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On the structure of Markov flows ☆

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Abstract

A new infinitesimal characterization of completely positive but not necessarily homomorphic Markov flows from a C^*-algebra to bounded operators on the boson Fock space over $L^2(R)$ is given. Contrarily to previous characterizations, based on stochastic differential equations, this characterization is universal, i.e., valid for arbitrary Markov flows. With this result the study of Markov flows is reduced to the study of four C_0-semigroups. This includes the classical case and even in this case it seems to be new. The result is applied to deduce a new existence theorem for Markov flows.

1. Introduction

Our goal is to understand the structure of dynamical evolutions, both classical and quantum. In both cases a reversible dynamical evolution is described by a one-parameter group of automorphisms of a certain algebra and an irreversible one by a one-parameter semigroup of completely positive maps on the same algebra.

Perturbations of reversible free evolutions lead us to introduce the so-called *interaction picture*. Mathematically this leads us to generalize the notion of one-parameter group into that of *flow*. Flows with an additional covariance property are 1-cocycles for the free evolution. Flows with an additional localization property are called *Markov flows* or *Markov cocycles* and were introduced in [1]. Summing up: the problem of studying the structure of nonlinear dynamical evolutions is equivalent to determine the structure of flows. Smooth deterministic equations lead to usual flows; stochastic or white noise equations lead to Markov flows.

A particular class of Markov flows are the Markov semigroups. Their structure, in the strongly continuous case, is determined, in the abstract context of Banach spaces, by the Hille–Yoshida theorem which characterizes their generators in terms of dissipations (derivations in the reversible case). In the present paper, we obtain a characterization of strongly continuous Markov flows in terms of infinitesimal characteristics. We solve this problem in the context of Markov flows from a C^*-algebra $\mathscr{B}_S$ to the algebra $\mathscr{B}_S \otimes \mathscr{B}(\Gamma(L^2(R)))$ of bounded operators on the boson Fock space over $L^2(R)$, identified to the standard Wiener space.

Since it is known that any stochastic process satisfying a (classical or quantum) stochastic differential equation admitting an existence, uniqueness and regularity theorem for a sufficiently large class of initial data, gives rise to a Markov flow, a corollary of our result is an infinitesimal characterization of such processes.

The standard way to tackle this problem up to now has been to show that a flow satisfies a stochastic differential equation, of the type first considered by Evans and Hudson [9], and to consider the structure maps defining such an equation as the infinitesimal characteristics of a flow.

Our approach is different: to every strongly continuous flow on a C^*-algebra $\mathscr{B}_S$ we associate a completely positive (but not identity preserving) C_0-semigroup on the 2×2 matrices with coefficients in $\mathscr{B}_S$. This gives four C_0-semigroups on $\mathscr{B}_S$: the infinitesimal characteristics of the flow are the generators of these semigroups.

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If the flow satisfies a stochastic differential equation, of Evans–Hudson type, then it is easy to express their structure maps in terms of our generators and conversely. However, our generators always exist, while the existence of the structure maps is constrained by strong analytical conditions that are neither easy nor natural to formulate in terms of the flow itself.

The structure of the present paper is the following. In Section 2 we remind some general properties of Markov flows. Starting from Section 3 we specialize our context to flows on the boson Fock space $\Gamma(L^2(R,\mathscr{K}))$ and we prove that such a flow is uniquely determined by a family $(P^{s,t}_{f,g})$ of C_0-evolutions, indexed by pairs of elements f,g in a totalizing set of $L^2(R,\mathscr{K})$ (cf. Definition 4 and Theorem 11 below). In Section 4, we show that the evolutions $(P^{s,t}_{f,g})$ reduce to semigroups $(P^{t}_{f,g})$ in the case of covariant flows. In Section 5, using a known result on exponential vectors, we show that the family $(P^{t}_{f,g})$ of semigroups can in fact be reduced to four semigroups which allows us to define a single completely positive C_0-semigroup on the algebra $M(2,\mathscr{B}_S)$ of 2×2 matrices with coefficients in $\mathscr{B}_S$. In Section 6, we further specialize to the class of flows satisfying a stochastic differential equation and prove a general existence theorem applicable to infinite lattice spin systems, whose analysis [5] motivated the present paper.

2. Markov flows

In this section, we recall some general properties of Markov flows. For more information we refer to [3,4].

