

and the function $b(z)$ is the solution of the canonical equation

$$b(z) = m^{-1} \sum_{k=1}^m \{-z + \lambda_k [1 - \gamma - z\gamma b(z)]\}^{-1}, \quad z = t + is, \quad s \neq 0.$$

There exists a unique solution $b(z)$ of the canonical equation in the class of analytic functions $L = \{b(z) : \text{Im}z \text{Im}b(z) > 0, \text{Im}z > 0\}$ and

$$c(z) = \int (u - z)^{-1} d\nu(u)$$

where $\nu(u)$ is some distribution function.

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Non adapted Stochastic Calculus as third quantization

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Received for ROSE 4 March 1995

Abstract—This note is not intended to give yet another definition of the anticipative stochastic integral, but rather to review existing definitions and try to put together what they have in common. We unify the definitions of stochastic integral on both Standard Wiener space and an abstract Quantum Mechanical Fock space. We end up considering a purely algebraic approach to the stochastic integral.

0. INTRODUCTION

In the *first quantization* one goes from the state space of classical mechanics to the Hilbert space of quantum mechanics. In the *second quantization* the elements of the first quantization space became associated to *operators* on a larger space (the Boson or Fermion Fock space). When the elements of second quantization spaces, or even functions with values in these spaces, become themselves test functions for the second quantized operators, it is reasonable to speak of *third quantization*. In the present note we show that this *third quantization procedure* provides a natural algebraic extension of the Hitsuda - Skorokhod *non causal stochastic calculus*.

When Wiener and Itô founded the stochastic calculus, they limited their considerations to integrals which are adapted, with respect to an integrating process. Initially these limitations were considered quite natural as corresponding to the *causality* principle in physics. As the time passed a growing evidence accumulated that this approach is in many cases too restrictive. For example, if you try to consider problems of optimal portfolio management in Mathematical Economics under *leaks of information* (or what they call *insider trading*) you eventually end up integrating processes that intrinsically depend on future.

On the other hand, mathematicians themselves were not quite happy with the strong measurability requirements on the integrand, which are not typical for the theories of integration [1]. All these considerations prompted the introduction of the notion of non adapted stochastic integral, which was pioneered by Hitsuda (1968) and Skorokhod (1975). This new integral, which is traditionally called a Skorokhod integral, was further developed by Gaveau, Trauber, Zakai, Protter, Nualart, Paradoux et al (for references cf. [1]).

At the same time a few more approaches to anticipative integration appeared in works of Hida, Kuo, Potthoff (the *white noise* approach) and Ogawa (Brownian motion decomposition approach). Most of these approaches have by now been related to each other.

The growth of Quantum Probability theory in recent decades prompted a development of yet another approach to stochastic calculus both non adapted and conventional. The latter approach is treated in works of Belavkin [3] and more recently Lindsay [4].

Both these authors, however, base their approaches to the definition of non adapted integral on a particular representation of the Fock space: the so called symmetric realization [4]. Since, however, the realizations of the Fock space are many, we feel that an approach which allows to separate the definition of the stochastic integral from a particular realization of it, is more satisfactory.

The aim of this paper is to propose such an approach to the definition of nonadapted stochastic integral. Although we use the quantum theoretical language extensively, we tried to make this paper completely *self-contained*, meaning that no prerequisites, neither in QP nor in *classical* noncausal calculus, are required for the reader.

Section 1 presents the highlights of calculus on Wiener space, following Nualart and Paradoux. In Sections 2-3 we recall the notion of Boson Fock space, Wiener-Segal isomorphism the Wiener space and the definition of the basic operators of creation and annihilation and in Section 4 the relation of the second quantization operators on the Fock space to the fundamental operations of integration and differentiation on the Wiener space. In Section 5 we introduce our algebraic approach to stochastic integration based on the so-called *operator of third quantization* procedure. In particular we do not need, as in [4] that our 1-particle Hilbert space is realized as an L^2 -space over some measurable space S and we do not need an order relation on S [1], [3]. We emphasize that our definition of stochastic integral does not depend on the particular realization of the Fock space.

The concluding sections present a brief exposition of Belavkin's [3] approach to construction of multiple anticipative quantum stochastic integral, along with the links to the approach of Nualart and Pardoux.

We heavily use in our approach the Wiener-Segal isomorphism between Boson Fock Space and Canonical Wiener space. But Boson Fock space is also canonically isomorphic to the Poisson space [5]. Therefore we can define an integral with respect to jump processes in the same fashion as we did it for Brownian motions.

