$$\operatorname{tr} Y\left(AXB - \frac{1}{2}\{X, AB\}\right) = \operatorname{tr}\left(BYA - \frac{1}{2}\{Y, AB\}\right)X$$

9

the master equation (25) will take the form

$$\sum_{\mu,\nu} \frac{d}{dt}\rho(\mu,\nu,t)|\mu\rangle\langle\nu| = \sum_{\mu,\nu}\rho(\mu,\nu,t)$$

$$\sum_{ij}\sum_{\omega\in F}\left(i\,\mathrm{Im}\,(g_i|g_j)_\omega^-[|\mu\rangle\langle\nu|, E_\omega^*(D_i)\,E_\omega(D_j)] - i\,\mathrm{Im}\,(g_i|g_j)_\omega^+[|\mu\rangle\langle\nu|, E_\omega(D_i)\,E_\omega^*(D_j)]+\right.$$

$$+2\mathrm{Re}\,(g_i|g_j)_\omega^-\left(E_\omega(D_j)\,|\mu\rangle\langle\nu|E_\omega^*(D_i) - \frac{1}{2}\{|\mu\rangle\langle\nu|, E_\omega^*(D_i)\,E_\omega(D_j)\}\right)+$$

$$+2\mathrm{Re}\,(g_i|g_j)_\omega^+\left(E_\omega^*(D_j)\,|\mu\rangle\langle\nu|E_\omega(D_i) - \frac{1}{2}\{|\mu\rangle\langle\nu|, E_\omega(D_i)\,E_\omega^*(D_j)\}\right)\right) =$$

$$= \sum_{\mu,\nu}\rho(\mu,\nu,t)\sum_{ij}\sum_{\omega\in F}\left(i\,\mathrm{Im}\,(g_i|g_j)_\omega^-\left(|\mu\rangle\langle\nu|\chi_\omega(\varepsilon_\nu)D_i^*P_{\varepsilon_\nu-\omega}D_jP_{\varepsilon_\nu} - \chi_\omega(\varepsilon_\mu)P_{\varepsilon_\mu}D_i^*P_{\varepsilon_\mu-\omega}D_j|\mu\rangle\langle\nu|\right) +\right.$$

$$+2\mathrm{Re}\,(g_i|g_j)_\omega^-\left(\chi_\omega(\varepsilon_\mu)\chi_\omega(\varepsilon_\nu)P_{\varepsilon_\mu-\omega}D_j|\mu\rangle\langle\nu|D_i^*P_{\varepsilon_\nu-\omega}-\right.$$

$$-\frac{1}{2}\left(|\mu\rangle\langle\nu|\chi_\omega(\varepsilon_\nu)D_i^*P_{\varepsilon_\nu-\omega}D_jP_{\varepsilon_\nu} + \chi_\omega(\varepsilon_\mu)P_{\varepsilon_\mu}D_i^*P_{\varepsilon_\mu-\omega}D_j|\mu\rangle\langle\nu|\right)\right)-$$

$$-i\,\mathrm{Im}\,(g_i|g_j)_\omega^+\left(|\mu\rangle\langle\nu|\chi_{-\omega}(\varepsilon_\nu+\omega)D_iP_{\varepsilon_\nu+\omega}D_j^*P_{\varepsilon_\nu} - \chi_{-\omega}(\varepsilon_\mu+\omega)P_{\varepsilon_\mu}D_iP_{\varepsilon_\mu+\omega}D_j^*|\mu\rangle\langle\nu|\right) +$$

$$+2\mathrm{Re}\,(g_i|g_j)_\omega^+\left(\chi_{-\omega}(\varepsilon_\mu+\omega)\chi_{-\omega}(\varepsilon_\nu+\omega)P_{\varepsilon_\mu+\omega}D_j^*|\mu\rangle\langle\nu|D_iP_{\varepsilon_\nu+\omega}-\right.$$

$$\left.\left.-\frac{1}{2}\left(|\mu\rangle\langle\nu|\chi_{-\omega}(\varepsilon_\nu+\omega)D_iP_{\varepsilon_\nu+\omega}D_j^*P_{\varepsilon_\nu} + \chi_{-\omega}(\varepsilon_\mu+\omega)P_{\varepsilon_\mu}D_iP_{\varepsilon_\mu+\omega}D_j^*|\mu\rangle\langle\nu|\right)\right)\right) \qquad (26)$$

where $\chi_\omega(\varepsilon_\mu) = 1$ if $\varepsilon_\mu \in F_\omega$ and equals to 0 otherwise.

4 Dynamics for generic systems

Let us investigate the behavior of a system with dynamics defined by (26). This dynamics will depend on the Hamiltonian of the system.

We will call the Hamiltonian H_S *generic*, if:
1) The spectrum Spec H_S of the Hamiltonian is non degenerate.
2) For any Bohr frequency ω there exists a unique pair of energy levels $\varepsilon,\,\varepsilon' \in$ Spec H_S such that:

$$\omega = \varepsilon - \varepsilon'$$

We investigate (26) for generic Hamiltonian. We also consider the case of one test function $g_i(k) = g(k)$, although this is not important. In this case

$$E_\omega(X) = |\sigma'\rangle\langle\sigma'|X|\sigma\rangle\langle\sigma| = |\sigma'\rangle\langle\sigma|\langle\sigma'|X|\sigma\rangle$$

where $\omega = \varepsilon_\sigma - \varepsilon_{\sigma'}$. The Markovian generator θ_0^* in (26) takes the form