Definition 1. Let $\mathscr{A}$ be a C^*-algebra. A localization in $\mathscr{A}$, based on the closed intervals of R is a two-parameter family $\mathscr{A}_{[s,t]}$ of subalgebras of $\mathscr{A}$, such that

$$[s,t]\subseteq[s',t'] \;\Rightarrow\; \mathscr{A}_{[s,t]}\subseteq\mathscr{A}_{[s',t']}. \tag{1}$$

We also require that $\bigcup_{[s,t]}\mathscr{A}_{[s,t]}$ is dense in $\mathscr{A}$ and we define $\mathscr{A}_{t]}$ as a norm closure of $\bigcup_{s\leqslant t}\mathscr{A}_{[s,t]}$.

The localization is called expected if, for each $\mathscr{A}_{t]}$ there is a completely positive norm 1 projection (*Umegaki conditional expectation*), denoted $E_{t]}$ from $\mathscr{A}$ onto $\mathscr{A}_{t]}$ satisfying for any $r\leqslant s<t\leqslant u$

$$E_{s]}E_{t]}=E_{s]} \quad \text{(projectivity)}$$

It is called *Markovian* if

$$E_{t]}\mathscr{A}_{[t,+\infty)}\subseteq\mathscr{A}_t,$$

where $\mathscr{A}_t:=\mathscr{A}_{[t,t]}$.

Definition 2. Let $\mathscr{A}$ be a C^*-algebra with a localization $\mathscr{A}_{[s,t]}$, based on the closed intervals of R and satisfying (1). A two-parameter family $j_{s,t}$ $(s\leqslant t)$ of maps of $\mathscr{A}$ into itself satisfying, for every $r\leqslant s\leqslant t$, the conditions

$$j_{r,s}\circ j_{s,t}=j_{r,t} \tag{2}$$

will be called a right flow (or multiplicative functional) on $\mathscr{A}$.

If the flow satisfies the identity $j_{s,t}(1)=1$ $\forall s,t$, then it is called unital (or conservative). We will consider in this paper only conservative flows. If the $j_{s,t}$ are completely positive identity preserving and

$$E_{t]}\circ j_{s,t}=j_{s,t}\circ E_{t]},\quad j_{s,t}(\mathscr{A}_{[r,u]})\subseteq\mathscr{A}_{[r,u]}, \tag{3}$$

for $r\leqslant s<t\leqslant u$, then the flow is called *Markovian*, or a Markov flow. If on $\mathscr{A}$ there is a *time shift*, i.e., a one-parameter semigroup u^0_t $(t\geqslant 0)$ of left invertible $*$-endomorphisms of $\mathscr{A}$ satisfying

$$u^0_r\mathscr{A}_{[s,t]}=\mathscr{A}_{[s+r,t+r]},\quad u^0_rE_{t]}=E_{r+t]}u^0_r$$

and the flow satisfies the condition

$$u^0_r\circ j_{s,t}=j_{s+r,t+r}\circ u^0_r \tag{4}$$

for all r,s,t, then it will be called *covariant*. In this case the one-parameter family

$$j_t:=j_{0,t} \tag{5}$$

satisfies the condition

$$j_{t+s} \circ u_s^0 = j_s \circ u_s^0 \circ j_t, \tag{6}$$

which shows that j_t is a right (u_t^0)-*cocycle* (more precisely a right (u_t^0)-1-*cocycle*). For this reason a covariant right Markov flow is also called a *right Markov cocycle*. The quantum Feynman–Kac formula of [1] states that, for any Markov flow $j_{s,t}$ and for any $s \leqslant t$, the two-parameter family

$$E_{s]} \circ j_{s,t} \circ u_t^0 =: P^{s,t}$$

is a *Markov evolution* on $\mathscr{A}$, i.e., the $P^{s,t}$ are completely positive identity preserving maps of $\mathscr{A}$ into itself satisfying

$$P^{r,s}P^{s,t} = P^{r,t}, \quad r \leqslant s \leqslant t.$$

If the flow $j_{s,t}$ is covariant, then

$$P^{s,t} = P^{0,t-s} =: P^{t-s}, \quad s \leqslant t,$$

and the one-parameter family P^t is a *Markov semigroup* on $\mathscr{A}$.