1. ANTICIPATIVE CALCULUS ON WIENER SPACE

In this section we review the Nualart-Paradoux approach [1] to non adapted stochastic calculus. Consider the standard Wiener probability space (Ω, \mathcal{F}, P) where $\Omega = C([0, 1])$ and P is the Wiener measure. The space $L^2(\Omega, P)$ admits a well-known Wiener-Itô decomposition

$$L^2 = \bigoplus_{n=0}^{\infty} \mathbb{Z}_n,$$

where $\mathbb{Z}_n = \{I_n(f_n) | f_n \in H_n^{\text{sym}}\}$ and H_n^{sym} denotes the space of symmetric functions in $L^2([0, 1]^n)$. Here I_n is an n -fold multiple Wiener integral.

Definition 1.1. For $h(t) \in L^2([0, 1])$ the stochastic derivative of $F \in L^2(\Omega)$ in the

direction h is defined by

$$D_h F = \sum_{n=1}^{\infty} \int_0^1 n I_{n-1}(f_n(\cdot, t)) h(t) dt$$

if $F = \sum_{n=0}^{\infty} I_n(f_n)$ and the series for $D_h F$ converges in $L^2(\Omega)$. The stochastic derivative of F is defined by

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)),$$

if the series converges in $L^2(\Omega \times [0, 1])$.

Definition 1.2. The stochastic integral of the function

$$u_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)), \quad f_n(\cdot, t) \in H_n^{\text{sym}} \quad \forall t \in [0, 1]$$

is defined by

$$\int_0^1 u_t dw_t := \sum_{n=0}^{\infty} I_{n+1}(f_n^{\text{sym}}),$$

where f_n^{sym} is the symmetrization of $f_n(\cdot, t)$ over all $(n+1)$ variables, and the series converges in $L^2(\Omega)$.

Definition 1.3. The operator $J_h : D \subset L^2(\Omega) \rightarrow L^2(\Omega)$ is defined for all $h \in L^2([0, 1])$ by

$$J_h F = \int_0^1 h(t) F dw_t, \quad (1)$$

where D denotes the linear subspace where the right hand side of (1) is well defined.

Remark 1.1. Although F does not depend on t , according to the Definition (3) of stochastic integral, it cannot be taken out of the integral sign. Moreover, the following equality holds [1], Lemma 4.1):

$$J_h F = B(h)F - D_h F,$$

where $B(h)$ denotes the multiplication operator by the brownian functional:

$$\int_0^1 h(t) dw_t.$$

Remark 1.2. Thus defined stochastic integral is, in a sense, "an adjoint operator" to D_t , i.e.,

$$\mathbf{E} \left(\int_0^1 u_t D_t F dt \right) = \mathbf{E} \left(F \int_0^1 u_t dw_t \right)$$

([1], Proposition 3.1).

2. BOSON FOCK SPACE AND WIENER-SEGAL ISOMORPHISM

Definition 2.1. For a Hilbert space H the symmetric (Boson) Fock space is defined as

$$\Gamma(H) = \bigoplus_{n=0}^{\infty} \otimes_{\text{sym}}^n H = \bigoplus_{n=0}^{\infty} \Gamma_n(H),$$

where $\otimes_{\text{sym}}^n H = H_n^{\text{sym}}$ is called the n -particle space. By definition

$$H_{\text{sym}}^0 = \mathcal{Q} \cdot \varphi, \quad \varphi\text{-vacuum vector.}$$

It is well-known that the Boson Fock space $\Gamma(L^2[0,1])$ is canonically isomorphic to the standard Wiener space $L^2(\Omega)$ and the unitary operator $U : L^2(\Omega) \rightarrow \Gamma(L^2[0,1])$ implementing this isomorphism is characterized by the property:

$$\frac{1}{\sqrt{n!}} I_n(\otimes^n f) = U^{-1}(\otimes^n f) \in \mathbb{Z}_n$$

for any function $\otimes^n f$ from the n -particle space Γ_n . This unitary operator U is called the *Wiener Segal isomorphism*.

Recall that the *exponential* (or *non-normalized coherent*) vectors on $\Gamma(H)$ are those of the form

$$\psi(f) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \otimes^n f.$$

They are linearly independent and total in $\Gamma(H)$. The inverse image, under the Wiener-Segal isomorphism, of the exponential vectors are the so called *Wick's exponentials*:

$$U(: \exp B_f :) = \psi(f) \in \Gamma(L^2[0,1])$$

defined by

$$: \exp B_f : \stackrel{\text{def}}{=} \exp \left\{ B(f) - \frac{1}{2} \|f\|^2 \right\} \in L^2(\Omega)$$

where

$$B(f) = \int_0^1 f(t) d\omega_t.$$

3. OPERATORS ON THE FOCK SPACE

The *annihilation* operator with test function $h \in H$ is a linear operator A_h on $\Gamma(H)$, defined on the exponential vectors by

$$A_h \psi(f) = \langle h, f \rangle \psi(f).$$

Notice that A_h is antilinear in h .

