UNITARITY CONDITIONS FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY NONLINEAR QUANTUM NOISE

Luigi Accardi

Centro Vito Volterra, Università di Roma Tor Vergata

00133 Roma, Italy

Andreas Boukas

American College of Greece

Athens 153 42, Greece

Contents

Abstract.

We prove the stochastic independence of the basic integrators of the renormalized square of white noise (SWN). We use this result to deduce the unitarity conditions for stochastic differential equations driven by the SWN.

1 Introduction

Linear quantum stochastic calculus on the Boson Fock space, as developed in [7], is associated with the stochastic differentials

$$dB(t) = b(t)dt \quad , \quad dB^+(t) = b^+(t)dt \quad , \quad dN(t) = b^+(t)b(t)dt$$

corresponding to functionals of the Boson Fock *white noise* b, b^+ satisfying the commutation relation $[b(t), b^+(s)] = \gamma \cdot \delta(t-s)$ where $\gamma > 0$ is the *variance* of the *quantum Brownian motion* defined by B and B^+, and δ is the delta function (cf. [2], [5]). A general, representation free, quantum stochastic calculus which included [7] and all other known examples of linear quantum noise was developed in [1] (see also [4], [5], [6]).

The theory has recently been extended (cf. [2], [3]) to include *normally ordered* nonlinear stochastic differentials of the form

$$dB_{(m,n)} = b^+(t)^m b(t)^n dt$$

where $m, n \in \{0, 1, \ldots\}$. This extension required the introduction, in classical probability theory, of renormalization techniques, widely used in quantum field theory.

The white noise b^+, b is defined as follows: let $L^2_{\text{sym}}(\mathbf{R}^n)$ denote the space of square integrable functions on $\mathbf{R}^n$ symmetric under permutation of their arguments, and let

$$F = \bigoplus_{n=0}^{\infty} L^2_{\text{sym}}(\mathbf{R}^n)$$

where: if $\psi = \{\psi^{(n)}\}_{n=1}^{\infty} \in F$, then $\psi^{(0)} \in \mathbf{C}$, $\psi^{(n)} \in L^2_{\text{sym}}(\mathbf{R}^n)$, and

$$\|\psi\|^2 = |\psi(0)|^2 + \sum_{n=1}^{\infty} \int_{R^n} |\psi^{(n)}(s_1, \ldots, s_n)|^2 ds_1 \ldots ds_n$$

Denote by $S \subset L^2(\mathbf{R}^n)$ the Schwartz space of smooth functions decreasing at infinity faster than any polynomial and let

$$D = \left\{ \psi \in F | \psi^{(n)} \in S, \ \sum_{n=1}^{\infty} n \|\psi^{(n)}\|^2 < \infty \right\}$$

For each $t \in \mathbf{R}$ define the linear operator $b(t) : D \to F$ by

$$(b(t)\psi)^{(n)}(s_1, \ldots, s_n) = \sqrt{n+1}\psi^{(n+1)}(t, s_1, \ldots, s_n)$$

and the operator valued distribution $b^+(t)$ by

$$(b^+(t)\psi)^{(n)}(s_1, \ldots, s_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\infty} \delta(t - s_i)\psi^{(n-1)}(s_1, \ldots, \hat{s}_i, \ldots, s_n)$$

Then

$$B(t) = \int_0^t b(s)ds \ , B^+(t) = \int_0^t b^+(s)ds \ , N(t) = \int_0^t b^+(s)b(s)ds$$

are, for each t, operators acting on D. The *renormalized Itô table*, proposed in [3], for the stochastic differentials dt, dB, dB^+, $dB_2 = dB_{(0,2)}$, $dB_2^+ = dB_{(2,0)}$, and dN is

	dt	dB	dB^+	dB_2	dB_2^+	dN
dt	0	0	0	0	0	0
dB	0	0	γdt	0	$2\gamma dB^+$	γdB
dB^+	0	0	0	0	0	0
dB_2	0	0	$2\gamma dB$	0	$4\gamma dN$	$2\gamma dB_2$
dB_2^+	0	0	0	0	0	0
dN	0	0	γdB^+	0	$2\gamma dB_2^+$	γdN

