Math. Z. 198, 451–477 (1988)

Mathematische Zeitschrift

Quantum Independent Increment Processes on Superalgebras

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1. Introduction

We introduce the notion of quantum independent stationary increment processes on superalgebras and prove a reconstruction theorem which establishes a one-to-one correspondence between these processes and their infinitesimal generators. In particular our result provides a new technique for constructing continuous tensor products of $\mathbb{Z}_2$-graded $*$-algebras which is not based on the use of representations of the canonical commutation (or anti-commutation) relations. We also obtain a quantum version of the Lévy-Khintchine formula and a full classification of the "continuous trajectories" quantum processes with independent stationary additive increments. Finally we prove that, both in the boson and the fermion case, solutions of quantum stochastic differential equations on the Fock space over $L^2(\mathbb{R}_+)$ give rise to quantum independent stationary increment processes in the sense of the previously developed theory. We derive a formula for the generators of these processes.

In classical probability theory a stochastic process, indexed by a set T, with values in a measurable space $(E, \mathscr{E})$, is a family $(X_t)_{t\in T}$ of measurable functions $X_t: \Omega \to E$ defined on a probability space $(\Omega, \mathfrak{A}, P)$. If E is a complete metric space with countable basis of its topology and $\mathscr{E}$ is the Boolean algebra of Borel subsets of E then there is a one-to-one correspondence between measurable functions X: $\Omega \to E$ and normal $*$-algebra homomorphisms $\hat{X}: L^\infty(E) \to L^\infty(\Omega)$ which is given by $\hat{X}(f) = f \circ X$, $f \in L^\infty(E)$; see [2, 17]. This establishes a one-to-one correspondence between stochastic processes $(X_t)_{t\in T}$ with values in E and families of normal $*$-algebra homomorphisms $(\hat{X}_t)_{t\in T}$ from $L^\infty(E)$ to $L^\infty(\Omega)$ where $L^\infty(\Omega)$ is also equipped with a normal state, still denoted by P, corresponding to the probability measure P on Ω, i.e.

$$P(F) = \int_\Omega dP(\omega)\, F(\omega); \quad F \in L^\infty(\Omega).$$

The $*$-algebras $L^\infty(\Omega)$ and $L^\infty(E)$ both are commutative. This justifies the following definition. A quantum stochastic process over a $*$-algebra $\mathscr{B}$ is a triplet $(\mathscr{A},(j_t)_{t\in T},\varphi)$ consisting of an other $*$-algebra $\mathscr{A}$, homomorphisms $j_t:\mathscr{B}\to\mathscr{A}$, and a state φ on $\mathscr{A}$. This definition differs from the one in [4] only in that, in the present paper, we work in a purely algebraic category (thus omitting any conditions of continuity on $\mathscr{B}$ or $\mathscr{A}$).

The algebra $\mathscr{B}$ in our definition plays the role of the algebra $L^\infty(E)$ in the considerations above and E is the "state space" of the process. The case in which the state space is t-dependent (i.e. each random variable X_t has its own state space E_t) can be easily reduced to the preceeding one by taking disjoint sums. Similarly, the case of a t-dependent algebra $\mathscr{B}_t$ can be reduced to our definition by taking direct sums.

Let $(F_t^\dagger, F_t)$ be a family of pre-closed operators defined on a common invariant dense subspace D of a Hilbert space $\mathscr{H}$ and such that, for each t, $F_t^\dagger$ is the adjoint of F_t on D (for example $\mathscr{H}$ is the Bose Fock space over $L^2(\mathbb{R}_+)$; for $t\geqq 0$, $F_t^\dagger=a^\dagger(\chi_{[0,t]})$, $F_t=a(\chi_{[0,t]})$ where $a^\dagger$ and a denote the creation and annihilation operators resp.; D is the space generated by the number eigenvectors). Let $\mathscr{A}$ be the polynomial algebra generated by all the $F_t^\dagger, F_t$, $t\in T$, and denote by $\mathbb{C}\langle x^*,x\rangle$ the polynomial algebra in two non-commuting indeterminates x^* and x. If we define $j_t:\mathbb{C}\langle x^*,x\rangle\to\mathscr{A}$ for $t\in T$ to be the homomorphism given by $j_t(x)=F_t$ then for any state φ on $\mathscr{A}$ the triplet $(\mathscr{A},j_t,\varphi)$ is a process over $\mathbb{C}\langle x^*,x\rangle$ in our sense. In the following, when no confusion can arise, we shall refer to this process simply as "the process $(F_t)_{t\in T}$".

If $(X_t)_{t\in\mathbb{R}_+}$ is real-valued stochastic process the random variables $X_{st}=X_t-X_s$, $s\leqq t$, are called the *increments* of the process. More generally, we call a stochastic process (X_{st}) indexed by pairs $(s,t)\in\mathbb{R}_+^2$, $s\leqq t$, with values in a topological semigroup G an *increment process* if

$$X_{rs}X_{st}=X_{rt},\quad r<s<t \tag{1.1}$$

$$X_{tt}\equiv e \tag{1.2}$$

where e denotes the unit element in G. In order to translate the properties (1.1) and (1.2) into the language of the corresponding quantum stochastic process $(L^\infty(\Omega), \hat{X}_{st}, P)$ over $L^\infty(G)$, notice that for any measurable set E the algebraic tensor product $L^\infty(E)\otimes L^\infty(E)$ can be embedded into $L^\infty(E\times E)$ by the formula $(f\otimes g)(a,b)=f(a)g(b)$, $a,b\in E$. For a semigroup H, let $\mathscr{F}(H)$ be the $*$-algebra of all complex-valued functions on H. Denote by $\Delta: L^\infty(G)\to L^\infty(G\times G)$ the restriction to $L^\infty(G)$ of the mapping which maps an element $f\in\mathscr{F}(G)$ to the element $\Delta f\in\mathscr{F}(G\times G)$ given by $\Delta f(x,y)=f(xy)$. Let $f\in L^\infty(G)$ and assume that Δf is an element of $L^\infty(G)\otimes L^\infty(G)$. Setting $j_{st}=\hat{X}_{st}$ we have

$$j_{rt}(f)(\omega)=(j_{rs}\otimes j_{st})\circ\Delta(f)(\omega,\omega),\quad r<s<t$$

$$j_{tt}(f)(\omega)=f(e).$$

If we write $j_{rs}*j_{st}(\omega)=(j_{rs}\otimes j_{st})\circ\Delta(\omega,\omega)$ the substitutes for equation (1.1) and (1.2) are

$$j_{rs}*j_{st}=j_{rt},\quad r<s<t \tag{1.3}$$

$$j_{tt}=\delta\mathbb{1} \tag{1.4}$$

where δ is the linear functional given by $\delta f=f(e)$. It is shown in [1] that $\mathscr{R}(G)=\Delta^{-1}(\mathscr{F}(G)\otimes\mathscr{F}(G))$ is a sub-$*$-algebra of $\mathscr{F}(G)$ and that Δ maps $\mathscr{R}(G)$ to $\mathscr{R}(G)\otimes\mathscr{R}(G)$. Moreover, $\mathscr{R}(G)$ is the space of all coefficient functions of the finite-dimensional representations of G. The triplet $(\mathscr{R}(G),\Delta,\delta)$ is an example of a coalgebra and the coalgebra structure of $\mathscr{R}(G)$ together with its $*$-algebra structure is an example of a $*$-*bialgebra*, which means that Δ and δ are $*$-algebra homomorphisms. If G is a locally compact abelian group or a compact group we can substitute for $L^\infty(G)$ the sub-$*$-bialgebra of $\mathscr{R}(G)$ consisting of all continuous functions in $\mathscr{R}(G)$ without loosing any information on the original stochastic increment process (X_{st}).

Hopf algebras, as considered in [1,21], and $*$-bialgebras have in common that they both are bialgebras. But instead of the Hopf algebra antipode a $*$-bialgebra possesses an involution; cf. [22]. If $\mathscr{B}$ is an arbitrary $*$-bialgebra and (j_{st}) is a quantum stochastic process over $\mathscr{B}$ indexed by pairs $(s,t)\in\mathbb{R}_+^2$, $s\leqq t$, the conditions (1.3) and (1.4) make sense, and we call (j_{st}) a quantum increment process over $\mathscr{B}$ if it satisfies (1.3) and (1.4).

A classical stochastic increment process (X_{st}) is called an *independent increment process* if the random variables $X_{t_1t_2},\ldots,X_{t_nt_{n+1}}$, are independent for all $t_1<\ldots<t_{n+1}$, i.e.

$$\begin{aligned}P(\hat{X}_{t_1t_2}(f_1)\ldots\hat{X}_{t_nt_{n+1}}(f_n))\\=P(\hat{X}_{t_1t_2}(f_1))\ldots P(\hat{X}_{t_nt_{n+1}}(f_n))\end{aligned} \tag{1.5}$$

for all $t_1<\ldots<t_{n+1}$, $f_1,\ldots,f_n\in L^\infty(G)$. The distributions P_{st} of X_{st} constitute a convolution evolution of probability measures on G. On the other hand, a given convolution evolution of probability measures on G gives rise to a projective family of probability measures which by the Kolmogorov reconstruction theorem determines a stochastic process. This process yields an independent increment process.

For the definition of a quantum independent increment process condition (1.5) easily can be transferred. But we add the condition of *physical independence*, that is we require the algebras $j_{st}(\mathscr{B})$ and $j_{s't'}(\mathscr{B})$ to commute if (s,t) and (s',t') are disjoint intervals in $\mathbb{R}_+$. (In a graded version we consider graded commutators.) The latter condition of course becomes trivial in the commutative case. We prove a reconstruction theorem for quantum independent increment processes over a (graded) $*$-bialgebra $\mathscr{B}$ which as in the classical case establishes an up to equivalence one-to-one correspondence between such processes and convolution evolutions of states on $\mathscr{B}$. In the case when the increments are also stationary the convolution evolution becomes a convolution semi-group. The convolution semi-groups $\{\varphi_t\}$ of states on a $*$-bialgebra which are pointwise continuous at the origin are exactly those of the form

$$\varphi_t=\exp_*(t\gamma)$$

where $\exp_*$ denotes the exponential with respect to convolution and γ is a conditionally positive, hermitian linear functional on $\mathscr{B}$ vanishing at the identity [19]. Thus, by our reconstruction theorem, the quantum independent stationary increment processes (j_{st}) over $\mathscr{B}$ (such that j_{st} converges to j_{tt} in law as $s\uparrow t$) are up to equivalence in one-to-one correspondence to linear functionals γ on $\mathscr{B}$ of the above type.

We give two non-commutative examples. If V is a complex vector space with a selfinverse antilinear mapping $v\mapsto v^*$ the tensor algebra $\mathscr{T}(V)$ can be turned into a $*$-bialgebra (cf. the notion of the enveloping Hopf algebra of a Lie algebra [16]). Quantum independent stationary increment processes over $\mathscr{T}(V)$ can be regarded as processes of operators $F(t)$, $t\geqq 0$, on a Hilbert space which, in a sense made precise in Section 4 of this paper, have independent stationary additive increments $F(t)-F(s)$. We prove that $F(t)$ must be a quantum Brownian motion in the sense of [9] if the fourth moments of $F(t)$ are of order $o(t)$. This result is related to the quantum Lévy martingale representation theorem of [3]. We also derive a formula for all conditionally positive, hermitian linear functionals on $\mathscr{T}(V)$, thus establishing a quantum version of the Lévy-Khintchine formula.

The other example is given by the unitary solutions of quantum differential equations as introduced in [6, 14]. In our example these solutions are families $U(t)$, $t\geqq 0$, of unitary operators on $\mathbb{C}^d\otimes\Gamma(L^2(\mathbb{R}_+))$ where $d\in\mathbb{N}$ and $\Gamma(L^2(\mathbb{R}_+))$ denotes the Bose or the Fermi Fock space over $L^2(\mathbb{R}_+)$. We prove that $U(t)$ gives rise to a quantum independent stationary increment process over a (graded) $*$-bialgebra which can be looked upon as the non-commutative analogue of the coefficient algebra of the group of unitary $d\times d$-matrices. We compute the generators of these processes.

2. Preliminaries

All the vector spaces and algebras will be over the complex numbers. The algebras are assumed to be associative and to have a unit element **1**. An algebra homomorphism maps **1** into **1**.