The adjoint operator to A_h is called *creation* operator and is defined by

$$A_h^+ \psi(f) = \frac{d}{dt} \Big|_{t=0} \psi(f + th).$$

Position and momentum operators are defined by

$$Q(h) = A_h + A_h^+ \\ P(h) = \frac{1}{i} [A_h - A_h^+] \quad \text{respectively.}$$

If R_h is some operator in Fock space with test function h , then we denote $R_t = R(\chi_{[0,t]})$ and we write formally

$$R_h = \int_0^1 h_t dR_t = \int_0^1 h_t r_t dt.$$

In this case operator valued distribution r_t is called a *density of operator* R .

B_h is a linear operator in $\Gamma(H)$ corresponding to the multiplication in $L^2(\Omega)$ by $B(h)$, i.e.

$$B_h U F \stackrel{\text{def}}{=} U(B(h) \cdot F).$$

LEMMA 3.1. $B_f = A_f + A_f^+$.

Proof. By definition $A_f^+ \psi(g) = \frac{d}{dt} \Big|_{t=0} \psi(g + tf)$. Any f can be represented as $f = \lambda g + f^\perp$, $\lambda = \frac{\langle f, g \rangle}{\|g\|^2}$ and $f^\perp \perp g$. Then, in the notations introduced at the end of Section 2,

$$U^{-1} A_f^+ U : \exp B_g : = \frac{d}{dt} \Big|_{t=0} : \exp B(g + tf) : \\ = \frac{d}{dt} \Big|_{t=0} \exp \left\{ B(g) + tB(f) - \frac{1}{2} \|g + tf\|^2 \right\} \\ =: (\exp B(g)) : \{ B(f) - \lambda \|g\|^2 \} \\ = \{ B(f) - \langle f, g \rangle \} (: \exp B(g) :)$$

i.e.

$$A_f^+ \psi(g) = [B_f - \langle f, g \rangle] \psi(g) = [B_f - A_f] \psi(g)$$

so $A_f^+ = B_f - A_f$.

4. ALGEBRAIC APPROACH TO NON ADAPTED CALCULUS

LEMMA 4.1. Under the Wiener-Segal isomorphism, the operators D_h and J_h on $L^2(\Omega)$ correspond to the annihilation and creation operators A_h and A_{+h} respectively on the total set $\{\psi(f) | f \in H\}$ in $\Gamma(L^2[0,1])$.

Proof. Consider

$$F = U^{-1}(\psi(f)) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\otimes^n f).$$

Then

$$D_h F = \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 n I_{n-1}(\otimes^{n-1} f) \cdot f(t) h(t) dt \\ = \sum_{m=0}^{\infty} \frac{1}{m!} I_m(\otimes^m f) \cdot \langle h, f \rangle \\ = \langle h, f \rangle U^{-1}(\psi(f)) \\ = U^{-1} A_h \psi(f).$$

$D_h \cong A_h$, i.e. $D_h = U^{-1}A_hU$.

Since (according to Lemma 4.1 [1])

$$J_h F = (B(h) - D_h)F$$

then passing to the Fock space and using the fact that

$$B_h := UB(h)U^{-1} = A_h + A_h^+$$

we obtain

$$UJ_hU^{-1} = B_h - A_h = A_h^+.$$

To construct a Fock space analogue of the stochastic integral, corresponding to integration of functions of the general form

$$u_t : [0, 1] \longrightarrow L^2(\Omega) \simeq \Gamma(L^2[0, 1])$$

we shall restrict our consideration, at the beginning, to square integrable $\Gamma(L^2[0, 1])$ -valued functions, i.e. to elements

$$u \in L^2([0, 1]; \Gamma(L^2[0, 1])) \cong L^2[0, 1] \otimes \Gamma(L^2[0, 1]).$$

The stochastic integral will then be considered as a linear map

$$L^2[0, 1] \otimes \Gamma(L^2[0, 1]) \mapsto \Gamma(L^2[0, 1]).$$

We shall construct such a map replacing $L^2[0, 1]$ by an arbitrary Hilbert space.

5. VECTOR STOCHASTIC INTEGRATION ON THE ABSTRACT FOCK SPACE

Let (S, μ) be measure space, $H = L^2(S, \mu)$ — a Hilbert space of *test functions*, and denote $\mathcal{H} = \Gamma(H)$ the symmetric (Boson) Fock space on H .