$$\theta_0^*(X) = \sum_{\sigma,\sigma'}|\langle\sigma'|D|\sigma\rangle|^2\left(i\,\mathrm{Im}\,(g|g)_{\sigma\sigma'}^-[X, |\sigma\rangle\langle\sigma|]+\right.$$

$$+2\mathrm{Re}\,(g|g)_{\sigma\sigma'}^{-}\left(|\sigma'\rangle\langle\sigma'|\langle\sigma|X|\sigma\rangle - \frac{1}{2}\{X,|\sigma\rangle\langle\sigma|\}\right)-$$

$$-i\,\mathrm{Im}\,(g|g)_{\sigma\sigma'}^{+}[X,|\sigma'\rangle\langle\sigma'|] + 2\mathrm{Re}\,(g|g)_{\sigma\sigma'}^{+}\left(|\sigma\rangle\langle\sigma|\langle\sigma'|X|\sigma'\rangle - \frac{1}{2}\{X,|\sigma'\rangle\langle\sigma'|\}\right)\right) \qquad (27)$$

We use here the notion

$$(g|g)_{\mu\sigma} = (g|g)_{\varepsilon_\mu-\varepsilon_\sigma}$$

Notice that the factors $\mathrm{Re}\,(g|g)_{\sigma\sigma'}^{\pm}$ are > 0 only for $\varepsilon_\sigma > \varepsilon_{\sigma'}$ and vanish for the opposite case.

It is easy to see that the terms in (27) of the form

$$|\sigma\rangle\langle\sigma|\langle\sigma'|X|\sigma'\rangle$$

for off–diagonal elements of the density matrix $X = |\mu\rangle\langle\nu|$ are equal to zero. We will show that in such case the equation (26) will predict fast damping of the states of the kind $|\mu\rangle\langle\nu|$.

In the non–generic case one can expect the fast damping of the state $|\mu\rangle\langle\nu|$ with different energies ε_μ and ε_ν.

With the given assumptions the action of θ_0^* on the off–diagonal matrix unit $|\mu\rangle\langle\nu|$, $\varepsilon_\mu \neq \varepsilon_\nu$ is equal to $A_{\mu\nu}|\mu\rangle\langle\nu|$ where the number $A_{\mu\nu}$ is given by the following

$$A_{\mu\nu} = \sum_\sigma \left(i\mathrm{Im}\,(g|g)_{\mu\sigma}^{-}|\langle\sigma|D|\mu\rangle|^2 - i\mathrm{Im}\,(g|g)_{\nu\sigma}^{-}|\langle\sigma|D|\nu\rangle|^2 - i\mathrm{Im}\,(g|g)_{\sigma\mu}^{+}|\langle\mu|D|\sigma\rangle|^2 + \right.$$

$$+i\mathrm{Im}\,(g|g)_{\sigma\nu}^{+}|\langle\nu|D|\sigma\rangle|^2 - \mathrm{Re}\,(g|g)_{\mu\sigma}^{-}|\langle\sigma|D|\mu\rangle|^2$$

$$\left.-\mathrm{Re}\,(g|g)_{\nu\sigma}^{-}|\langle\sigma|D|\nu\rangle|^2 - \mathrm{Re}\,(g|g)_{\sigma\mu}^{+}|\langle\mu|D|\sigma\rangle|^2 - \mathrm{Re}\,(g|g)_{\sigma\nu}^{+}|\langle\nu|D|\sigma\rangle|^2\right) \qquad (28)$$

The map θ_0^* multiplies off–diagonal matrix elements of the density matrix $\hat\rho_S$ by a number $A_{\mu\nu}$. Let us note that

$$\mathrm{Re}\,A_{\mu\nu} \leq 0$$

Moreover, for generic Hamiltonian the map θ_0^* mixes diagonal elements of the density matrix but does not mix diagonal and off–diagonal elements (the action of θ_0^* on diagonal element is equal to the linear combination of diagonal elements).

The equation (26) for the generic case takes the form

$$\sum_{\mu,\nu} \frac{d}{dt}\rho(\mu,\nu,t)|\mu\rangle\langle\nu| = \sum_{\mu\neq\nu} A_{\mu\nu}\rho(\mu,\nu,t)|\mu\rangle\langle\nu|+$$

$$+\sum_\sigma |\sigma\rangle\langle\sigma| \sum_{\sigma'}\left(\rho(\sigma',t)\left(2\mathrm{Re}\,(g|g)_{\sigma'\sigma}^{-}|\langle\sigma|D|\sigma'\rangle|^2 + 2\mathrm{Re}\,(g|g)_{\sigma\sigma'}^{+}|\langle\sigma'|D|\sigma\rangle|^2\right)-\right.$$

$$\left.-\rho(\sigma,t)\left(2\mathrm{Re}\,(g|g)_{\sigma'\sigma}^{+}|\langle\sigma|D|\sigma'\rangle|^2 + 2\mathrm{Re}\,(g|g)_{\sigma\sigma'}^{-}|\langle\sigma'|D|\sigma\rangle|^2\right)\right) \qquad (29)$$

with $A_{\mu\nu}$ given by (28) and $\rho(\sigma,t) = \rho(\sigma,\sigma,t)$.

For instance we get

$$j_t^*(|\mu\rangle\langle\nu|) = \exp(A_{\mu\nu}t)|\mu\rangle\langle\nu|$$

We see that if any of $\mathrm{Re}\,(g|g)|\langle\beta|D|\alpha\rangle|^2$ in (28) is non–zero then the corresponding off–diagonal matrix element of the density matrix decays. We obtain an effect of the diagonalization of the

density matrix. This gives an effective criterium for quantum decoherence in the stochastic approximation: the system will exhibit decoherence if the constants $\mathrm{Re}\,(g|g)^{\pm}$ are non–zero.