Denote by $\mathbb{Z}_2$ the field $\mathbb{Z}/2\mathbb{Z}$ with two elements 0 and 1. A *graded vector space* V is a vector space together with a pair (V^0, V^1) of subspaces such that $V=V^0\oplus V^1$. The elements of V^0 are called *even* and the elements of V^1 are called *odd*. If v is an element of V^ι, $\iota\in\mathbb{Z}_2$, then v is called *homogeneous* and ι is called the *degree* of v. We write $\iota=g(v)$. If V is any vector space one always can define the trivial graduation (V^0, V^1) on V by $V^0=V$, $V^1=\{0\}$. We consider the vector space $\mathbb{C}$ of complex numbers as a graded vector space with the trivial graduation. If V and W are graded vector spaces the space $A(V,W)$ of all additive mappings from V to W becomes a graded vector space by the definition

$$A(V,W)^\iota=\{R\in A(V,W)\colon RV^\kappa\subset W^{\kappa+\iota},\ \kappa=0,1\}.$$

The algebraic tensor product $V\otimes W$ of V and W can be turned into a graded vector space, which is sometimes denoted by $V\hat{\otimes} W$, by setting

$$(V\hat{\otimes} W)^\iota=\bigotimes_{\kappa+\kappa'=\iota} V^\kappa\otimes W^{\kappa'}.$$

If $R\colon V\to V$ and $S\colon W\to W$ are linear operators, S homogeneous, we define $R\hat{\otimes} S$: $V\otimes W\to V\otimes W$ to be the linear operator given by

$$R\hat{\otimes} S(v\otimes w)=(-1)^{g(S)g(v)}Rv\otimes Sw$$

for $v\in V$, $w\in W$, v homogeneous. A *graded algebra* is an algebra $\mathscr{A}$ which is graded as a vector space such that

$$\mathscr{A}^\iota\mathscr{A}^\kappa=\mathscr{A}^{\iota+\kappa};\quad \iota,\kappa\in\mathbb{Z}_2.$$

Any algebra becomes a graded algebra with the trivial graduation. A $*$-algebra is an algebra $\mathscr{A}$ with an involution $a\mapsto a^*$. A homomorphism $R\colon\mathscr{A}\mapsto\mathscr{B}$ from a $*$-algebra $\mathscr{A}$ to a $*$-algebra $\mathscr{B}$ is an algebra homomorphism satisfying $R(a^*)=R(a)^*$ for all $a\in\mathscr{A}$. A *graded $*$-algebra* is a graded algebra and a $*$-algebra such that the involution is even. For two graded algebras $\mathscr{A}$ and $\mathscr{B}$ we define the graded algebra tensor product $\mathscr{A}\hat{\otimes}\mathscr{B}$ to be the graded vector space $\mathscr{A}\otimes\mathscr{B}$ with multiplication given by

$$(a\otimes b)(a'\otimes b')=(-1)^{g(b)g(a')}aa'\otimes bb'$$

for $a,a'\in\mathscr{A}$ and $b,b'\in\mathscr{B}$ where a' and b are homogeneous. If $\mathscr{A}$ and $\mathscr{B}$ both are graded $*$-algebras $\mathscr{A}\hat{\otimes}\mathscr{B}$ becomes a graded $*$-algebra with the involution given by

$$(a\otimes b)^*=(-1)^{g(a)g(b)}a^*\otimes b^*$$

for homogeneous elements $a\in\mathscr{A}$ and $b\in\mathscr{B}$.

An algebra can be regarded as a triplet $(\mathscr{A},M,m)$ where $\mathscr{A}$ is a vector space and $M\colon\mathscr{A}\otimes\mathscr{A}\to\mathscr{A}$ and $m\colon\mathbb{C}\to\mathscr{A}$ are linear mappings satisfying

$$M\circ(M\otimes\mathrm{id})=M\circ(\mathrm{id}\otimes M) \tag{2.1}$$

and

$$M\circ(m\otimes\mathrm{id})=M\circ(\mathrm{id}\otimes m)=\mathrm{id}. \tag{2.2}$$

In the usual notation $M(a\otimes a')=aa'$ and $m(\lambda)=\lambda\mathbf{1}$, and (2.1) is the associativity law and (2.2) is the property of the unit element. By dualizing we get the notion of a *coalgebra*; see [1, 8, 16, 21]. A coalgebra is a triplet $(\mathscr{C},\Delta,\delta)$ consisting of a vector space $\mathscr{C}$, a linear map $\Delta\colon\mathscr{C}\to\mathscr{C}\otimes\mathscr{C}$, called the comultiplication, satisfying the coassociativity identity

$$(\Delta\otimes\mathrm{id})\circ\Delta=(\mathrm{id}\otimes\Delta)\circ\Delta$$

and a linear map $\delta\colon\mathscr{C}\to\mathbb{C}$, called the counit, satisfying the identity

$$(\delta\otimes\mathrm{id})\circ\Delta=(\mathrm{id}\otimes\delta)\circ\Delta=\mathrm{id}.$$

If we define Δ_n, $n\in\mathbb{N}$, inductively by

$$\Delta_1=\mathrm{id}$$
$$\Delta_{n+1}=(\mathrm{id}\otimes\Delta_n)\circ\Delta$$

then the general coassociativity law

$$(\Delta_{n_1}\otimes\ldots\otimes\Delta_{n_k})\circ\Delta_k=\Delta_m$$

with $m=\sum_{l=1}^{k} n_l$ holds. If $(\mathscr{A}, M, m)$ is an algebra and $(\mathscr{C}, \Delta, \delta)$ is a coalgebra and R, S: $\mathscr{C}\to\mathscr{A}$ are linear maps the *convolution* $R*S$ is the linear map from $\mathscr{C}$ to $\mathscr{A}$ defined by

$$R*S=M\circ(R\otimes S)\circ\Delta.$$

The vector space $L(\mathscr{C}, \mathscr{A})$ of all linear maps from $\mathscr{C}$ to $\mathscr{A}$ is an algebra with respect to convolution with unit $m\circ\delta$. A *graded coalgebra* $(\mathscr{C}, \Delta, \delta)$ is a coalgebra which is also a graded vector space such that Δ and δ are even. A *graded *-bialgebra* is a set $\mathscr{B}$ with both the structure of a graded *-algebra and the structure $(\mathscr{B}, \Delta, \delta)$ of a graded coalgebra such that Δ: $\mathscr{B}\to\mathscr{B}\hat{\otimes}\mathscr{B}$ and δ: $\mathscr{B}\to\mathbb{C}$ are *-algebra homomorphisms.

To a pair $(\mathscr{A}, \varphi)$ consisting of a *-algebra $\mathscr{A}$ and a state φ on $\mathscr{A}$ we associate the GNS-construction which yields a Hilbert space $\mathscr{H}$, a linear mapping ϑ: $\mathscr{A}\to\mathscr{H}$ with dense range, a representation π of $\mathscr{A}$ on $\mathscr{H}$ with cyclic vector $\Phi=\vartheta(\mathbf{1})\in\mathscr{H}$ characterized by the relations

$$\langle\vartheta(a)|\pi(b)\vartheta(c)\rangle=\langle\Phi|\pi(a^*bc)\Phi\rangle=\varphi(a^*bc);\quad a,b,c\in\mathscr{A}.$$

We call $(\mathscr{H}, \pi, \vartheta)$ the *GNS-triplet associated to* $(\mathscr{A}, \varphi)$.

3. Independent Increment Processes

In the following we simply say "stochastic process" instead of "quantum stochastic process".

Definition 3.1. A *stochastic process*, indexed by a set T, over a *-algebra $\mathscr{B}$, is a triplet $(\mathscr{A}, (j_t)_{t\in T}, \varphi)$ consisting of a *-algebra $\mathscr{A}$, a state φ on $\mathscr{A}$, and for each t in T a homomorphism j_t: $\mathscr{B}\to\mathscr{A}$. A stochastic process $(\mathscr{A}, (j_t)_{t\in T}, \varphi)$ is called *minimal* if $\mathscr{A}$ is algebraically generated by its elements $j_t(b)$, $t\in T$, $b\in\mathscr{B}$. Two stochastic processes $(\mathscr{A}^{(i)}, (j_t^{(i)})_{t\in T}, \varphi^{(i)})$, $i=1, 2$, over the same *-algebra $\mathscr{B}$, indexed by the same set T are said to be *equivalent* if

$$\varphi^{(1)}(j_{t_1}^{(1)}(b_1)\ldots j_{t_n}^{(1)}(b_n))=\varphi^{(2)}(j_{t_1}^{(2)}(b_1)\ldots j_{t_n}^{(2)}(b_n))$$

for all choices of $n\in\mathbb{N}$, $b_1, \ldots, b_n\in\mathscr{B}$, and $t_1, \ldots, t_n\in T$.

Two classical stochastic processes $(X_t^{(i)})_{t\in T}$, $i=1, 2$, with values in E are stochastically equivalent, i.e. they have the same finite-dimensional distributions, if and only if the associated processes $(L^\infty(\Omega^{(i)}), (\hat{X}_t^{(i)})_{t\in T}, P^{(i)})$ over $L^\infty(E)$ are equivalent in the sense of Definition 3.1.

Proposition 3.1. (cf. Propos. 1.1 in [4]) *Let* $(\mathscr{A}^{(i)}, (j_t^{(i)})_{t\in T}, \varphi^{(i)})$, $i=1, 2$, *be two minimal stochastic processes over a *-algebra* $\mathscr{B}$ *and denote by* $(\mathscr{H}^{(i)}, \pi^{(i)}, \vartheta^{(i)})$ *the GNS-triplet associated to* $(\mathscr{A}^{(i)}, \varphi^{(i)})$. *Then the two processes are equivalent if and only if there exists a unitary operator* $\mathscr{U}$: $\mathscr{H}^{(1)}\to\mathscr{H}^{(2)}$ *such that* $\mathscr{U}\Phi^{(1)}=\Phi^{(2)}$ (where $\Phi^{(i)}=\vartheta^{(i)}(\mathbf{1})$) *and*

$$\mathscr{U}(\pi^{(1)}(j_t^{(1)}b))\mathscr{U}^{-1}=\pi^{(2)}(j_t^{(2)}b)$$

for all $t\in T$ *and* $b\in\mathscr{B}$. □

Denote by T the subset of $\mathbb{R}_+^2$ of all pairs (s, t) with $s\leqq t$.

Definition 3.2. Let $\mathscr{B}$ be a graded *-bialgebra. *An increment process* over $\mathscr{B}$ is a stochastic process $(\mathscr{A}, (j_{st})_{(s,t)\in T}, \varphi)$ over $\mathscr{B}$, considered as a *-algebra, such that

(a) $j_{rs}*j_{st}=j_{rt}$, $r<s<t$

(b) $j_{tt}=\delta\mathbf{1}$.

Definition 3.3. An *independent* increment process over a graded *-bialgebra $\mathscr{B}$ is an increment process $(\mathscr{A}, (j_{st}), \varphi)$ over $\mathscr{B}$ such that

(a) $\varphi(j_{t_1t_2}(b_1)\ldots j_{t_nt_{n+1}}(b_n))$
$=\varphi(j_{t_1t_2}(b_1))\ldots\varphi(j_{t_nt_{n+1}}(b_n))$
for all $n\in\mathbb{N}$, $t_1, \ldots, t_{n+1}\in\mathbb{R}_+$ with $t_1<\ldots<t_{n+1}$, and $b_1, \ldots, b_n\in\mathscr{B}$.

(b) $j_{st}(b)j_{s't'}(b')=(-1)^{g(b)g(b')}j_{s't'}(b')j_{st}(b)$ for all disjoint intervals (s, t) and (s', t') in $\mathbb{R}_+$ and all homogeneous $b, b'\in\mathscr{B}$.

A set $\{\psi_{st}$: $(s, t)\in T\}$ of linear functionals ψ_{st} on a coalgebra is called a *convolution evolution* if $\psi_{rs}*\psi_{st}=\psi_{rt}$ and $\psi_{tt}=\delta$ for all $r<s<t$. If (j_{st}) is an independent increment process over $\mathscr{B}$ we denote by φ_{st} the state $\varphi\circ j_{st}$ on $\mathscr{B}$. Property (a) of Definition 3.3 yields

$$\varphi\circ(j_{rs}\otimes j_{st})=\varphi_{rs}\otimes\varphi_{st},\quad r<s<t$$

and thus

$$\varphi_{rt}=\varphi\circ j_{rt}=\varphi\circ(j_{rs}*j_{st})=\varphi\circ(j_{rs}\otimes j_{st})\circ\Delta$$
$$=(\varphi_{rs}\otimes\varphi_{st})\circ\Delta=\varphi_{rs}*\varphi_{st}$$

which, together with (b) of Definition 3.3, means that $\{\varphi_{st}$: $(s, t)\in T\}$ is a convolution evolution of states on $\mathscr{B}$. The aim of this section is to associate a "canonical" independent increment process to a given convolution evolution of states. Let $\mathscr{B}$ be a graded *-bialgebra. Denote by $\mathfrak{L}_{\mathscr{B}}$ the category of all objects $(\mathscr{A}, (j_{st})_{(s,t)\in T})$ where $\mathscr{A}$ is a *-algebra and j_{st} are homomorphisms from $\mathscr{B}$ to $\mathscr{A}$ such that (a) and (b) of Definition 3.2 and (b) of Definition 3.3 hold. A morphism from $(\mathscr{A}^{(1)}, j_{st}^{(1)})$ to $(\mathscr{A}^{(2)}, j_{st}^{(2)})$ is a homomorphism η: $\mathscr{A}^{(1)}\to\mathscr{A}^{(2)}$ such that $j_{st}^{(2)}=\eta\circ j_{st}^{(1)}$. According to the general definition of [15] a universally repelling object $(C(\mathscr{B}), h_{st})$ in $\mathfrak{L}_{\mathscr{B}}$ is characterised by the following property. For all objects $(\mathscr{A}, j_{st})$ in $\mathfrak{L}_{\mathscr{B}}$ there exists a unique homomorphism η: $C(\mathscr{B})\to\mathscr{A}$ such that $j_{st}=h_{st}\circ\eta$ for all $(s, t)\in T$. We give a construction of a universally repelling object of $\mathfrak{L}_{\mathscr{B}}$ as an inductive limit of *-algebras. Denote by $\mathscr{D}$ the set of all subsets of T of the form $\{(t_1, t_2), \ldots, (t_n, t_{n+1})\}$, $n\in\mathbb{N}$, $t_1<\ldots<t_{n+1}$. To every element $\alpha=\{(t_1, t_2), \ldots, (t_n, t_{n+1})\}$ we associate the ordered $n+1$-tuple $\bar{\alpha}=(t_1, \ldots, t_{n+1})$ of non-negative real numbers. We set $\alpha_0=t_1$, $\alpha_1=t_{n+1}$ and $\bar{\alpha}(t_k, t_l)=(t_k, \ldots, t_l)$ for $k, l\in\{1, \ldots, n+1\}$, $k<l$. For $\alpha, \beta\in\mathscr{D}$, $\bar{\alpha}=(t_1, \ldots, t_{n+1})$, $\bar{\beta}=(s_1, \ldots, s_{m+1})$, we write $\alpha<\beta$ if $\{t_1, \ldots, t_{n+1}\}\subset\{s_1, \ldots, s_{m+1}\}$, turning $\mathscr{D}$ into an ordered set. Fix for each $(s, t)\in T$ an isomorphic copy $\mathscr{B}_{st}$ of $\mathscr{B}$, i.e. a pair $(\mathscr{B}_{st}, I_{st})$ where $\mathscr{B}_{st}$ is a graded *-algebra and I_{st}:

