Can mathematics help solving the interpretational problems of quantum theory?

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1 Introduction

It is well known that the crux of the interpretation of quantum theory is the superposition principle and it is a common place among physicists that the only way to exorcize the interference terms, related to this principle, is to accept the statement: *observables not looked at cannot have definite values*. According to this point of view, definite values arise from measurements and the term *objectification*, invented to describe this transition adds a metaphysical touch to contemporary physics.

Statements like the above one (and its consequences) have plagued the interpretation of quantum theory since its origins. Therefore, if one would be able to deduce the superposition principle from physically meaningful axioms which involve only observable quantities and experimentally performable operations and do not include any metaphysical assumptions, then quantum theory shall be made free of interpretations and statements which constitute a source of embarassment and uneasiness for all those who believe that the inner coherence of physics should not be saved at the price of accepting statements such as the above one.

The widespread attitude of dismissing these problems by claiming that ... *they have no practical implications* ... are the expression of a too narrow vision of science, tending to reduce scientific research to the production of algorithms which should be used for so called *practical purposes*. But many people have been, and probably shall always be, fascinated by theoretical physics not as a branch of applied research, but as a creative and speculative adventure. For them the pretense to close a speculative challenge by claiming that ... *it has no practical implications* ... , is not very convincing.

The present paper is an attempt to meet this challenge with the usual strategy adopted to clarify the foundations of a theory: to axiomatize it. The idea is very simple to state: *To formulate a set of axioms in terms of statements which are, at least in principle, physically meaningful, model independent and whose content is generally accepted by the physical community. Then to proceed by means of pure mathematical deduction and prove that the whole mathematical apparatus of quantum theory is a necessary consequence of the stated axioms.*

The two main ideas introduced here to achieve this goal are:

(i) to extrapolate from the Heisenberg indeterminacy principle, a *universal, information theoretic statement* which should govern any form of acquisition of information through experiments. This shall require an *epuration* of all the specifically physical notions (such as *mass, Planck constant, energy,...*) from the formulation of this principle.

(ii) to perform a similar operation on Schwinger's proposal of taking *measurements* rather than *events* as starting point for the description of the conceptual structure of quantum theory. We propose to use the notion of *Schwinger algebra of measurements* as the basic framework of a new axiomatic approach to probability theory in which the *platonic events* of pre–quantum physics are replaced by the *measurement operations*.

Within this new axiomatization of probability the information theoretic formulation of the Heisenberg principle appears as an optional axiom, just as Euclid's parallel postulate in the modern approaches to geometry. The introduction of this new, apparently purely qualitative, axiom has some very strong mathematical implications whose investigation is the main goal of the present paper.

It is a non trivial fact that **all the basic features of quantum theory can be deductively obtained within this axiomatic approach**: from the necessity to introduce *amplitudes* rather than probabilities, to the superposition principle; from the Schrödinger equation to the symmetry of the transition probabilities; from the emergence of complex numbers to that of unitary representations of groups .... It is worth to draw the reader's attention on the fact that also the *reversible* character of the Schrödinger equation has a purely statistical and information–theoretic origin, stemming from the symmetric role that two maximal (nondegenerate) observables play in their mutual conditioning: it is a kind of information theoretic relativity principle, asserting that, from the point of view of statistical predictions *there exist no priviliged set of compatible observables* (cf. the remark after Theorem 122).

The fact that such a rich and detailed structure emerges from very general, qualitative and (most of all) **physically meaningful** axioms, is responsible for the lengthy mathematical procedure. This deductive chain is not only an amusing intellectual game: the conceptual implications of the final result are relevant for the interpretation of quantum physics because of the following reasons. Suppose one accepts the axioms at the basis of the present approach, then, since *no theory requires, for its interpretation, more than it is required for the interpretation of its axioms* and since the axioms themselves do not require the introduction of strange properties such as *non reality, objectification, non locality, non separability,...* it follows that the interpretation of quantum theory can do without these things and students shall no longer be obliged to learn them.

Another advantage of the present axiomatization is that it suggests some nontrivial geometric generalizations of the quantum formalism: an example is described in Section (8.) below, other shall be discussed elsewhere.

The information theoretic formulation of the Heisenberg principle which we propose is the following:

There exist pairs of observables which cannot be simultaneously measured with arbitrary precision on the same system.

We interpret this statement not as a specific statement of quantum physics, but as a universal principle of any experimental science. The first conceptual implication of this statement is that the identification *event* $\equiv$ *measured event* , implicit in classical probability, should not be taken for granted in nature and one should distinguish between the two statements:

The observable A has the value a (*platonic statement*).

The measurement of the observable A has given the value a (*experimental statement*).

In other words: while in classical probability the experiment is **neutral** and there is no distinction between platonic statements and experimental statements, in quantum probability the experiment is **active** and the consequences of this distinction have to find some expression in the mathematical model.

All this has been well understood since a long time in the literature on the foundations of quantum theory, but the recognition that from these statements one can deduce some very detailed informations on the mathematical structure of quantum theory seems to be a characteristic of the present approach which, even if absorbing ideas and terminology of practically all the previously developed attempts to deduce the quantum mechanical formalism from physically meaningful requirements (axioms), is based on quite different ideas. The basic construction, described in the following was first discussed in [Ac82] for observables assuming only a finite set of values. A preliminary formulation of the axioms (but without proofs) was discussed in [Ac94]. In the present paper we concentrate on the deduction of the quantum model from physically meaningful axioms; a detailed discussion of the implications of this deduction for the interpretational problems of quantum theory, including the so–called *paradoxes*, the hidden variable question, the *statistical invariants* and the related Bell (and two slit) inequalities, the probabilistic meaning of the use of complex numbers in quantum theory, ..., can be found in [Ac82], [Ac84], [Ac86], [Ac93].

2 Algebras of measurements: heuristic considerations

The main difference between classical and quantum probability is that in the former theory acquisition of information is considered a cumulative process (new informations do not destroy old ones) while in the latter the process may not be cumulative, in the sense that acquisition of new informations may alter previously acquired informations. Since alterations may occur because we act upon a system by a measurement, it is natural to set up our mathematical model as an idealization of the various operations which are present in the measurement procedures. The usual Boolean-Kolmogorovian structure will be recovered as the limiting case, in which the informations acquired in a measurement process do not affect those acquired by previous measurements. The heuristic considerations of this Section are inspired by the essay [Schw70] of J. Schwinger and are aimed at justifying, on a physical level, the notion of **algebra of measurements** introduced by him. On the other hand, the mathematical analysis of the algebra of measurements, contained in the following Sections, was first developed in [Ac82].

In this Section we shall use terms such as *measurement, ensemble,* ..., in their intuitive meaning. The axioms are formally stated in Section (3.).

Let n be an integer or $+\infty$ and let A be a discrete observable whose values we denote a_j ($j = 1, \dots, n$). With the pair $\{A, (a_\alpha)\}$ we associate an idealized measurement instrument, denoted A_α, which from an ensemble of independent similar systems selects those for which the value of the observable A is a_α schematically:

$$[A \ = ?] \ \Rightarrow \ A_\alpha \ \Rightarrow \ [A = a_\alpha]$$

Such an instrument will be called an **elementary filter**.

We shall call an **apparatus** any object obtained by repeated application to a set of elementary filters of the basic operations to be described below (how these operations can be effectively realized realized is described in Feynman [Fey66], Chapters (5), (6)).

The symbol $A_j \cdot B_k$ will be associated with the apparatus corresponding to the consecutive application, to the same ensemble, first of the filter A_j and then of B_k. The operation $A_j, B_k \mapsto A_j \cdot B_k$, called **multiplication**, is associative, with an identity, denoted 1 and consisting of the apparatus which does not filter away anything.

The multiplication between elementary filters is extended to a multiplication between apparata simply by the requirement of associativity.

In the present idealization, the interaction with any apparatus is instantaneous, and between two interactions the system is isolated. The only role played by time in the present discussion is through its order: the symbol $A_j B_k$ means that we first apply A_j and then B_k. When a chain of measurements is considered, any consecutive pair is thought to be separated from the others by a very short time (the colloquial expression that *one takes place immediately after the other* is often used in the literature).

Our apparata are idealizations of the so-called **first-kind measurements**. This means that the measured system is not destroyed by the apparatus but emerges from it and can again be subjected to measurements. Elementary filters A_j have the additional property that, in the apparatus $A_j \cdot A_j$, each particle which passed the first filter will also pass the second (i.e., if the observable A has the value a_j at certain moment $t-0$, it will also have this value also at $t+0$) schematically:

$$\Rightarrow\ A_\alpha\ \Rightarrow\ A_\alpha\ \Rightarrow \qquad \equiv \qquad \Rightarrow\ A_\alpha\ \Rightarrow$$

Thus elementary filters act both as measurement apparata and as preparing apparata. This symmetry will be reflected by the formalism (cf. Theorem 20).

Not all measurements in nature are of the first kind, but in this and the following Section our considerations will be limited to this class. Since the measurements which do not disturb the system at all are of the first kind, it follows that all the events considered in the classical theory are included in the present discussion.

Two apparata X, Y are called **compatible** if

$$X \cdot Y = Y \cdot X \tag{1}$$

Two observables A, B are called **compatible** if for any pair of values a_j of A and b_k of B

$$A_j \cdot B_k = B_k \cdot A_j \tag{2}$$

The commutativity of the observables A, B means that *acquisition of information on the observable A does not alter previous information acquired on B, and conversely.*

Two apparata (observables) X, Y are called **mutually exclusive** if they are compatible and

$$X \cdot Y = 0 \tag{3}$$

where 0 denotes the filter which does not allow any particle to pass.

Elementary filters are mutually exclusive in the sense that, if an observable A has the value a_j at time t, then it cannot have any other value a short time (*immediately*) after t. In symbols:

$$A_j \cdot A_k = \delta_{jk} A_j \tag{4}$$

where $\delta_{jk} = 0$ if $j \neq k$ and =1 if $j = k$. Another natural operation on filters is the **time reversal**, which will be denoted $*$ and which corresponds to the applications of the filters A_j and B_k in reversed time order. More generally, if $A(x_1), \ldots, (x_n)$ are observables and $A_j(x_k)$ denotes the elementary filter corresponding to the j-th value of the observable $A(x_k)$, then

$$\Big(A_{j_1}(x_1) \cdot \ldots \cdot A_{j_n}(x_n)\Big)^* = A_{j_n}(x_n) \cdot \ldots \cdot A_{j_1}(x_1) \tag{5}$$

in particular, for n=1, and for any elementary filter A_j:

$$A_j^* = A_j \tag{6}$$

The self–adjointness denotes reversibility of the apparatus:

$$\Rightarrow \ A_\alpha \ \Rightarrow \ = \ \Leftarrow \ A_\alpha \ \Leftarrow$$

If p is a number in $[0, 1]$ and A_j an elementary filter, the symbol $p \cdot A_j$ will be associated with an apparatus which, from an ensemble of particles identically prepared so that $A = a_j$, selects at random a fraction p of these particles. In symbols:

$$(p \cdot A_j) \cdot A_k = A_k \cdot (p \cdot A_j) = \delta_{jk} p \cdot A_j \tag{7}$$

Theorem (5.9) below shows how to realize experimentally such an apparatus.

Compatible mutually exclusive filters A_j, B_k can be applied in parallel. The resulting apparatus, denoted

$$A_j + B_k \tag{8}$$

is characterized by the fact that it will allow the passage of those particles for which either

$$A = a_j \tag{9}$$

or

$$B = b_k \tag{10}$$

Elementary filters corresponding to different values of the same observable are always compatible and, moreover:

$$A_1 + \ldots + A_n = 1 \tag{11}$$

By standard mathematical procedures (quotienting a free algebra by certain relations) this structure of *partial algebra* – i.e. one in which addition, multiplication, and scalar multiplication are not everywhere defined – can be embedded in a real associative $*$-algebra with identity.

These algebras generalize the usual Boolean structure of the Kolmogorovian model. The following section shows that this generalization is not too wide, in the sense that in some significant cases it allows a complete classification of the new models which can arise.

3 Axioms of probability

In the present Section we shall formulate a set of axioms which account for the intuitive properties of measurements, described in the previous Section.

Let $\mathcal{M}$ be a set whose elements shall be denoted $X, Y, Z, \ldots \in \mathcal{M}$ and called indifferently *measurements, measurement apparata, apparata* or *instruments.*

(**A1.**) *(Composition axiom)* There exists a binary composition law

$$(X,Y) \in \mathcal{M} \times \mathcal{M} \Longrightarrow X \cdot Y \in \mathcal{M}$$

called *multiplication.* Interpretation: The multiplication of two instruments X, Y corresponds to the consecutive performance of each of them (in series):

$$\textit{input} \;\Longrightarrow\; X \;\Longrightarrow\; Y \;\Longrightarrow\; \textit{output}$$

For non 1–st kind measurements multiplication is meaningless.