Now we estimate the velocity of decay of the density matrix $|\mu\rangle\langle\nu|$ for a quantum system with N particles. The eigenstate $|\mu\rangle$ of the Hamiltonian of such a system can be considered as a tensor product over degrees of freedom of the system of some substates. Let us estimate from below the number of degrees of freedom of the system by the number of particles that belong to the system (for each particle we have few degrees of freedom). To get the estimate from below for the velocity of decay we assume that $|\langle\sigma|D|\mu\rangle|^2$ in (28) is non–zero only if the state σ differs from the state μ only for one degree of freedom.

Then the summation over ω (or equivalently over σ) in (28) can be estimated by the summation over the degrees of freedom, or over particles belonging to the system. If we have total decoherence, i.e. all $\mathrm{Re}\,(g|g)$ are non–zero, then, taking all corresponding $|\langle\sigma|D|\mu\rangle|^2 = 1$, we can estimate (28) as $-N\mathrm{Re}\,(g|g)$, where N is the number of particles in the system, or

$$j_t^*(|\mu\rangle\langle\nu|) = \exp(-N\mathrm{Re}\,(g|g)t)|\mu\rangle\langle\nu| \tag{30}$$

The off–diagonal element of the density matrix decays exponentially, with the exponent proportional to the number of particles in the system. Therefore for macroscopic (large N) systems with decoherence the quantum state will collapse into the classical state very quickly.

This observation clarifies, why macroscopic quantum systems usually behave classically. The equation (30) describes such type of behavior, predicting that the quantum state damps at least as quickly as $\exp(-N\mathrm{Re}\,(g|g)t)$. Therefore a macroscopic system (large N) will become classical in a time of order $(N\mathrm{Re}\,(g|g))^{-1}$.

Let us estimate the constant $\mathrm{Re}\,(g|g)$ for the equilibrium state of the reservoir with the temperature $\beta^{-1} = kT$. In this case

$$\mathrm{Re}\,(g|g)_\omega^- = \frac{1}{\hbar}\frac{\pi}{1 - e^{-\beta\omega}}\int dk\,|g(k)|^2\delta(\omega(k) - \omega)$$

Taking $g(k) = 1$ and using that the dispersion function $\omega(k)$ depends only on $|k|$ we get

$$\int dk\,|g(k)|^2\delta(\omega(k) - \omega) = \int d\Omega\int_0^\infty d\rho\,\delta(\omega(\rho) - \omega) = 4\pi\int_0^\infty d\rho\,\delta(\omega(\rho) - \omega)$$

where $\int d\Omega$ is the integration over angles. If we take the dispersion function $\omega(k) = |k|$, then for this integral we get $4\pi\omega$.

Therefore for this choice of dispersion function we get

$$\mathrm{Re}\,(g|g)_\omega^- = \frac{1}{\hbar}\frac{4\pi^2\omega}{1 - e^{-\beta\omega}}$$

and analogously

$$\mathrm{Re}\,(g|g)_\omega^+ = \frac{1}{\hbar}\frac{4\pi^2\omega}{e^{\beta\omega} - 1}$$

In the limit of small β (or high temperatures) $\beta^{-1} = kT >> \omega$ both these integrals tend to

$$4\pi^2\frac{kT}{\hbar}$$

Summing up, we get that the constant $\mathrm{Re}\,(g|g)_\omega^\pm$ for the case of high temperature will be equal to $\frac{kT}{\hbar}$ up to multiplication by a dimensionless constant depending on the model.

This means that every degree of freeedom that energetically admissible ($kT >> \omega$) and not forbidden by the model ($|\langle\nu|D|\mu\rangle|^2 \neq 0$) gives the term of order $\frac{kT}{\hbar}$ in the exponent for dumping of off–diagonal matrix elements.

The off–diagonal matrix element will dump as $\exp(-tN\frac{kT}{\hbar})$, where N is the number of degrees of freedom (that for a generic system can be taken proportional to the number of particles). Off–diagonal matrix elements describe the quantum interference. Our result for the dumping of off–diagonal matrix elements (30) gives us a possibility to distinguish between microscopic systems (where quantum effects such as quantum interference are important) and macroscopic system which can be described by classical mechanics. The macroscopic system is a system satisfying

$$N\frac{kT}{\hbar} >> 1 \tag{31}$$

Actually the value $N\frac{kT}{\hbar}$ is of dimension of t^{-1}, and (31) means that $\left(N\frac{kT}{\hbar}\right)^{-1}$ is much less than the time of observation.

In the last section of the present paper we will illustrate the collapse phenomenon (30) using the quantum extension of the Glauber dynamics for a system of spins.

We see that the stochastic limit predicts the collapse of a quantum state into a classical state and, moreover, allows us to estimate the velocity of the collapse (30). One can consider (30) as a more detailed formulation of the Fermi golden rule: the Fermi golden rule predicts exponential decay of quantum states; formula (30) also relates the speed of the decay to the dimensions (number of particles) of the sstem.

Consider now the system density matrix $\hat\rho_S \in \mathcal{C}$, where $\mathcal{C}$ is the algebra generated by the spectral projections of the system Hamiltonian H_S, and consider the master equation (27) (we consider the generic case). We will find that the evolution defined by this master equation will conserve the algebra $\mathcal{C}$ and therefore will be a classical evolution. We will show that this classical evolution in fact describes quantum phenomena.