(1)

Since $L^2_{\text{sym}}(\mathbf{R}^n) = L^2_{\text{sym}}(\mathbf{R}^{\otimes n})$ we can identify F with the symmetric (Boson) Fock space over S. In the case when the elements of S are defined on $[0, +\infty)$ we denote the Fock space by $\Gamma(S_+)$. If $\psi = \{(n!)^{-1/2} f^{\otimes n}\}$ we denote ψ by $\psi(f)$. With these notations:

$$b_t\psi(f) = f(t)\psi(f),\ b_t^2\psi(f) = f(t)^2\psi(f),\ \langle\psi(g), b_t^+ b_t\psi(f)\rangle = \overline{g(t)}f(t)\langle\psi(g), \psi(f)\rangle \quad (2)$$

Throughout this paper b_t^{+2} will be interpreted as a quadratic form defined on the linear span of the exponential vectors.

We couple $\Gamma(S_+)$ with an *initial* Hilbert space H_0 and we define an *adapted process* $A = \{A(t) \ : \ t \geq 0\}$ to be a family of operators on $H_0 \otimes \Gamma(S_+)$ such that for each t, $A(t) = A_t \otimes 1$ where A_t acts on $H_0 \otimes \Gamma(S_+^{t]})$ and 1 is the identity operator on $\Gamma(S_+^{(t})$, where $S_+^{t]} = \{f \cdot \chi_{[0,t]}/f \in S\}$ and $S_+^{(t} = \{f \cdot \chi_{(t,+\infty)}/f \in S\}$. If, for each t, $A(t) = A \otimes 1$, where A is on operator on H_0 and 1 is the identity on $\Gamma(S_+)$, then A is a *constant process*. If, for each t, $A(t)$ is a (locally) bounded operator then A is a *(locally) bounded process*.

In what follows we identify $B(t)$, $B^+(t)$, $B_2(t)$, $B_2^+(t)$, and $N(t)$ with $1 \otimes B(t)$, $1 \otimes B^+(t), \ldots, 1 \otimes N(t)$ where 1 is the identity on H_0. For a constant adapted process $A = \{A(t)/t \geq 0\}$ we denote $A(t)$ simply by A.

Once a quantum stochastic calculus has been constructed, one usually considers the problem of finding conditions under which stochastic differential equations driven by quantum noise admit unitary solutions. It is well known (cf. [7]) that the unique solution $U = \{U(t) \ : \ t \geq 0\}$ of the initial value problem

$$dU(t) = \left[\left(iH - \frac{1}{2}\, L^* L\right) dt - L^* W dB(t) + L dB^+(t) + (W-1) dN(t)\right] U(t) \quad (3)$$

$$U(0) = 1 \ , \quad 0 \leq t \leq T < +\infty$$

where L, H, W are bounded, constant adapted processes with H self–adjoint and W is unitary i.e. $U(t)U^*(t) = U^*(t)U(t) = 1$ for each t.

In this note we discuss, in Sections 2 and 3, the unitarity of the solution U of the initial value problem

$$dU(t) = [A_1 dt + A_2 dB(t) + A_3 dB^+(t) + A_4 dB_2(t) + A_5 dB_2^+(t) + A_6 dN(t)] U(t) \quad (4)$$

$$U(0) = 1 \ , \qquad 0 \leq t \leq T < +\infty$$

where the coefficients A_1, $A_2, \ldots, A_6$ are bounded, constant adapted processes.

The derivation of the unitarity conditions depends on the linear independence of the stochastic differentials which is established in Section 4.

For an operator K we denote its adjoint by K^* while its real part is $\mathrm{Re}\ K = \frac{K+K^*}{2}$.