If all the algebras $\mathscr{A}_t$ are isomorphic to a single algebra $\mathscr{B}_S$, and this always happens in the covariant case, then the evolutions $P^{s,t}$ (respectively the semigroup P^t) can all be realized as maps of $\mathscr{B}_S$ into itself. In this case, the term *flow* also for the maps $j_{s,t} : \mathscr{B}_S \to \mathscr{A}$. If $\mathscr{A} = \mathscr{B}_S \otimes \mathscr{B}(\mathscr{F})$, $\mathscr{F} = \Gamma(L^2(R, \mathscr{K}))$, which will be the only case considered in our paper starting from Section 3 on, there is a well-known technique to give a meaning to the flow equation also in this case [2,3]. This technique is discussed in Lemma 10 below (cf. formula (25)), and includes the extension of the map $j_{r,s} : \mathscr{B}_S \to \mathscr{B}_S \otimes \mathscr{B}(\mathscr{F}_{[s,t]})$ to a map from $\mathscr{B}_S \otimes \mathscr{B}(\mathscr{F}_{[s,t]})$ to $\mathscr{B}(\mathscr{H}_S \otimes \mathscr{F}_{[r,t]})$ through the prescription

$$j_{r,s}(x \otimes X_{s,t}) = j_{r,s}(x) \otimes X_{s,t} \equiv j_{r,s}(x)X_{s,t}$$

for any $x \in \mathscr{B}_S$ and $X_{s,t} \in \mathscr{B}(\mathscr{F}_{[s,t]})$.

3. Evolutions associated to Markov flows

In the present paper, we consider the flows $j_{s,t}$, where $j_{s,t}$ are completely positive maps from the C^*-algebra $\mathscr{B}_S = \mathscr{B}(\mathscr{H}_S)$ of all the bounded operators in the Hilbert space $\mathscr{H}_S$, called the system space, with values in the bounded operators in the Hilbert space $\mathscr{H}_S \otimes \mathscr{F}$, where $\mathscr{F} = \Gamma(L^2(R, \mathscr{K}))$ is a Bose Fock space (the reservoir space in physical terminology).

For $f \in L^2(R, \mathscr{K})$ the exponential vector ψ_f is defined by

$$\psi_f = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} f^{\otimes k}$$

and enjoys the factorization property

$$\psi_f = \psi_{f_{t]}} \otimes \psi_{f_{[t}},$$

where $f_{[t,s]} = \chi[t,s]f$ and similarly for $f_{t]}$, $f_{[t}$.

Lemma 3. *Let $j_{s,t}$ be a Markov flow and, for any pair f, g of test functions in $L^2(R, \mathscr{K})$ and $s, t \in R$, $s \leqslant t$, define*

$$P_{f,g}^{s,t}(x) := \langle \psi_{f_{[s,t]}}, j_{s,t}(x)\psi_{g_{[s,t]}} \rangle, \quad x \in \mathscr{B}_S. \tag{7}$$

Then each $P_{f,g}^{s,t}$ is a linear map of $\mathscr{B}_S$ into itself and

$$P_{f,g}^{r,t} = P_{f,g}^{r,s}P_{f,g}^{s,t}, \quad r < s < t. \tag{8}$$

Proof. Clearly the $P_{f,g}^{s,t}$ map $\mathscr{B}_S$ into itself. Moreover, in the above notations the factorization identity for exponential vectors and the flow (2) imply that for $r < s < t$ and $x \in \mathscr{B}_S$ one has

$$P_{f,g}^{r,t}(x) = \left\langle \psi_{f_{[r,t]}}, j_{r,t}(x)\psi_{g_{[r,t]}} \right\rangle = \left\langle \psi_{f_{[r,t]}}, j_{r,s}j_{s,t}(x)\psi_{g_{[r,t]}} \right\rangle = \left\langle \psi_{f_{[r,s]}}, j_{r,s}\left(\left\langle \psi_{f_{[s,t]}}, j_{s,t}(x)\psi_{g_{[s,t]}} \right\rangle \right)\psi_{g_{[r,s]}} \right\rangle = P_{f,g}^{r,s}P_{f,g}^{s,t}(x).$$

Our goal is to reconstruct the flow in terms of the evolutions $(P_{f,g}^{s,t})$ when f, g vary in a suitably chosen set of test functions. We shall see that, in the covariant case and for suitably choosen test functions, the evolutions $(P_{f,g}^{s,t})$ are in fact semigroups. This will allow us to reduce the theory of flows to the highly developed theory of semigroups.