$\mathscr{B} \to \mathscr{B}_{st}$ is an isomorphism. We form the time ordered graded tensor product

$$\mathscr{B} = \vec{\bigotimes_{(s,t)\in\alpha}} \mathscr{B}_{st} = \mathscr{B}_{t_1 t_2} \hat{\otimes} \ldots \hat{\otimes} \mathscr{B}_{t_n t_{n+1}}.$$

If $\alpha, \beta \in \mathscr{D}$ with $\alpha < \beta$ and $\beta(\alpha_0, \alpha_1) = \alpha$ we define

$$\eta_{\beta,\alpha}: \mathscr{B}_\alpha \to \mathscr{B}_\beta$$

to be the imbedding which maps an element B of $\mathscr{B}_\alpha$ to the element $\mathbf{1}^{\beta(\beta_0,\alpha_0)} \otimes B \otimes \mathbf{1}^{\beta(\alpha_1,\beta_1)}$ of $\mathscr{B}_\beta$ where for $\gamma \in \mathscr{D}$ we denote by $\mathbf{1}^\gamma$ the unit element of $\mathscr{B}_\gamma$. In the general case $\alpha < \beta$ we set

$$\eta_{\beta,\alpha} = \eta_{\beta,\beta(\alpha_0,\alpha_1)} \circ \left(\vec{\bigotimes_{(s,t)\in\beta(\alpha_0,\alpha_1)}} I_{st} \right) \circ \left(\vec{\bigotimes_{(s',t')\in\alpha}} \Delta_{\#\beta(s',t')} \right) \circ \left(\vec{\bigotimes_{(s',t')\in\alpha}} I_{s't'}^{-1} \right)$$

where $\# X$, X a finite set, denotes the number of elements of X.

Proposition 3.2. *The $*$-algebras $\mathscr{B}_\alpha$, $\alpha \in \mathscr{D}$, together with the maps $\eta_{\beta,\alpha}$, $\alpha, \beta \in \mathscr{D}$, $\alpha < \beta$, constitute an inductive system of $*$-algebras. The $\eta_{\beta,\alpha}$ are injective.*

Proof. We must prove $\eta_{\gamma,\beta} \circ \eta_{\beta,\alpha} = \eta_{\gamma,\alpha}$ for $\alpha < \beta < \gamma$. Because of $\Delta_m \mathbf{1} = \mathbf{1}^{\otimes m}$ we have for $\bar{\alpha} = (t_1, \ldots, t_{n+1})$, $\alpha < \beta$, and $k, l \in \{1, \ldots, n+1\}$, $k < l$

$$\eta_{\beta,\alpha} \circ \eta_{\alpha,\alpha(t_k,t_l)} = \eta_{\beta,\alpha(t_k,t_l)}.$$

This yields for $\alpha < \beta < \gamma$

$$\eta_{\gamma,\beta} \circ \eta_{\beta,\alpha} = \eta_{\gamma,\gamma(\alpha_0,\alpha_1)} \circ \eta_{\gamma(\alpha_0,\alpha_1),\beta(\alpha_0,\alpha_1)} \circ \eta_{\beta(\alpha_0,\alpha_1),\alpha}$$

which means that we may restrict ourselves to the case $\gamma = \gamma(\alpha_0, \alpha_1)$, $\beta = \beta(\alpha_0, \alpha_1)$. We have

$$\begin{aligned} \eta_{\gamma,\beta} \circ \eta_{\beta,\alpha} &= \left(\vec{\bigotimes_{(s,t)\in\gamma}} I_{st} \right) \circ \left(\vec{\bigotimes_{(s',t')\in\beta}} \Delta_{\#\gamma(s',t')} \right) \circ \left(\vec{\bigotimes_{(s'',t'')\in\alpha}} \Delta_{\#\beta(s'',t'')} \right) \circ \left(\vec{\bigotimes_{(s'',t'')\in\alpha}} I_{s''t''}^{-1} \right) \\ &= \left(\vec{\bigotimes_{(s,t)\in\gamma}} I_{st} \right) \circ \left(\vec{\bigotimes_{(s'',t'')\in\alpha}} \Delta_{\#\gamma(s'',t'')} \right) \circ \left(\vec{\bigotimes_{(s'',t'')\in\alpha}} I_{s''t''}^{-1} \right) \\ &= \eta_{\gamma,\alpha}. \end{aligned}$$

By the counit property of δ the mapping $\delta \otimes \mathrm{id}$ is a left inverse of Δ. This means that Δ and also Δ_n, $n \in \mathbb{N}$, are injective which gives the injectivity of the homomorphisms $\eta_{\beta,\alpha}$. □

Denote by $(C(\mathscr{B}), \eta_\alpha)$ the inductive limit of the sets $\mathscr{B}_\alpha$ and the maps $\eta_{\alpha,\beta}$; see [7]. The η_α are injective. There is a unique way to introduce a $*$-algebra structure on $C(\mathscr{B})$ such that $(C(\mathscr{B}), \eta_\alpha)$ has the universal property of an inductive limit of $*$-algebras. We define the mappings h_{st}, $(s,t) \in T$, by $h_{tt} = \delta \mathbf{1}$ and $h_{st} = \eta_{\{(s,t)\}} \circ I_{st}$ for $s < t$.

Proposition 3.3. *The pair $(C(\mathscr{B}), (h_{st})_{(s,t)\in T})$ is a universally repelling object in $\mathfrak{L}_{\mathscr{B}}$.*

Proof. First we must prove that $(C(\mathscr{B}), h_{st})$ is an object in $\mathfrak{L}_{\mathscr{B}}$. Denote by M the multiplication in $C(\mathscr{B})$. If $\alpha, \beta \in \mathscr{D}$ with $\alpha_1 = \beta_0$ then $\mathscr{B}_{\alpha\cup\beta} = \mathscr{B}_\alpha \hat{\otimes} \mathscr{B}_\beta$ and

$$M \circ (\eta_\alpha \otimes \eta_\beta) = \eta_{\alpha\cup\beta}. \tag{3.1}$$

This yields for $r < s < t$, $\bar{\gamma} = (r, s, t)$

$$\begin{aligned} & h_{rs} * h_{st} \\ &= M \circ (h_{rs} \otimes h_{st}) \circ \Delta = \eta_\gamma \circ (I_{rs} \otimes I_{st}) \circ \Delta \\ &= \eta_\gamma \circ \eta_{\gamma,(r,t)} \circ I_{rt} \\ &= h_{rt}. \end{aligned}$$

Equation (3.1) gives that for $\alpha, \beta \in \mathscr{D}$, $\alpha_1 < \beta_0$ the mapping

$$M \circ (\eta_\alpha \otimes \eta_\beta): \mathscr{B}_\alpha \hat{\otimes} \mathscr{B}_\beta \to C(\mathscr{B})$$

is a homomorphism which means that (b) of Definition 3.3 holds. To prove that $C(\mathscr{B}), h_{st})$ is universally repelling, let $(\mathscr{A}, j_{st})$ be an object in $\mathfrak{L}_{\mathscr{B}}$. We must show that there exists a unique homomorphism η: $C(\mathscr{B}) \to \mathscr{A}$ such that $j_{st} = \eta \circ h_{st}$. For $\alpha \in \mathscr{D}$ define the homomorphism η'_α: $\mathscr{B}_\alpha \to \mathscr{A}$ by

$$\eta'_\alpha = M^{\mathscr{A}}_{\#\alpha} \circ \left(\vec{\bigotimes_{(s,t)\in\alpha}} j_{st} \right) \circ \left(\vec{\bigotimes_{(s,t)\in\alpha}} I_{st}^{-1} \right)$$

where for $n \in \mathbb{N}$ we denote by $M_n^{\mathscr{A}}$ the n-fold multiplication in $\mathscr{A}$. The properties of the j_{st} yield that $\eta'_\beta \circ \eta_{\beta,\alpha} = \eta'_\alpha$ for $\alpha, \beta \in \mathscr{D}$, $\alpha < \beta$. By the universal property of the inductive limit $(C(\mathscr{B}), \eta_\alpha)$ there exists a unique homomorphism η: $C(\mathscr{B}) \to \mathscr{A}$ such that $\eta'_\alpha = \eta \circ \eta_\alpha$ for all $\alpha \in \mathscr{D}$. The special case $\bar{\alpha} = (s, t)$ gives $j_{st} = \eta \circ h_{st}$. Conversely, if a homomorphism η': $C(\mathscr{B}) \to \mathscr{A}$ fulfills $j_{st} = \eta' \circ h_{st}$ then it also fullfills $\eta'_\alpha = \eta' \circ \eta_\alpha$, thus $\eta = \eta'$. □

It is clear that $(C(\mathscr{B}), h_{st})$ is also minimal in the sense that $h_{st}(b)$, $(s,t) \in T$, $b \in \mathscr{B}$ generate the algebra $C(\mathscr{B})$.

Theorem 3.1. *Let $\{\varphi_{st}: (s,t) \in T\}$ be a convolution evolution of states on a graded $*$-bialgebra $\mathscr{B}$. Then there exists a uniquely determined state φ on $C(\mathscr{B})$ such that $(C(\mathscr{B}), h_{st}, \varphi)$ is an independent increment process over $\mathscr{B}$ with convolution evolution of states $\{\varphi_{st}\}$. Two independent increment processes over $\mathscr{B}$ are equivalent if and only if they have the same convolution evolution of states.*

Proof. For $\alpha \in \mathscr{D}$ we define the state φ_α on $\mathscr{B}_\alpha$ by

$$\varphi_\alpha = \left(\vec{\bigotimes_{(s,t)\in\alpha}} \varphi_{st} \right) \circ \left(\vec{\bigotimes_{(s,t)\in\alpha}} I_{st}^{-1} \right).$$

For $\alpha, \beta \in \mathscr{D}$, $\alpha < \beta$, $\beta(\alpha_0, \alpha_1) = \beta$ we have

$$\begin{aligned}
&\varphi_\beta \circ \eta_{\beta,\alpha}\\
&= \Big(\vec{\bigotimes}_{(s,t)\in\beta} \varphi_{st}\Big) \circ \Big(\vec{\bigotimes}_{(s',t')\in\alpha} \Delta_{\#\beta(s',t')}\Big) \circ \Big(\vec{\bigotimes}_{(s',t')\in\alpha} I_{s't'}^{-1}\Big)\\
&= \Big(\vec{\bigotimes}_{(s',t')\in\alpha} \varphi_{s't'}\Big) \circ \Big(\vec{\bigotimes}_{(s',t')\in\alpha} I_{s't'}^{-1}\Big)\\
&= \varphi_\alpha
\end{aligned}$$

by the evolution property of $\{\varphi_{st}\}$. As $\varphi_{st}(\mathbf{1}) = 1$ for all $(s, t) \in T$ the equation $\varphi_\beta \circ \eta_{\beta,\alpha} = \varphi_\alpha$ also holds in the general case $\alpha < \beta$. It follows the existence of a unique state φ on $C(\mathscr{B})$ such that $\varphi \circ \eta_\alpha = \varphi_\alpha$ for all $\alpha \in \mathscr{D}$. By construction φ fulfills property (a) of Definition 3.3. Conversely, if φ' is a state on $C(\mathscr{B})$ having this property and satisfying $\varphi' \circ \tau_{st} = \varphi_{st}$ it also fulfills $\varphi' \circ \eta_\alpha = \varphi_\alpha$ which means $\varphi = \varphi'$. For the second part of the theorem we prove that an independent increment process $(\mathscr{A}, j_{st}, \psi)$ over $\mathscr{B}$ with convolution evolution equal to $\{\varphi_{st}\}$ is equivalent to $(C(\mathscr{B}), h_{st}, \varphi)$. As $(\mathscr{A}, j_{st})$ is an object in $\mathfrak{L}_{\mathscr{B}}$ there is a homomorphism η: $C(\mathscr{B}) \to \mathscr{A}$ satisfying $j_{st} = \eta \circ h_{st}$. The state $\psi \circ \eta$ on $C(\mathscr{B})$ has property (a) of Definition 3.3 and satisfies $\psi \circ \eta \circ h_{st} = \varphi_{st}$ which gives $\psi \circ \eta = \varphi$. □

We now establish the connection with continuous tensor products. We call an independent increment process $(\mathscr{A}, j_{st}, \varphi)$ over a graded $*$-bialgebra $\mathscr{B}$ *even* if all the states $\varphi_{st} = \varphi \circ j_{st}$ are even. A *graded Hilbert space* is a Hilbert space $\mathscr{H}$ together with a pair $(\mathscr{H}^0, \mathscr{H}^1)$ of orthogonal subspaces of $\mathscr{H}$ such that $\mathscr{H} = \mathscr{H}^0 \oplus \mathscr{H}^1$. Denote by $\mathscr{O}$ the system of open subsets of $\mathbb{R}_+$.