Let A_h (resp A_h^+) be the annihilation (resp. creation) operator on \mathcal{H} with a test function $h \in H$. We will denote by \mathcal{D}_h^- (resp \mathcal{D}_h^+) its domain in \mathcal{H} . Then

$$\mathcal{D}^- := \bigcap_{h \in H} \mathcal{D}_h^-$$

$$\mathcal{D}^+ := \bigcap_{h \in H} \mathcal{D}_h^+$$

$$\mathcal{D} := \mathcal{D}^- \cap \mathcal{D}^+$$

are total subsets of $\Gamma(H)$ because they contain the exponential vectors.

Let us consider the space of Fock-valued functions

$$L^2(S, \mu; \Gamma(H)) \cong H \otimes \Gamma(H).$$

Definition 5.1. The map \mathcal{A}^+ , defined by

$$\mathcal{A}^+(h \otimes F) := A_h^+ F$$

can be extended by linearity to the whole linear span \mathcal{L}_H of the set $\{h \otimes F \mid h \in H, F \in \mathcal{D}\}$ which is a dense subspace of $H \otimes \Gamma(H)$. This extension, still denoted \mathcal{A}^+ will be called *partial creation map*.

THEOREM 5.1. The operator $\mathcal{A}^+ : H \otimes \Gamma(H) \rightarrow \Gamma(H)$ is preclosed and its adjoint, denoted $\mathcal{A}_+^* : \Gamma(H) \rightarrow H \otimes \Gamma(H)$, is defined on the linear span of the exponential vectors by linear extension of

$$\mathcal{A}_+^* \psi(g) := g \otimes \psi(g); \quad g \in H.$$

Proof. Let be given $k := \sum_{n=1}^m h_n \otimes F_n \in \mathcal{L}_H$ and $g \in H$. Then:

$$\begin{aligned} \langle \mathcal{A}^+ k, \psi(g) \rangle &= \sum_{n=1}^m \langle \mathcal{A}^+ h_n \otimes F_n, \psi(g) \rangle \\ &= \sum_{n=1}^m \langle F_n, A_{h_n} \psi(g) \rangle \\ &= \sum_{n=1}^m \langle F_n, \psi(g) \rangle \langle h_n, g \rangle \\ &= \sum_{n=1}^m \langle h_n \otimes F_n, g \otimes \psi(g) \rangle \\ &= \langle k, g \otimes \psi(g) \rangle \end{aligned}$$

in particular

$$|\langle \mathcal{A}^+ k, \psi(g) \rangle| \leq \|k\| \cdot \|g \otimes \psi(g)\| \quad (2)$$

for any k in \mathcal{L}_H .

Remark 5.1. The estimate (2) allows to extend the partial creation map \mathcal{A}^+ to the whole space $H \otimes \Gamma(H)$ by continuity from the norm topology on $H \otimes \Gamma(H)$ to the topology of weak convergence on the linear combinations of exponential vectors.

THEOREM 5.2. If $(S, \mu) = ([0, 1], \lambda)$ and u_t is some stochastic process, such that Uu_t -its image under Wiener-Segal isomorphism belongs to the domain of \mathcal{A}^+ , then u_t is integrable in Skorokhod sense and

$$\int_0^1 u_t dw_t = U^{-1} \mathcal{A}^+ U u_t.$$

Proof. The statement was proven in the previous section for $u_t = h_t \cdot F$. The general case follows from linearity and continuity of \mathcal{A}^+ .

Similarly one can define the *partial annihilation map*

$$\mathcal{A} : H^* \otimes \Gamma(H) \rightarrow \Gamma(H),$$

where H^* denotes the *conjugate space* of H , i.e. the space of the continuous linear functionals on H with scalar product given by

$$\langle h, k \rangle_{H^*} = \langle k, h \rangle_H; \quad h, k \in H.$$

The map \mathcal{A} is characterized by extension (anti-linear in h , linear in F) of

$$\mathcal{A}(h \otimes F) = A_h F; \quad h \in H; \quad F \in \mathcal{D}.$$

Clearly for any $F, G \in \mathcal{D}$ and $h \in H$:

$$\langle \mathcal{A}^+(h \otimes F), G \rangle = \langle F, \mathcal{A}(h \otimes G) \rangle.$$

Moreover, the maps $\mathcal{A}, \mathcal{A}^+$ are related by the following Proposition, which generalizes the Skorokhod isometry (called in this way even if it is not an isometry):