For $\hat\rho_{S,t} \in \mathcal{C}$ we define the evolved density matrix of the system

$$\hat\rho_{S,t} = \sum_\sigma \rho(\sigma,t)|\sigma\rangle\langle\sigma|$$

For this density matrix the master equation (25) takes the form

$$\frac{d}{dt}\rho(\sigma,t) = \sum_{\sigma'}\Big(\rho(\sigma',t)\big(2\mathrm{Re}\,(g|g)_{\sigma'\sigma}^-|\langle\sigma|D|\sigma'\rangle|^2 + 2\mathrm{Re}\,(g|g)_{\sigma\sigma'}^+|\langle\sigma'|D|\sigma\rangle|^2\big) -$$

$$-\rho(\sigma,t)\big(2\mathrm{Re}\,(g|g)_{\sigma'\sigma}^+|\langle\sigma|D|\sigma'\rangle|^2 + 2\mathrm{Re}\,(g|g)_{\sigma\sigma'}^-|\langle\sigma'|D|\sigma\rangle|^2\big)\Big) \tag{32}$$

Let us note that if $\rho(\sigma,t)$ satisfies the detailed balance condition

$$\rho(\sigma,t)2\mathrm{Re}\,(g|g)_{\sigma\sigma'}^- = \rho(\sigma',t)2\mathrm{Re}\,(g|g)_{\sigma\sigma'}^+ \tag{33}$$

then $\rho(\sigma,t)$ is the stationary solution for (32).

Let us investigate (32), (33) for the equilibrium state of the field. In this case

$$2\text{Re}\,(g|g)^-_{\sigma\sigma'} = 2\pi \int dk\,|g(k)|^2\delta(\omega(k) + \varepsilon_{\sigma'} - \varepsilon_\sigma)\frac{1}{1 - e^{-\beta\omega(k)}} =$$

$$= 2\pi \int dk\,|g(k)|^2\delta(\omega(k) + \varepsilon_{\sigma'} - \varepsilon_\sigma)\frac{1}{1 - e^{-\beta(\varepsilon_\sigma-\varepsilon_{\sigma'})}} = \frac{C_{\sigma\sigma'}}{1 - e^{-\beta(\varepsilon_\sigma-\varepsilon_{\sigma'})}}$$

$$2\text{Re}\,(g|g)^+_{\sigma\sigma'} = \frac{C_{\sigma\sigma'}}{e^{\beta(\varepsilon_\sigma-\varepsilon_{\sigma'})} - 1}$$

The equation (32) takes the form

$$\frac{d}{dt}\rho(\sigma,t)e^{\beta\varepsilon_\sigma} = \sum_{\sigma'}\frac{C_{\sigma\sigma'}|\langle\sigma'|D|\sigma\rangle|^2 - C_{\sigma'\sigma}|\langle\sigma|D|\sigma'\rangle|^2}{1 - e^{-\beta(\varepsilon_\sigma-\varepsilon_{\sigma'})}}\left(\rho(\sigma',t)e^{\beta\varepsilon_{\sigma'}} - \rho(\sigma,t)e^{\beta\varepsilon_\sigma}\right) \qquad (34)$$

Let us note that $C_{\sigma\sigma'}$ are non–zero (and therefore positive) only if denominators in (34) are positive and $C_{\sigma'\sigma}$ are non–zero only if the corresponding denominators are negative.

If the system possesses decoherence then $C_{\sigma\sigma'}$, $C_{\sigma'\sigma}$ are non–zero and the solution of equation (34) for $t \to \infty$ tends to the stationary solution given by the detailed balance condition (33)

$$\frac{\rho(\sigma,t)}{1 - e^{-\beta(\varepsilon_\sigma-\varepsilon_{\sigma'})}} = \frac{\rho(\sigma',t)}{e^{\beta(\varepsilon_\sigma-\varepsilon_{\sigma'})} - 1}$$

or

$$\rho(\sigma,t)e^{\beta\varepsilon_\sigma} = \rho(\sigma',t)e^{\beta\varepsilon_{\sigma'}}$$

This means that the stationary solution (33) of (32) describes the equilibrium state of the system

$$\rho(\sigma,t) = \frac{e^{-\beta\varepsilon_\sigma}}{\sum_{\sigma'} e^{-\beta\varepsilon_{\sigma'}}}$$

For a system with decoherence the density matrix will tend, as $t \to \infty$, to the stationary solution (33) of (32). In particular, as $t \to \infty$, the density matrix collapses to the classical Gibbs distribution.

The phenomenon of a collapse of a quantum state into a classical state is connected with the quantum measurement procedure. The quantum uncertainty will be concentrated at the degrees of freedom of the quantum field and vanishes after the averaging procedure. One can speculate that the collapse of the wave function is a property of open quantum systems: we can observe the collapse of the wave function of the system averaging over the degrees of freedom of the reservoir interacting with the system. Usually the collapse of a wave function is interpreted as a projection onto a classical state (the von Neumann interpretation). The picture emerging from our considearations is more general: the collapse is a result of the unitary quantum evolution and conditional expectation (averaging over the degrees of freedom of quantum field). This is a generalization of the projection: it is easy to see that every projection P generates a (non identity preserving) conditional expectation $E_P(X) = PXP$, more generally a set of projections P_i generates the conditional expectation

$$\sum_i \alpha_i E_{P_i}, \qquad \alpha_i \geq 0$$

but not every conditional expectation could be given in this way.