Definition 3.4. (cf. [18]) Let $(Y_I)_{I\in\mathscr{O}}$ be a family of graded $*$-algebras (of graded Hilbert spaces) such that for two disjoint open sets I_1 and I_2 in $\mathbb{R}_+$ there is an even isomorphism (an even unitary map)

$$\tau(I_1, I_2)\colon Y_{I_1} \hat{\otimes} Y_{I_2} \to Y_{I_1 \cup I_2}.$$

If the condition

$$\begin{aligned}
&\tau(I_1 \cup I_2, I_3) \circ (\tau(I_1, I_2) \otimes \mathrm{id})\\
&= \tau(I_1, I_2 \cup I_3) \circ (\mathrm{id} \otimes \tau(I_2, I_3))
\end{aligned} \tag{3.2}$$

is fulfilled for any three disjoint open sets I_1, I_2 and I_3 in $\mathbb{R}_+$ then $Y_{\mathbb{R}_+}$ is said to be a *continuous tensor product.*

Theorem 3.2. *Let $(\mathscr{A}, j_{st}, \varphi)$ be an even independent increment process over a graded $*$-bialgebra $\mathscr{B}$ and denote by $(\mathscr{H}, \pi, \vartheta)$ the GNS-triplet associated to $(\mathscr{A}, \varphi)$. Then $\pi(\mathscr{A})$ and $\mathscr{H}$ are continuous tensor products.*

Proof. By Proposition 3.1 and Theorem 3.1 we may assume that $\mathscr{A} = C(\mathscr{B})$ and $j_{st} = h_{st}$. The algebra $C(\mathscr{B}) = C$ is an inductive limit of graded $*$-algebras and therefore carries a natural graded $*$-algebra structure (C^0, C^1) such that the homomorphisms j_{st}: $\mathscr{B} \to C$ are even. Moreover, as the states φ_{st} are even by assumption, the states φ_α on $\mathscr{B}_\alpha$, $\alpha \in \mathscr{D}$, as defined in the proof of Theorem 3.1, are even which means that φ is even as an inductive limit of even states. For I in $\mathscr{O}$ we define C_I to be the subalgebra of C generated by all elements of the form $h_{st}(b)$ where b runs through all elements of $\mathscr{B}$ and (s, t) runs through all open intervals contained in I. As the h_{st} are even, we have $C_I = C_I^0 \oplus C_I^1$ where $C_I^\iota = C_I \cap C^\iota$, $\iota \in \mathbb{Z}_2$, and the graduation of C induces the graduation (C_I^0, C_I^1) of C_I. We set $\tilde{\mathscr{A}}_I = \pi(C_I)$, $\tilde{\mathscr{A}}_I^\iota = \pi(C_I^\iota)$, $\mathscr{H}_I = \overline{\vartheta(C_I)}$ and $\mathscr{H}_I^\iota = \overline{\vartheta(C_I^\iota)}$ for $I \in \mathscr{O}$ and $\iota \in \mathbb{Z}_2$. If $\tilde{a}$ is in $\tilde{\mathscr{A}}_I^0 \cap \tilde{\mathscr{A}}_I^1$ then $\tilde{a} = \pi(c^0) = \pi(c^1)$ for some $c^\iota \in C_I^\iota$. Thus for $c_1, c_2 \in C$

$$\langle \vartheta(c_1) | \tilde{a}\vartheta(c_2)\rangle = \varphi(c_1^* c^0 c_2) = \varphi(c_1^* c^1 c_2)$$

and if c_1 and c_2 are homogeneous either $c_1^* c^0 c_2$ or $c_1^* c^1 c_2$ is odd, thus $\langle \vartheta(c_1) | \tilde{a}\vartheta(c_2)\rangle = 0$ because φ is even. But this means $\tilde{a} = 0$. It follows that $(\tilde{\mathscr{A}}_I^0, \tilde{\mathscr{A}}_I^1)$ is a graduation of the $*$-algebra $\tilde{\mathscr{A}}_I$. Similarly it can be shown that $(\mathscr{H}_I^0, \mathscr{H}_I^1)$ is a graduation of the Hilbert space $\mathscr{H}_I$. For disjoint $I_1, I_2 \in \mathscr{O}$ we define the linear mapping

$$\tau(I_1, I_2)\colon \tilde{\mathscr{A}}_{I_1} \hat{\otimes} \tilde{\mathscr{A}}_{I_2} \to \tilde{\mathscr{A}}_{I_1 \cup I_2}$$

by

$$\tau(I_1, I_2)(\tilde{a}_1 \otimes \tilde{a}_2) = \tilde{a}_1 \tilde{a}_2;\ \tilde{a}_1 \in \tilde{\mathscr{A}}_{I_1},\ \tilde{a}_2 \in \tilde{\mathscr{A}}_{I_2}.$$

By property (b) of Definition 3.3 the map $\tau(I_1, I_2)$ is an isomorphism. It is also even by construction. The associativity of multiplication in $\pi(C)$ yields that the mappings $\tau(I_1, I_2)$ fulfill condition (3.2). We proved that $\pi(C) = \tilde{\mathscr{A}}_{\mathbb{R}_+}$ is a continuous tensor product.

If we define the linear mapping

$$\sigma(I_1, I_2)\colon \vartheta(C_{I_1}) \otimes \vartheta(C_{I_2}) \to \vartheta(C_{I_1 \cup I_2})$$

by

$$\sigma(I_1, I_2)(\vartheta(c_1) \otimes \vartheta(c_2)) = \vartheta(c_1 c_2);\ c_1 \in C_{I_1},\ c_2 \in C_{I_2},$$

we have for homogeneous elements $c_1, c_1' \in C_{I_1}$, $c_2, c_2' \in C_{I_2}$

$$\begin{aligned}
&\langle \sigma(\vartheta(c_1) \otimes \vartheta(c_2)) | \sigma(\vartheta(c_1') \otimes \vartheta(c_2'))\rangle\\
&= \langle \vartheta(c_1 c_2) | \vartheta(c_1' c_2')\rangle = \varphi(c_2^* c_1^* c_1' c_2')\\
&= (-1)^{(\partial(c_1) + \partial(c_1'))\partial(c_2)} \varphi(c_1^* c_1') \varphi(c_2^* c_2')\\
&= \varphi(c_1^* c_1') \varphi(c_2^* c_2')\\
&= \langle \vartheta(c_1) \otimes \vartheta(c_2) | \vartheta(c_1') \otimes \vartheta(c_2')\rangle
\end{aligned}$$

where we used properties (a) and (b) of Definition 3.3 and the fact that φ is even. It follows that $\sigma(I_1, I_2)$ can be extended to an even unitary map from $\mathscr{H}_{I_1} \otimes \mathscr{H}_{I_2}$ to $\mathscr{H}_{I_1 \cup I_2}$. Again by the associativity of multiplication in $\pi(C)$ we have that the maps $\sigma(I_1, I_2)$ fulfill condition (3.2), and $\mathscr{H} = \mathscr{H}_{\mathbb{R}_+}$ is a continuous tensor product. □

4. Generators of Convolution Semi-Groups

Definition 4.1. An increment process $(\mathscr{A}, j_{st}, \varphi)$ over a graded $*$-bialgebra $\mathscr{B}$ is called a *stationary* increment process if $\varphi \circ j_{st} = \varphi \circ j_{0,t-s}$ for all $(s, t) \in T$.

If $(\mathscr{A}, j_{st}, \varphi)$ is an independent stationary increment process over $\mathscr{B}$ the associated convolution evolution $\{\varphi_{st}\}$ of states on $\mathscr{B}$ can be replaced by the convolution semi-group $\{\varphi_t\}$ of states on $\mathscr{B}$ given by $\varphi_t = \varphi_{0,t}$.

Definition 4.2. An independent stationary increment process $(\mathscr{A}, j_{st}, \varphi)$ over a graded $*$-bialgebra $\mathscr{B}$ is called *continuous* if the associated convolution semi-group of states $\{\varphi_t\}$ is continuous, i.e.

$$\lim_{t \downarrow 0} \varphi_t(b) = \delta(b) \tag{4.1}$$

for all $b \in \mathscr{B}$.

Let $\mathscr{C}$ be a coalgebra. To every linear functional ν on $\mathscr{C}$ one can associate the linear operator T_ν on $\mathscr{C}$ defined by

$$T_\nu = (\nu \otimes \mathrm{id}) \circ \Delta.$$

Making use of the identity

$$\Delta \circ (\rho \otimes \mathrm{id}) = (\rho \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)$$

which holds for any linear functional ρ one $\mathscr{C}$, and using the coassociativity of $\mathscr{C}$, we have for any two linear functionals ν and ρ on $\mathscr{C}$

$$\begin{aligned} T_\nu \circ T_\rho &= (\nu \otimes \mathrm{id}) \circ \Delta \circ (\rho \otimes \mathrm{id}) \circ \Delta \\ &= (\nu \otimes \mathrm{id})(\rho \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \Delta \\ &= (\rho \otimes \nu \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \\ &= ((\rho * \nu) \otimes \mathrm{id}) \circ \Delta = T_{\rho * \nu}. \end{aligned}$$

We see that for every 1-parameter convolution semi-group $\{\varphi_t : t \in \mathbb{R}_+\}$ on $\mathscr{C}$ the 1-parameter family $\{T_{\varphi_t}\}$ is a semi-group of linear operators on $\mathscr{C}$. The following example illustrates how $\{T_{\varphi_t}\}$ for a convolution semi-group of states $\{\varphi_t\}$ on a graded $*$-bialgebra plays the role of the Markov semi-group in classical probability theory. Let G be a topological semi-group. Then for a function f in $L^\infty(G) \cap \mathscr{R}(G)$ and a convolution semi-group $\{\varphi_t : t \in \mathbb{R}_+\}$ of probability measures on G we have

$$(T_{\varphi_t} f)(x) = (\varphi_t \otimes \mathrm{id}) \circ \Delta f(x) = \int_G d\varphi_t(y) f(yx).$$

Proposition 4.1. *Let ν be an even linear functional on a graded $*$-bialgebra $\mathscr{B}$. Then the following are equivalent*

(i) *ν is positive*
(ii) *T_ν is positive*
(iii) *T_ν is completely positive.*

Proof. Clearly (iii)$\Rightarrow$(ii) and the identity $\nu = \delta \circ T_\nu$ shows that (ii)$\Rightarrow$(i). Thus it remains to prove (i)$\Rightarrow$(iii) and, since Δ as a homomorphism is completely positive, this will follow from the complete positivity of $\nu \otimes \mathrm{id}$: $\mathscr{B} \hat{\otimes} \mathscr{B} \to \mathscr{B}$ (which is a well known fact in case the graduation on $\mathscr{B}$ is trivial). For $n, N \in \mathbb{N}$ let a_k, $k = 1, \ldots, n$, be $N \times N$-matrices and let b_{kl}, b'_{kl}, $k, l = 1, \ldots, n$, be homogeneous elements of $\mathscr{B}$.

Then

$$\begin{aligned} &\mathrm{id} \otimes \nu \otimes \mathrm{id} \left(\sum_{k,l=1}^n a_k \otimes b_{kl} \otimes b'_{kl} \right)^* \left(\sum_{u,v=1}^n a_u \otimes b_{uv} \otimes b'_{uv} \right) \\ &= \sum_{k,l,u,v=1}^n (-1)^{g(b'_{kl})(g(b_{kl}) + g(b_{uv}))} \nu(b^*_{kl} b_{uv}) a^*_k a_u \otimes b'_{kl} b'_{uv}. \end{aligned} \tag{4.2}$$

Since ν is even, the only non-zero elements in the sum (4.2) are those for which the $\pm$ 1-factor is equal to 1. But then the complete positivity of $\nu \otimes \mathrm{id}$ follows by the same argument as in the trivial graduation case. □

As a consequence of Proposition 4.1 an even convolution semi-group of states $\{\varphi_t : t \in \mathbb{R}_+\}$ on a graded $*$-bialgebra $\mathscr{B}$ gives rise to a semi-group $\{T_{\varphi_t}\}$ of linear operators on $\mathscr{B}$ with the properties

$$T_{\varphi_t} \text{ is completely positive}$$

$$T_{\varphi_t}(\mathbf{1}) = 1$$

for all $t \in \mathbb{R}_+$.