The semigroups $(P^t_{f,g})$ were first introduced by Fagnola and Sinha [10] and the papers were extensively used by Lindsay and Parthasarathy [12] and Lindsay and Wills [13,14] who gave a new proof, different from Belavkin's original one [6], of the characterization, in terms of structure maps, of completely positive flows, satisfying a stochastic differential equation. The present paper goes in a different direction, its main goal being to provide a new infinitesimal characterization of quantum flows which does not rely on the assumption that the flow satisfies a stochastic equation. Some recent results of Skeide [22] suggest that most of the results of the present paper, at least up to Section 5 included, should continue to hold in the more general framework of tensor product systems of Hilbert modules. □

Definition 4. A set $\mathscr{S}_0 \subseteq L^2(R)$ such that the exponential vectors $\{\psi_f : f \in \mathscr{S}_0\}$ are total in $\mathscr{F} = \Gamma(L^2(R;\mathscr{K}))$ is called totalizing.

The following lemma extends a well-known property of exponential vectors.

Lemma 5. *Let $\mathscr{S}_0 \subseteq L^2(R)$ be a totalizing set. Then the set of all linear combinations of the form*

$$\sum_{\alpha\in F} \xi_\alpha \otimes \psi_{f_\alpha} = \psi \tag{9}$$

with F a finite set, $\xi_\alpha \in \mathscr{H}_S$ and $f_\alpha \in \mathscr{S}_0$ is a dense subspace of $\mathscr{H}_S \otimes \mathscr{F}$. Moreover, the representation (9) *of a vector $0 \neq \psi \in \mathscr{H}_S \otimes \mathscr{F}$ is unique if the f_α are mutually different and we agree to eliminate from the summation all the ξ_α which are zero. We shall denote $\mathscr{D}(\mathscr{S}_0)$ the subspace of $\mathscr{H}_S \otimes \mathscr{F}$ of vectors of the form* (9).

Proof. Let $\mathscr{F}_0$ denote the algebraic linear span of the vectors ψ_f with $f \in \mathscr{S}_0$. Then $\mathscr{H}_S \otimes \mathscr{F}_0$ is a dense subspace of $\mathscr{H}_S \otimes \mathscr{F}$ and it is clear that any vector in this subspace can be written in the form (9). Suppose now that

$$\sum_{\alpha\in F} \xi_\alpha \otimes \psi_{f_\alpha} = \sum_{\beta\in G} \theta_\beta \otimes \psi_{g_\beta} \tag{10}$$

are two different representations of a vector $\psi \neq 0$. A vector $\xi \in \mathscr{H}_S$, which is orthogonal to all ξ_α will satisfy

$$\sum_{\beta\in G} \langle\theta_\beta, \xi\rangle \psi_{g_\beta} = 0,$$

so it must be also orthogonal to all the θ_β. Therefore, we can assume that the ξ_α and the θ_β generate the same subspace S_ψ. If ξ is a nonzero vector in this subspace, then we have

$$\sum_{\alpha\in F_\xi} \langle\xi_\alpha, \xi\rangle \psi_{f_\alpha} = \sum_{\beta\in G_\xi} \langle\theta_\beta, \xi\rangle \psi_{g_\beta}, \tag{11}$$

where $\emptyset \neq F_\xi$ is the set of indices α such that $\langle\xi_\alpha, \xi\rangle \neq 0$ and similarly for G_ξ. The linear independence of the exponential vectors then implies that identity (11) is possible only if the cardinality of F_ξ is equal to that of G_ξ. So up to relabeling the indices we can assume that $F_\xi = G_\xi$. In this case we must have $\forall \alpha \in F_\xi = G_\xi$

$$f_\alpha = g_\alpha, \quad \langle\xi_\alpha, \xi\rangle = \langle\theta_\alpha, \xi\rangle. \tag{12}$$

Since this must be true for all vectors ξ in the subspace S_ψ, it follows that

$$\xi_\alpha = \theta_\alpha \quad \forall \alpha \in F_\xi. \tag{13}$$

Therefore, identity (10) is equivalent to

$$\sum_{\alpha\in F\setminus F_\xi} \xi_\alpha \otimes \psi_{f_\alpha} = \sum_{\beta\in G\setminus F_\xi} \theta_\beta \otimes \psi_{g_\beta}.$$

Since F and G are finite sets, iterating this argument we see that they must have the same cardinality and (up to relabeling) (12) must hold for all indices α. □