LEMMA 5.1. For any $h_1, h_2 \in H$ and $F_1, F_2 \in \mathcal{D} \subseteq \Gamma(H)$ one has:

$$\begin{aligned} \langle \mathcal{A}^+(h_1 \otimes F_1), \mathcal{A}^+(h_2 \otimes F_2) \rangle \\ = \langle h_1 \otimes F_1, h_2 \otimes F_2 \rangle + \langle \mathcal{A}(h_2 \otimes F_1), \mathcal{A}(h_1 \otimes F_2) \rangle. \end{aligned}$$

Proof. In the above notations, and using the Canonical Commutation Relation (CCR), one has:

$$\begin{aligned} \langle \mathcal{A}^+(h_1 \otimes F_1), \mathcal{A}^+(h_2 \otimes F_2) \rangle &= \langle A_{h_1}^+ F_1, A_{h_2}^+ F_2 \rangle \\ &= \langle F_1, [A_{h_1}, A_{h_2}^+] F_2 \rangle + \langle F_1, A_{h_2}^+ A_{h_1} F_2 \rangle \\ &= \langle h_1, h_2 \rangle \langle F_1, F_2 \rangle + \langle A_{h_2} F_1, A_{h_1} F_2 \rangle \\ &= \langle h_1 \otimes F_1, h_2 \otimes F_2 \rangle + \langle \mathcal{A}(h_2 \otimes F_1), \mathcal{A}(h_1 \otimes F_2) \rangle. \end{aligned}$$

Remark 5.2. The partial annihilation map $\mathcal{A} : H^* \otimes \Gamma(H) \rightarrow \Gamma(H)$ has no densely defined adjoint

$$\mathcal{A}^* : \Gamma(H) \rightarrow H^* \otimes \Gamma(H).$$

In fact one easily verifies that for any $n \in \mathbb{N}, h_1, \dots, h_n, f_1, \dots, f_n \in H, G \in \mathcal{D}$, one has:

$$\langle \mathcal{A} \left(\sum_{j=1}^n h_j \otimes \psi(f_j) \right), G \rangle = \sum_{j=1}^n \langle h_j \otimes \psi(f_j), f_j \otimes G \rangle.$$

So, unless $n = 1$, it will be impossible in general to obtain an inequality of the form (2).

6. OPERATOR STOCHASTIC INTEGRATION

Let \mathcal{B} denote a $*$ -algebra of linear operators acting on the Fock space $\Gamma(H)$ and suppose that \mathcal{B} leaves the domain \mathcal{D} invariant, i.e.

$$B\mathcal{D} \subseteq \mathcal{D}; \quad \forall B \in \mathcal{B}.$$

Definition 6.1. On the linear span \mathcal{L} of tensor products of the form $h \otimes D$, with $h \in H$, and $D \in \mathcal{B}$ which is isomorphic to the algebraic tensor product $H \otimes \mathcal{B}$ we define an operator, still denoted

$$\mathcal{A}^+ : \mathcal{L} \rightarrow \text{Linear operators on } \Gamma(H)$$

by an action

$$[\mathcal{A}^+(h \otimes D)]F := \mathcal{A}_+(h \otimes DF), \quad \forall F \in \mathcal{D} \subseteq \Gamma(H).$$

We shall see, that for each $h \in H$ and $F \in \Gamma(H)$, the element $\mathcal{A}^+(h \otimes F)$ of $\Gamma(H)$ will correspond to a non adapted vector valued stochastic integral, while the operator $\mathcal{A}^+(h \otimes D)$ ($D \in \mathcal{L}$ on $\Gamma(H)$) to an operator valued stochastic integral. Notice, that if $D, D' \in \mathcal{B}$, then for any vector $F \in \mathcal{D}$ and $h \in H$ one has

$$\mathcal{A}^+(h \otimes D)D'F = \mathcal{A}^+(h \otimes DD'F).$$

Therefore, for any $h \in H, D, D' \in \mathcal{B}$

$$\mathcal{A}^+(h \otimes DD') = \mathcal{A}^+(h \otimes D)D'.$$

It is however not true that

$$\mathcal{A}^+(h \otimes DD') = D\mathcal{A}^+(h \otimes D'). \quad (3)$$

The only case in which (3) holds is when D commutes with the creation operator A_h^+ . The situation strongly reminds what happens in Lu's theory of stochastic integration over an Hilbert module. Computation of the adjoint of $\mathcal{A}^+(h \otimes D)$ leads to the identities

$$\mathcal{A}^+(h \otimes D)F = \mathcal{A}(h \otimes DF)$$

$$\begin{aligned} \langle \mathcal{A}^+(h \otimes D)F, G \rangle &= \langle \mathcal{A}^+(h \otimes DF), G \rangle \\ &= \langle DF, \mathcal{A}(h \otimes G) \rangle \\ &= \langle F, D^* \mathcal{A}(h \otimes G) \rangle \\ &= \langle F, D^* A_h G \rangle. \end{aligned}$$

Thus, if D^* commutes with A_h , then

$$\langle \mathcal{A}^+(h \otimes D)F, G \rangle = \langle F, \mathcal{A}(h \otimes D^*)G \rangle.$$

If $H = L^2(S, \mu)$ it is easy to construct a subspace of vectors $h \otimes D$.