We have found the effect of the collapse of density matrix for $\rho(t) = \langle U_t \rho U_t^* \rangle$, where $U_t = \lim_{\lambda \to 0} e^{itH_0} e^{-itH}$ is the stochastic limit of interacting evolution. The same effect of collapse will be valid for the limit of the full evolution e^{-itH}, because the full evolution is the composition of interacting and free evolution. The free evolution leaves invariant the elements of diagonal subalgebra and multiplies the considered above nondiagonal element $|\sigma'\rangle\langle\sigma|$ by the oscillating factor $e^{it(\varepsilon_{\sigma'} - \varepsilon_\sigma)}$. Therefore for the full evolution we get the additional oscillating factor, and the collapse phenomenon will survive.

5   Control of coherence

In this section we generalize the approach of [3] and investigate different regimes of qualitative behavior for the considered model.

The master equation (32) at first sight looks completely classical. In the present paper we derived this equation using quantum arguments. Now we will show that (32) in fact describes a quantum behavior. To show this we consider the following example.

Let us rewrite (32) using the particular form (16) of $(g|g)^\pm$. Using (15), (16) we get

$$\frac{d}{dt}\rho(\sigma, t) = \sum_{\sigma'} 2\pi \int dk \, |g(k)|^2 \Big((N(k) + 1)$$

$$\Big(\rho(\sigma', t)\delta(\omega(k) + \varepsilon_\sigma - \varepsilon_{\sigma'})|\langle\sigma|D|\sigma'\rangle|^2 - \rho(\sigma, t)\delta(\omega(k) + \varepsilon_{\sigma'} - \varepsilon_\sigma)|\langle\sigma'|D|\sigma\rangle|^2 \Big) +$$

$$+ N(k)\Big(\rho(\sigma', t)\delta(\omega(k) + \varepsilon_{\sigma'} - \varepsilon_\sigma)|\langle\sigma'|D|\sigma\rangle|^2 - \rho(\sigma, t)\delta(\omega(k) + \varepsilon_\sigma - \varepsilon_{\sigma'})|\langle\sigma|D|\sigma'\rangle|^2 \Big) \Big) \qquad (35)$$

The first term (integrated with $N(k) + 1$) on the RHS of this equation describes the emission of bosons and the second term (integrated with $N(k)$) describes the absorption of bosons. For the emission term the part with $N(k)$ describes the induced emission and the part with 1 the spontaneous emission of bosons.

Let us note that the Einstein relation for probabilities of emission and absorption of bosons with quantum number k

$$\frac{\text{probability of emission}}{\text{probability of absorption}} = \frac{N(k) + 1}{N(k)}$$

is satisfied in the stochastic approximation.

The formula (35) describes a macroscopic quantum effect. To show this let us take the spectrum of a system Hamiltonian (the set of system states $\Sigma = \{\sigma\}$) as follows: let Σ contain two groups Σ_1 and Σ_2 of states with the energy gap between these groups (or, for simplicity, two states σ_1 and σ_2 with $\varepsilon_{\sigma_2} > \varepsilon_{\sigma_1}$). This type of Hamiltonian was considered in different models of quantum optics, see for review [10] (for the case of two states we get the spin–boson Hamiltonian investigated in [3] using the stochastic limit). Let the state $\langle \cdot \rangle$ of the bosonic field be taken in such a way that the density $N(k)$, of quanta of the bosonic field, has support in a set of momentum variables k such that

$$0 < \omega(k) < \omega_0 < |\varepsilon_{\sigma_1} - \varepsilon_{\sigma_2}|, \qquad k \in \operatorname{supp} N(k) \qquad (36)$$

This means that high–energetic bosons are absent. It is natural to consider the state $\langle \cdot \rangle$ as a sum of equilibrium state at temperature β^{-1} and non–equilibrium part. Therefore the density $N(k)$ will be non–zero for small k because the equilibrium state satisfies this property.

Under the considered assumption (36) the integral of δ–function $\delta(\omega(k) + \varepsilon_{\sigma_1} - \varepsilon_{\sigma_2})$ with $N(k)$ in (35) equals identically to zero. Therefore the RHS of (35) will be equal to

$$\sum_{\sigma'} 2\pi \int dk \, |g(k)|^2 \Big(\rho(\sigma', t)\delta(\omega(k) + \varepsilon_\sigma - \varepsilon_{\sigma'})|\langle \sigma|D|\sigma' \rangle|^2 -$$

$$-\rho(\sigma, t)\delta(\omega(k) + \varepsilon_{\sigma'} - \varepsilon_\sigma)|\langle \sigma'|D|\sigma \rangle|^2 \Big)$$

It is natural to consider this value (corresponding to the spontaneous emission of bosons by the system) as small with respect to the induced emission (for $N(k) >> 1$). In this case the density matrix $\rho(\sigma, t)$ will be almost constant in time. This is an effect of conservation of quantum coherence: in the absence of bosons with the energy $\omega(k)$ equal to $\varepsilon_{\sigma_1} - \varepsilon_{\sigma_2}$ the system cannot jump between the states σ_1 and σ_2 (or, at least, this transition is very slow), because in the stochastic limit such jump corresponds to quantum white noise that must be on a mass shell.

At the same time, the transitions between states inside the groups Σ_1 and Σ_2 are not forbidden by (36), because these transitions are connected with the soft bosons (with small k) that are present in the equilibrium part of $\langle \cdot \rangle$. In the above assumptions equation (35) describes the transition of the system to intermediate equilibrium, where the transitions between groups of states Σ_1 and Σ_2 are forbidden.

If the state $\langle \cdot \rangle$ does not satisfy the property (36), then the system undergoes fast transitions between states σ_1 and σ_2. We can switch on such a transition by switching on the bosons with the frequency $\omega(k) = \varepsilon_{\sigma_2} - \varepsilon_{\sigma_1}$.

In conclusion: equation (35) describes a macroscopic quantum effect controlled by the distribution of bosons $N(k)$ which can be physically controlled for example by filtering.