2 Necessary and sufficient unitarity conditions

In this note we suppose that equation (4) has a solution, defined as a quadratic form on the exponential vectors and we also assume that the expression

$$\langle U(t)\psi(f), U(t)\psi(g)\rangle$$

has a meaning as a quadratic form on the exponential vectors. Under these assumptions we study under which conditions on the coefficients of equation (1.4), the solution of this equation is unitary, in the sense that the identity between quadratic forms on the exponential vectors

$$\langle U(t)\psi(f), U(t)\psi(g)\rangle = \langle \psi(f), \psi(g)\rangle$$

takes place.

Proposition 1 The solution U of (4) is unitary if and only if

$$A_1 + A_1^* + A_2A_2^*\gamma = 0 \tag{5}$$

$$A_2 + A_3^* + A_4A_2^*2\gamma + A_2A_6^*\gamma = 0 \tag{6}$$

$$A_4 + A_5^* + A_4A_6^*2\gamma = 0 \tag{7}$$

$$A_6 + A_6^* + A_4A_4^*4\gamma + A_6A_6^*\gamma = 0 \tag{8}$$

and

$$A_1^* + A_1 + A_3^*A_3\gamma = 0 \tag{9}$$

$$A_3^* + A_2 + A_3^*A_6\gamma + A_5^*A_3 2\gamma = 0 \tag{10}$$

$$A_5^* + A_4 + A_5^*A_6 2\gamma = 0 \tag{11}$$

$$A_6^* + A_6 + A_5^*A_5 4\gamma + A_6^*A_6\gamma = 0 \tag{12}$$

Proof: U is unitary if and only if for each $t \in [0,T]$, $U(t)U^*(t) = U^*(t)U(t) = 1$. Since $U(0) = U^*(0) = 1$, $U(t)U^*(t) = 1 \Leftrightarrow d(U(t)U^*(t)) = 0 \Leftrightarrow dU(t)\cdot U^*(t) + U(t)\cdot dU^*(t) + dU(t)\cdot dU^*(t) = 0$ which by Itô's table (1.1) is equivalent to

$$(A_1+A_1^*+A_2A_2^*\gamma)dt+(A_2+A_3^*+A_4A_2^*2\gamma+A_2A_6^*\gamma)dB(t)+(A_3+A_2^*+A_2A_4^*2\gamma+ A_6A_2^*\gamma)dB^+(t) + (A_4 + A_5^* + A_4A_6^*2\gamma)dB_2(t) + (A_5 + A_4^* + A_6A_4^*2\gamma)dB_2^+(t)+ (A_6 + A_6^* + A_4A_4^*4\gamma + A_6A_6^*\gamma)dN(t) = 0 \tag{13}$$

By the linear independence of the stochastic differentials (see Proposition 38), (13) is equivalent to (5) – (8).

Similarly $U^*(t)U(t) = 1$ is equivalent to (9) – (12). □

Remark: By (6) and (10), A_2 and A_3 are either both zero or both nonzero. The same is true, by (7) and (11), for A_4 and A_5.