Let $\mathscr{C}$ be a coalgebra. The sum $\mathscr{D} + \mathscr{D}' = \{d + d' : d \in \mathscr{D}, d' \in \mathscr{D}'\}$ of two subcoalgebras $\mathscr{D}, \mathscr{D}'$ of $\mathscr{C}$ is again a subcoalgebra of $\mathscr{C}$. By the fundamental theorem on coalgebras [21] the subcoalgebra of $\mathscr{C}$ generated by a single element in $\mathscr{C}$ is finite-dimensional. It follows that $\mathscr{C}$ is the inductive limit of its system of finite-dimensional subcoalgebras. For a linear functional ν on $\mathscr{C}$ the operator T_ν leaves invariant every subcoalgebra of $\mathscr{C}$. If $\mathscr{D}$ is a finite-dimensional subcoalgebra of $\mathscr{C}$ we define the linear functional $\rho(\mathscr{D})$ on $\mathscr{D}$ by

$$\rho(\mathscr{D})(d) = \sum_{n=0}^{\infty} \frac{\nu^{*n}}{n!}(d) \tag{4.3}$$

for $d \in \mathscr{D}$. The inductive limit of the family $(\rho(\mathscr{D}))$ is a linear functional on $\mathscr{C}$ which we denote by $\exp_* \nu$. Moreover, if $\{\varphi_t : t \in \mathbb{R}_+\}$ is a convolution semi-group on $\mathscr{C}$ fulfilling the continuity condition (4.1) then the restriction $T_{\varphi_t}|\mathscr{D}$ of T_{φ_t} to a subcoalgebra $\mathscr{D}$ is a semi-group of linear operators on $\mathscr{D}$, and if $\mathscr{D}$ is finite-dimensional $T_{\varphi_t}|\mathscr{D}$ is of the form

$$T_{\varphi_t}|\mathscr{D} = \exp(t\, G(\mathscr{D}))$$

for some linear operator $G(\mathscr{D})$ on $\mathscr{D}$. We define the linear functional γ on $\mathscr{C}$ to be the inductive limit of the family $(\delta \circ G(\mathscr{D}))$. Clearly

$$\varphi_t = \exp_*(t\gamma).$$

We see that the continuous semi-groups $\{\varphi_t : t \in \mathbb{R}_+\}$ on a coalgebra $\mathscr{C}$ are exactly those of the form $\{\exp_*(t\gamma)\}$ for some linear functional γ on $\mathscr{C}$. Positivity conditions on $\{\varphi_t\}$ impose some further restrictions on γ. One has [19, 20]:

Theorem 4.1. *For an even linear functional γ on a graded $*$-bialgebra $\mathscr{B}$ the following are equivalent*

(i) *γ is conditionally positive, i.e. $\gamma(b^* b) \geqq 0$ for all $b \in \mathscr{B}$ with $\delta b = 0$, and γ is hermitian*
(ii) *$\exp_*(t\gamma)$ is positive for all $t \in \mathbb{R}_+$.*

By Theorem 3.1 and Theorem 4.1 an even, continuous independent stationary increment process over $\mathscr{B}$ is (up to equivalence) uniquely determined by the generator of its convolution semi-group of states on $\mathscr{B}$, which is a conditionally positive, hermitian linear functional on $\mathscr{B}$ vanishing at $\mathbf{1}$. Conversely every linear functional on $\mathscr{B}$ satisfying these properties gives rise to an even, continuous independent stationary increment process over $\mathscr{B}$.

We give an example of a non-commutative graded $*$-bialgebra. Let $(X_t)_{t\in T}$, $X_t=(X_t^1, \ldots, X_t^d)$, be a $\mathbb{C}^d$-valued, $d\in\mathbb{N}$, classical stochastic process such that all moments of the process exist. We associate to this classical process a stochastic process $(\hat{X}_t)_{t\in T}$ over the polynomial algebra $\mathbb{C}[x_n^*, x_n; n=1, \ldots, d]$ in $2d$ *commuting indeterminates* x_n^* *and* x_n by setting

$$\hat{X}_t((x_1^*)^{k_1} \ldots (x_d^*)^{k_d} x_1^{l_1} \ldots x_d^{l_d})(\omega) = \overline{X_t^1(\omega)^{k_1}} \ldots \overline{X_t^d(\omega)^{k_d}} X_t^1(\omega)^{l_1} \ldots X_t^d(\omega)^{l_d},$$

$k_n, l_n=0, 1, \ldots$. The algebra $\mathbb{C}[x_n^*, x_n]$ is a sub-$*$-bialgebra of $\mathscr{R}(\mathbb{C}^d)$; the comultiplication is given by $\Delta x_n = x_n\otimes\mathbf{1}+\mathbf{1}\otimes x_n$, the counit by $\delta x_n=0$, and the involution by $(x_n)^*=x_n^*$. The non-commutative analogue of $\mathbb{C}[x_n^*, x_n]$ is the polynomial algebra $\mathbb{C}\langle x_n^*, x_n\rangle$ in $2d$ *non-commuting* indeterminates; see [10]. More generally, also including the graded case, let $V=V^0\oplus V^1$ be a graded vector space and let $v\mapsto v^*$ be a selfinverse, antilinear, even map on V. Denote by $\mathscr{T}(V)$ the graded tensor algebra of V; see [8]. The space $\mathscr{T}(V)$ is equal to the direct sum $\bigoplus_{n=0}^{\infty}\mathscr{T}(V)^{(n)}$ where $\mathscr{T}(V)^{(n)}$, $n\in\mathbb{N}$, denotes the n-fold tensor product of $\mathscr{T}(V)$ with itself and $\mathscr{T}(V)^{(0)}=\mathbb{C}$. We denote by $\mathscr{T}(V)_1$ the subspace $\bigoplus_{n=1}^{\infty}\mathscr{T}(V)^{(n)}$ of $\mathscr{T}(V)$. We turn $\mathscr{T}(V)$ into a graded $*$-algebra with the involution given by extension of $v\mapsto v^*$. The $*$-algebra $\mathscr{T}(V)$ becomes a graded $*$-bialgebra by defining $\Delta:\mathscr{T}(V)\to\mathscr{T}(V)\hat{\otimes}\mathscr{T}(V)$ and $\delta:\mathscr{T}(V)\to\mathbb{C}$ to be the homomorphisms given by $\Delta v=v\otimes\mathbf{1}+\mathbf{1}\otimes v$ and $\delta v=0$.

As an illustration of the above results let us determine the form of the most general conditionally positive, hermitian linear functional on $\mathscr{T}(V)$. We introduce the following notation. If β is a linear functional on V which is hermitian, i.e. $\beta(v^*)=\overline{\beta(v)}$ for all $v\in V$, we define the conditionally positive, hermitian linear functional d_β on $\mathscr{T}(V)$ by $d_\beta|\mathscr{T}(V)^{(n)}=0$ for $n\neq 1$ and $d_\beta(v)=\beta(v)$ for $v\in V=\mathscr{T}(V)^{(1)}$.

Let $(x_i)_{i\in I}$ be a vector space basis of V, which without loss of generality can be assumed to consist of hermitian elements. An element F in $\mathscr{T}(V)$ can be written in a unique way in the form

$$F=c(F)\mathbf{1}+\sum_{i\in I} c_i(F)x_i+\sum_{i,j\in I} x_iF_{ij}x_j$$

where $c(F)=\delta(F)$ and $c_i(F)$, $i\in I$, are complex numbers and F_{ij}, $i,j\in I$, are in $\mathscr{T}(V)$. We denote by $M(I)$ the algebra of matrices $(c_{ij})_{i,j\in I}$, $c_{ij}\in\mathbb{C}$, for which $c_{ij}\neq 0$ only for a finite number of pairs $(i,j)\in I\times I$. If $\mathscr{A}$ is any algebra an element $A\in\mathscr{M}(I)\otimes\mathscr{A}$ can be regarded as a matrix $(A_{ij})_{i,j\in I}$, $A_{ij}\in\mathscr{A}$, for which $A_{ij}\neq 0$ only for a finite number of pairs $(i,j)\in I\times I$.

Theorem 4.2. *Let* $v\colon\mathscr{T}(V)\to\mathbb{C}$ *be a mapping. Then the following statements are equivalent:*

(i) *v is a conditionally positive, hermitian linear functional*
(ii) *there exist a real number α, a hermitian linear functional β on V and a positive linear functional N on $\mathscr{M}(I)\otimes\mathscr{T}(V)$ such that*

$$v(F)=\alpha\delta(F)+d_\beta(F)+N((F_{ij})_{i,j\in I}) \tag{4.3}$$

for all $F\in\mathscr{T}(V)$.

If v is a conditionally positive, hermitian linear functional the quantities α, β and N in (4.3) are uniquely determined.

Proof. (i)$\Rightarrow$(ii): We set $\alpha=v(\mathbf{1})$, $\beta(v)=v(v)$ for $v\in V$, and

$$N((G_{ij})_{i,j})=\sum_{i,j} v(x_iG_{ij}x_j)$$

for $(G_{ij})_{i,j}\in\mathscr{M}(I)\otimes\mathscr{T}(V)$. Then (4.3) holds and, since v is hermitian, α is real and β is hermitian. We still have to prove that N is positive. We have for $(G_{ij})_{i,j}\in\mathscr{M}(I)\otimes\mathscr{T}(V)$

$$N\left(\left(\sum_n G_{ni}^*G_{nj}\right)_{i,j}\right) = \sum_{i,j,n} v(x_iG_{ni}^*G_{nj}x_j) = \sum_n v\left(\left(\sum_i G_{ni}x_i\right)^*\left(\sum_j G_{nj}x_j\right)\right) \geqq 0$$

because v is conditionally positive. (ii)$\Rightarrow$(i): We may assume $\alpha=\beta=0$. We must prove that v of the form (4.3) is conditionally positive, hermitian if N is positive. For $F\in\mathscr{T}(V)$ we have $v(F^*)=N((F_{ji}^*)_{i,j})=\overline{N((F_{ij})_{i,j})}=\overline{v(F)}$, so v is hermitian. Now let F be an element in $\mathscr{T}(V)$ such that $\delta(F)=c(F)=0$. Then we have $v(F^*F)=N((G_i^*G_j)_{i,j})$ where $G_i=c_i(F)+\sum_n x_nF_{ni}$, $k=1, \ldots, d$. As $(G_i^*G_j)_{i,j}$ is a positive element in $\mathscr{M}(I)\otimes\mathscr{T}(V)$, it follows that $v(F^*F)\geqq 0$. If v is a linear functional on $\mathscr{T}(V)$ of the form (4.3) then $\alpha=v(\mathbf{1})$ and $\beta(v)=v(v)$, $v\in V$, which means that α, β and N are uniquely determined by v. $\square$

If Q is a sesquilinear form on V we define the linear functional g_Q on $\mathscr{T}(V)$ by $g_Q|\mathscr{T}(V)^{(n)}=0$ for $n\neq 2$ and $g_Q(vw)=Q(v^*, w)$ for $v, w\in V$. It is immediate to show that g_Q is conditionally positive, hermitian if Q is positive, i.e. $Q(v)=Q(v,v)\geqq 0$ for all $v\in V$.

Theorem 4.3. *Let ν be a conditionally positive, hermitian linear functional on $\mathcal{T}(V)$. Then ν can be represented in the form*

$$\nu = \alpha\delta + d_\beta + g_Q + p \tag{4.4}$$

where

(i) *α is a real number*
(ii) *β is a hermitian linear functional on V*
(iii) *Q is a positive sesquilinear form on V with the following property:*

If P is a positive sesquilinear form on V such that $\nu - g_P$ is conditionally positive then $Q \geqq P$ (4.5)

(iv) *p is a conditionally positive, hermitian linear functional on $\mathcal{T}(V)$ vanishing on $\mathcal{T}(V)^{(0)} \oplus \mathcal{T}(V)^{(1)}$.*

The quantities α, β, Q and p are uniquely determined by ν.