7. KERNEL APPROACH TO STOCHASTIC INTEGRATION

In view of our general definition we consider now a construction of QS integral with respect to a more general class of integrators due to Belavkin [2]. In order to do so we shall slightly change our view on the Boson Fock space.

We shall restrict our consideration of the QS integral to the simplest case of processes indexed by "time" variable on the \mathbb{R}_+ . In [2] Belavkin deals with much more general framework of an abstract Borel space X with a measure μ . In order to achieve this goal he introduces a partial order on X , and a measure μ on X , which is nonatomic in the following sense: for any $n > 0$

$$\mu^{\otimes n}(X_{\neq}^{\otimes n}) = 0,$$

where $\mu^{\otimes n}$ is the product-measure on $X^{\otimes n}$ given by n copies of μ and

$$X_{\neq}^{\otimes n} := \{(x_1, \dots, x_n) \in X^{\otimes n} | \exists i \neq j : x_i = x_j\}.$$

The choice $X = \mathbb{R}_+ \times \mathbb{R}^d$ with Lebesgue measure satisfies these conditions.

This condition guarantees that operators like Malliavin derivative would be densely defined on the appropriate spaces.

Denote by Ω^+ a set of all possible finite chains $\tau = (t_1, \dots, t_n)$, $t_k \in \mathbb{R}_+$, $t_k < t_{k+1}$. Then the Fock space $\mathcal{F} = \Gamma(L^2(\mathbb{R}_+))$ can be regarded as a space of square-integrable functions

$$\Omega^+ \ni \tau \rightarrow \xi(\tau) \in L^2(\mathbb{R}_+)^{\otimes |\tau|} = \Gamma_{|\tau|}(L^2(\mathbb{R}_+))$$

with scalar product

$$\langle \xi | \eta \rangle = \int_0^\infty \langle \xi(\tau) | \eta(\tau) \rangle d\tau = \sum_{n=0}^\infty \int_{0 < t_1 < \dots < t_n} \langle \xi | \eta \rangle(t_1 \dots t_n) dt_1 \dots dt_n$$

and $\langle \xi | \eta \rangle(t_1, \dots, t_n)$ is a tensor product in $L^2(\mathbb{R}_+)^{\otimes n}$. In other words $\mathcal{F} = L^2(\Omega^+)$.

Define the Hilbert scale of Fock spaces $\mathcal{F}(\xi)$ for $\xi > 0$:

$$\mathcal{F}(\xi) = \{k : \Omega^+ \ni \tau \rightarrow k(\tau) \in L^2(\mathbb{R}_+)^{\otimes |\tau|}\}$$

with a norm:

$$\|k\|^2(\xi) = \sum_{n=0}^\infty \xi^n \int_{0 < t_1 < \dots < t_n} \|k\|^2(t_1 \dots t_n) dt_1 \dots dt_n.$$

For some fixed $\xi^+ \geq 1 \geq \xi_-$ denote

$$\mathcal{F}^+ = \mathcal{F}(\xi^+) \subset \mathcal{F} \subset \mathcal{F}(\xi_-) = \mathcal{F}_-$$

Consider a table $D = (D_\nu^\mu(x))_{\nu=0,+}^{\mu=-,0}$, $x \in \mathbb{R}_+$ of operator-valued functions

$$D_\nu^\mu(x) : \mathcal{F}^+ \rightarrow \mathcal{F}_-$$

— integrands, which generalize a concept of Fock-valued process that is to be integrated.

Definition 7.1. The point Malliavin derivative $\dot{a}(t)$ of $a \in \mathcal{F}$ is

$$\dot{a}(t) = \int_{\Omega^+}^{\oplus} a(\tau \sqcup t) d\tau$$

i.e. $\dot{a}(t)(\tau) = a(\tau \sqcup t)$ where $\tau \sqcup t = \{\tau, t | t \notin \tau\}$.