Corollary 2.1: The solution U of

$$dU(t) = [A_1 dt + A_4 dB_2(t) + A_5 dB_2^+(t) + A_6 dN(t)]U(t) \tag{14}$$

$$U(0) = 1 \ , \qquad 0 \le t \le T < +\infty$$

is unitary if and only if

$$A_1 + A_1^* = 0 \tag{15}$$

$$A_4 + A_5^* + A_4 A_6^* 2\gamma = 0 \tag{16}$$

$$A_6 + A_6^* + A_4 A_4^* 4\gamma + A_6 A_6^* \gamma = 0 \tag{17}$$

$$A_5^* + A_4 + A_5^* A_6 2\gamma = 0 \tag{18}$$

$$A_6^* + A_6 + A_5^* A_5 4\gamma + A_6^* A_6 \gamma = 0 \tag{19}$$

Proof: The proof follows from Proposition 5 by letting $A_2 = A_3 = 0$. □

Corollary: The solution U of

$$dU(t) = [A_1 dt + A_2 dB(t) + A_3 dB^+(t) + A_6 dN(t)]U(t) \tag{20}$$

$$U(0) = 1 \ , \quad 0 \le t \le T < +\infty$$

is unitary if and only if

$$A_1 + A_1^* + A_2 A_2^* \gamma = 0 \tag{21}$$

$$A_2 + A_3^* + A_2 A_6^* \gamma = 0 \tag{22}$$

$$A_6 + A_6^* + A_6 A_6^* \gamma = 0 \tag{23}$$

$$A_1^* + A_1 + A_3^* A_3 \gamma = 0 \tag{24}$$

$$A_3^* A_2 + A_3^* A_6 \gamma = 0 \tag{25}$$

Proof: The proof follows from Proposition 5 by letting $A_4 = A_5 = 0$. □

Remark: Conditions (21) – (25), for $\gamma = 1$, are well known (cf. [7]) and they are certainly satisfied if $A_1 = \frac{W-1}{\gamma}$, $A_2 = -L^*W$, $A_3 = L$, and $A_1 = iH - \frac{\gamma}{2} L^*L$ where L, W, H are bounded operators with W unitary and H self–adjoint. For $\gamma = 1$ we obtain (3).

3 Sufficient unitarity conditions

Proposition 2 Let L, H, W, M be bounded operators with H self–adjoint and W, M unitary. Suppose also that

$$L^*(1-W)+\sqrt{2}(1-\text{ Re } W)^{1/2}ML=0 \tag{26}$$

Then $A_1 = iH - \frac{\gamma}{2}\,L^*L$, $A_2 = -L^*W$, $A_3 = L$

$$A_4 = -\left(\frac{1-\text{ Re } W}{8\gamma^2}\right)^{1/2} MW\ , \quad A_5 = M^*\left(\frac{1-\text{ Re } W}{8\gamma^2}\right)^{1/2} \quad \text{and} \quad A_6 = \frac{W-1}{2\gamma}$$

satisfy (5) – (12). Therefore, the solution U of

$$dU(t) = \Big[\Big(iH - \frac{\gamma}{2}\,L^*L\Big)\,dt - L^*W dB(t) + LdB^+(t)$$

$$-\left(\frac{1-\text{ Re } W}{8\gamma^2}\right)^{1/2} MWdB_2(t) + M^*\left(\frac{1-\text{ Re } W}{8\gamma^2}\right)^{1/2} dB_2^+ + \frac{W-1}{2\gamma}\,dN(t)\Big]U(t) \tag{27}$$

$$U(0)=1\ , \qquad 0\le t\le T<+\infty$$

is unitary.

Proof:

$$A_1 + A_1^* + A_2A_2^*\gamma = iH - \frac{\gamma}{2}\,L^*L - iH - \frac{\gamma}{2}\,L^*L + (-L^*W)(-W^*L)\gamma = 0$$

$$A_2 + A_3^* + 2\gamma A_4A_2^* + \gamma A_2A_6^* = -L^*W + L^* + 2\gamma\left(-\left(\frac{1-\text{ Re } W}{8\gamma^2}\right)^{1/2}\right)MW(-W^*L) +$$

$$\gamma(-L^*W)\frac{W^*-1}{2\gamma} = -L^*W + L^* + \frac{1}{\sqrt{2}}\,(1-\text{ Re } W)^{1/2}ML + \frac{L^*W-L^*}{2} =$$

$$\frac{1}{2}\,[L^*(1-W)-\sqrt{2}(1-\text{ Re } W)^{1/2}ML] = 0 \text{ by (28).} \tag{28}$$

Finally,

$$A_6 + A_6^* + A_4A_4^*4\gamma + A_6A_6^*\gamma = \frac{W-1}{2\gamma} + \frac{W^*-1}{2\gamma} +$$

$$\left(\frac{1-\text{ Re } W}{8\gamma^2}\right)^{1/2} MWW^*M\left(\frac{1-\text{ Re } W}{8\gamma^2}\right)^{1/2} 4\gamma$$

$$+\frac{W-1}{2\gamma}\frac{W^*-1}{2\gamma}\gamma=\frac{\text{Re }W-1}{\gamma}+\frac{1-\text{ Re }W}{2\gamma}+\frac{1-\text{ Re }W}{2\gamma}=0$$

thus proving (5) – (8).