(**A2.**) Associativity of the multiplication

$$(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$$

Interpretation: The definition of *instrument* is largely arbitrary: one can always consider two consecutive measurements as a single one according to the following scheme:

$$\Longrightarrow\; (X \;\Longrightarrow\; Y) \;\Longrightarrow\; Z \;\Longrightarrow = \Longrightarrow\; X \;\Longrightarrow\; (Y \;\Longrightarrow\; Z) \;\Longrightarrow$$

(**A3.**) There exist a measurement, denoted 1, characterized by the property:

$$X \cdot 1 = 1 \cdot X = X \quad ; \quad \forall X \in \mathcal{M}$$

Interpretation: 1 is the trivial measurement in which there is no interaction between the system and the apparatus therefore every system *passes through* the apparatus.

(**A4.**) There exists a measurement, denoted 0, caracterized by the property:

$$X \cdot 0 = 0 \cdot X = 0 \quad ; \quad \forall X \in \mathcal{M}$$

Interpretation: 0 is the trivial measurement which destroys every system, therefore no system *passes through* the apparatus.

(**A5.**) There exist an operation

$$* : \mathcal{M} \Longrightarrow \mathcal{M}$$

called *time reversal*, such that

$$(X \cdot Y)^* = Y^* X^*$$

$$(X^*)^* = X$$

Moreover

$$1^* = 1 \quad ; \quad 0^* = 0$$

Interpretation: The time reversal of X corresponds to do in reverse order all the physical operations corresponding to the measurement X.

$$(\Longrightarrow X \Longrightarrow)^* \qquad = \qquad \Longleftarrow X^* \Longleftarrow$$

A *symmetric instrument* is one in which the order of the sequential operations is irrelevant.

$$X^* = X$$

(**A6.**) **Randomization axiom** For each instrument X and each number $p \in [0,1]$ there exists an instrument, denoted pX with the following properties:

$$(pX) \cdot Y = p(X \cdot Y) = X \cdot (pY)$$

$$(1 \cdot X) = X \quad ; \quad (0 \cdot X) = 0$$

Interpretation: For any input, pX either produces the same output as X or no output. In many trials of X and pX with the same preparation the ratio

$$\frac{\#\text{outputs of } p\,X}{\#\text{outputs of } X}$$

is approximatively p.

Definition 1 *Two instruments X, Y are called compatible if:*

$$X \cdot Y = Y \cdot X$$

Interpretation: (the information in the two experiments is cumulative):

$$\Rightarrow\ X\ \Rightarrow\ Y\ \Rightarrow \qquad = \qquad \Rightarrow\ Y\ \Rightarrow\ X\ \Rightarrow$$

The previous axioms concerned measurements in series. Now we discuss measurements in parallel. Because of the indetedrminacy principle not all measurements can be performed in parallel. This means that the corresponding composition law cannot be everywhere defined.

(**A7.**) (Sum Axiom) There exist a binary composition law among compatible measurements

$$X \cdot Y = Y \cdot X \Rightarrow X + Y \in \mathcal{M}$$

satisfying the following conditions:

(**A7.**1) COMMUTATIVITY:

$$X + Y = Y + X$$

(**A7.**2) ASSOCIATIVITY:

$$(X + Y) + Z = X + (Y + Z)$$

(**A7.**3) NEUTRALITY OF ZERO

$$X + 0 = 0 + X = X$$

(**A7.**4) CANCELLATION LAW:

$$X + Y = X + Z \Rightarrow Y = Z$$

The following Lemma shows that the distributive property is meaningful:

Lemma 1 *If X_o, is compatible with $X_1, \ldots, X_n$ then it is also compatible with*
$X_1 \cdot X_2 \cdot \cdots \cdot X_n$.

Proof. Simple computation.

(**A7.**5) Distributivity: if X, Y, Z are pairwise compatible, then

$$(X + Y) \cdot Z = X \cdot Z + Y \cdot Z$$

The interpretation of the axioms concerning the addition is straightforward.

Definition 2 *An algebra of measurements is a quintuple:*

$$\{\mathcal{M}, \cdot, +, *, \textit{multiplication} \quad \textit{by } p \in [0, 1]\}$$

where $\mathcal{M}$ is a set and the operations $\cdot, +, *$ and the multiplication by $p \in [0, 1]$ satisfy the Axioms (**A1**.), ..., (**A7**.).

Now we begin to explore the mathematical consequences of the axioms. The first main remark is that the physical operations can be uniquely extended to all measurements and give rise to a $*$–algebra.

Theorem 1 *Given an algebra of measurements $\mathcal{M}$, there exists a, unique up to isomorphism, $*$–algebra $\mathcal{A}$ over the reals and an injective map $j : \mathcal{M} \Rightarrow \mathcal{A}$ which preserves the algebraic structure, i.e. such that*

$$j(X)^* = j(X)^*$$

$$j(XY) = j(X) \cdot j(Y)$$

$$j(pX) = pj(X) \qquad , \qquad \forall p \in [0, 1]$$

$$j(1) = 1_{\mathcal{A}} \qquad ; \qquad j(0) = 0$$

$$j(X + Y) = j(X) + j(Y) \qquad \text{if} \qquad XY = YX$$

And such that if $\mathcal{B}$ is a $*$–algebra and $k : \mathcal{M} \Rightarrow \mathcal{B}$ is a map satisfying the above identities, then there exists a $*$–homomorphism $\alpha : \mathcal{A} \Rightarrow \mathcal{B}$ such that:

$$k = \alpha \circ j$$

The pair $\{\mathcal{A}, j\}$ is called the $*$–algebra generated by $\mathcal{M}$. Since $\mathcal{A}$ is determined by $\mathcal{M}$ up to isomorphism, in the following we shall also call $\mathcal{A}$ a measurement algebra.

4 Classical probability

What characterizes the structure of the classical events in terms of measurements is the following axiom:

Universal Compatibility. All measurements are mutually compatible.

The implication of this axiom on the mathematical model is that the measurement algebra is commutative.

Definition 3 *An algebra of measurements $\mathcal{M}$ is called classical if the multiplication in $\mathcal{M}$ is commutative.*

Theorem 2 *If $\mathcal{M}$ is a commutative measurement algebra, then there exist a topological space S and an $*$–homorphism*

$$j : \mathcal{M} \Rightarrow \mathcal{C}(S)$$

Where $\mathcal{C}(S)$ denotes the $*$–algebra of continuous complex valued functions on S with the pointwise operations. If the states on $\mathcal{M}$ separate the points, then the homomorphism can be taken to be injective.

Proof. Simple application of the GNS construction.

Definition 4 *Let $\mathcal{A}$ be a $*$–algebra of measurements. A subalgebra $\mathcal{A}_o$ of $\mathcal{A}$ is called maximal abelian if it is abelian and not properly contained in any abelian subalgebra of $\mathcal{A}$.*

Remark (1.) If $\mathcal{A}_o \subseteq \mathcal{A}$ is maximal abelian then necessarily $\mathcal{A}_o$ contains the center of $\mathcal{A}$.

Remark (2.) Any abelian subalgebra $\mathcal{A}_o \subseteq \mathcal{A}$ is contained in a maximal abelian subalgebra (by Zorn's lemma).

Definition 5 *$A \in \mathcal{M}$ is called a projection if*

$$A = A^* \quad ; \quad A^2 = A$$

In the following we shall use the notation $Proj(\mathcal{M})$ to denote the family of all projections on $\mathcal{M}$.

Lemma 2 *Let $\mathcal{M}$ be a classical measurement algebra generated (algebraically) by its projections. Then $Proj(\mathcal{M})$ is a Boolean algebra with the operations:*

$$p \wedge q = pq$$

$$p \vee q = p + q - pq$$

Theorem 3 *The two categories*

i) Boolean algebras

ii) Classical algebras of measurements generated by projections are isomorphic.

Proof. By Stone's representation theorem for Boolean algebras.

Remark. No probability has been introduced up to now! Only Boolean algebras. This is natural since a classical theory needs not be a priori a statistical one. We shall see that the situation with a quantum theory is quite different.

Once recovers the usual framework of classical probability by introducing the axioms, characterizing the probability measures on Boolean algebras. These are the usual ones. In this sense our extension of the axioms is a conservative one: classical probability is included in the new formulation, but this is a *proper* inclusion because the structure, as we shall see, is richer. This last statement is clarified by the fact that any self–adjoint element X in a measurement algebra $\mathcal{M}$ generates a classical subalgebra of measurements $\mathcal{M}_o \subseteq \mathcal{M}$ hence, by Zorn's lemma, is contained in a maximal classical subalgebra $\mathcal{M}$. This property shall be called *local Kolmogorovianity.*

At the moment there are no examples of models, with reasonable probabilistic properties, which do not satisfy this condition.

Thus the new probabilistic models can be looked at as a *bunch of classical probabilistic model lumped together in a more or less fancy way*: as long as we remain in a classical submodel, classical probability applies, but when we consider the mutual relationships among observables in different (incompatible) submodels, the new features emerge.

Not all measurement algebras contain some projectors. Our further axioms shall be formulated for the class of algebras that are algebraically generated by their projectors.

5 Schwinger Algebras

Now we want to formulate in terms of measurements the fundamental new qualitative feature of quantum physics, i.e. the Heisenberg principle.

As stated in the introduction, we want to formulate this principle as a general
information–theoretic statement, whose application should not be restricted to the domain of micro–physics. More specifically, we individuate the qualitative essence of the Heisenberg principle in the mere statement of existence of incompatible observables. In order to translate this statement in terms of measurements we need to introduce some more notations.

Throughout this Section, as well as in Sections (3.) and (4.), n will denote a fixed natural integer or $+\infty$ and, unless the contrary is explicitly specified, $\mathcal{A}$ will denote a topological $*$-algebra over the reals such that addition and multiplication are jointly continuous in $\mathcal{A}$ and the involution $*$ is continuous. If $n = +\infty$, a sum of the form

$$\sum_{j=1}^{n} a_j \qquad ; \qquad a_j \in \mathcal{A} \tag{12}$$

is meant in the topology of $\mathcal{A}$. If $n < \infty$, the topology of $\mathcal{A}$ will play no role in what follows. The center of $\mathcal{A}$ will be denoted κ, thus

$$\kappa = \{c \in \mathcal{A} \ : \ ca = ac \qquad ; \ \forall a \in \mathcal{A}\} \tag{13}$$

Definition 6 *Let $\mathcal{A}$ be an associative real algebra and let n be an integer or $+\infty$. A* **partition of the identity** *of rank n in $\mathcal{A}$ is a set $A = \{A_1, ..., A_n\}$ of elements of $\mathcal{A}$ satisfying the following relations*

$$A_j = A_j^2 \tag{14}$$

$$A_j \cdot A_k = \delta_{jk} A_k \qquad ; \qquad j = 1, ..., n \tag{15}$$

$$\sum_{j=1}^{n} A_j = 1 \tag{16}$$

Condition (15) implies that in any moment an observable assumes only one value. Condition (16) means that in any moment an observable assumes at least one value.

If $\mathcal{A}$ is a $*$-algebra it is also required that A_j is self- adjoint, i.e.

$$A_j = A_j^* \tag{17}$$

Moreover, if $A = \{A_1, ..., A_n\}$ is a partition of the identity in $\mathcal{A}$ we denote $\mathcal{B}_A$ the abelian $*$-algebra generated by A and κ i.e., the set of all linear combinations of the form

$$\mathcal{B}_A = \{\sum_{j=1}^{n} \gamma_j A_j : \gamma_j \in \kappa \ ; \ j = 1, ..., n\} \tag{18}$$

Definition 7 *A partition of the identity* (A_j) *in an associative real algebra* $\mathcal{A}$ *is called* **maximal** *if the algebra* $\mathcal{B}_A$ *coincides with its commutant i.e., if for any* X *in* $\mathcal{A}$

$$XA_j = A_jX \qquad \forall j = 1, ..., n \Longleftrightarrow X \in \mathcal{B}_A \tag{19}$$

Summing up, within the general correspondence among physical quantities and mathematical objects, one has the more specific correspondences:

– apparatus ≡ element of $\mathcal{A}$
– Filter ≡ projection
– elementary filter ≡ atomic projection (precise value of an observable)
– an observable assumes one value at the time ≡ $\alpha \neq \beta \Rightarrow A_\alpha A_\beta = 0$
– all values are present ≡ $\sum_\alpha A_\alpha = 1$
– concatenation of filters ≡ multiplication
– simultaneous action of compatible filters ≡ addition
– observable ≡ maximal partition of the identity

In the classical case there can be at most one maximal partition of the identity and conversely if this is the case then the measurement algebra is classical in the sense of the previous definition. Thus a necessary condition for a measurement algebra to describe a non trivial quantum situation is that it admits at least two different maximal partitions of the identity. We shall see that, to a large extent, this conditionis also sufficient. So we formulate the distinctive axiom of quantum probability as follows:

(**A8.**) **HEISENBERG PRINCIPLE (weak form)**
There exist two different maximal partitions of the identity $(A_\alpha), (B_\beta)$.