Definition 7.2. Quantum Stochastic (QS) integral $\Lambda^t(D)$ is defined by

$$\Lambda^t(D) = \sum_{\mu,\nu} \Lambda_\mu^\nu(t, D_\nu^\mu)$$

and $\Lambda_\mu^\nu(t, D)$ are operators on \mathcal{F} acting by:

$$[\Lambda_0^0(T, D_+^0)a](\tau) = \sum_{t \in \tau^T} [D_0^0(t)\dot{a}(t)](\tau \setminus t)$$

—integral with respect to the number operator

$$[\Lambda_0^+(T, D_+^0)a](\tau) = \sum_{t \in \tau^T} [D_+^0(t)a](\tau \setminus t)$$

—integral with respect to the creation operator

$$[\Lambda_-^0(T, D_0^-)a](\tau) = \int_0^T [D_0^-(t)\dot{a}(t)](\tau) dt$$

—integral with respect to the annihilation operator

$$[\Lambda_-^+(T, D_+^-)a](\tau) = \int_0^T [D_+^-(t)a](\tau) dt$$

—integral with respect to time.

Here $\tau^T = \{x \in \tau | x < T\} = \tau \cap [0, T)$.

Operators Λ_μ^ν are densely defined in \mathcal{F} as (ξ^+, ξ_-) continuous operators from \mathcal{F}^+ to \mathcal{F}_- for (nonadapted) D , satisfying local QS-integrability conditions:

$$\begin{aligned} \|D_0^0\|_\infty^t < \infty \quad \|D_+^0\|_2^t < \infty \\ \|D_0^-\|_2^t < \infty \quad \|D_+^-\|_1^t < 1 \end{aligned} \quad \forall t > 0,$$

where

$$\begin{aligned} \|D\|_p^t &= \left[\int_0^t (\|D(x)\|)^p dx \right]^{1/p} \\ \|D(x)\| &= \sup_{a \in \mathcal{F}^+} \{ \|D(x)a\|(\xi_-) / \|a\|(\xi^+) \}. \end{aligned}$$

Remark 7.1. It is clear, that thus defined QS integral with respect to creation process generalizes the notion of anticipative S.I. in Skorokhod–Nualart–Pardoux sense.

Indeed, in [1] case we ought to integrate a Fock-space valued process. In Belavkin's setup we integrate a Fock-automorphism-valued process. But if we restrict our consideration only to the action of the integrand on the $1 \in \Gamma(H)$, then we get

$$(\Lambda_0^+(t, D_+^0) \cdot 1)(\tau) = \sum_{t \in \tau^T} [D_+^0(t) \cdot 1](\tau \setminus t) \quad (4)$$

if we denote $D_+^0(t) \cdot 1 =: u(t) \in \Gamma(H)$, $\forall t \geq 0$, then (4) can be written in the form

$$\Lambda_0^+(t, u_t)(\tau) = \sum_{t \in \tau^T} u(t)(\tau \setminus t).$$

It is clear, that the right hand side corresponds to symmetrization operation over all $|\tau| + 1$ variables in every $|\tau|$ -particle subspace of $\Gamma(H)$, i.e., $\Lambda_0^+(t, u_t)$ is an image under Wiener–Segal isomorphism of $\int_0^T U^{-1} u_t dw_t$ in Nualart–Pardoux sense.

Let us compare now sufficient conditions for existence of QS integral and AS integral. The local integrability condition means, in particular, that

$$\int_0^T \|D_+^0(t) \cdot 1\|^2(\xi_-) dt < \infty \quad \forall T > 0. \quad (5)$$

If

$$D_+^0(t) \cdot 1 = U \left(\sum_{n=0}^\infty I_n(f_n(\cdot, t)) \right)$$

then

$$D_+^0(t) \cdot 1 = \sum_{u=0}^\infty \sqrt{n!} f_n(\cdot, t).$$

Then we can rewrite (5) in the form

$$\sum_{n=0}^\infty n! |\xi_-|^n \|f_n\|_{L^2([0,1]^{n+1})}^2 < \infty. \quad (6)$$

According to [1] (Def. 3.3), for the process to be integrable the sufficient condition is that it should belong to class

$$\mathbb{L}^{2,1} = \left\{ \sum_{n=0}^{\infty} n! n \|f_n\|_{L_2([0,1]^{n+1})}^2 < \infty \right\}.$$

Hence, if $U^{-1}(D_+^0(t) \cdot 1) \in \mathbb{L}^{2,1}$, then (6) is clearly satisfied and $D_+^0(t)$ is locally integrable in QS sense.

Clearly, $\mathbb{L}^{2,1}$ does not exhaust the class of all Skorokhod integrable processes, but it is much more difficult to write estimates for the wider class of processes.

8. MULTIPLE ANTICIPATIVE QS INTEGRAL

The natural generalization of the introduced definition of QS integral is a notion of the multiple QS integral.