The proof of (9) – (12) is similar. □

Remark: Equation (28) connects the linear case (20) with the non–linear case (14). Several examples of L, M, W satisfying (28) are given in the following corollary to Proposition 28.

Corollary 3.1: Let L, M, W, H be bounded operators with H self–adjoint and M, W unitary. The solution $U=\{U(t)\ :\ 0\leq t\leq T<+\infty\}$ of each of the following initial value problems, with initial value $U(0)=1$, is unitary.

$$dU(t)=[iHdt]U(t) \tag{29}$$

$$dU(t)=\left[\left(iH-\frac{\gamma}{2}L^*L\right)dt-L^*dB(t)+LdB^+(t)\right]U(t) \tag{30}$$

$$dU(t)=\left[iHdt+\frac{W-1}{2\gamma}\,dN(t)\right]U(t) \tag{31}$$

$$\begin{aligned} dU(t) &= \Big[iHdt-\left(\tfrac{1-\text{ Re }W}{8\gamma^2}\right)^{1/2}MWdB_2(t)+M^*\left(\tfrac{1-\text{ Re }W}{8\gamma^2}\right)^{1/2}dB_2^+(t) && (32)\\ &\qquad +\tfrac{W-1}{2\gamma}\,dN(t)\Big]U(t) && (33)\end{aligned}$$

$$\begin{aligned} dU(t) &= \Big[\left(iH-\tfrac{\gamma}{2}\,|\gamma|^2\right)dt-\overline{\gamma}e^{i\theta}dB(t)+\lambda dB^+(t) && (34)\\ &\quad -\tfrac{(e^{i\theta}-1)e^{i\theta}}{4\gamma}\,\tfrac{\overline{\lambda}}{\lambda}\,dB_2(t)+\tfrac{(e^{i\theta}-1)}{4\gamma}\,\tfrac{\lambda}{\overline{\lambda}}\,dB_2^+(t)+\tfrac{e^{i\theta}-1}{2\gamma}\,dN(t)\Big]U(t) && (35)\end{aligned}$$

where $\theta\in\mathbf{R}-\{2k\pi/k=0,\pm1,\pm2,\ldots\}$, $\lambda\in\mathbf{C}-\{0\}$.

$$\begin{aligned} dU(t) &= \Big[\left(iH-\tfrac{\gamma}{2}\right)dt-e^{i\theta}L^*dB(t)+LdB^+(t)-\tfrac{(e^{i\theta}-1)}{4\gamma}\,(L^2)^*dB_2(t) && (36)\\ &\qquad +\tfrac{e^{-i\theta}-1}{4\gamma}\,L^2dB_2^+(t)+\tfrac{e^{i\theta}-1}{2\gamma}\,dN(t)\Big]U(t) && (37)\end{aligned}$$

where $\theta\in\mathbf{R}-\{2k\pi/k=0,\pm1,\pm2,\ldots\}$ and $LL^*=L^*L=1$.

Proof: The proof follows from Proposition 28 by taking

$$\text{(a)} \qquad L = 0, \quad W = 1$$

$$\text{(b)} \qquad W = 1$$

$$\text{(c)} \qquad L = M = 0$$

$$\text{(d)} \qquad L = 0$$

$$\text{(e)} \qquad L = \lambda 1\,,\ W = e^{i\theta},\ M = \frac{e^{i\theta} - 1}{\sqrt{2}(1 - \cos\theta)^{1/2}\lambda}\,\frac{\overline{\lambda}}{\lambda}\,1$$

$$\text{(f)} \qquad W = e^{i\theta}1,\ M = \frac{e^{i\theta} - 1}{\sqrt{2}(1 - \cos\theta)^{1/2}}\,(L^2)^* \qquad \Box$$