6   The Glauber dynamics

In the present section we apply the master equation (32) to the derivation of the quantum extension of the classical Glauber dynamics. The Glauber dynamics is a dynamics for a spin lattice with nearest neighbor interaction, see [11], [12]. We will prove that the Glauber dynamics can be considered as a dynamics generated by the master equation of the type (32) derived from a stochastic limit for a quantum spin system interacting with a bosonic quantum field.

We take the bosonic reservoir space $\mathcal{F}$ corresponding to the bosonic equilibrium state at temperature β^{-1}. Thus the reservoir state is Gaussian with mean zero and correlations given by

$$\langle a^*(k)a(k') \rangle = \frac{1}{e^{\beta\omega(k)} - 1}\delta(k - k')$$

For simplicity we only consider the case of a one dimensional spin lattice, but our considerations extend without any change to multi–dimensional spin lattices.

The spin variables are labeled by integer numbers Z, and, for each finite subset $\Lambda \subseteq Z$ with cardinality $|\Lambda|$, the system Hilbert space is

$$\mathcal{H}_S = \mathcal{H}_\Lambda = \otimes_{r \in \Lambda} C^2$$

and the system Hamiltonian has the form

$$H_S = H_\Lambda = -\frac{1}{2} \sum_{r,s \in \Lambda} J_{rs} \sigma_r^z \sigma_s^z$$

where σ_r^x, σ_r^y, σ_r^z are Pauli matrices ($r \in \Lambda$) at the r-th site in the tensor product

$$\sigma_r^i = 1 \otimes \cdots \otimes 1 \otimes \sigma^i \otimes 1 \otimes \cdots \otimes 1$$

For any r, $s \in \Lambda$

$$J_{rs} = J_{sr} \in R, \qquad J_{rr} = 0$$

We consider for simplicity the system Hamiltonian that describes the interaction of spin with the nearest neighbors (Ising model):

$$J_{rs} = J_{r,r+1}$$

The interaction Hamiltonian H_I (acting in $\mathcal{H}_S \otimes \mathcal{F}$) has the form

$$H_I = \sum_{r \in \Lambda} \sigma_r^x \otimes \psi(g_r), \quad \psi(g) = A(g) + A^*(g), \quad A(g) = \int dk\, \overline{g}(k) a(k),$$

where ψ is a field operator, $A(g)$ is a smeared quantum field with cutoff function (form factor) $g(k)$.

The eigenvectors $|\sigma\rangle$ of the system Hamiltonian H_S can be labeled by spin configurations σ (sequences of ± 1), which label the natural basis in $\mathcal{H}_S$ consisting of tensor products of eigenvectors of σ_r^z (spin up and spin down vectors $|\varepsilon_r\rangle$, corresponding to eigenvalues $\varepsilon_r = \pm 1$)

$$|\sigma\rangle = \otimes_{r \in \Lambda} |\varepsilon_r\rangle$$

In the present section we denote ε_r the energy of the spin at site r, and denote as $E(\sigma)$ the energy of the spin configuration σ

$$E(\sigma) = -\frac{1}{2} \sum_{r,s \in \Lambda} J_{rs} \varepsilon_r \varepsilon_s$$

The action of the operator σ_r^x on the spin configuration σ is defined using the action of σ_r^x on the corresponding eigenvector $|\sigma\rangle$: so the operator σ_r^x flips the spin at the r–th site in the sequence σ (i.e. it maps the vector $|\varepsilon_r\rangle$ in the tensor product into the vector $|-\varepsilon_r\rangle$). From the form of H_S and H_I it follows that, in (32), the matrix element $\langle \sigma | D | \sigma' \rangle$ of any two eigenvectors, corresponding to the spin configurations σ, σ', will be non–zero only if the configurations σ, σ' differ exactly at one site. If the configurations σ, σ' differ exactly at one site then $\langle \sigma | D | \sigma' \rangle = 1$.

The (classical) Glauber dynamics will be given by the master equation for the density matrix laying in the algebra of spectral projections of the system Hamiltonian (32)

$$\frac{d}{dt}\rho(\sigma,t) = \sum_{r \in \Lambda} \left(\rho(\sigma_r^x \sigma, t) \left(2\mathrm{Re}\,(g|g)_{\sigma_r^x \sigma, \sigma}^- + 2\mathrm{Re}\,(g|g)_{\sigma, \sigma_r^x \sigma}^+ \right) - \right.$$

$$\left. -\rho(\sigma, t) \left(2\mathrm{Re}\,(g|g)_{\sigma_r^x \sigma, \sigma}^+ + 2\mathrm{Re}\,(g|g)_{\sigma, \sigma_r^x \sigma}^- \right) \right) \tag{37}$$

that gives the Glauber dynamics of a system of spins, see [11], [12]. Here

$$2\text{Re}\,(g|g)^-_{\sigma,\sigma^x_r\sigma} = 2\pi \int dk\,|g(k)|^2 \delta(\omega(k) - J_{r-1,r}\varepsilon_{r-1} - J_{r,r+1}\varepsilon_{r+1})\frac{1}{1 - e^{-\beta\omega(k)}} \tag{38}$$

and analogously all the other $(g|g)^\pm$.