In order to explore the consequences of this axiom we introduce the following defintion:

Definition 8 *A* **Schwinger algebra** *of rank* n *over a set* T *is a triple*

$$\{\mathcal{A}, T, (A(x))_{x\in T}\} \tag{20}$$

where $\mathcal{A}$ *is a real associative* $*$*-algebra,* T *is a set and, for every* x *in* T*,* $A(x) = \{A_1(x), ..., A_n(x)\}$ *is a maximal partition of the identity in* $\mathcal{A}$ *such that for any* x, y *in* T*, for any* $j, k = 1, ..., n$ *and for any* $\gamma \in \kappa$*, the following conditions hold:*

$$\gamma A_j(x)A_k(y) = 0 \Longleftrightarrow \gamma = 0 \tag{21}$$

$$\gamma A_j(x) \geq 0 \Longleftrightarrow \gamma \geq 0 \tag{22}$$

Recall that, by definition, a positive element in a $*$–algebra is a linear combination of elements of the form a^*a or a limit thereof.

Writing $A_j(x)$ as $A_j(x) \cdot A_j(x)$, it follows that, because of (21) equality holds in the right hand side of (22) if and only if $\gamma = 0$.

Remark. The conditions (21) and (22) are of a generic nature. Dropping them from the definition of Schwinger algebra should not change too–much the overall picture.

Example (4). Let n be a natural integer or $+\infty$. Let $H = \mathbf{C}^n$ with the usual Hermitean product if n is finite and let H be any separable complex Hilbert space if $n = \infty$. Let T denote the set of all self-adjoint operators on H with non degenerate spectrum (discrete spectrum, if $n = \infty$), and let, for each $x \in T$, $(\phi_j(x))$ be the orthonormal basis of H of the eigenvectors of x. Denote $A_j(x)$ the rank one projector:

$$A_j(x) : \psi \in H \rightarrow < \phi_j(x), \psi > \cdot \phi_j(x) = A_j(x)$$

where $< \cdot \, , \, \cdot >$ denotes the the scalar product in H. In this case the Schwinger algebra $\mathcal{A}$, generated by the $A_j(x)$, is the algebra of all $n \times n$ matrices if n is finite and, if $n = +\infty$ a natural choice for $\mathcal{A}$ is the $*$–algebra generated by all the countable linear combinations of the form $\sum_j \gamma_j A_j(x)$ with $\gamma_j \in \mathbf{C}$ and

$$\sup_j |\gamma_j| < \infty \tag{23}$$

Remark. In the classical case on the same Boolean structure one can put several probability measures, i.e. the Boolean algebra model cannot be intrinsically related to a single set of probabilities. However, if we are dealing with two different maximal observables, then by definition the acquirement of an exact information on the values of one of them implies that the information on the values of the other one cannot be but statistical. Therefore, in this case we expect a set of privileged probabilities intrinsically associated to the observables. It is a remarkable feature of the Schwinger algebra of measurements that they provide a mathematical support for this intuition in the sense made precise by Theorem 20 below which shows that, intrinsic in the very definition of Schwinger algebra, there is a well defined form of *stochasticity*.

Definition 9 *Let $\mathcal{A}$ be an associative topological real algebra and let S be an at most countable set. A family (a_j) $(j \in S)$ of elements of $\mathcal{A}$ is called* **linearly independent over** κ *if for each family (λ_j) $(j \in S)$ of elements of κ one has*

$$\sum_{j \in S} \lambda_j a_j = 0 \Longleftrightarrow \lambda_j = 0 \qquad ; \qquad \forall j \in S \tag{24}$$

*$(a_j)_{j \in S}$ is called a κ-***basis** *of $\mathcal{A}$ if any element x of $\mathcal{A}$ can be written in the form*

$$x = \sum_{j \in S} \lambda_j a_j \tag{25}$$

for some $\lambda_j \in \kappa$. In the following, unless otherwise stated, the term linear combination shall mean linear combination with coefficients in the centre n of A.

Proposition 1 *Let $A = (A_j)$ be a partition of the identity in $\mathcal{A}$ and let $\mathcal{B}_A$ denote the algebra generated by it. The map*

$$E : X \in \mathcal{A} \to \sum_j A_j \cdot X \cdot A_j \tag{26}$$

is a projection onto the commutant of $\mathcal{B}_A$. Moreover, if the partition A is maximal then for each $a \in \mathcal{A}$ and $j = 1, \ldots, n$ there exists an element $p_j(a) \in \kappa$ such that

$$A_j \cdot a \cdot A_j = p_j(a) A_j \qquad ; \qquad j = 1, \ldots, n \tag{27}$$

Proof. Clearly E maps $\mathcal{A}$ into commutant of $\mathcal{B}_{\mathcal{A}}$ and if X is in the commutant of $\mathcal{B}_{\mathcal{A}}$ then E $(X) = X$. Thus E is onto. Hence, if A (hence $\mathcal{B}_A$) is maximal, E is a projection onto $\mathcal{B}_A$ and therefore, for each element $a \epsilon \mathcal{A}$, there exist elements $p_j(a) \in \kappa$ such that:

$$E(a) = \sum_j A_j a A_j = \sum_j p_j(a) A_j$$

Multiplying both sides of this identity by a fixed A_j one finds (27).

Definition 10 *Let κ be a real $*$-algebra and let n be an integer or $+\infty$. A n-dimensional κ-valued* **stochastic matrix** *is a matrix $P = (p_{ij})$ $(i, j = 1, ..., n)$ such that*

$$p_{ij} \in \kappa \qquad ; \qquad p_{ij} \geq 0 \qquad ; \qquad \sum_{j=1}^{n} p_{ij} = 1 \tag{28}$$

where 1 denotes the identity in κ. If also the condition

$$\sum_{i=1}^{n} p_{ij} = 1 \tag{29}$$

is satisfied, then we say that P is a κ-valued **bi-stochastic matrix.**

Remark (8). If κ is a commutative C^*-algebra, as it will be the generic case for us, then we can realize it as the algebra $\mathcal{C}(S)$ of continuous complex valued functions on a compact Hausdorff space S. In this case a κ–valued stochastic matrix is simply a function on S with values in the usual real valued stochastic matrices. If the indices $i, j = 1, ..., n$ are thought to label the states of a discrete system, then the points $s \in S$ should be thought of as additional (superselection) parameters to be fixed in order to obtain a well defined transition probability $i \to j$ for any pair of these states.

Theorem 4 *Let* $\{\mathcal{A}, T, (A(x))_{x\in T}\}$ *be a Schwinger algebra. Then for any pair of elements* x, y *of* T *there exists a* κ*-valued* n *-dimensional bi-stochastic matrix* $P = (p_{ij})$ $(i, j = 1, ..., n)$ *such that for any* $i, j = 1, ..., n$ *one has*

$$A_i(x)A_j(y)A_i(x) = p_{ij}(x, y)A_i(x) \tag{30}$$

$$p_{ij}(x, y) = p_{ji}(y, x) \tag{31}$$

Proof. Let $x, y \in T$ and $i, j = 1, ..., n$. Then, because of Proposition 17, and of the maximality of $A(x)$, $A_i(x)A_j(y)A_i(x)$ must have the form

$$A_i(x)A_j(y)A_i(x) = \sum_{i'} p_{i'j}(x, y)A_{i'}(x) \tag{32}$$

for some elements $p_{ij}(x, y) \in \kappa$. Multiplying both sides of (5.20) on the right by $A_i(x)$ $(i = 1, ..., n)$ and using (15) one finds that

$$p_{ij}(x, y)A_i(x) = A_i(x)A_j(y)A_i(x) = \Big(A_j(y)A_i(x)\Big)^* \Big(A_j(y)A_i(x)\Big) \geq 0$$

and because of (22) this implies that $p_{ij}(x, y)$ is positive. Summing (32) over j and using (15), (16) leads to

$$A_i(x) = \Big(\sum_{j=1}^{n} p_{ij}(x, y)\Big)A_i(x) \tag{33}$$

and this implies (28) because of (22). The symmetry relation (31) follows from associativity in fact:

$$p_{ij}(x,y)A_i(x)A_j(y) = A_i(x)\Big(A_j(y)A_i(x)A_j(y)\Big) = p_{ji}(y,x)A_i(y)A_j(x)$$

which implies (31) because of (21).

We can now formulate the structure axiom which defines the natural mathematical model for the first kind measurements.

SAQ Let T be a set and let $\{\bar{A}(x) : x \in T\}$ be a family of n-valued maximal (physical) observables. Denote $a_j(x)$ $(j = 1, ..., n)$ the values of $A(x)$ and let

$$Prob\Big\{A(y) = a_j(y)|A(x) = a_i(x)\Big\} = p_{ij}(x,y) \qquad ; \; x, y \in T \; , \; i, j = 1, \ldots, n$$

the (experimentally measured) transition probabilities among their values.

Then there exists a Schwinger algebra $\{\mathcal{A}, T, (A(x))_{x\in T}\}$ with the property that for any pair of elements x, y in T and for any $i, j = 1, ..., n$ one has

$$Prob\Big\{A(y) = a_j(y)|A(x) = a_i(x)\Big\} = p_{ij}(x,y) \tag{34}$$

where $P(x,y) = (p_{ij}(x,y))$ is the transition matrix associated to the pair of partitions of the identity $A(x), A(y)$ according to Theorem 20.

This means that the empirical data are said to satisfy the above (quantum structure) axiom if there exists a mathematical model which is related to the empirical data through the identity (**??**).

In view of the structure axiom above we shall identify, in the following, the maximal discrete observables with their mathematical models in the Schwinger algebra (i.e., the partitions of the identity) and for two such observables $A(x), A(y)$ the associated bistochastic matrix $P(x,y) = (p_{ij}(x,y))$ will be called the **transition probability matrix** between $A(x)$ and $A(y)$. The following theorem shows that the structure axiom **SAQ** above extends in a natural way the corresponding structure axiom of the Kolmogorovian model.

Theorem 5 *Let $\mathcal{A}$ be a real $*$- algebra such that the triple $\{\mathcal{A}, T, (A(x))_{x\in T}\}$ satisfies all the axioms of a Schwinger algebra with the exception of maximality. If $\mathcal{A}$ is abelian. Then there exist*
(i) a set Ω and a Boolean algebra $\mathcal{F}$ of subsets of Ω
(ii) for any observable $A(x) = (A_j(x))$ an n -valued $\mathcal{F}$–measurable function $\bar{A}(x) : \Omega \longrightarrow \mathbf{R}$ such that, denoting $a_j(x)$ the j-th value of $\bar{A}(x)$ and

$$\bar{A}_j(x) := \chi_{a_j(x)}(\bar{A}(x)) \quad ; \quad \chi_{a_j(x)}(r) = 1 \text{ if } a_j(x) = r \ ; \ = 0 \text{ if } a_j(x) \neq r \qquad i, j = 1, ..., n \tag{35}$$

the correspondence

$$A_j(x) \in \mathcal{A} \mapsto \bar{A}_j(x) \tag{36}$$

establishes an isomorphism between the Schwinger subalgebra of $\mathcal{A}$, generated by the $A_j(x)$ and the Schwinger algebra of all finite real linear combinations of the $\bar{A}_j(x)$.

Conversely any pair $\{(\Omega, \mathcal{F}), \{\bar{A}(x)\}\ \}$ where Ω and $\mathcal{F}$ are as in (i) above and $\{\bar{A}(x)\}$ is a family of n-valued random variables, is obtained from a Schwinger algebra in the way described above.
Proof. Let, in the statement of the theorem, $\mathcal{A}$ be abelian. Then $\mathcal{A}$ coincides with its center κ and the Boolean algebra generated by the $A_j(x)$ with $j = 1, ..., n$ and x in T with the operations

$$X \wedge Y = XY \qquad ; \qquad X \vee Y = X + Y - X \wedge Y \tag{37}$$

can be realized, by Stone' s theorem, as a Boolean algebra of subsets of a certain set Ω. Therefore $\mathcal{A}$ itself can be realized as the family of real linear combinations of charachteristic functions of sets in this Boolean algebras.
The converse is trivial because if $\{(\Omega, \mathcal{F}), \{\bar{A}(x)\}\ \}$ is a pair with the properties described in the theorem, then the complex combinations of the characteristic functions of the events $[\bar{A}(x) = a_j(x)]$ define he required Schwinger algebra.

Now, just as Stone's theorem provides a standard mathematical model for the Boolean algebras of classical probability, we would like to have a standard representation theorem for Schwinger algebras in order to obtain a classification of all the possible mathematical models of first kind measurements.

Lemma 3 *Let $\{\mathcal{A}, T, (A(x))_{x\in T}\}$ be a Schwinger algebra. For any pair of elements x, y of T the set*

$$\{A_i(x)A_j(y) : i, j = 1, ..., n\} \tag{38}$$

is linearly independent over κ in the sense of Definition (6).