4 Linear independence of the stochastic differentials

Proposition 3 For each $i = 1, 2, \ldots, 6$ let $A_i = \hat{A}_i \otimes 1$ be a constant adapted process acting on $H_0 \otimes \Gamma(S_+)$ and suppose that for all $t \geq 0$

$$A_1 dt + A_2 dB(t) + A_3 dB^+(t) + A_4 dB_2(t) + A_5 dB_2^+(t) + A_6 dN(t) = 0 \quad (38)$$

Then

$$A_1 = A_2 = \ldots = A_6 = 0 \quad (39)$$

on the exponential domain.

Proof: By (38), for all $f,\ g \in S_+$, $u,\ v \in H_0$, and $t \geq 0$

$$\langle u \otimes \psi(f), [A_1 dt + \ldots + A_6 dN(t)] v \otimes \psi(g)\rangle = 0 \quad (40)$$

By (2), (5) implies

$$\langle u \otimes \psi(f), A_1 v \otimes \psi(g)\rangle + g(t)\langle u \otimes \psi(f), A_2 v \otimes \psi(g)\rangle +$$

$$\overline{f(t)}\langle u \otimes \psi(f), A_3 v \otimes \psi(g)\rangle + g(t)^2\langle u \otimes \psi(f), A_4 v \otimes \psi(g)\rangle +$$

$$\overline{f(t)}^2\langle u \otimes \psi(f), A_5 v \otimes \psi(g)\rangle + \overline{f}(t) g(t)\langle u \otimes \psi(f), A_6 v \otimes \psi(f)\rangle = 0 \quad (41)$$

Since, for each $i = 1, 2, \ldots, 6$, $\langle u \otimes \psi(f),\, A_i v \otimes \psi(g)\rangle = \langle u, \hat{A}_i v\rangle \langle \psi(f), \psi(g)\rangle$ and $\langle \psi(f),\, \psi(g)\rangle = \exp(\langle f, g\rangle) \neq 0$, (41) implies

$$\langle u, \hat{A}_1 v\rangle + g(t)\langle u, \hat{A}_2 v\rangle + \overline{f(t)}\langle u, \hat{A}_3 v\rangle + g(t)^2\langle u, \hat{A}_4 v\rangle + \overline{f(t)}^2\langle u, \hat{A}_5 v\rangle +$$

$$\overline{f(t)}g(t)\langle u, A_6 v\rangle = 0 \tag{42}$$

Taking f and g to be such that $g(t) = 0$ and $f(t) \neq 0$ we obtain

$$\langle u, \hat{A}_3 v\rangle + \overline{f(t)}\langle u, \hat{A}_5 v\rangle = 0 \tag{43}$$

for all $u,\, v \in H_0$. Since $f(t)$ can be fixed, while u, v are arbitrary, it follows that $\langle u, \hat{A}_3 v\rangle = \langle u, \hat{A}_5 v\rangle = 0$ and so $\hat{A}_3 = \hat{A}_5 = 0$.

Similarly, if g is such that $g(t) = 0$ then

$$\langle u, \hat{A}_2 v\rangle + g(t)\langle u, \hat{A}_4 v\rangle = 0 \tag{44}$$

from which we obtain $\hat{A}_2 = \hat{A}_4 = 0$.