Up to now we have investigated the dynamics for the diagonal part of the density matrix. The master equation for the off–diagonal part of the density matrix (25) will give the quantum extension of the Glauber dynamics. We consider now this off–diagonal part:

$$\sum_{\mu\neq\nu} \rho(\mu,\nu,t)|\mu\rangle\langle\nu|$$

From (25), (28) we obtain the equation for the off–diagonal elements of the density matrix

$$\frac{d}{dt}\rho(\mu,\nu,t) = A_{\mu\nu}\rho(\mu,\nu,t) \tag{39}$$

$$A_{\mu\nu} = \sum_{r\in\Lambda}\Bigg(i\text{Im}\,(g|g)^-_{\mu,\sigma^x_r\mu} - i\text{Im}\,(g|g)^-_{\nu,\sigma^x_r\nu} - i\text{Im}\,(g|g)^+_{\sigma^x_r\mu,\mu} + i\text{Im}\,(g|g)^+_{\sigma^x_r\nu,\nu}$$

$$-\text{Re}\,(g|g)^-_{\mu,\sigma^x_r\mu} - \text{Re}\,(g|g)^-_{\nu,\sigma^x_r\nu} - \text{Re}\,(g|g)^+_{\sigma^x_r\mu,\mu} - \text{Re}\,(g|g)^+_{\sigma^x_r\nu,\nu}\Bigg) \tag{40}$$

Equations (37), (39), (40) describe the quantum extension of the classical Glauber dynamics (37). As it was already noted in section 4, the coefficient $A_{\mu\nu}$ in (40) is proportional to $|\Lambda|$ (the number of particles in the system). Due to the summation on $r \in \Lambda$ the coefficient $A_{\mu\nu}$ will diverge for large $|\Lambda|$ (the real part of $A_{\mu\nu}$ will tend to $-\infty$). Therefore the density matrix will collapse to the diagonal subalgebra (the classical distribution function) very quickly.

Let us consider now the particular case of one dimensional system with translationally invariant Hamiltonian:

$$J_{rs} = J_{r,r+1} = J > 0$$

The translationally invariant Hamiltonian does not satisfy the generic non degeneracy conditions on the system spectrum that we have used in the derivation of equations (37), (39) and therefore we cannot apply these equations to describe the dynamics for this Hamiltonian.

However in the translation invariant one–dimensional case we can investigate these equations by direct methods.

In this case the $(g|g)^\pm$, given by (38), are non–zero only if $\varepsilon_{r-1} = \varepsilon_{r+1} = 1$, and we get for (38)

$$2\text{Re}\,(g|g)^-_{\sigma,\sigma^x_r\sigma} = 2\pi \int dk\,|g(k)|^2 \delta(\omega(k) - 2J)\frac{1}{1 - e^{-2\beta J}} = \frac{C}{1 - e^{-2\beta J}} \tag{41}$$

Therefore for one–dimensional translation invariant Hamiltonians we get for (37), compare with [11], [12]

$$\frac{d}{dt}\rho(\sigma,t) = \frac{C}{1 - e^{-2\beta J}}\Bigg(\sum_{r\in\Lambda;E(\sigma)>E(\sigma^x_r\sigma)}\Big(e^{-2\beta J}\rho(\sigma^x_r\sigma,t) - \rho(\sigma,t)\Big) +$$

$$+ \sum_{r\in\Lambda;E(\sigma)<E(\sigma^x_r\sigma)}\Big(\rho(\sigma^x_r\sigma,t) - e^{-2\beta J}\rho(\sigma,t)\Big)\Bigg) \tag{42}$$

18

The detailed balance stationary solution of (42) satisfy the following: for two spin configurations σ, $\sigma_r^x\sigma$ that differ by the flip of spin at site r the energy of corresponding configurations differ by $2J$. The expectation $\rho(\mu)$, $\mu = \sigma, \sigma_r^x\sigma$ of configuration with the higher energy will be $e^{-2\beta J}$ times less.

For the off–diagonal part of the density matrix for the case of one–dimensional translation invariant Hamiltonian the terms in the imaginary part of (40) cancel and using (41) we get for (40)

$$A_{\mu\nu} = -\sum_{r\in\Lambda}\left(\frac{2C}{1-e^{-2\beta J}} + \frac{2C}{e^{2\beta J}-1}\right) = -2C\sum_{r\in\Lambda}\frac{1+e^{-2\beta J}}{1-e^{-2\beta J}}$$

This sum, over r, of equal terms diverges with $|\Lambda| \to \infty$. Therefore the off–diagonal elements of the density matrix that satisfy (39) will decay very quickly and for sufficiently large t, $|\Lambda|$ the dynamics of the system will be given by the classical Glauber dynamics.

For the master equation considered above we used the master equation for generic (non–degenerate) Hamiltonian. This gives us the Glauber dynamics. But the translation invariant Hamiltonian is degenerate. Therefore in the translation invariant case we will get some generalization of the Glauber dynamics. To derive this generalization let us consider the general form (26) of the master equation. For the considered spin system this gives

$$\sum_{\mu,\nu}\frac{d}{dt}\rho(\mu,\nu,t)|\mu\rangle\langle\nu| = \sum_{\mu,\nu}\rho(\mu,\nu,t)\sum_{a,b\in\Lambda}\left(i\left(|\mu\rangle\langle P_{E(\nu)}\sigma_a^x P_{\nu-}\sigma_b^x\nu| - |P_{E(\mu)}\sigma_a^x P_{\mu-}\sigma_b^x\mu\rangle\langle\nu|\right) + \right.$$

$$+\frac{C}{1-e^{-2\beta J}}\left(|P_{E(\mu)-J}\sigma_a^x\mu\rangle\langle P_{E(\nu)-J}\sigma_b^x\nu| - \frac{1}{2}\left(|\mu\rangle\langle P_{E(\nu)}\sigma_a^x P_{E(\nu)-J}\sigma_b^x\nu| + |P_{E(\mu)}\sigma_a^x P_{E(\mu)-J}\sigma_b^x\mu\rangle\langle\nu|\right)\right) -$$