Proof. If, for given $x, y \in T$, there exist elements $\gamma_{ij} \in \kappa$ (i,j = 1,..., n) such that

$$\sum_{i,j=1}^{n} \gamma_{ij} A_i(x)A_j(y) = 0 \tag{39}$$

then fixing i and j in $\{1, ..., n\}$ and multiplying (39) on the left by $A_i(x)$ and on the right by $A_j(y)$ one finds

$$\gamma_{ij} A_i(x)A_j(y) = 0 \qquad \forall i, j = 1, ..., n$$

Hence all the γ_{ij} are zero by condition (21).

<u>Corollary (12) </u>. In the conditions of Lemma 8, if T is a finite set then

$$(n \mid T \mid -1)! > dim_{\kappa}(\mathcal{A}) \geq n^2 \tag{40}$$

where $\mid T \mid$ denotes the cardinality of T. In particular if $n < \infty$, then $\mathcal{A}$ is finite dimensional as a module over κ.

Proof. By definition $\mathcal{A}$ is generated by the elements $A_k(x)$ with x in T and $k = 1, ..., n$. This means that $\mathcal{A}$ coincides with the set of all linear combinations of the form

$$A_{j_1}(x_1)....A_{j_M}(x_M) \qquad ; \qquad M \in \mathbf{N}\,; j_1, \ldots, \; j_M \in \{1, ..., n\}\;;\; x_1, ..., x_M \in T \tag{41}$$

Let $N < \infty$ be the cardinality of T. If $M \geq n \cdot N$, then in the string (5.30) at least two of the $A_j(x)$ must be equal. So the string (5.30) contains a substring of the form

$$A_j(x)A_{k_1}(y_1)....A_{k_m}(y_m)A_j(x) \tag{42}$$

which belongs to the commutant of $A(x)$. Hence, by Proposition 16 such a string must be of the form $\gamma A_j(x)$ for some $\gamma \in \kappa$. Thus the strings of the form (41) with $M \geq nN$ are multiples of strings of the same type with $M < nN$. If $M < nN - 1$, then we can use the identity

$$\sum_{j=1}^{n} A_j(x) = 1$$

to express the string (41) as a sum of strings of length $nN - 1$. Thus every element in $\mathcal{A}$ is a linear combination, with coefficients in κ of strings of the type (41) with $M = nN - 1$. And this proves the left inequality in (40). The right inequality has been proved in Lemma 20 and is independent of the cardinality of T.

Summing up we have shown that if $A(x)$ and $A(y)$ $(x, y \in T)$ are two maximal observables in a Schwinger algebra $\mathcal{A}$, then there exists a bistochastic matrix $P(x, y) = (p_{ij}(x, y))$ with values in the center of $\mathcal{A}$ such that:

$$A_i(x) \cdot A_j(y) \cdot A_i(x) = p_{ij}(x, y) \cdot A_i(x) \tag{43}$$

$$p_{ij}(x, y) = p_{ji}(y, x) \tag{44}$$

Thus: not only two maximal observables in a Schwinger algebra canonically define a transition probability matrix, but this matrix has necessarily the symmetry property (44) which is found in the usual quantum mechanical (Hilbert space) model where the explicit form of the transition probabilities is (in obvious notations):

$$p_{ij}(x, y) = |\langle a_i(x), a_j(y)\rangle|^2 = |\langle a_j(y), a_i(x)\rangle|^2 = p_{ji}(y, x)$$

Therefore the following problem arises quite naturally: given a set

$$\{P(x, y) \ : \ x, y \in T\}$$

of $n \times n$ κ-valued bistochastic matrices (the experimental data), determine under which conditions there exists a Schwinger algebra $\mathcal{A}$ associated with a family $\{A(x) \ : \ x \in T\}$ of n-valued observables, such that

(i) each $A(x)$ is maximal in $\mathcal{A}$

(ii) for each $x, y \in T$ the bistochastic matrix canonically associated to the pair $A(x), A(y)$ according to Theorem 20 is $P(x, y)$

Remark. In the following we shall characterize the Schwinger algebras which have minimal dimension over their center. They shall be called Heisenberg algebras.

Definition 11 *Let n be a natural integer or $+\infty$; let κ be a commutative $*$-algebra and let $P = (p_{ab})$ be a κ-valued $n \times n$ bistochastc matrix. An* **Heisenberg algebra** *with center κ and transition probability P is an associative $*$-algebra $\mathcal{A}$ such that:*

i) The center of $\mathcal{A}$ is isomorphic to κ.

ii) There exist two maximal partitions of the identity in $\mathcal{A}$

$$A = (A_a) \qquad ; \qquad B = (B_b) \qquad ; \qquad a, b = 1, \ldots, n \tag{45}$$

such that:

$$A_a B_b \cdot A_a = p_{ab} A_a \qquad ; \qquad B_b A_a B_b = p_{ba} B_b \tag{46}$$

iii) Any element $x \in \mathcal{A}$ can be written in the form

$$x = \sum_{ab} \lambda_{ab} A_a B_b \tag{47}$$

for some $\lambda_{ab} \in \kappa$.

Although not conceptually necessary we shall include, in the definition of Heisenberg algebra, the condition that *the transition probabilities are strictly positive*, i.e.

$$p_{ab} > 0 \qquad \forall a, b \tag{48}$$

Notice that, because of (46) this implies

$$A_\alpha B_\beta \neq 0 \qquad \forall \alpha, \beta = 1, \cdots, n$$

Condition (48) is a genericity requirement which makes several computations much more transparent. The qualitative picture emerging from the classification theorem should not change in an essential way if one drops this condition.

6 Deduction of the superposition principle

We can summarize what achieved up to now as follows: Let T be a set; $\{\bar{A}(x) : x \in T\}$ a family of n-valued maximal observables; $\{a_j(x) \ : \ j = 1, ..., n\}$ the values of $\bar{A}(x)$. Suppose that there exists a Schwinger algebra $\{\mathcal{A}, T, (A(x))_{x\in T}\}$ such that, for any element x of T there exist a maximal partition of the identity $\{A_j(y) \ : \ i, j = 1, ..., n\}$ in $\mathcal{A}$ and these partitions generate $\mathcal{A}$ over its center κ. Then, for any pair of elements x, y in T the transition matrix $P(x,y) = (p_{ij}(x,y))$, associated to the partitions of the identity $A(x), A(y)$ according to Theorem 20, is uniquely defined.

But, since the transition probabilities are model independent quantities, which could be obtained in a set of experiments, completely independent of any mathematical representation, it is quite natural to ask the converse question, namely: given a family $\bar{A}(x)$ of observable quantities, their values $a_\alpha(x)$ and, for any pair of elements x, y in T, a transition probability matrix

$$Prob\left\{ \bar{A}(y) = a_j(y) | \bar{A}(x) = a_i(x) \right\} = p_{ij}(x,y)$$

when does there exist a Schwinger algebra and, for any observable $\bar{A}(x)$, a partition of the identity $A_j(x)$, with the property that for any $x, y \in T$ the **transition probability matrix** , associated to $A(x)$ and $A(y)$ is precisely $P(x,y) = (p_{ij}(x,y))$?

It can be proved that the Heisenberg principle is not contained in the transition probabilities (i.e. transition probabilities among incompatible observables may, in some cases, admit a Kolmogorov model): it is a genuine physical principle that cannot be read off from the only knowledge of them. Therefore it is natural to ask oneself: *Beyond the Heisenberg indeterminacy principle (in weak form), what else is needed to deduce the structure of complex Hilbert space and the statistical interpretation of its vectors?* The answer ([Ac82]) is: *Essentially Nothing*! In this Section we shall prove this statement.

Let $\mathcal{A}$ be an Heisenberg algebra in the sense of Definition 26. Then there exist elements $\gamma_{ab}^{cd} \in \kappa$ $(a,b,c,d = 1,...,n)$ such that

$$B_b \cdot A_a = \sum_{c,d=1}^{n} \gamma_{ab}^{cd} A_c \cdot B_d \tag{49}$$

these elements will be called the **structure constants** of $\mathcal{A}$ in the $(A_a B_b)$-basis.

Theorem 6 *Let $\mathcal{A}$ be an associative algebra with identity and let κ denote its center. Let $(A_a), (B_b)$ $(a,b = 1,...,n)$ be partitions of the identity in $\mathcal{A}$*

such that the set $\{A_a \cdot B_b : a, b = 1, ..., n\}$ *is a* κ*-basis of* $\mathcal{A}$*, and let* γ_{ab}^{cd} $(a, b = 1, ..., n)$ *be elements of* κ *such that the identity (6) holds. Then*

$$\sum_{a=1}^{n} \gamma_{ab}^{a'b'} = \delta_{bb'} \tag{50}$$

$$\sum_{b=1}^{n} \gamma_{ab}^{a'b'} = \delta_{aa'} \tag{51}$$

$$\gamma_{a'b}^{ab'} \gamma_{a''b'}^{ab''} = \gamma_{a'b}^{ab''} \gamma_{a''b'}^{a'b''} \tag{52}$$

If moreover $\mathcal{A}$ *is a* $*$*-algebra then*

$$\sum_{c,d=1}^{n} (\gamma_{ab}^{cd})^* \cdot \gamma_{cd}^{c'd'} = \delta_{ac'} \cdot \delta_{bd'} \tag{53}$$

$$(\gamma_{a'b}^{ab'})^* \cdot \gamma_{ab'}^{a''b''} = \sum_{e,d=1}^{n} \gamma_{a'b'}^{a''d} \cdot \gamma_{ed}^{a''b''} \cdot \gamma_{ab}^{eb''} \tag{54}$$

Conversely, if κ *is a commutatve associative real* $*$*-algebra with identity and* γ_{ab}^{cd} $(a, b = 1, ..., n)$ *are elements of* κ *satisfying (2), (51), (52), then there exist an associative algebra* $\mathcal{A}$ *with center* κ *and two partitions of the identity in* $\mathcal{A}$*,* $A = (A_a)$ $B = (B_b)$*, such that* $A_a \cdot B_b$ *is a basis of* $\mathcal{A}$ *over* κ *and (6) holds. If moreover (53) and (54) hold then* $\mathcal{A}$ *has a unique structure of* $*$*-algebra whose involution is characterized by the property that its restriction on* κ *coincides with the original involution on* κ *and for all* a, b

$$A_a = A_a^* \qquad ; \qquad B_b = B_b^* \tag{55}$$

Proof. **Necessity.** Associativity and (6) imply that for any $a, b, a', b' = 1, \ldots, n$

$$A_{a'} B_b A_a B_{b'} = \gamma_{a'b'}^{ab} A_{a'} B_{b'} \tag{56}$$

Summing (12) over b (resp a) and using (x.10) one finds (51) (resp. (2)). Using twice (6) in the identity

$$[A_a B_b A_{a'} B_{b'}] \cdot A_{a''} B_{b''} = A_a B_b \cdot [A_{a'} B_{b'} A_{a''} B_{b''}] \tag{57}$$

the identity (52) follows from (58). Taking adjoints of both sides of (6) we obtain

$$A_a B_b = \sum_{cd} [\sum_{c'd'} (\gamma_{ab}^{cd})^* \gamma_{cd}^{c'd'}] \cdot A_{c'} B_{d'}$$

which is equivalent to (53) because of the independence of the A_aB_b over κ. Moreover

$$(A_aB_bA_{a'}B_{b'})^* = (\gamma_{a'b}^{ab'})^* \cdot (A_aB_{b'})^* = \sum_{a''b''}(\gamma_{a'b}^{ab'})^* \cdot \gamma_{ab'}^{a''b''} \cdot A_{a''}B_{b''} \tag{58}$$

On the other hand

$$(A_aB_bA_{a'}B_{b'})^* = (A_{a'}B_{b'})^* \cdot (A_aB_b)^* = \Big(\sum_{a''d}\gamma_{a'b'}^{a''d} \cdot A_{a''} \cdot B_b\Big) \cdot \Big(\sum_{eb''}\gamma_{ab}^{eb''} \cdot A_e \cdot B_{b''}\Big) =$$

$$= \sum_{a''b''}\big[\gamma_{a'b'}^{a''d} \cdot \gamma_{ab}^{eb''} \cdot \gamma_{ed}^{a''b''} A_{a''} \cdot B_{b''}\big] \tag{59}$$

and (54) follows from the independence of the products A_aB_b over κ. **Sufficiency.**