First, for any chain τ we can consider it as being a 4-fold table of subchains $\tau_\nu^\mu \subset \tau$, $\nu = 0, +, \mu = -, 0$. In this case we write $\tau = \sqcup \tau_\nu^\mu$, meaning that different subsets in this union do not overlap.

It is clear that any table τ can be represented as a union

$$\tau = \sqcup_{t \in \tau} x(t)$$

where $x(t)$ takes value in one of the three elementary tables

$$\begin{aligned} x_0^0 &= \begin{pmatrix} \varphi & \varphi \\ t & \varphi \end{pmatrix} & x_+^0 &= \begin{pmatrix} \varphi & \varphi \\ \varphi & t \end{pmatrix} \\ x_0^- &= \begin{pmatrix} t & \varphi \\ \varphi & \varphi \end{pmatrix} & x_+^- &= \begin{pmatrix} \varphi & t \\ \varphi & \varphi \end{pmatrix}. \end{aligned}$$

Assume now that $B : \Omega^+ \rightarrow \mathcal{L}(\mathcal{F}(\eta^+), \mathcal{F}(\eta_-))$ is an operator-valued function to be integrated.

Definition 8.1. The multiple QS integral $\Lambda_{[0,T]}^\otimes(B) = \int_0^T \Lambda^\otimes(d\tau, B)$ is defined as an operator in \mathcal{F} with the action

$$[\Lambda_{[0,T]}^\otimes(B)a](\tau) = \sum_{v_0^- \sqcup v_+^0 \subset \tau} \int_0^T \int_0^T [B(v) \dot{a}(v_0^- \sqcup v_+^0)](v_0^-) dv_0^- dv_+^0.$$

Here \dot{a} denotes a multy-point Malliavin derivative, defined for almost all $\tau, \nu \in \Omega^+$, $\tau \cap \nu = \psi$ as $[\dot{a}(\tau)](v) = a(\tau \sqcup v)$.

Also in the definition

$$v = \begin{pmatrix} v_0^- & v_+^- \\ v_0^0 & v_+^0 \end{pmatrix}$$

and $v_0^- = \tau \setminus v_0^0 \setminus v_+^0$. As usually, we assume that subchains involved are disjoint.

Definition 8.2. The function B is locally QS integrable if for any $T \geq 0 \exists \eta^0 = (\eta^-, \eta^0, \eta^+)$ of positive numbers for which

$$\begin{aligned} \|B\|(t) &\stackrel{\text{def}}{=} \int_0^T dv_+^- \left[\int_0^T \int_0^T \frac{(\eta_+)^{|v_+^0|}}{(\eta_-)^{|v_0^-|}} \text{esssup}_{v_0^0 < T} \frac{(\eta_0)^{|v_0^0|}}{(\eta^0)^{|v_0^0|}} \right. \\ &\quad \left. \times (\|B(v)\|_{\eta_+}^2)^2 dv_+^0 dv_0^- \right]^{1/2} < \infty. \end{aligned}$$

Definition 8.3. Let

$$v^* = \begin{pmatrix} v_+^0 & v_+^- \\ v_0^0 & v_0^- \end{pmatrix},$$

define $B^*(v) = B(v^*)^*$. Then we define a formal adjoint to $\Lambda_{[0,t]}^\otimes(B)$ by $\Lambda_{[0,t]}^\otimes(B)^* := \Lambda_{[0,t]}^\otimes(B^*)$.

Definition 8.4. For a fixed integrand B we shall denote by U^t an operator

$$\Lambda_{[0,t]}^\otimes(B) : \mathcal{F}(\xi^+) \rightarrow \mathcal{F}(\xi_-).$$

THEOREM 8.1. If B is locally QS -integrable then for $\xi^+ \geq \Sigma_\mu \eta^\mu$, $(\xi_-)^{-1} \geq \Sigma_\nu (\eta_\nu)^{-1}$ operators U^t and $(U^t)^*$ are continuous in $\mathbb{L}(\mathcal{F}(\xi^+), \mathcal{F}(\xi_-))$ and $\mathbb{L}(\mathcal{F}(\frac{1}{\xi_-}), \mathcal{F}(\frac{1}{\xi^+}))$ respectively. Moreover,

$$\|\Lambda_{[0,t]}^\otimes(B)a\|(\xi_-) \leq \|B\|(t)\|a\|(\xi^+)$$

and

$$\|U^* \|_{1/\xi_-}^{1/\xi^+} = \|U\|_{\xi^+}^{\xi_-}.$$

Acknowledgements. L.A. acknowledges partial support from the Human Capital and Mobility programme, contract number: erbchrxt930094.

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