Corollary 6. *Let $\mathscr{S}_0$ be as in Lemma* 5. *Then any bounded operator $X \in \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})$ is uniquely determined by the $\mathscr{B}(\mathscr{H}_S)$-valued matrix elements*

$$\langle\psi_f, X\psi_g\rangle, \quad f, g \in \mathscr{S}_0. \tag{14}$$

Proof. Let $n \in N$, $\xi_1, \ldots, \xi_n \in \mathscr{H}_S$, $f_1, \ldots, f_n \in \mathscr{S}_0$. Then

$$\left\langle \sum_\alpha \xi_\alpha \otimes \psi_{f_\alpha}, X \sum_\alpha \xi_\alpha \otimes \psi_{f_\alpha} \right\rangle = \sum_{\alpha,\beta} \left\langle \xi_\alpha, \left\langle \psi_{f_\alpha}, X\psi_{f_\beta} \right\rangle \xi_\beta \right\rangle.$$

So the matrix elements (14) allow us to define all the matrix elements $\langle \psi, X\psi \rangle$ for $\psi \in \mathscr{D}(\mathscr{S}_0)$ hence, by polarization, all the matrix elements $\langle \psi, X\psi' \rangle$ with ψ, $\psi' \in \mathscr{D}(\mathscr{S}_0)$. Since $\mathscr{D}(\mathscr{S}_0)$ is a dense subspace and X is bounded, these matrix elements determine X uniquely. □

Definition 7. Let $\mathscr{S}_0$ be a set and $\mathscr{A}$, $\mathscr{B}$ C^*-algebra. A *completely positive kernel* from $\mathscr{A}$ to $\mathscr{B}$, based on $\mathscr{S}_0$ is a family

$$\{P_{f,g} : f, g \in \mathscr{S}_0\} \tag{15}$$

of C-linear maps $P_{f,g} : \mathscr{A} \to \mathscr{B}$ such that for any $n \in N$, any $f_1, \ldots, f_n \in \mathscr{S}_0$, and any $b_1, \ldots, b_n \in \mathscr{B}$, the map

$$x \in \mathscr{A} \mapsto \sum_{j,k=1}^{n} b_j^* P_{f_j,f_k}(x) b_k \in \mathscr{B} \tag{16}$$

is completely positive. If the maps $P_{f,g}$ are maps from $\mathscr{B}$ into itself, then we speak of a completely positive kernel on $\mathscr{B}$.

Theorem 8. *Let $\mathscr{B}_S = \mathscr{B}(\mathscr{H}_S)$ be a C^*-algebra, $\mathscr{S}_0 \subseteq L^2(R)$ a totalizing set and*

$$P_{f,g} : \mathscr{B}_S \to \mathscr{B}_S; \quad f, g \in \mathscr{S}_0$$

a family of linear maps. Then the following are equivalent:

(i) *There exists a completely positive map $j : \mathscr{B}_S \to \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})$ such that*

$$j(1) = 1, \tag{17}$$

$$P_{f,g}(x) = \langle \psi_f, j(x)\psi_g \rangle \quad \forall f, g \in \mathscr{S}_0, \ \forall x \in \mathscr{B}_S. \tag{18}$$

(ii) *The family $\{P_{f,g} : f, g \in \mathscr{S}_0\}$ is a completely positive kernel on $\mathscr{B}_S$ based on $\mathscr{S}_0$ with the property that*

$$P_{f,g}(1) = \mathrm{e}^{\langle f,g \rangle}. \tag{19}$$

Proof. (i) $\Rightarrow$ (ii) Let n, b_j, f_j be as in (16). Then for any $\xi \in \mathscr{H}_S$ and $x \in \mathscr{B}_S$

$$\left\langle \xi, \sum_{j,k} b_j^* P_{f_j,f_k}(x) b_k \xi \right\rangle = \sum_{j,k} \left\langle b_j\xi, \left\langle \psi_{f_j}, j(x)\psi_{f_k} \right\rangle b_k\xi \right\rangle = \left\langle \sum_j b_j\xi \otimes \psi_{f_j}, j(x) \sum_k b_k\xi \otimes \psi_{f_k} \right\rangle. \tag{20}$$

and, as a function of x, the right-hand side of (20) is completely positive because j has this property. From (18) with $n = 1$, $b = 1_{\mathscr{B}}$ using $j(1) = 1$, we obtain (19).