Proof. Suppose we have a representation of ν of the form (4.4). As p and g_Q both vanish on $\mathcal{T}(V)^{(0)} \oplus \mathcal{T}(V)^{(1)}$ it follows that $\alpha = \nu(\mathbf{1})$ and $\beta(v) = \nu(v)$, $v \in V$, so α and β are uniquely determined. Of course Q is uniquely determined by property (4.5). But then p must be equal to the difference $\nu - (\alpha\delta + d_\beta + g_Q)$ which proves uniqueness of the representation (4.4). For the proof of existence we must find a positive sesquilinear form Q on V satisfying (4.5) and such that $\nu - g_Q$ is conditionally positive. To this goal let $(x_i)_{i \in I}$ be a fixed vector space basis of V. Denote by H the vector space direct sum

$$H = \bigoplus_{i \in I} \mathcal{T}(V).$$

The elements of H are written in the form $(F_i)_{i \in I} = (F_i)$, $F_i \in \mathcal{T}(V)$. We define a sesquilinear form S on H by

$$S((F_i), (G_i)) = \sum_{i,j \in I} \nu(x_i^* F_i^* G_j x_j).$$

As ν is conditionally positive, S is positive. We denote by $\mathcal{H}$ the completion of the pre-Hilbert space H/S and denote by $\vartheta: H \to \mathcal{H}$ the canonical mapping. Let $H_1 \subset H$ be the direct sum

$$H_1 = \bigoplus_{i \in I} \mathcal{T}(V)_1$$

and denote by E the projection of $\mathcal{H}$ onto $\vartheta(H_1)^\perp$. We define the positive sesquilinear form Q on V by

$$Q\Big(\sum_{i \in I} c_i x_i, \sum_{i \in I} c_i' x_i\Big) = \langle (E \circ \vartheta)(c_i \mathbf{1}) | \vartheta(c_i' \mathbf{1}) \rangle_{\mathcal{H}}.$$

We claim that $\nu - g_Q$ is conditionally positive. To see this, let F be an element of $\mathcal{T}(V)_1$. Then there are complex numbers c_i, $i \in I$, and an element (F_i) in H_1 such that

$$F = \sum_{i \in I} (c_i \mathbf{1} + F_i) x_i$$

and we have

$$\begin{aligned}
&\nu(F^* F) \\
&= \sum_{i,j \in I} \nu(x_i^* (c_i \mathbf{1} + F_i)^* (c_j \mathbf{1} + F_j) x_j) \\
&= \| \vartheta(c_i \mathbf{1} + F_i) \|_{\mathcal{H}}^2 \\
&\geqq \inf_{(G_i) \in H_1} \| \vartheta(c_i \mathbf{1} + G_i) \|_{\mathcal{H}}^2 \\
&= \min_{\xi \in \overline{\vartheta(H_1)}} \| \vartheta(c_i \mathbf{1}) + \xi \|_{\mathcal{H}}^2 \\
&= \| (E \circ \vartheta)(c_i \mathbf{1}) \|_{\mathcal{H}}^2 \\
&= Q(\Sigma c_i x_i) \\
&= g_Q(F^* F).
\end{aligned}$$

If P is another positive sesquilinear form on V such that $\nu - g_P$ is conditionally positive then for every $\Sigma c_i x_i \in V$ and every $\varepsilon > 0$ there is an element $(G_i^{(\varepsilon)})$ in H_1 such that

$$\begin{aligned}
&Q(\Sigma c_i x_i) \\
&\geqq \| \vartheta(c_i \mathbf{1} + G_i^{(\varepsilon)}) \|_{\mathcal{H}}^2 - \varepsilon \\
&= \nu\Big(\sum_{i,j} x_i^* (c_i \mathbf{1} + G_i^{(\varepsilon)})^* (c_j \mathbf{1} + G_j^{(\varepsilon)}) x_j \Big) - \varepsilon \\
&\geqq P(\Sigma c_i x_i) - \varepsilon
\end{aligned}$$

which means $Q \geqq P$. □

5. Processes with Independent Additive Increments

In view of the applications it is useful to look at quantum stochastic processes as families of operators on some Hilbert space. In this section we will see that for an interesting class of quantum stochastic processes this point of view is included in Definition 3.3.

Definition 5.1. Let $F_v(t)$, $v \in V$, $t \in \mathbb{R}_+$, be linear operators on a Hilbert space $\mathcal{H}$ with common dense domain D such that the domain of the adjoint $F_v^\dagger(t)$ of $F_v(t)$ includes D and $F_v^\dagger(t)$ and $F_v(t)$ leave D invariant. For $(s, t) \in T$ we denote by $\mathcal{A}_{st}$ the polynomial algebra of operators on D generated by $F_v(s, t) = F_v(t) - F_v(s)$ and $F_v^\dagger(s, t)$, $v \in V$, and by $\mathcal{A}$ the polynomial algebra of operators on D generated by all $F_v^\dagger(s, t)$ and $F_v(s, t)$, $(s, t) \in T$, $v \in V$. Let there also be given a unit vector $\Phi \in D$ such that $\mathcal{A}\Phi = D$. We call $(F_v(t))_{v \in V, t \in \mathbb{R}_+}$ a *process with independent additive increments* (with cyclic vector Φ) if

(a) the mapping $v \mapsto F_v(t)$ is linear and a $*$-map, i.e. $F_{v^*}(t) = F_v^\dagger(t)$

(b) $\langle \Phi | G_1 \ldots G_n \Phi \rangle = \langle \Phi | G_1 \Phi \rangle \ldots \langle \Phi | G_n \Phi \rangle$ for all $n \in \mathbb{N}$, $t_1, \ldots, t_{n+1} \in \mathbb{R}_+$ with $t_1 < \ldots < t_{n+1}$, and $G_m \in \mathscr{A}_{t_m t_{m+1}}$, $m = 1, \ldots, n$

(c) $F_v(s, t) F_{v'}(s', t') = (-1)^{g(v)g(v')} F_{v'}(s', t') F_v(s, t)$ on D for all disjoint intervals (s, t) and (s', t') in $\mathbb{R}_+$ and all homogeneous $v, v' \in V$.

Proposition 5.1. *Let $(F_v(t))$ be a process with independent additive increments. Then $(\mathscr{A}, j_{st}, \Phi)$ is an independent increment process over $\mathscr{T}(V)$ where the cyclic vector Φ is identified with the state $a \mapsto \langle \Phi | a \Phi \rangle$ on $\mathscr{A}$ and $j_{st}: \mathscr{T}(V) \to \mathscr{A}$ is the homomorphism given by*

$$j_{st}(v) = F_v(s, t).$$

Conversely, if $(\mathscr{A}', j'_{st}, \varphi)$ is an independent increment process over $\mathscr{T}(V)$ and $(\mathscr{H}, \pi, \vartheta)$ denotes the GNS-triplet associated to $(\mathscr{A}', \varphi)$ then $(F_v(t))$ with

$$F_v(t) = (\pi \circ j'_{0,t})(v)$$

is a process with independent additive increments. □

We say that two processes $(F_v^{(i)}(t))$, $i = 1, 2$, with independent additive increments are *equivalent* if there is a unitary operator $\mathscr{U}: \mathscr{H}^{(1)} \to \mathscr{H}^{(2)}$ such that $\mathscr{U}\Phi^{(1)} = \Phi^{(2)}$ and $\mathscr{U} F_v^{(1)} \mathscr{U}^{-1} = F_v^{(2)}$. As a consequence of the above proposition and of Theorem 3.1 together with Proposition 3.1 a process with independent additive increments is determined up to equivalence by its convolution evolution of states $\{\varphi_{st}\}$ which is given by

$$\varphi_{st}(v_1 \ldots v_n) = \langle \Phi | F_{v_1}(s, t) \ldots F_{v_n}(s, t) \Phi \rangle.$$

We say that a process with independent additive increments is *continuous* (*even*, *has stationary increments*) if the corresponding process over $\mathscr{T}(V)$ is continuous (even, is a stationary increment process). A continuous process with independent stationary additive increments is determined by the generator of its convolution semi-group of states on $\mathscr{T}(V)$ and can be reconstructed by Theorem 3.1 and Proposition 5.1.

In the following we restrict ourselves to the case where V is two dimensional and is spanned as a vector space by elements x^* and x. We set $F_x(t) = F(t)$ and write $(F(t))$ for $(F_v(t))$. There are two possible graduations of V such that $v \mapsto v^*$ is even, namely the trivial one and the one where $V^0 = \{0\}$, $V^1 = V$. Let Γ denote the Bose or the Fermi Fock space over $L^2(\mathbb{R}_+)$ and $A^\dagger(f)$ and $A(f)$, $f \in L^2(\mathbb{R}_+)$, the creation and annihilation operators on Γ. We set $A(t) = A(\chi_{[0,t]})$ where $\chi_{[0,t]}$ denotes the characteristic function of the interval $[0, t]$ in $\mathbb{R}_+$. For positive real numbers λ and μ we denote by $A_{\lambda,\mu}(t)$ the operators on $\Gamma \otimes \Gamma$ given by

$$A_{\lambda,\mu}(t) = \lambda A(t) \hat{\otimes} \mathrm{id} + \mu \, \mathrm{id} \hat{\otimes} A^\dagger(t).$$

If λ and μ satisfy

$$(\lambda, \mu) = (\cosh x, \sinh x) \tag{5.1}$$

in the Bose case and

$$(\lambda, \mu) = (\cos x, \sin x) \tag{5.2}$$

in the Fermi case, the stochastic process $(A_{\lambda,\mu}(t))$ has been called a quantum Brownian motion of variance $\sigma^2 = \lambda^2 \pm \mu^2$ where the plus sign stands in the Bose case and the minus sign stands in the Fermi case; see [9, 5].

We call a linear functional ν on $\mathscr{T}(V)$ *gaussian* if it is of the form

$$\nu = \exp_*(d_\beta + g_Q)$$

where β is a linear functional on V and Q is a sesquilinear form on V; see [10, 23]. In the trivial graduation case a gaussian functional is positive if β is hermitian and Q is positive by Theorem 4.1. If the graduation is non-trivial g_Q is even, but d_β is not even if $\beta \neq 0$. Thus in this case a gaussian functional is positive if $\beta = 0$ and Q is positive, again by Theorem 4.1.

Theorem 5.1. *Let $(F(t))$ be an even, continuous process with independent stationary additive increments such that*

$$\lim_{t \downarrow 0} t^{-1} \langle \Phi | F_{v_1}(t) F_{v_2}(t) F_{v_3}(t) F_{v_4}(t) \Phi \rangle = 0 \tag{5.3}$$

for any choice of v_1, v_2, v_3, v_4 in V. Then the convolution semi-group $\{\varphi_t = \exp_*(t\gamma)\}$ of the process is gaussian, that is all the states φ_t are gaussian, and have the form

$$\varphi_t = \exp_* t(d_\beta + g_Q)$$

where $\beta(v) = \gamma(v)$ and $Q(v, w) = \gamma(v^* w)$, $v, w \in V$. Moreover, one (and only one) of the following statements holds:

(i) *$\mathscr{H} \cong \mathbb{C}$ and there exists a complex number z such that $F(t) = zt$*

(ii) *there exist complex numbers z, z_1, z_2, $|z_1| + |z_2| \neq 0$, such that $(F(t))$ is equivalent to*

$$(zt + z_1 A(t) + z_2 A^\dagger(t)) \tag{5.4}$$

with cyclic vector the vacuum in Γ

(iii)[1] *there exist complex numbers $z, z_1, z_2, |z_1| > |z_2|$, such that $(F(t))$ is equivalent to*

$$(zt + z_1 A_{1,1}(t) + z_2 A_{1,1}^\dagger(t)) \tag{5.5}$$

with cyclic vector the vacuum in $\Gamma \otimes \Gamma$

(iii)[2] *there exist complex numbers $z, z_1, z_2, |z_1| + |z_2| \neq 0$, and uniquely determined positive real numbers $\lambda, \mu, \lambda > \mu$, satisfying (5.1) in the trivial graduation case and (5.2) in the non-trivial graduation case, such that $(F(t))$ is equivalent to*

$$(zt + z_1 A_{\lambda,\mu}(t) + z_2 A_{\lambda,\mu}^\dagger(t)) \tag{5.6}$$

with cyclic vector the vacuum in $\Gamma \otimes \Gamma$.

The constant z is uniquely determined and equal to 0 in the non-trivial graduation case. The pair (z_1, z_2) in (ii) and (iii) is uniquely determined up to a transformation of the form $(z_1, z_2) \mapsto (e^{i\vartheta} z_1, e^{i\vartheta} z_2)$, $\vartheta \in \mathbb{R}$.