Let κ be a commutative associative real $*$-algebra with identity, let the $\gamma_{ij}^{i'j'}$ satisfy the conditions (2), (51), (52), (53), (54) and let $\mathcal{A}$ be the real vector space of all formal series of the form

$$\sum_{a,b=1}^{n}\lambda_{ab}A_aB_b + \sum_{a=1}^{n}\lambda_aA_a + \sum_{b=1}^{n}\nu B_b + \lambda \cdot 1 \tag{60}$$

with $\lambda_{ab}, \lambda_a, \nu_b, \lambda \in \kappa$ and with relations

$$\sum_{a=1}^{n}A_a = \sum_{b=1}^{n}B_b = 1 \tag{61}$$

On $\mathcal{A}$ we define a multiplication by

$$1 \cdot x = x \cdot 1 = x \qquad ; \qquad x \in \mathcal{A} \tag{62}$$

$$(\sum_{ab}\lambda_{ab}A_aB_b) \cdot (\sum_{a'b'}\mu_{a'b'}A_{a'}B_{b'}) = \sum_{a'b'ab}\lambda_{ab}\mu_{a'b'}\gamma_{a'b}^{ab'}A_aB_{b'} \tag{63}$$

and an involution $*$ by

$$(\sum_{ab}\lambda_{ab}A_aB_b)^* = \sum_{a'b'ab}\lambda_{ab}^*\gamma_{ab}^{a'b'}A_{a'}B_{b'} \tag{64}$$

One easily checks the distributivity of the multiplication. This and the relations (66) imply that conditions (62), (63) above uniquely define a multiplication on $\mathcal{A}$. In particular (62), (63) imply that

$$A_aB_bA_{a'}B_{b'} = \gamma_{a'b}^{ab'}A_aB_{b'} \tag{65}$$

Summing (65) over b and using (66) and (51) we find

$$A_a A_{a'} B_{b'} = \delta_{aa'} A_a B_{b'} \tag{66}$$

hence, again by

$$A_a A_{a'} = \delta_{aa'} A_a \tag{67}$$

From (66) and (51) we also deduce that

$$A_a^* = (\sum_b A_a B_b)^* = \sum_{a'b'b} \gamma_{ab}^{a'b'} A_{a'} B_{b'} = \sum_{a'b'} \delta_{a'a} A_{a'} B_{b'} = A_a \tag{68}$$

Hence $A = (A_a)$ is a partition of the identity in $\mathcal{A}$ in the sense of Definition (x.1). Similarly for $B = (B_b)$. For the associativity of $\mathcal{A}$ it will be sufficient to check the identity (4); but due to (65) and (52) this is immediate. It remains to be shown that $\mathcal{A}$ is a $*$-algebra. According to (64) one has

$$(\sum_{ab} \lambda_{ab} A_a B_b)^{**} = \sum_{aba'b'} \lambda_{ab} (\gamma_{ab}^{a^{'}b^{'}})^* \cdot \gamma_{a''b''}^{a'b'} A_{a''} B_{b''} \tag{69}$$

which, because of (53), is equal to

$$\sum_{ab} \lambda_{ab} A_a B_b$$

So for any $x \in \mathcal{A}$ one has $(x^*)^* = x$. Finally,

$$[(\sum_{ab} \lambda_{ab} A_a B_b) \cdot (\sum_{a'b'} \mu_{a'b'} A_{a'} B_{b'})]^* = [\sum_{a'b'ab} \lambda_{ab} \mu_{a'b'} \gamma_{a'b}^{ab'} A_a B_{b'})]^* =$$

$$= \sum_{a',b',a,b,a'',b''} \lambda_{ab}^* \mu_{a'b'}^* (\gamma_{a'b}^{ab'})^* \gamma_{ab'}^{a^{''}b^{''}} A_{a''b''} B_{b''} \tag{70}$$

and, because of (54) this is equal to

$$(\sum_{a',b'} \mu_{a',b'} A_{a'}, \cdot B_{b'})^* \cdot (\sum_{a,b} \lambda_{ab} A_a B_b)^* = (\sum_{a',b'} \mu_{a'b'}^* \gamma_{a'b'}^{a^{''}d} A_{a''} \cdot B_d) \cdot (\sum_{a,b} \lambda_{ab}^* \gamma_{ab}^{eb^{''}}) =$$

$$= \sum_{a,b,a',b',a'',b''} \lambda_{ab}^* \mu_{a'b'}^* \gamma_{a'b'}^{a^{''}d} \gamma_{ab}^{eb^{''}} \gamma_{ed}^{a^{''}b^{''}} A_{a''} \cdot B_{b''} \tag{71}$$

So for any $x, y \in \mathcal{A}$ one has $(xy)^* = y^* x^*$, and this ends the proof.

The following Proposition shows that, under generic conditions ((AI) below always holds if κ is a field and (AII) if $\mathcal{A}$ is a C^*-algebra) the strict positivity of the transition probabilities implies the linear independence of the A_jB_j over κ.

Proposition 2 *Let $\mathcal{A}$ be an associative real algebra and let $(A_i), (B_j)$ be a pair of maximal partitions of the identity in $\mathcal{A}$. Consider the following statements:*

i) Condition (59) holds in $\mathcal{A}$ with equality if $\gamma = 0$ and for each $i, j \in 1, \ldots, n$ one has

$$p_{ij}(B|A) > 0 \tag{72}$$

ii) Condition (58) holds in $\mathcal{A}$ (iii) Condition (58) holds in $\mathcal{A}$ and the products $A_i \cdot B_j$ are independent over κ. Then $(i) \Rightarrow (ii) \Rightarrow (iii)$ and (iii) implies that

$$p_{ij}(B|A) = \gamma_{ij}^{ij'} = \gamma_{ij}^{i'j} \qquad ; \ \forall i, i', j, j' \in 1, \ldots, n \tag{73}$$

If moreover

– (AI) The product of two positive elements in κ is zero if and only if one of them is zero.

– (AII) For every $a \in \mathcal{A}$ one has

$$a^* a = 0 \Longleftrightarrow a = 0 \tag{74}$$

then $(iii) \Longrightarrow (i)$ and

$$\gamma_{ij}^{i'j'} \neq 0 \qquad ; \quad \forall i, j, i', j' \in 1, \ldots, n \tag{75}$$

Proof. $(i) \Rightarrow (ii)$. Assume that (**??**) holds and that, for some pair $i, j \in 1, \ldots, n$, one has $A_i B_j = 0$. Then

$$0 = A_i B_j A_i = p_{ij}(B|A) A_i$$

which contradicts (**??**) by (6.11). The implication $(ii) \Rightarrow (iii)$ has been proved in Lemma 60.

$(iii) \Rightarrow (i)$. Assume that for some $i, j, i', j' \in 1, \ldots, n$, one has

$$\gamma_{ij}^{i'j'} = 0$$

Then, using (6) one finds

$$A_{i'}B_jA_iB_{j'} = 0$$

which implies, multiplying on the right by A_i and on the left by B_j, that:

$$p_{ji'}(A|B)p_{ij'}(B|A) \cdot B_jA_i = 0 \tag{76}$$

hence taking adjoints of (76) and using (58)

$$p_{ji'}(A|B)p_{ij'}(B|A) = 0$$

Therefore, by the assumption (AI) one of the two factors must be zero. Assuming that $p_{ij'}(B|A) = 0$, one has:

$$0 = p_{ij'}(B|A)A_i = A_i \cdot B_{j'} \cdot A_i = (B_{j'} \cdot A_i)^* \cdot (B_{j'} \cdot A_i)$$

By assumption (AII) this implies that $B_{j'} \cdot A_i = 0$ hence $A_iB_j = 0$ which contradicts the independence of the products A_iB_j.

Corollary (6.3). If $\mathcal{B}_A$ and $\mathcal{B}_B$ are maximal Abelian with transition matrix $P = (p_{ab})$ then

$$p_{a'b'}p_{ab}\gamma_{ab'}^{a''b''} = \gamma_{ab}^{a'b}\gamma_{a'b'}^{a''b}\gamma_{ab}^{a''b''} \tag{77}$$

$$p_{a'b'}p_{ab}p_{ab'} = \gamma_{ab'}^{a'b}\gamma_{a'b'}^{a''b}\gamma_{ab}^{a''b'} \tag{78}$$

Proof. Since κ is commutative, the expression

$$\gamma_{ab'}^{a'b}\gamma_{a'b'}^{a''b}\gamma_{ab}^{a''b''} \tag{79}$$

is equal to

$$\left(\gamma_{a'b'}^{a''b}\gamma_{ab'}^{a'b}\right)\gamma_{ab}^{a''b''} \tag{80}$$

and applying (52) to the expression in parentheses in (79), this becomes equal to

$$\gamma_{a'b'}^{a''b'}\left(\gamma_{ab'}^{a''b}\gamma_{ab}^{a''b''}\right) \tag{81}$$

Applying again (52) to the expression in parentheses in (81), this becomes equal to

$$\gamma_{a'b'}^{a''b'}\gamma_{ab'}^{a''b''}\gamma_{ab}^{ab''}$$

which, because of (73), is equal to

$$p_{a'b'}p_{ab}\gamma_{ab'}^{a''b''}$$

and this proves (75). Since (76) is obtained from (75) by putting $b'' = b'$ and using (**??**), the Lemma is proved.

Remark (6.4). From (79) it follows that, if for each $a, b = 1, ..., n$ one has

$$p_{ab} > 0$$

(i.e. $p_{ab} \geq 0$ and invertible) then each of the structure constants $\gamma_{ab}^{a''b'}$ is invertible. In fact, assuming the contrary would lead to the contradiction that, for some choices of a, b, a', b', a'', the left hand side of (79) is invertible while the right hand side is not.

Theorem 7 *Let $\mathcal{A}$ be an associative real algebra generated by the maximal partitions of the identity $A = (A_a)$; $B = (B_b)$. Assume that the transition probability matrix $P = (p_{ab})$ between A and B is strictly positive in the sense of (**??**) and denote γ_{ab}^{cd} the structure constants of $\mathcal{A}$ in the (A_aB_b)-basis. Then there exists a κ-valued matrix $U = (u_{ab})$ such that*

$$\sum_{b=1}^{n} u_{a'b}(\frac{p_{ab}}{u_{ab}}) = \delta_{a,a'} \qquad ; \qquad a, a' = 1, ..., n \tag{82}$$

$$\sum_{a=1}^{n}(\frac{p_{ab}}{u_{ab}})u_{ab'} = \delta_{bb'} \qquad ; \qquad b, b' = 1, ..., n \tag{83}$$

$$\gamma_{ab}^{a'b'} = \frac{u_{ab'}u_{a'b}}{u_{ab}u_{a'b'}}p_{ab} \qquad ; \qquad a, b, a', b' = 1, ..., n \tag{84}$$

Conversely if κ is a real commutative $*$-algebra then, given a κ-valued strictly positive bi-stochastic matrix $P = (p_{ab})$ and a κ-valued matrix $U = (u_{ab})$ satisfying (82), (83), (84) then there exist:
- an associative real algebra $\mathcal{A}$ with center κ
- two maximal partitions of the identity $A = (A_a)$ $B = (B_b)$ in $\mathcal{A}$ with transition matrix P such that the γ_{ab}^{cd}, defined by the right hand side of (84), are the structure constants of $\mathcal{A}$ in the (A_aB_b)-basis.

Proof. **Necessity** *Since each p_{ab} is invertible, we can define the normalized structure constants*

$$\Gamma_{ab}^{a'b'} = \gamma_{ab}^{a'b'} \Big/ p_{ab} \qquad a, b, a'b = 1, ...n \tag{85}$$

which, because of (73) satisfy

$$\Gamma_{ab}^{ab'} = \Gamma_{ab}^{a'b} = 1 \qquad \forall a, a', b, b' \tag{86}$$

Now, using (52) twice and (73), we obtain

$$\left(\gamma_{a'b'}^{a''b}\gamma_{ab'}^{a'b}\right)\gamma_{ab}^{a''b''} = \gamma_{a'b'}^{a''b'}\left(\gamma_{ab'}^{a''b}\gamma_{ab}^{a''b''}\right) = \gamma_{a'b'}^{a''b'}\gamma_{ab'}^{a''b''}\gamma_{ab}^{ab''} = p_{ab'}p_{ab}\gamma_{ab'}^{a''b''} \tag{87}$$

hence, dividing both sides of (6.38) by $p_{ab'}$ one finds

$$\Gamma_{a'b'}^{a''b}\Gamma_{ab'}^{a'b}\Gamma_{ab}^{a''b''} = \Gamma_{ab'}^{a''b''} \tag{88}$$

Choosing $b'' = b$ in (88) and using (86), we find

$$\Gamma_{a'b'}^{a''b}\Gamma_{ab'}^{a'b} = \Gamma_{ab'}^{a''b''} \tag{89}$$

In particular, letting $a'' = a$ in (89) we obtain

$$\Gamma_{a'b'}^{ab} = (\Gamma_{ab'}^{a'b})^{-1} \tag{90}$$

Now fix an index a_o arbitrarily and denote

$$\gamma\left(a; \begin{matrix} b \\ b' \end{matrix}\right) = \Gamma_{ab'}^{a_ob} \tag{91}$$