Thus $\hat{A}_i = 0$ and so $A_i = 0$ for all $i = 1, 2, \ldots, 6$. $\Box$

Proposition 4 For each $i = 1, 2, \ldots, 6$ and $t \geq 0$, let $A_i = \{A_i(t) = \hat{A}_i(t) \otimes 1 \;:\; t \geq 0\}$ be an adapted process acting on $H_0 \otimes \Gamma(S_+)$ and assume that:
(a) The map $t \in [0, +\infty) \to \langle u \otimes \psi(f),\, A_i(t)v \otimes \psi(g)\rangle$ is continuous for all $u,\, v \in H_0$ and $f,\, g \in S_+$.
(b) The map $(v, g) \in H_0 \times S_+ \to A_i(t)v \otimes \psi(g)$ is continuous for all $t \geq 0$.
(c) For all $t \geq 0$, $u,\, v \in H_0$, and $f,\, g \in S_+$ with $\mathrm{Re}\, f \cdot\, \mathrm{Re}\, g = 0$, $\mathrm{Im}\, \langle u \otimes \psi(f),\, A_i(t)v \otimes \psi(g)\rangle = 0$ for all $i = 2, 3, \ldots, 6$.

Then

$$A_1(t)dt + A_2(t)dB(t) + A_3(t)dB^+(t) + A_4 dB_2(t) + A_5(t)dB_2^+(t) + A_6(t)dN(t) = 0 \tag{45}$$

for all $t \geq 0$, implies

$$A_1 \equiv A_2 \equiv \ldots \equiv A_6 \equiv 0$$

on the exponential domain.

Proof: As in the proof of Proposition 38, in view of (45), for all $f,\, g \in S_+$, $u,\, v \in H_0$, and $t \geq 0$

$$\langle u \otimes \psi(f), A_1(t)v \otimes \psi(g)\rangle + g(t)\langle u \otimes \psi(f), A_2(t)v \otimes \psi(g)\rangle$$

$$+\overline{f(t)}\langle u\otimes\psi(f),A_3(t)v\otimes\psi(g)\rangle+g(t)^2\langle u\otimes\psi(f),A_4(t)v\otimes\psi(g)\rangle$$

$$+\overline{f(t)}^2\langle u\otimes\psi(f),A_5(t)v\otimes\psi(g)\rangle+\overline{f(t)}g(t)\langle u\otimes\psi(f),A_6(t)v\otimes\psi(g)\rangle=0 \quad (46)$$

Allowing f, g to vary over continuous functions with compact support contained in $[0,t)$, (46) implies $\langle u\otimes\psi(f),\ A_1(t)v\otimes\psi(g)\rangle=0$. By the adaptedness of A_1 and the totality of $\{u\otimes\psi(f)/f$ continuous, $\sup pf\subset[0,t]\}$ in $H_0\otimes\Gamma(S_+)$ it follows that $\hat{A}_i(t)=0$ and so $A_1\equiv 0$.

Allowing g to vary over continuous functions with compact support contained in $[0,t)$, and taking f such that $\mathrm{Re}\ f=0$ and $f(t)\neq 0$ we obtain from (46), with $A_1(t)=0$,

$$i\langle u\otimes\psi(f),A_3(t)v\otimes\psi(g)\rangle+\ \mathrm{Im}\ f(t)\langle u\otimes\psi(f),A_5(6)v\otimes\psi(g)\rangle=0 \quad (47)$$

By assumption (c) and the fact that $\mathrm{Im}\ f(t)=f(t)\neq 0$, (47) yields

$$\langle u\otimes\psi(f),A_3(t)v\otimes\psi(g)\rangle=\langle u\otimes\psi(f),A_5(t)v\otimes\psi(g)\rangle=0$$

and so, as before, $A_3\equiv A_5\equiv 0$.

Repeating the above argument interchanging the roles of f and g we obtain $A_2\equiv A_4\equiv 0$.

Finally, if we vary f, g over continuous functions such that $\overline{f(t)}g(t)\neq 0$, we conclude that $A_6\equiv 0$. □

Proposition 5 Let $A_i=\{A_i(t)/t\geq 0\}$ be, for each $i=1,2,\ldots,6$, an adapted process in $H_0\otimes\Gamma(S_+)$ such that
(a) For fixed f, $g\in S_+$ and u, $v\in H_0$ the map $t\in[0,+\infty)\to\langle u\otimes\psi(f),$ $A_i(t)v\otimes\psi(g)\rangle$ is continuous
(b) For fixed $t\in[0,+\infty)$ the map $(v,g)\in H_0\times S_+\to A_i(t)v\otimes\psi(g)$ is continuous.