Proof. Because of

$$\varphi_{s+t}(x) = (\varphi_s * \varphi_t)(x) = \varphi_s(x) + \varphi_t(x)$$

and the continuity of $\{\varphi_t\}$ the function $t \mapsto \varphi_t(x)$ must be of the form zt for some

$z \in \mathbb{C}$. As $\{\varphi_t\}$ is even we must have $z=0$ in the Fermi case. We set $M(t)=F(t)-zt$ and get another even, continuous family $(M(t))$ of operators with independent and stationary additive increments. The generator γ' of $(M(t))$ coincides with γ on all monomials in $\mathscr{T}(V)$ not of length 1 and vanishes on x^* and x. As $\lim_{t\downarrow 0} t^{-1}\varphi_t(F) = \gamma(F)$ *for all* $F \in \mathscr{T}(V)$ we have by (5.3)

$$0=\lim_{t\downarrow 0} t^{-1}\varphi_t(v_1\, v_2\, v_3\, v_4)=\gamma'(v_1\, v_2\, v_3\, v_4)$$

for all $v_1, v_2, v_3, v_4 \in V$. As γ' is conditionally positive its restriction to the sub-$*$-algebra of $\mathscr{T}(V)$ consisting of all monomials of length greater than 0 is positive. By Schwartz inequality the vanishing of the fourth moments of γ' implies that γ' vanishes on all monomials of order greater than 2. Thus γ' vanishes on all monomials not of length 2, and γ' is equal to g_Q where Q is represented by the 2×2-matrix Q given by $Q_{kl}=\gamma(x_k^* x_l)$, $k, l=1, 2$, where $x_1=x^*$ and $x_2=x$. The matrix Q is positive definite, so it is of the form

$$\begin{pmatrix} r & m \\ \bar{m} & s \end{pmatrix} \tag{5.7}$$

where r and s are non-negative real numbers and $m=|m|e^{i\vartheta_0}$ is a complex number such that $rs \geqq |m|^2$. If Q is equal to 0 then $F(t)=zt$, $D=\mathbb{C}\Phi$, and we are in case (i). Before we continue, we observe that the processes in (ii), (iii)^1 and (iii)^2 are indeed even, continuous processes with independent stationary additive increments in the sense of Definition 5.1. If we assume $z=0$ the three cases can be dealed with simultaneously by looking at processes of the form

$$F(t)=z_1 A_{\lambda,\mu}(t)+z_2 A^{\dagger}_{\lambda,\mu}(t)$$

with $z_1, z_2 \in \mathbb{C}$, $\lambda, \mu \in \mathbb{R}_+$. The process (5.4) is obtained by setting $\lambda=1$, $\mu=0$, and if we set $\lambda=\mu=1$ we are in case (iii)^1. For $z=0$ the generators of the processes (5.4), (5.5) and (5.6) are of the form d_P where P is the positive definite 2×2-matrix given by

$$\begin{pmatrix} |z_1|^2\lambda^2+|z_2|^2\mu^2 & z_1 z_2(\lambda^2+\mu^2) \\ \bar{z}_1\bar{z}_2(\lambda^2+\mu^2) & |z_1|^2\mu^2+|z_2|^2\lambda^2 \end{pmatrix}. \tag{5.8}$$

For a given positive definite matrix Q of the form (5.7) we now try to find constants z_1, z_2, λ, μ such that the matrix P of the form (5.8) is equal to Q. First we assume that $\lambda^2+\mu^2=1$. By comparing the entries of the matrices P and Q we arrive at the equations

$$ab=|m| \tag{5.9}$$

$$a^2+b^2=r+s \tag{5.10}$$

$$\tau(a^2-b^2)=r-s \tag{5.11}$$

where $a=|z_1|$, $b=|z_2|$ and $\tau=\lambda^2-\mu^2$. If we have a solution of these equations for a, b and τ we set $z_1=a$, $z_2=be^{i\vartheta_0}$, $\lambda=(\frac{1}{2}(1+\tau))^{\frac{1}{2}}$ and $\mu=(1-\lambda^2)^{\frac{1}{2}}$ which is a solution of our problem. Equations (5.9) and (5.10) give a quadratic equation and there is always a solution for a and b in $\mathbb{R}_+$, which is unique up to interchanging a and b.

Assume that rank $Q=1$. Then $a=r^{\frac{1}{2}}$, $b=s^{\frac{1}{2}}$ solves (5.9) and (5.10), and if we put $\tau=1$ equation (5.11) is fulfilled, too. We also have $a+b\neq 0$ which shows that we are in case (ii). Now assume that Q is non-singular. Then $a\neq b$ must hold. Equation (5.11) is solved by setting $\tau=(r-s)(a^2-b^2)^{-1}$. If $r=s$ we have $\tau=0$ and this is case (iii)^1. If $r\neq s$ we may assume $\tau>0$. (If $\tau<0$ interchange a and b!) If $r\neq s$ and the graduation is non-trivial we are in case (iii)^2. If $r\neq s$ and the graduation is trivial we set $z_1'=\tau^{\frac{1}{2}}z_1$, $z_2'=\tau^{\frac{1}{2}}z_2$, $\lambda'=\tau^{-\frac{1}{2}}\lambda$ and $\mu'=\tau^{-\frac{1}{2}}\mu$. Then $(\lambda')^2-(\mu')^2=1$ and, substituting $z_1', z_2', \lambda', \mu'$ for z_1, z_2, λ, μ, we get the same matrix P. This is (iii)^2 for the trivial graduation case.

The matrix P remains unchanged if we substitute $(e^{i\vartheta}z_1, e^{i\vartheta}z_2)$, $\vartheta\in\mathbb{R}$, for (z_1, z_2). On the other hand, given a matrix of the form (5.8), the quantities $|z_1|, |z_2|, \lambda, \mu$ must satisfy equations (5.9), (5.10), (5.11) which yields the uniqueness statement for (z_1, z_2). □

From Definition 5.1 it follows that for the validity of Theorem 5.1 it is sufficient to require that (5.3) holds only when each of the vectors v_1, v_2, v_3, v_4 coincides either with x^* or with x. Condition (5.3) becomes the condition of "continuity of the trajectories" introduced in [3]. The "canonical forms" (5.4), (5.5) and (5.6) correspond to the canonical forms deduced in [3].

6. Generators of Processes Given by Solutions of Quantum Stochastic Differential Equations

Let $\mathscr{U}(d)$, $d\in\mathbb{N}$, be the group of unitary $d\times d$-matrices. Denote by f_{kl}, $k, l=1,\ldots,d$, the complex-valued function on $\mathscr{U}(d)$ given by $f_{kl}(U)=U_{kl}$, $U=(U_{mn})_{m,n=1,\ldots,d}\in\mathscr{U}(d)$, and by f_{kl}^* the complex conjugate function of f_{kl}, i.e. $f_{kl}^*(U)=\overline{U_{kl}}$. The subalgebra $K[d]$ of the $*$-algebra $\mathscr{F}(\mathscr{U}(d))$ of all complex-valued functions on $\mathscr{U}(d)$ generated by the functions f_{kl}^* and f_{kl} is a sub-$*$-bialgebra of $\mathscr{R}(\mathscr{U}(d))$ and can be identified with the space of all coefficients of continuous, irreducible unitary representations of $\mathscr{U}(d)$. The $*$-algebra $K[d]$ is often called the coefficient algebra of $\mathscr{U}(d)$ [12] and is an example of a Krein algebra [11]. The coalgebra structure of $K[d]$ induced by $\mathscr{R}(\mathscr{U}(d))$ is given by the comultiplication

$$\Delta f_{kl}=\sum_{n=1}^{d} f_{kn}\otimes f_{nl}$$

and the counit

$$\delta f_{kl}=\delta_{kl}\ \text{(Kronecker delta)}.$$

Let $(X_t)_{t\in T}$ be a classical stochastic process with values in $\mathscr{U}(d)$ and consider the stochastic process $(L^\infty(\Omega), (\hat{X}_t)_{t\in T}, P)$ over $K[d]$ given by $\hat{X}_t(f)=f\circ X_t$, $f\in K[d]$. The process (X_t) is uniquely determined by $(\hat{X}_t)$ via the equation

$$(X_t(\omega))_{kl}=\hat{X}_t(f_{kl})(\omega).$$

Denote by $\mathscr{M}(d)$ the $*$-algebra of complex $d\times d$-matrices. If $\mathscr{A}$ is any algebra an element $A\in\mathscr{M}(d)\otimes\mathscr{A}$ can be regarded as a $d\times d$-matrix $(A_{kl})_{k,l=1,\ldots,d}$ with entries

$A_{kl} \in \mathscr{A}$. The classical stochastic process (X_t) can also be identified with the family $(U_t)_{t\in T}$ of unitary elements in $\mathscr{M}(d)\otimes L^\infty(\Omega)$ given by

$$(U_t)_{kl}(\omega)=(U_t(\omega))_{kl}.$$

The non-commutative analogue $K\langle d\rangle$ of $K[d]$ is the polynomial algebra $\mathbb{C}\langle x_{kl}^*, x_{kl}; k, l=1,\ldots,d\rangle$ in the $2d^2$ non-commuting indeterminates x_{kl}^* and x_{kl} divided by the $*$-ideal J generated by the elements $\sum_{n=1}^{d} x_{kn}x_{ln}^* - \delta_{kl}$ and $\sum_{n=1}^{d} x_{nk}^* x_{nl} - \delta_{kl}$; see [24]. If $(U_t)_{t\in T}$ is a family of unitary elements in $\mathscr{M}(d)\otimes\mathscr{A}$ where $\mathscr{A}$ is a $*$-algebra, and if φ is a state on $\mathscr{A}$ then a stochastic process $(\mathscr{A}, (j_t)_{t\in T}, \varphi)$ over $K\langle d\rangle$ can be defined by

$$j_t(x_{kl}+J)=(U_t)_{kl}.$$

We describe the construction of a graded version of $K\langle d\rangle$; see [20].

Let $\mathscr{H}$ be a finite-dimensional graded Hilbert space; $\dim\mathscr{H}=d$. Then $\mathscr{H}$ is the orthogonal sum $\mathscr{H}^0\oplus\mathscr{H}^1$ of its subspaces $\mathscr{H}^0$ and $\mathscr{H}^1$ of even and odd elements. We denote by $L(\mathscr{H})$ the graded vector space of linear operators on $\mathscr{H}$ and by $L(\mathscr{H})^c$ the complex conjugate graded vector space of $L(\mathscr{H})$. As a set $L(\mathscr{H})^c$ consists of elements a^c, $a\in L(\mathscr{H})$. The vector space structure of $L(\mathscr{H})^c$ is given by $a^c+b^c=(a+b)^c$ and $\lambda a^c=(\bar{\lambda}a)^c$. We form the graded tensor algebra $\mathscr{T}_{\mathscr{H}}$ of the graded vector space $L(\mathscr{H})^c\oplus L(\mathscr{H})$. The algebra $\mathscr{T}_{\mathscr{H}}$ is a graded $*$-algebra with the involution given by $a^*=a^c\in L(\mathscr{H})^c$ for $a\in L(\mathscr{H})$. If $\{e(k): k=1,\ldots,d\}$ is an orthonormal basis of $\mathscr{H}$ adapted to the graduation of $\mathscr{H}$ we define the homomorphisms $\Delta: \mathscr{T}_{\mathscr{H}}\to\mathscr{T}_{\mathscr{H}}\hat{\otimes}\mathscr{T}_{\mathscr{H}}$ and $\delta: \mathscr{T}_{\mathscr{H}}\to\mathbb{C}$ by

$$\Delta e(k,l)=\sum_{n=1}^{d} e(k,n)\otimes e(n,l)$$

and

$$\delta e(k,l)=\delta_{kl}$$

where $e(k,l)$, $k,l=1,\ldots,d$, denote the matrix units associated with $\{e(k)\}$. The algebra $\mathscr{T}_{\mathscr{H}}$ is a graded $*$-bialgebra with comultiplication Δ and counit δ. We form the $*$-ideal J of $\mathscr{T}_{\mathscr{H}}$ generated by the elements

$$\sum_{n=1}^{d} e(k,n)\otimes e(l,n)^* - \delta_{kl}$$

$$\sum_{n=1}^{d} e(n,k)^*\otimes e(n,l) - \delta_{kl}.$$

A simple computation shows that J is a graded coideal in $\mathscr{T}_{\mathscr{H}}$, so $K\langle\mathscr{H}\rangle=\mathscr{T}_{\mathscr{H}}/J$ is a graded $*$-bialgebra with the structure induced by $\mathscr{T}_{\mathscr{H}}$. The construction of $K\langle\mathscr{H}\rangle$ does not depend on the choice of the adapted orthonormal basis of $\mathscr{H}$. We denote by π_J the canonical homomorphism from $\mathscr{T}_{\mathscr{H}}$ to $K\langle\mathscr{H}\rangle$.

Let Γ again be the Bose or Fermi Fock space over $L^2(\mathbb{R}_+)$ depending on whether the graduation of $\mathscr{H}$ is trivial or not. We consider the Bose Fock space as a graded Hilbert space with the trivial graduation and turn the Fermi Fock space

into a graded Hilbert space by setting

$$\Gamma^0=\bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+)^{\otimes_a 2n}$$

$$\Gamma^1=\bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+)^{\otimes_a 2n+1}$$

where $\otimes_a m$ denotes the m-fold anti-symmetric tensor product of Hilbert spaces. For $N\in\mathbb{N}$ let b_r, $r=1,\ldots,N$, and h be elements in $L(\mathscr{H})$, h hermitian. Let b_r be odd and let h be even in the non-trivial graduation case. We consider the N-dimensional quantum stochastic differential equation in the sense of [14] or [6] of the form

$$dU=U\left(\left(\sum_{r=1}^{N} dA_r^\dagger b_r - b_r^\dagger dA_r\right)+\left(ih-\frac{1}{2}\sum_{r=1}^{N} b_r^\dagger b_r dt\right)\right);$$

$$U(0)=\mathrm{id};$$

see also [13]. The solution $U(t)$ consists of unitary operators on $\mathscr{H}\otimes\Gamma^{\otimes N}$. A bounded operator A on $\mathscr{H}\otimes\Gamma^{\otimes N}$ is regarded as a matrix $(A_{kl})_{k,l=1,\ldots,d}$, $A_{kl}=A(k,l)\in\mathscr{B}(\Gamma^{\otimes N})$, with respect to the orthonormal basis $\{e(k)\}$ of $\mathscr{H}$. By Φ we denote the vacuum state on $\mathscr{B}(\Gamma^{\otimes N})$. We define homomorphisms j_{st}: $K\langle\mathscr{H}\rangle\to\mathscr{B}(\Gamma^{\otimes N})$ by

$$j_{st}(\pi_J e_{kl})=(U_{st})_{kl}$$

where $U_{st}=U_s^+U_t$, $s\leqq t$.