With these notations, using (89) and (90), we find that every $\Gamma_{ab'}^{a'b}$ can be written as

$$\Gamma_{ab'}^{a'b} = \Gamma_{ab'}^{a_ob}\Gamma_{a_ob}^{a'b} = \frac{\gamma\left(a; {b \atop b'}\right)}{\gamma\left(a'; {b \atop b'}\right)} \tag{92}$$

and therefore (88) becomes

$$\frac{\gamma\left(a; {b'' \atop b'}\right)}{\gamma\left(a''; {b'' \atop b'}\right)} = \frac{\gamma\left(a; {b \atop b'}\right)}{\gamma\left(a'; {b \atop b'}\right)} \cdot \frac{\gamma\left(a; {b'' \atop b}\right)}{\gamma\left(a''; {b \atop b'}\right)} \cdot \frac{\gamma\left(a; {b'' \atop b}\right)}{\gamma\left(a''; {b'' \atop b}\right)} \tag{93}$$

This implies that the expression

$$\frac{\gamma\left(a; {b \atop b'}\right)\gamma\left(a; {b'' \atop b}\right)}{\gamma\left(a; {b'' \atop b'}\right)} \tag{94}$$

does not depend on a and therefore it must be necessarily equal to 1, which is the value obtained by choosing $a = a_o$. *Therefore, fixing arbitrarily an index* b_o *and denoting*

$$u_{ab} = \gamma\Big(a; \begin{matrix} b \\ b_o \end{matrix}\Big) = \Gamma^{a_o b}_{ab_o} \tag{95}$$

we have that u_{ab} *is invertible because of Remark (52) and, using (94):*

$$\gamma\Big(a; \begin{matrix} b' \\ b \end{matrix}\Big) = \gamma\Big(a; \begin{matrix} b_o \\ b \end{matrix}\Big)\gamma\Big(a; \begin{matrix} b' \\ b_o \end{matrix}\Big) = \frac{u_{ab'}}{u_{ab}} \tag{96}$$

so that from (85), (92) and (96) we conclude

$$\gamma^{a'b'}_{ab} = p_{ab}\Gamma^{a'b'}_{ab} = p_{ab}\frac{\gamma\Big(a; \begin{smallmatrix} b' \\ b \end{smallmatrix}\Big)}{\gamma\Big(a'; \begin{smallmatrix} b' \\ b \end{smallmatrix}\Big)} = p_{ab}\frac{u_{ab'}u_{a'b}}{u_{ab}u_{a'b'}} \tag{97}$$

and this proves (84). Once proved (84), (82) and (83) are immediate consequences of (2) and (51) respectively.

Sufficiency. *It is a straightforward verification to check that, if* (u_{ab}) *is a* κ*-valued matrix satisfying (82) and (83) then the* $\gamma^{a'b'}_{ab}$ *defined by (84) satisfy the equations (2), (51), (52) and (**??**). The statement then follows from Theorem 6 and Proposition 2.*

Remark (6). Introducing the matrices $U(A|B) = (u_{ab}(A|B))$ *and* $U(B|A) = (u_{ba}(B|A))$ *defined respectively by*

$$u_{ab}(A|B) = u_{ab} \qquad ; \qquad u_{ba}(B|A) = \frac{p_{ab}}{u_{ab}} \tag{98}$$

where (u_{ab}) *is the matrix introduced in Theorem 7, the orthogonality relations (82) and (83) become respectively*

$$U(B|A)U(A|B) = 1 \qquad ; \qquad U(A|B)U(B|A) = 1$$

Remark (7) Let $P = (p_{ab})$ *be a* κ*-valued bistochastic matrix and let* $U = (u_{ab})$ *be a solution of the equations (82) and (83). Then one immediately verifies that for any invertible* q_b, r_b $(b = 1, \ldots, n)$ *the* κ*-valued matrix*

$$V_{ab} = q_a u_{ab} r_b \tag{99}$$

is also a solution of the equations (82), (83) and

$$\frac{v_{ab'}v_{a'b}}{v_{ab}v_{a'b'}} = \frac{u_{ab'}u_{a'b}}{u_{ab}u_{a'b'}} \tag{100}$$

Definition 12 *Two κ-valued matrices (u_{ab}) (v_{ab}), related by a transformation of the form (99) will be called* **equivalent** . *Thus, by definition, two equivalent κ-valued matrices $(\kappa_{ab}),(v_{ab})$ generate the same family of structure constants by formula (84).*

Lemma 4 *Suppose that in κ any positive element has a square root and let $P = (p_{ab})$ be a κ-valued bistochastic matrix; let (u_{ab}) be a solution of the equations (82) and (83); and $\gamma_{ab}^{a'b'}$ be defined by (84). Then the $\gamma_{ab}^{a'b'}$ satisfy the equations (2), (51), (52), (53), (54) if and only if (u_{ab}) is equivalent, in the sense of Definition* **??** *to a matrix (v_{ab}) satisfying:*

$$p_{ab} = |v_{ab}|^2 \qquad ; \qquad a, b = 1, \ldots, n \tag{101}$$

Proof. *Because of Theorem 7 We have only to check (7) and (1). If the $\gamma_{ab}^{a'b'}$ are defined by (84) then writing equation (54) in terms of the u_{ab} and their adjoints one finds, after trivial simplifications*

$$\frac{u^*_{a'b'}u^*_{ab}}{u^*_{a'b}u^*_{ab'}}\frac{p_{a'b}p_{ab'}}{u_{ab'}} = \frac{p_{a'b'}p_{ab}}{u_{a'b'}u_{ab}}\sum_d u_{a'd}\sum_e u_{eb}(\frac{p_{ed}}{u_{ed}}) \tag{102}$$

*Using (83), (**??**) can be written in the form*

$$\frac{p_{a'b}}{|u_{a'b}|^2}\frac{p_{ab'}}{|u_{ab'}|^2} = \frac{p_{a'b'}}{|u_{a'b'}|^2}\frac{p_{ab}}{|u_{ab}|^2} \tag{103}$$

Denoting q_{ab} the ratio

$$q_{ab} = p_{ab}/|u_{ab}|^2 \tag{104}$$

equation (54) means that the quotient

$$\lambda_a^{a'} = q_{a'b}/q_{ab} \tag{105}$$

is independent of b and therefore we can take

$$\lambda_a^{a'} = q_{a'a}/q_{aa} \tag{106}$$

*From (**??**) one immediately deduces that for any $a, a', a'' = 1, \ldots, n$*

$$\lambda_{a''}^{a'} = \lambda_a^{a'} \cdot \lambda_{a''}^{a} \qquad ; \qquad (\lambda_a^{a'})^{-1} = \lambda_{a'}^{a} \tag{107}$$

and therefore there exist positive invertible elements $s_a \in \kappa$ such that

$$\lambda^a_{a'} = s_a/s_{a'} \qquad ; \ \forall a, a' = 1. \dots, n \tag{108}$$

*Since κ admits square roots, (**??**) and (**??**) imply that*
if, for each $a = 1, \dots, n$, q_a is a square root of s_a in the sense that

$$q^*_a q_a = s_a \tag{109}$$

then the quotients:

$$\frac{p_{a'b}}{|q_{a'}u_{a'b}|^2} = \frac{p_{ab}}{|q_a u_{ab}|^2} = v_b \tag{110}$$

are independant of a, σ and therefore, if r_b is a square root of v_b and if we define

$$v_{ab} := q_\sigma u_{ab} \ r_b \tag{111}$$

*then (v_{ab}) is equivalent to (u_{ab}) and equation (**??**) is satisfied.*

*Lemma 4 shows that, up to replacement of (u_{ab}) with an equivalent solution of the equations (82) and (83) (in the sense of Definition **??**), we can always assume that*

$$\frac{p_{a'b}}{|u_{a'b}|^2} = \frac{p_{ab}}{|u_{ab}|^2} \qquad ; \qquad \forall a, a', b = 1, \dots, n \tag{112}$$

Denoting t_b the common (positive invertible) value of the expressions (112) for a fixed $b = 1, \dots, n$, one obtains:

$$p_{ab} = t_b |u_{ab}|^2 \qquad ; \qquad a, b = 1, \dots, n \tag{113}$$

and, with the same argument as before, we can suppose that, up to equivalence

$$p_{ab} = |u_{ab}|^2 \qquad ; \qquad \forall a, b = 1, \dots, n \tag{114}$$

*Conversely, if (**??**) is satisfied then (**??**), and therefore (1), hold in an obvious way. Finally, expressing (7) in terms of the (u_{ab}) through (84), one finds, after simplifications:*

$$\frac{|u_{ab}|^2}{p_{ab}} \delta_{ac'} \delta_{bd'} = \sum_{cd} u^*_{cb} u_{cd'} \frac{p_{cd}}{|u_{cd}|^2} u^*_{cd} u_{c'd} \tag{115}$$

Definition 13 *Let κ be a commutative $*$-algebra and let $P = (p_{ab})$ $(a, b = 1, \dots, n)$ be a $n \times n$ κ-valued bistochastic matrix. A κ-valued* **transition amplitude** *for P is a κ-valued unitary matrix $U = (u_{ab})$ such that, if U^* is defined by*

$$(U^*)_{ba} := \frac{p_{ab}}{u_{ab}} =: u_{ba} \quad \forall a, b = 1, \dots, n \tag{116}$$

then

$$UU^* = U^*U = 1 \tag{117}$$

Notice that the notation u_{ba} might be misleading. When confusion can arise it is better to use $u_{ab}(A \mid B)$ for u_{ab} and $u_{ba}(B \mid A)$ for u_{ba}.

Theorem 8 *Let κ be a commutative $*$-algebra such that any positive element in κ has a square root and let $P = (p_{ab})$ be a κ-valued bistochastic matrix with invertible elements. An Heisenberg algebra with center κ and transition probability P exist if an only if P admits a κ-valued transition amplitude $U = (u_{ab})$. In this case the structure constants of $\mathcal{A}$ have the form (84).*

Proof. *The proof follows easily from the above discussion.*

7 Deduction of the Schödinger equation and of the Hilbert space

Now we want to extend the above theorem to the case of an arbitrary family of maximal partitions of the identity. More precisely, the present section is devoted to the solution of a problem which is the quantum probabilistic analogue of the following, well known, problem in classical probability: given a family of transition probability matrices $\{P(s,t) \ : \ s < t, s, t\}$, when does there exist a Markovian process $(A(t))$ such that for each $s < t$, the transition matrix canonically associated to the pair of random variables $A(s), A(t)$ is $P(s,t)$? It is well known that the classical probabilistic problem has a positive solution if and only if the family of transition probability matrices $(P(s,t))$ satisfies the Chapman–Kolmogorov equation:

$$P(r,s) \cdot P(s,t) = P(r,t) \qquad ; \ r < s < t$$

A quantum generalization of this problem can be formulated as follows: given a natural integer $n = 1, 2, \ldots,$ (or $+\infty$), a a commutative associative real $*$-algebra with identity κ_o, a set T and a family $\{P(x,y) : x, y\epsilon T\}$ of κ_o–valued $n \times n$ transition probability matrices, find:

i) an Heisenberg algebra $\mathcal{A}$ of dimension n^2 over its center κ and such that κ contains a subalgebra isomorphic to κ_o.

ii) for each $x \in T$, a maximal abelian partition of the identity $\{A_a(x) \ : = 1, ..., n\}$ in $\mathcal{A}$, such that for each $x, y \in T$ the transition probability matrix associated to the pair $(A_a(x))$ $(A_b(y))$ in the sense of Theorem 20 is $P(x,y)$ Notice that, since the symmetry condition

$$p_{ab}(x,y) = p_{ba}(y,x)$$

has been shown in Proposition 13 to be a necessary condition for the solution of the problem, we can assume that it is satisfied. Moreover we will assume that:

$$P(x,x) = 1 \qquad ; \qquad \forall x \in T$$

$$p_{ab}(x,y) > 0 \qquad ; \qquad \forall x \in T \qquad \forall a, b = 1, ..., n$$

and we shall look only for generic solutions (i.e. such that the structure constants of $\mathcal{A}$ in all the $(A_a(x)B_b(y))$–bases are invertible.

A rather surprising fact is that the above stated quantum–probabilistic problem has a positive solution if and only if to each transition matrix $P(x,y)$ one can associate an *amplitude matrix $U(x,y)$* **with coefficients in an abelian algebra κ containing a subalgebra isomorphic to κ_o, so that the family $U(x,y)$ with $x, y \in T$ satisfies a generalization of the Schrödinger equation (in integral form)** *(cf. Theorem 122 below and the remarks following it).*

In order to state precisely, and then prove, the above mentioned result we begin to determine the structure of the maximal abelian partitions of the identity (C_a) of $\mathcal{A}$ which are **generic** *in the sense that if*

$$C_g = \sum_{ab} C^g_{ab} A_a B_b \qquad ; \qquad C^g_{ab} \in \kappa \tag{118}$$

then all the C^g_{ab} are invertible.