$$-i\left(|\mu\rangle\langle P_{E(\nu)}\sigma_a^x P_{\nu+}\sigma_b^x\nu| - |P_{E(\mu)}\sigma_a^x P_{\mu+}\sigma_b^x\mu\rangle\langle\nu|\right) +$$

$$\left.+\frac{C}{e^{2\beta J}-1}\left(|P_{E(\mu)+J}\sigma_a^x\mu\rangle\langle P_{E(\nu)+J}\sigma_b^x\nu| - \frac{1}{2}\left(|\mu\rangle\langle P_{E(\nu)}\sigma_a^x P_{E(\nu)+J}\sigma_b^x\nu| + |P_{E(\mu)}\sigma_a^x P_{E(\mu)+J}\sigma_b^x\mu\rangle\langle\nu|\right)\right)\right)$$

$$(43)$$

Here C is given by (41), operator $P_{E(\mu)}$ is a projector onto the states with the energy $E(\mu)$, operator $P_{\nu-}$ is given by

$$P_{\nu-} = \text{Im}\ (g|g)_{-1}^- P_{E(\nu)-J} + \text{Im}\ (g|g)_0^- P_{E(\nu)} + \text{Im}\ (g|g)_1^- P_{E(\nu)+J}$$

$$\text{Im}\ (g|g)_a^- = \text{P.P.}\int dk\,|g(k)|^2\frac{1}{\omega(k)+aJ}\frac{1}{1-e^{-\beta\omega(k)}}, \qquad a = -1, 0, 1$$

For the operator $P_{\nu+}$ we get the analogous expression

$$P_{\nu+} = \text{Im}\ (g|g)_{-1}^+ P_{E(\nu)-J} + \text{Im}\ (g|g)_0^+ P_{E(\nu)} + \text{Im}\ (g|g)_1^+ P_{E(\nu)+J}$$

with the coefficients $(g|g)_a^+$:

$$(g|g)_a^+ = \text{P.P.}\int dk\,|g(k)|^2\frac{1}{\omega(k)-aJ}\frac{1}{e^{\beta\omega(k)}-1}, \qquad a = -1, 0, 1$$

The equation (43) gives the quantum generalization of the Glauber dynamics. The matrix elements $\rho(\mu,\nu,t)$ of the density matrix corresponding to the states μ, ν with different energies will decay quickly. But for the translation invariant Hamiltonian there exist different μ, ν with equal energies. Corresponding matrix element will decay with the same speed as the diagonal elements of the density matrix. Moreover one can expect non–ergodic behavior for this model. Therefore the generalization (43) of the Glauber dynamics is non–trivial.

19

6.1   Evolution for subalgebra of local operators

In this section to compare with the results of [4] we consider the dynamics of spin systems, described in the previous section, for Hamiltonian with non necessarily finite set of spins Λ but for local observable X.

The observable X is local if it belongs to the local algebra, that is UHF–algebra (uniformly hyperfinite algebra)

$$\mathcal{A} = \bigcup_{\Lambda \text{ is finite}} \mathcal{A}_\Lambda$$

where $\mathcal{A}_\Lambda$ is the $*$–algebra generated by the elements

$$\otimes_i X_i, \qquad X_i = 1 \text{ for } i \notin \Lambda$$

Consider now the action of θ_0 on local X

$$\theta_0(X) = \sum_{ij} \sum_{\omega \in F} \Big(-i \operatorname{Im} (g_i|g_j)_\omega^- [X, E_\omega^*(D_i) E_\omega (D_j)] + i \operatorname{Im} (g_i|g_j)_\omega^+ [X, E_\omega(D_i) E_\omega^*(D_j)] +$$

$$+ 2\operatorname{Re}(g_i|g_j)_\omega^- \left(E_\omega^*(D_i) X E_\omega(D_j) - \frac{1}{2}\{X, E_\omega^*(D_i) E_\omega(D_j)\} \right) +$$

$$+ 2\operatorname{Re}(g_i|g_j)_\omega^+ \left(E_\omega(D_i) X E_\omega^*(D_j) - \frac{1}{2}\{X, E_\omega(D_i) E_\omega^*(D_j)\} \right) \Big) \tag{44}$$

For $E_\omega(D_i)$ we get

$$E_\omega(D_i) = \sum_{E(r) \in F_\omega} P_{E(r)-\omega} D_i P_{E(r)} =$$

$$= 1 \otimes |\varepsilon_{i-1}\rangle\langle\varepsilon_{i-1}| \otimes |-\varepsilon_i\rangle\langle\varepsilon_i| \otimes |\varepsilon_{i+1}\rangle\langle\varepsilon_{i+1}| \otimes 1 + 1 \otimes |-\varepsilon_{i-1}\rangle\langle-\varepsilon_{i-1}| \otimes |\varepsilon_i\rangle\langle-\varepsilon_i| \otimes |-\varepsilon_{i+1}\rangle\langle-\varepsilon_{i+1}| \otimes 1; \tag{45}$$

with the frequency ω of the following form

$$\omega = J_{i-1,i}\varepsilon_{i-1} + J_{i,i+1}\varepsilon_{i+1}$$

Therefore the operator E_ω given by (45) is local and moreover, corresponding map θ_0 given by (44) maps $\mathcal{A}$ into itself.

The formula (45) explains the physical meaning of the operator $E_\omega(D_i)$. For positive ω this it flips the spin at site i along the direction of the mean field of its neighbors (for negative ω it flips the same spin into the opposite direction).

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