Theorem 6.1. *The stochastic process $(\mathscr{B}(\Gamma^{\otimes N}), (j_{st})_{(s,t)\in T}, \Phi)$ over $K\langle\mathscr{H}\rangle$ is an even, continuous independent stationary increment process.*

Proof. We restrict ourselves to the case $N=1$ and $h=0$. We denote by Γ_s^t, $s\leqq t$, the Fock space over $L^2[s,t]$. For an element a of a $*$-algebra $\mathscr{A}$ we set $a^0=a$ and $a^1=a^*$ where 0, 1 are considered as elements of $\mathbb{Z}_2$. By Theorem 2.5 of [13] (the theorem only deals with the Bose case, but the proof is almost the same in the Fermi case) the unitary operators U_{st} commute with all operators of the form $\mathrm{id}_{\mathscr{H}}\hat{\otimes} T\hat{\otimes}\mathrm{id}_s^\infty$ with $T\in\mathscr{B}(\Gamma_0^s)$. This implies that $U_{st}(k,l)^\iota$, $\iota\in\mathbb{Z}_2$, is of the form

$$\mathrm{id}_0^s\hat{\otimes}S\hat{\otimes}\mathrm{id}_t^\infty$$

for some $S\in\mathscr{B}(\Gamma_s^t)$. As U_{st} is even we have $g(U_{st}(k,l)^\iota)=g(e(k,l))$ and for $(s,t)\cap(s',t')=\varnothing$

$$U_{st}(k,l)^\iota U_{s't'}(k',l')^{\iota'}=\varepsilon(k,l,k',l')U_{s't'}(k',l')^{\iota'}U_{st}(k,l)^\iota$$

where we set $\varepsilon(k,l,k',l')=(-1)^{g(e(k,l))g(e(k',l'))}$. This gives property (b) of Definition 3.3. Using this result, the increment properties (a) and (b) of Definition 3.2 follow by an easy computation. It is clear that the vacuum Φ has the property (a) of Definition 3.3. For the proof of stationarity denote by σ_r, $r\in\mathbb{R}_+$, the shift operator on $L^2(\mathbb{R}_+)$ given by $\sigma_r f(t)=0$ for $t<r$ and $\sigma_r f(t)=f(t-r)$ for $t\geqq r$ and by $\mathscr{S}_r$ its second quantisation. By Theorem 2.5 of [13] (which can be proved in exactly the

same manner for the Fermi case) we have

$$\mathscr{S}_s^\dagger U_{st}(k,l)^{\iota}\mathscr{S}_s = U_{t-s}(k,l)^{\iota}.$$

Using this cocycle property, one immediately arrives at $\Phi \circ j_{st} = \Phi \circ j_{0,t-s}$. The continuity of the process follows by [6], page 486, and by [14], Corollary 2. As the operators U_{st} are even and as Φ is an even state all states $\Phi \circ j_{st}$ must be even. □

By Section 4 the process $(\mathscr{B}(\Gamma^{\otimes N}), j_{st}, \Phi)$ is determined by its generator up to equivalence. We proceed to compute this generator.

For a matrix $a=(a(k,l))_{k,l=1,\ldots,d}$ in $\mathscr{M}(d)$ define a $*$-derivation (that is a derivation which is also a $*$-map) $\tilde{D}_a$ on $\mathscr{T}_{\mathscr{H}}$ by

$$\tilde{D}_a e(k,l) = \sum_{n=1}^{d} a(k,n)e(n,l).$$

If a is skew hermitian $\tilde{D}_a$ leaves invariant the ideal I and gives rise to a $*$-derivation D_a on $K\langle\mathscr{H}\rangle$. For two matrices $a, b \in \mathscr{M}(d)$ define the linear operator $\tilde{L}_{ab}$ on $\mathscr{T}_{\mathscr{H}}$ by

$$\tilde{L}_{ab} e(k,l) = \sum_{n=1}^{d} c(k,n)e(n,l)$$

where $c = i[a,b]$ and by requiring $\tilde{L}_{ab} = \tilde{L}$ to satisfy the functional equation

$$\tilde{L}(F\otimes G) = (\tilde{L}F)\otimes G + F\otimes(\tilde{L}G) + 2i(\tilde{D}_a F\otimes \tilde{D}_b G - \tilde{D}_b F\otimes \tilde{D}_a G)$$

for all $F, G \in \mathscr{T}_{\mathscr{H}}$. The operator $\tilde{L}_{ab}$ leaves I invariant if a and b are skew hermitian and, in this case, induces a linear operator L_{ab} on $K\langle\mathscr{H}\rangle$. Denote by $\Theta \in L(\mathscr{H})$ the parity operator given by $\Theta v = (-1)^{g(v)} v$ for $v \in \mathscr{H}$ homogeneous. Notice that $\Theta a = (-1)^{g(a)} a\,\Theta$ for $a \in L(\mathscr{H})$ homogeneous. Define the linear operator S on $K\langle\mathscr{H}\rangle$ by

$$S = \frac{1}{2}\left(\sum_{r=1}^{N} D_{p_r}^2 + D_{q_r}^2 + L_{p_r q_r}\right) + D_{ih}$$

where

$$p_r = \tfrac{1}{2}(\Theta b_r - b_r^\dagger \Theta)$$
$$q_r = -\tfrac{1}{2}(\Theta b_r + b_r^\dagger \Theta).$$

Theorem 6.2. *The generator γ of $(\mathscr{B}(\Gamma^{\otimes N}), j_{st}, \Phi)$ is equal to $\delta \circ S$.*

Proof. We again assume $N=1$ and $h=0$. We substitute the operator $\Theta(p+iq)$ for b in the quantum stochastic differential equation

$$dU = U(dA^\dagger b - b^\dagger dA - \tfrac{1}{2} b^\dagger b\,dt)$$

and get the equation

$$dU = U(p\,dF + q\,dG + \tfrac{1}{2}c\,dt) \tag{6.1}$$

where

$$dF = dA^\dagger\Theta + \Theta\,dA$$
$$dG = i(dA^\dagger\Theta - \Theta\,dA)$$
$$c = p^2 + q^2 + i[p,q].$$

Now let $R, T \in \mathscr{B}(\Gamma)$ be homogeneous and let T be odd. Then we have for $k,l=1,\ldots,d$

$$\begin{aligned}&(e(k,l)\otimes R)(\Theta\,\hat{\otimes}\,T)\\ &=(-1)^{g(e(l))g(R)}(e(k,l)\,\hat{\otimes}\,R)(\Theta\,\hat{\otimes}\,T)\\ &=(-1)^{g(e(l))g(R)}\,e(k,l)\Theta\,\hat{\otimes}\,RT\\ &=(-1)^{g(e(l))g(R)+g(e(l))}\,e(k,l)\,\hat{\otimes}\,RT\\ &=e(k,l)\otimes RT\end{aligned}$$

which yields together with (6.1)

$$\begin{aligned}&d\left(\sum_{k,l=1}^{d} e(k,l)\otimes U(k,l)\right)\\ &=\left(\sum_{k,l=1}^{d} e(k,l)\otimes U(k,l)\right)(p\Theta\,\hat{\otimes}\,d(A^+ + A) + q\,\Theta\,\hat{\otimes}\,i\,d(A^+ - A) + \tfrac{1}{2}c\otimes dt)\\ &=\sum_{k,l=1}^{d} e(k,l)\otimes\sum_{n=1}^{d} U(k,n)(p(n,l)d(A^+ + A) + q(n,l)i\,d(A^+ - A)\\ &\quad+\tfrac{1}{2}c(n,l)dt)\end{aligned}$$

or

$$dU(k,l) = \sum_{n=1}^{d} U(k,n)\,dM(n,l)$$

with

$$dM(k,l) = p(k,l)d(A^+ + A) + q(k,l)i\,d(A^+ - A) + \tfrac{1}{2}c(k,l)dt.$$

Using the quantum Ito formula [6, 14] we obtain

$$\begin{aligned}&dM(k,l)^{\iota}\,dM(k',l')^{\iota'}\\ &=(p(k,l)^{\iota}\,p(k',l')^{\iota'} + q(k,l)^{\iota}\,q(k',l')^{\iota'}\\ &\quad+ip(k,l)^{\iota}\,q(k',l')^{\iota'} - iq((k,l)^{\iota}\,p(k',l')^{\iota'})dt.\end{aligned} \tag{6.2}$$

For an element $F = e(k_1,l_1)^{\iota_1}\otimes\ldots\otimes e(k_n,l_n)^{\iota_n}$ of $\mathscr{T}_{\mathscr{H}}$ we compute the expression

$$\begin{aligned}&(\gamma\circ\pi_J)(F)\\ &=\frac{d}{dt}\langle\Phi|U_t(k_1,l_1)^{\iota_1}\ldots U_t(k_n,l_n)^{\iota_n}\Phi\rangle|_{t=0}.\end{aligned} \tag{6.3}$$

We set $Y_r = U_t(k_r,l_r)^{\iota_r}$. The value of (6.3) is equal to the d_t-part at $t=0$ of $d(Y_1\ldots Y_n)$. Using again the quantum Ito formula, we have

$$\begin{aligned}&d(Y_1\ldots Y_n)\\ &=(dY_1)Y_2\ldots Y_n + \ldots + Y_1\ldots Y_{n-1}(dY_n)\\ &\quad+(dY_1)(dY_2)Y_3\ldots Y_n + \ldots + (dY_1)Y_2\ldots Y_{n-1}(dY_n)\\ &\quad+\ldots+\ldots+Y_1\ldots Y_{n-2}(dY_{n-1})(dY_n).\end{aligned} \tag{6.4}$$

Taking under account the initial condition $U(0)=\mathrm{id}$, we get that the terms of first order in (6.4) contribute

$$\tfrac{1}{2}(\delta \circ \tilde{D}_c)(F).$$

As U_t is non-anticipating and even we have for $r<s$

$$\begin{aligned}&(dY_r)\,Y_{r+1}\ldots Y_{s-1}(dY_s)\\ &=\sum_{v,w}(-1)^{\iota_r g(e(k_r,w))+\iota_s g(e(k_s,v))}\\ &\quad \varepsilon(k_{r+1},l_{r+1})\ldots\varepsilon(k_{s-1},l_{s-1})\,\varepsilon(k_s,v)\\ &\quad U_t(k_r,w)^{\iota_r}\,U_t(k_{r+1},l_{r+1})^{\iota_{r+1}}\\ &\quad \ldots U_t(k_{s-1},l_{s-1})^{\iota_{s-1}}\,U_t(k_s,v)^{\iota_s}\\ &\quad dM(w,l_r)^{\iota_r}\,dM(v,l_s)^{\iota_s}\end{aligned}$$

where we put $\varepsilon(k,l)=(-1)^{g(e(k,l))}$. Using (6.2) we have that the second order terms in (6.4) contribute

$$\tfrac{1}{2}(\delta\circ(\tilde{D}_p^2+\tilde{D}_q^2+\tilde{L}_{p,q}-\tilde{D}_c))(F)$$

which completes the proof. □

If in the case $N=1$ the operator b is hermitian or skew hermitian then $p=0$ or $q=0$ and S is equal to a second order derivation D^2 on $K\langle\mathscr{H}\rangle$.

In the Bose case we have $S=(\gamma\otimes\mathrm{id})\circ\Delta$ and S is the generator of the semi-group of operators associated to the process while this is not true in general in the Fermi case.

We treat a special case of Theorem 6.2. Let $N=2$ and $\lambda,\ \mu\in\mathbb{R}, b,h\in\mathscr{M}(d)$, h hermitian, b odd and h even if the graduation of $\mathscr{H}$ is not trivial. We consider the quantum stochastic differential equation on $\mathscr{H}\otimes\Gamma\otimes\Gamma$

$$dU=U\left(dA^{\dagger}_{\lambda,\mu}b-b^{\dagger}dA_{\lambda,\mu}+\left(ih-\frac{\lambda^2}{2}b^{\dagger}b-\frac{\mu^2}{2}bb^{\dagger}\right)dt\right);$$

see [13] and Section 5. The generator of the process associated with the solution of this equation is equal to $\delta\circ S$ where

$$S=\tfrac{1}{2}(\lambda^2+\mu^2)(D_p^2+D_q^2)+\tfrac{1}{2}(\lambda^2-\mu^2)L_{pq}+D_{ih}.$$

The generators of processes constructed in [24] by a method called the multiplicative Ito integral are of the same type.

Support by the Office of Naval Research under the Contract No. 00014-84-K-4421 is gratefully acknowledged by L. A. – The authors acknowledge support by the Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 123 "Stochastische Mathematische Modelle".

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Received January 20, 1987