Lemma 5 *Let* (C_a) (D_b) $(a, b = 1, \ldots n)$ *be two maximal generic partitions of the identity in* $\mathcal{A}$*. Denote* $\delta_{ab}^{a'b'}$ *the structure constants of* $\mathcal{A}$ *in the* $(C_a \cdot D_b)$*-basis, and* $P(C|D) = (p_{ab})$*, the* κ*-valued bistochastic matrix associated to the pair* $(C_a), (D_b)$*. If* p_{ab} *is invertible for any* a, b*, then for any set* $x(a, b)$ *of invertible elements of* κ*, the following two identities are equivalent:*

$$\delta_{ab}^{a'b'} = \frac{x(a', b)x(a, b')}{x(a', b')x(a, b)} p_{ab} \tag{119}$$

$$E_{ab} \cdot E_{a'b'} = \frac{p_{a'b}}{x(a', b)} E_{ab'} \tag{120}$$

where, by definition $E_{ab} = C_a D_b / x(a, b)$

Proof. *By definition of structure constants:*

$$C_{a'} D_b C_a D_{b'} = \delta_{ab}^{a'b'} C_{a'} D'_b \tag{121}$$

or equivalently:

$$E_{a'b} \cdot E_{ab'} = \delta_{ab}^{a'b'} \frac{x(a', b')}{x(a', b)x(a, b')} E_{a'b'} \tag{122}$$

thus (10) is equivalent to:

$$\delta_{ab}^{a'b'} \frac{x(a', b')x(a, b)}{x(a', b)x(a, b')} \frac{1}{p_{ab}} = 1 \tag{123}$$

which is (9).

Theorem 9 *In the notations and the assumptions of Lemma (7.1) let* (C_a) *be a generic maximal abelian partition of the identity in* $\mathcal{A}$*. Then there exists two* κ*-valued matrices* $U(A|C) = (\rho(a, b))$ *and* $U(B|C) = (\tau(b, c))$ $(a, , = 1, ..., n)$ *satisfying*

$$\sum_{c=1}^{n} \rho(a, c)\tau(c, b) = u_{ab} \tag{124}$$

$$\sum_{a,b=1}^{n} \tau(c, b)u_{ba}\rho(a, c') = \delta_{cc'} \tag{125}$$

$$\sum_{a,b=1}^{n} \frac{\rho(a,)\tau(c, b)}{u_{ab}} A_a B_b = C_c \tag{126}$$

Conversely, any pair of κ*-valued matrices* $U(A|C) = (\rho_{ab})$ $U(B|C) = (\tau_{bc})$ *satisfying (124) and (125) defines, through (126) a generic maximal abelian partition of the identity in* $\mathcal{A}$*.*

Proof. **Sufficiency.** *Let* (ρ_{ab}) (τ_{bc}) *be* κ*-valued matrices satisfying (124), (125). Defining* (C_c) *by (126), the conditions (72) are easily verified by direct calculation. Denote* $\mathcal{A}(C)$ *the* $*$*-algebra generated by* κ *and the* C_c*. To prove that* $\mathcal{A}(C)$ *is maximal abelian, it will be sufficient show that it is the image of* $\mathcal{A}$ *for the conditional expectation* $X \in \mathcal{A} \mapsto \sum_c C_c X C_c$ *is* $\mathcal{A}(C)$ *or, equivalently, that for each* a, b, c *one has*

$$C_c A_a B_b C_c = \lambda C_c \tag{127}$$

for some $\lambda \in \kappa$*. The identity (127) is verified by direct computation using (124) (125).* **Necessity.** *Let* (C_c) *be a generic partition of the identity in* $\mathcal{A}$*, denote* $E_{ab} = A_a B_b / u_{ab}$ *and let*

$$C_c = \sum_{a,b=1}^{n} C^c_{ab} E_{ab} \qquad ; \qquad C^c_{ab} \in \kappa \tag{128}$$

Then:

$$(A_a C_c B_b) \cdot (A_{a'} C_c B_b) = C^c_{ab} C^c_{a'b} u_{ba'} E_{ab} \tag{129}$$

and, by the maximal abelianity of (C_c)*:*

$$C_c B_b A_{a'} C_c = \Gamma_{c,a',b} C_c \tag{130}$$

for some $\Gamma_{c,a',b} \in \kappa$*. Comparing (129) and (130), one deduces:*

$$C_c B_b A_{a'} C_c = C^c_{a'b} u_{ba'} C_c \tag{131}$$

But one has also:

$$C_c B_b A_{a'} C_c = \sum_{a,b''=1}^{n} C^c_{ab} u(b, a') C^c_{a'b''} E_{ab''} \tag{132}$$

Thus, comparing (131) and (132) one obtains:

$$C^c_{ab} (C^c_{a'b})^{-1} = C^c_{ab'} (C^c_{a'b'})^{-1} \tag{133}$$

independently of a, b, a', b'*. Hence, by the same arguments as in Theorem* **??** *there exist two* κ *-valued matrices* $(\rho(a, c))$ $(\tau_o(c, b))$*, such that:*

$$C^c_{ab} (C^c_{a'b})^{-1} = \frac{\rho(a, c)}{\rho(a', c)} \tag{134}$$

Solving (133) for the combination $C^c_{ab}(C^c_{ab'})^{-1}$ one finds:

$$C^c_{ab}(C^c_{ab'})^{-1} = \frac{\tau_o(c,b)}{\tau_o(c,b')}$$

In particular the quantity:

$$t_c := \frac{C^c_{ab}}{\rho(a,c)\cdot\tau_o(c,b)}$$

is independent of $a,b = 1,...,n$ and denoting $\tau(c,b) := t_c\tau_o(c,b)$, one obtains

$$C^c_{ab} = \rho(a,c)\cdot\tau(c,b) \tag{135}$$

and this proves (126). Moreover:

$$1 = \sum_{c=1}^{n} C_c = \sum_{a,b=1}^{n}[\sum_{c=1}^{n}\frac{\rho(a,c)\cdot\tau(c,b)}{u_{ab}}]A_aB_b$$

and this implies (124). Finally using the notation (116) and (10) one sees that the orthogonality relations $C_cC_{c'} = \delta_{c,c'}C_c$ are equivalent to:

$$\sum_{a',b=1}^{n} C^c_{ab}u_{ba'}C^{c'}_{a'b'} = C^c_{ab'}\ \delta_{c,c'} \tag{136}$$

and this, due to (135) and the genericity assumption is equivalent to (125). Keeping into account (135) and the definition of E_{ab}, Equation (126) is just a rewriting of (128). The theorem is proved.

Theorem 10 *Let $\mathcal{A}$ be an Heisenberg algebra of dimensions n^2 over its center κ. For any triple $(A_a),(B_b),(C_c)$ of maximal abelian generic partitions of the identity in $\mathcal{A}$ there exist κ-valued matrices $\{U(X|Y) = u_{ab}(X|Y)\ :\ X,Y = A,B,C\}$ satisfying:*

$$U(X|X) = 1 \tag{137}$$

$$U(X|Y)\cdot U(Y|Z) = U(X|Z) \tag{138}$$

and such that, if $X \neq Y$, then the matrix $P(X|Y) = (p_{ab}(X|Y))$ defined by:

$$p_{ab}(X|Y) = u_{ab}(X|Y)u_{ba}(Y|X) \tag{139}$$

is the transition probability matrix canonically associated to the $(X_a\cdot Y_b)$-basis according to Proposition 2. Conversely, any set $\{U(X|Y)\ :\ X,Y = A,B,C\}$ of κ-valued matrices satisfying (137), (138), (139) can be obtained in this way.

Proof. **Necessity.** *Because of Proposition* **??** *the matrices* $P(A|C), P(B|C), \ldots$ *are characterized by the properties:*

$$C_c A_a C_c = p_{ca}(A|C) C_c \qquad ; \qquad A_a C_c A_a = p_{ac}(C|A) \mathcal{A}_a \tag{140}$$

$$C_c B_b C_c = p_{cb}(B|C) C_c \qquad ; \qquad B_b C_c B_b = p_{bc}(C|B) |cal B_b \tag{141}$$

Using (140) and (126) one obtains:

$$p_{ac}(C|A) A_a = A_a C_c A_a = \sum_{b=1}^{n} \rho(a,c) \cdot \tau(c,b) \frac{p_{ab}(A|B)}{u_{ab}(A|B)} A_a$$

and this, due to (136) is equivalent to:

$$\sum_{b=1}^{n} \tau(c,b) \cdot u_{ba}(B|A) = \frac{p_{ab}(A|C)}{\rho(a,c)} \tag{142}$$

In a similar way, using (141), one shows that:

$$\sum_{a=1}^{n} u_{ba}(B|A) \cdot \rho(a,c) = \frac{p_{cb}(C|B)}{\tau(c,b)} \tag{143}$$

Thus, defining the κ*-valued matrices:*

$$u_{ac}(A|C) = \rho(a,c) \qquad ; \qquad u_{ca}(C|A) = p_{ac}(A|C)/\rho(a,c) \tag{144}$$

$$u_{cb}(C|B) = \tau(c,b) \qquad ; \qquad u_{bc}(B|C) = p_{cb}(C|B)/\tau(c,b) \tag{145}$$

(142) and (143) become respectively:

$$U(C|B) \cdot U(B|A) = U(C|A) \qquad ; \qquad U(B|A) \cdot U(A|C) = U(B|C) \tag{146}$$

which, in view of (137) imply:

$$U(C|B) = U(C|A) \cdot U(A|B) \qquad ; \qquad U(A|C) = U(A|B) \cdot U(B|C) \tag{147}$$

and this proves (138) for $X \neq Z$*. Now for* $X, Y = A, B, C$ *denote:*

$$E^{X,Y}_{ab} = X_a Y_b / u_{cb}(X|Y) \tag{148}$$

Using (126) and (146) one finds:

$$C_c B_b \cdot C_{c'} B_{b'} = \sum_{a,a'=1}^{n} u_{ac}(A|C) u_{cb}(C|B) u_{a'c'}(A|C) u_{c'b'}(C|B) E^{AB}_{ab} E^{AB}_{a'b'}$$

$$= \sum_{a=1}^{n} u_{ac}(A \mid C) u_{cb}(C \mid B) u_{bc'}(B|C) u_{bc'} u_{c'b'}(C \mid B) E^{AB}_{ab'}$$

or, equivalently, using again (126) in appropriate notations:

$$E^{CB}_{cb} E^{CB}_{c'b'} = [\sum_{a',b=1}^{n} u_{a'c}(A|C) u_{cb}(C|B) E^{AB}_{a'b} \quad] B_{b'} = u_{bc'}(B|C) E^{CB}_{cb'}$$

Therefore, from Lemma 118 we conclude that

$$\frac{u_{c'b}(C|B) u_{cb'}(C|B)}{u_{c'b'}(C|B) u_{cb}(C|B)} p_{cb}(C|B) = \delta^{c'b'}_{cb}$$

are the structure constants of $\mathcal{A}$ in the $(C_c B_b)$-basis and, in view of Remark (53), this implies, in particular:

$$U(C|B) \cdot U(B|C) = U(B|C) U(C|B) = 1$$

Similarly one shows that:

$$U(A|C) \cdot U(C|A) = U(C|A) \cdot U(A|C) = 1$$

Sufficiency. *Let $U(X|Y), P(X|Y) \quad (X, Y, = A, B, C)$ be as in the formulation of the theorem. Denote $\mathcal{A}$ the Heisenberg algebra generated over its center κ by the partitions of the identity $(A_a), (B_b)$ $(a, b = 1, ..., n)$, whose structure constants in the $(A_a B_b)$-basis are:*

$$\gamma^{a'b'}_{ab} = \frac{u_{ab'}(A|B) u_{a'b}(A|B)}{u_{a'b'}(A|B) u_{ab}(A|B)} p_{ab}(A|B)$$

Because of (138) (with $X = Z$) and (83) and of Theorem 51, these are the structure constants of an Heisenberg algebra with center κ. Define, for $c = 1, ..., n$:

$$C_c = \frac{u_{ac}(A|C) u_{cb}(C|B)}{u_{ab}(A|B)} A_a B_b$$

Then, according to Theorem 9, (C_c) is a maximal abelian partition of the identity in $\mathcal{A}$ and it is easy to convince oneself that the transition amplitude and probability matrices, associated to the triple $(A_a), (B_b), (C_c)$ according to the first part of the theorem are the given ones.

The above results can be summarized as follows: let be given a family of transition probability matrices $\{P(x, y) \ : \ x, y \in T\}$ satisfying the conditions listed at the beginning of the present Section.