Then the stochastic differentials dt, dB, dB^+, dB_2, dB_2^+, and dN are linearly independent i.e.

$$A_1(t)dt+\ldots+A_6(t)dN(t)=0 \text{ for all } t\geq 0 \quad \text{implies} \quad A_1\equiv\ldots\equiv A_6\equiv 0$$

on the exponential domain.

Proof: For all u, $v\in H_0$ and f, $g\in S_+$

$$\langle u\otimes\psi(f),[A,(t)dt+A_2(t)dB(t)+A_3(t)dB^+(t)+A_4(t)dB_2(t)+A_5(t)dB_2^+(t)$$

$$+A_6(t)dN(t)]v \otimes \psi(g)\rangle = 0$$

implies

$$\langle u \otimes \psi(f), [A_1(t) + g(t)A_2(t) + \overline{f(t)}A_3(t) + g(t)^2 + \overline{f(t)}^2 A_5(t) +$$
$$f(t)\overline{g(t)}A_6(t)]v \otimes \psi(g)\rangle = 0 \tag{48}$$

Allowing f, g to vary over continuous functions with compact support contained in $[0, t)$ we obtain from (4.11) $\langle u \otimes \psi(f),\ A_1(t)v \otimes \psi(g)\rangle = 0$. By the adaptedness of A_1 and the totality of $\{u \otimes \psi(f)/u \in H_0,\ f$ continuous, $\sup pg \subset [0, t)\}$ in $H_0 \otimes \Gamma(S_+)$ it follows that $A_1 \equiv 0$.

Letting f, g vary over continuous functions with $f(t) \neq 0$ and $\sup pg \subset [0, t)$ (48) implies

$$\langle u \otimes \psi(f), [\overline{f(t)}A_3(t) + \overline{f(t)}^2 A_5(t)]v \otimes \psi(g)\rangle = 0$$

and so

$$\langle u \otimes \psi(f), [A_3(t) + \overline{f(t)}A_5(t)]v \otimes \psi(g)\rangle = 0$$

Thus

$$\langle u \otimes \psi(f), [A_3(t)dt + \overline{f(t)}A_5(t)dt]v \otimes \psi(g)\rangle = 0$$

i.e.

$$\langle u \otimes \psi(f), [A_3(t)dt + A_5(t)dB^+(t)]v \otimes \psi(g)\rangle = 0 \tag{49}$$

By a totality argument and by the arbitrariness of t, (49) implies

$$A_3(t)dt + A_5(t)dB^+(t) = 0 \tag{50}$$

But dt, dB, dB^+ and dN are known to be linearly independent (cf. [8]). Thus (50) implies $A_3 \equiv A_5 \equiv 0$.

Similarly $A_2 \equiv A_4 \equiv 0$.

Now letting f, g be such that $f(t)\overline{g(t)} \neq 0$ we conclude that $A_6 \equiv 0$. □*Remark*: A formal proof of the linear independence of dt, dB, dB^+, dB_2, dB_2^+, and dN can be obtained with the use of Itô's table as follows: suppose that

$$A_1dt + A_2dt + A_3dB^+ + A_4dB_2 + A_5dB_2^+ + A_6dN = 0 \tag{51}$$

Multiplying (51) from the left by dB we obtain by (1)

$$A_3dt + 2\gamma A_5dB^+ + \gamma A_6dN = 0 \tag{52}$$

By the linear independence of dt, dB, dB^+, and dN (52) yields

$$A_3 \equiv A_5 \equiv A_6 \equiv 0$$

Multiplying (4.14) from the right by dB^+ we obtain

$$A_2 dt + 2\gamma A_4 dB = 0 \tag{53}$$

As before, (53) yields $A_2 \equiv A_4 \equiv 0$.
Finally $A_1 dt = 0$ implies $A_1 \equiv 0$.

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