Theorem 11 *Under the assumptions stated at the beginning of the present Section, the following assertions are equivalent:*

i) There exists an Heisenberg algebra $\{\mathcal{A}, T, (A(x))_{x\in T}\}$ *with center* κ *such that for each* $x, y \in T$, $P(x, y)$ *is the transition probability matrix canonically associated to the pair* $A(x), A(y)$.

ii) For each $x, y \in T$ *there exists a* κ*-valued matrix* $U(x, y)$ *such that for each* $x, y, z \in T$ $i, j, k = 1, ..., n$.

$$\sum_{i=1}^{n} \left(\frac{p_{ij}(x,y)}{u_{ij}(x,y)} \right) \cdot u_{ik}(x,y) = \delta_{jk} \tag{149}$$

$$\sum_{j=1}^{n} \left(\frac{p_{ij}(x,y)}{u_{ij}(x,y)} \right) \cdot u_{kj}(x,y) = \delta_{ik} \tag{150}$$

$$U(x,x) = 1 \tag{151}$$

$$U(x,y) \cdot U(y,z) = U(x,z) \tag{152}$$

Moreover, in this case, the Heisenberg algebra $\mathcal{A}$ *can be identified with the Algebra of all* $n \times n$ *matrices with coefficients in* κ.

Even more completely:

Theorem 12 *The following assertions are equivalent:*

i1) There exist an Heisenberg algebra with centre κ *satisfying conditions (i), (ii) of Theorem 10.*

i2) For each $x, y \in T$ *there exists a* κ*-valued transition amplitude matrix* $U(x, y)$ *for* $P(x, y)$ *such that:*

$$U(x,x) = 1 \qquad ; \qquad \forall x \in T \tag{153}$$

$$U(x,y) \cdot U(y,z) = U(x,z) \qquad ; \qquad x,y,z \in T \tag{154}$$

i3) There exists a κ*–module* H *and, for each* $x \in T$, *a* κ*–basis* $(a_j(x))$ $(j = 1, ..., n)$ *of* H *such that the operators* $A_j(x)$ *defined by*

$$A_j(x) a_k(x) = \delta_{jk} a_j(x) \tag{155}$$

satisfy

$$A_j(x) A_k(y) A_j(x) = p_{jk}(x,y) A_j(x) \tag{156}$$

Proof. *The implication* $(i1) \Rightarrow (i2)$ *follows from Theorem 10. To prove the implication* $(i2) \Rightarrow (i3)$*, fix* $x_o \in T$ *arbitrarily and denote* H *the free* κ*-module generated by the symbols* $a_1(x_o), ..., a_n(x_o)$*. Define, for every* $x \in T$ *and* $j = 1, .., n$*, the vector:*

$$a_j(x) := U(x_o, x)a_j(x_o) := \sum_{k=1}^{n} u_{kj}(x_o, x)a_k(x_o) \in H \tag{157}$$

and the operator $A_j(x) \ : \ H \to H$*:*

$$A_j(x)a_k(x_o) := u_{jk}(x, x_o)a_j(x)$$

Since the matrix $(u_{kj}(x_o, x))$ *is invertible, it follows that also* $a_1(x), ..., a_n(x)$ *is a basis for* H*. Then it is easy to verify that for each* $x \in T$ $(a_j(x))$ *is a* κ*-basis of* H *and (155) and (156) hold. Finally, if (i3) holds, then, because of (**??**) and (156), for each* $x, y \in T$ *and* $j, k = 1, ...n$ $A_j(x)A_k(y) \neq 0$*, hence by Lemma (1) the set* $\{A_j(x)A_k(y)\}$ $(j, k = 1, ..., n)$ *is a* κ*-basis of the Heisenberg algebra of all* κ*-linear operators on* H*. But then, due to Proposition 17, condition (156) implies the maximal abelianity of the* κ*-algebra generated by* $A_j(x) \ : \ j = 1, ..., n$*. Thus* $(i3) \Rightarrow (i1)$ *and the theorem is proved.*

<u>*Remark*</u>*. Notice that the reversibility of the generalized evolution* $U(x, y)$*, implicit in equation (154), has a purely statistical origin, stemming from the symmetric role that two maximal observables* $A(x)$ *and* $A(y)$ *play in their mutual conditioning.*

In the notations of Theorem 122 above, we say that the family of transition probability matrices $\{P(x, y) \ : \ x, y \in T\}$ **admits a** κ**-Hilbert space model** *if in the* κ*-module* H*, defined in point (i3) of Theorem 13 one can define a* κ*-valued scalar product* $< \cdot \cdot >$ *for which all the* κ*-bases* $(a_j(x))$ *are orthonormal bases. That is, if there is a map* $u, v \in H \times H \mapsto < u, v >$*, such that* $\forall u, v' \in H$ *one has*

$$< u, v + v' >=< u, v > + < u, v' > \tag{158}$$

$$< u, \lambda v >= \lambda < u, v > \qquad ; \qquad \lambda \in \kappa \tag{159}$$

$$< u, v >=< v, u >^* \tag{160}$$

$$< u, u >\geq 0 \tag{161}$$

$$< a_j(x), a_k(x) >= \delta_{jk} \qquad ; \ \forall x \in T \tag{162}$$

If this is the case, by (162), the transition amplitude matrices $U(x, y)$ are defined by:

$$< a_j(x), a_k(y) >= u_{jk}(x, y) \tag{163}$$

Theorem 13 *The following assertions are equivalent:*

i1) The family of transition probability matrices $\{P(x, y) \ : \ x, y \in T\}$ admits a κ–Hilbert space model.

i2) The family $\{P(x, y) \ : \ x, y \in T\}$ admits an Heisenberg algebra model with centre κ and, if $U(x, y) = (u_{jk}(x, y))$ is the κ–valued transition amplitude associated to $P(x, y)$ according to Theorem 13, then

$$u_{jk}(x, y)^* = u_{kj}(y, x) \qquad ; \ \forall x, y \in T \quad , \ \forall j, k = 1, ..., n \tag{164}$$

Proof. *It is sufficient to notice that, because of (157) and of the orthogonality assumption, one has, for any $x, y \in T$*

$$< U(x, y)a_j(x)U(x, y)a_k(x) >= \sum_h u_{hj}^* u_{hk}$$

so, by the uniqueness of the inverse matrix, (164) follows. Conversely, if (164) is satisfied, then condition (154) implies that $U(x, y)$ is unitary in the usual Hilbert space sense.

<u>Remark</u>. Notice that, if a κ–Hilbert space model exists, then one has

$$p_{ij}(x, y) = |U_{ij}(x, y)|^2 \tag{165}$$

The usual quantum model is recovered when $\kappa = \mathbf{C}$.

<u>Remark</u>. Equations (153), (154) are a generalization of Schrödinger's evolution, which is recovered when $T = \mathbf{R}$, interpreted as time. In our theory this equations appear as compatibility conditions for a set of transition probability matrices $\{P(x, y)\}$ to admit an Heisenberg algebra model. If the index set T is acted upon by a group G so that probabilities are preserved (i.e. $(P(x, y) = P(gx, gy))$), one might study the corresponding generalized unitary representation of G on H. If moreover one has that the amplitudes themselves are G–invariant, i.e.

$$U(x, y) = U(gx, gy) \tag{166}$$

then one can fix a $x_o \in T$ and define

$$U_g := U(x_o, gx_o)$$

Correspondingly one has $U_e = 1$, where e is the identity in G and 1 the identity operator on the κ–Hilbert space H. Moreover equation (154) implies that

$$U_g^{-1} U_h := U(gx_o, x_o)U(x_o, hx_o) = U(gx_o, hx_o) = U_{g^{-1}h} \tag{167}$$

Thus, under the invariance condition (166), equation (154) is also a generalization of the notion of *unitary representation.* Finally, if G acts transitively on T, then equation (167) plus the condition $U_e = 1$, becomes equivalent to the pair of equations (153), (154).

8 Geometric extensions of the quantum probabilistic formalism: gauge theories

We keep the notations of sections (3) and (4). To fix the ideas our considerations will be restricted to complex Hilbert space models (i.e. $\kappa = \mathbf{C}$-the complex numbers, and we assume that the set of transition probability matrices $\{P(x, y) \ : \ x, y \in T\}$ admits a complex Hilbert space model, i.e. - cf. Theorem **??** - that there exists a complex Hilbert space H and for each $x \in T$ - an orthonormal basis $\{a_j(x) \ : = 1, ..., n\}$ of H, for any $x, y \in T$ - a unitary operator $U(x, y)$ with complex coefficients satisfying:

$$U(x, y) \ : \ H \mapsto H \tag{168}$$

$$U(x, x) = 1 \tag{169}$$

$$U(x, y) \cdot U(y, z) = U(x, z) \qquad ; \qquad x, y, z \in T \tag{170}$$

$$| < a_k(y), U(x, y)a_j(x) > |^2 = p_{jk}(x, y) \qquad ; \qquad j, k = 1, ..., n \tag{171}$$

If T is a manifold, it is natural to introduce a path dependent generalization of the *evolution equation* (3) along the following lines: one considers a complex Hilbert bundle, i.e. a fibre bundle $H(T)$ with base T and fiber $H(x)$ $(x \in T)$ isomorphic to a complex Hilbert space H. Introducing the space $\Omega(T)$ of all piecewise smooth paths $[0, 1] \to T$, for each pair of points $x, y \in T$, we denote γ_{xy} an element of $\Omega(T)$ such that

$$\gamma_{xy}(0) = x \ \ \gamma_{xy}(1) = y$$

With these notations the notion of Heisenberg algebra model for a set of transition probability matrices $\{P(x,y)\}$ can be generalized as follows:

Definition 14 *Let T be a manifold and n be an integer or $+\infty$. A family of $n\times n$ transition probability matrices $\{P(\gamma_{xy}) \ : \ \gamma_{xy}\in\Omega(T) \ \ x,y\in T\}$ is said to admit a* **Hilbert bundle model** *if there exist:*

i) A Hilbert bundle $H(T)$ with base T

ii) A unitary parallel transport on $H(T)$, i.e. a map

$$U:\gamma_{xy}\in\Omega(T)\to U(\gamma_{xy})\in\{\ \textit{Unitaries}\ \ H(x)\to H(y)\}=U_n(H_x,H_y)$$

such that denoting $\gamma^{-1}(t)=\gamma(1-t)$ and $\gamma\circ\gamma'(t)=\gamma(2t)$ if $0\le t<1/2$ and $\gamma\circ\gamma'(t)=\gamma'(2t-1)$, if $1/2\le t<1$ one has

$$U(\gamma_{xy})\cdot U(\gamma_{yz})=U(\gamma_{xy}\circ\gamma_{yz}) \qquad ; \qquad x,y,z\in T \tag{172}$$

$$U(\gamma^{-1})=U(\gamma)^{-1} \tag{173}$$

Remark. Notice that we are not requiring that

$$U(\gamma)=U(\gamma') \tag{174}$$

if γ' is a riparametrization of γ.

iii) For any $x\in T$ an orthonormal basis $a_j(x):j=1,...,n$ of $H(x)$ such that, for any $\gamma_{xy}\in\Omega(T)$ $(x,y\in T)$

$$|<a_k(y),U(\gamma_{xy})a_j(x)>|^2=p_{jk}(\gamma_{xy}) \tag{175}$$

Definition 15 *Two Hilbert bundle models $\{H(T),U(\ \cdot)\}$ $\{H'(T),U'(\ \cdot)\}$ are called* **isomorphic** *if there exists a vector bundle isomorphism $V:H(T)\to H'(T)$ which intertwines the parallel transports.*

It is not clear at the moment to what extent the transition probabilities $p_{jk}(\gamma_{xy})$ fix the isomorphic type of the bundle. However, for a trivial bundle, the triviality of the holonomy group of the connection $U(\ \cdot\ )$ is easily seen to be a necessary and sufficient condition for an Hilbert bundle model to be isomorphic to a usual quantum (i.e. complex Hilbert space) model. This suggests the conjecture that also in the general case the statistical invariants of the transition probabilities $p_{jk}(\gamma_{xy})$, or part of them, should be expressible in terms of the topological and geometrical invariants of the pair $\{H(T),U(\ \cdot)\}$.

Finally, let us remark that point (iii) in Definition 12 means that we are fixing a cross section into the frame bundle $F(H(T))$, i.e. the bundle of orthonormal frames of $H(T)$, and consequently an identification of $F(H(T)$ with the principal bundle $P(T, U(n; \mathbf{C}))$ where $U(n; \mathbf{C})$ denotes the unitary group with coefficients in $\mathbf{C}$. Once the cross section $a : T \to F(H(T))$ is fixed, the assignment of a connection $U(\ \cdot)$ on $H(T)$ becomes equivalent to the assignment of a connection $V(\ \cdot)$ on $F(H(T))$ or, through the formula

$$V(\gamma) = P \exp^{\int_\gamma A_a} \qquad ; \ \gamma \in \Omega(T)$$

(where $P \exp$ means path-ordered exponential), to the assignment of a matrix valued 1-form A_a - the connection matrix of the connection $V(\ \cdot)$ in the frame field a (or simply the "potential"). The curvature form associated to A (i.e. $F = dA + (1/2)A \wedge A$) is called a **gauge field**. Thus as the Heisenberg models deduced in Theorem **??** extend the usual quantum model, the corresponding bundle models, obtained with an obvious modification of Definition 12, generalize, in the same direction, the gauge field theories.

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