(ii) $\Rightarrow$ (i). For any $x \in \mathscr{B}_S^+$ we define a quadratic form $q_x(\psi, \psi)$ on the space $\mathscr{D}(\mathscr{S}_0)$ by

$$q_x(\psi, \psi) := \sum_{\alpha,\beta} \left\langle \xi_\alpha, P_{f_\alpha,f_\beta}(x) \xi_\beta \right\rangle, \tag{21}$$

where ψ is a vector of the form (9). The complete positivity property of (16) implies that, for a positive $x \in \mathscr{B}_S$

$$q_x(\psi, \psi) \leqslant \|x\| q_1(\psi, \psi) = \|x\| \sum_{\alpha,\beta} \langle \xi_\alpha, \xi_\beta \rangle \mathrm{e}^{\langle f_\alpha, f_\beta \rangle} = \|x\| \left\| \sum \xi_\alpha \otimes \psi_{f_\alpha} \right\|^2 = \|x\| \|\psi\|^2.$$

This implies that the sesquilinear form, defined by $q_x(\cdot, \cdot)$ through polarization

$$q_x(\psi, \psi') = \sum_{n=0}^{3} i^n q_x(\psi' + i^n\psi, \psi' + i^n\psi) \tag{22}$$

is continuous and therefore there exists a unique bounded positive operator $j(x)$ such that

$$q_x(\psi, \psi) = \langle \psi, j(x)\psi \rangle \quad \forall \psi \in \mathscr{D}(\mathscr{S}_0)$$

and this proves (18).

The map $x \in \mathscr{B}_S^+ \mapsto j(x)$ is extended to all $\mathscr{B}_S$ by complex linearity. We want to prove that this map is completely positive. To this goal it is sufficient to show that for any vector $\psi \in \mathscr{H}_S \otimes \mathscr{F}$, for any $n \in N$ and for any $a_1, \ldots, a_n \in \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})$ and $x_1, \ldots, x_n \in \mathscr{B}_S$, one has

$$\sum_{j,k=1}^{n} \left\langle \psi, a_j^* j(x_j^* x_k) a_k \psi \right\rangle = \sum_{j,k=1}^{n} \left\langle a_j \psi, j(x_j^* x_k) a_k \psi \right\rangle \geqslant 0. \tag{23}$$

Since the vectors of the form (9) are dense in $\mathscr{H}_S \otimes \mathscr{F}$, for each $j = 1, \ldots, n$ there is a sequence $(\sum_{\alpha \in F_{j,m}} \xi_{j,\alpha}^{(m)} \otimes \psi_{f_{j,\alpha}^{(m)}})_m$, where $F_{j,m}$ is a finite set, $\xi_{j,\alpha}^{(m)} \in \mathscr{H}_S$ and $f_{j,\alpha}^{(m)} \in \mathscr{S}_0$, such that

$$a_j \psi = \lim_{m \to +\infty} \sum_{\alpha \in F_{j,m}} \xi_{j,\alpha}^{(m)} \otimes \psi_{f_{j,\alpha}^{(m)}}.$$

Moreover, since j runs over the finite set $1, \ldots, n < +\infty$, then, possibly by defining some $\xi_{j,\alpha}^{(m)}$ to be equal to zero, we can suppose that the index set $F_{j,m}$ does not depend on j, i.e.,

$$F_{j,m} = F_m \text{ (finite set)} \quad \forall j = 1, \ldots, n, \ \forall m.$$

With this convention, the left-hand side of (23) is equal to

$$\lim_{m \to \infty} \sum_{j,k=1}^{n} \sum_{\alpha \in F_m} \sum_{\beta \in F_m} \left\langle \xi_{j,\alpha}^{(m)} \otimes \psi_{f_{j,\alpha}^{(m)}}, j_{s,t}(x_j^* x_k) \xi_{k,\beta}^{(m)} \otimes \psi_{f_{k,\beta}^{(m)}} \right\rangle = \lim_{m \to \infty} \sum_{(j,\alpha) \in \{1,\ldots,n\} \times F_m} \sum_{(k,\beta) \in \{1,\ldots,n\} \times F_m} \left\langle \xi_{j,\alpha}^{(m)}, P_{f_{j,\alpha}^{(m)}, f_{k,\beta}^{(m)}}(x_j^* x_k) \xi_{k,\beta}^{(m)} \right\rangle,$$

which is $\geqslant 0$ because of the complete positivity property (16).

Finally, having proved (23), (18) follows from (21) and Corollary 6. □

Corollary 9. *Let $s, t \in R$ with $s < t$ and let $\mathscr{S}_{0,[s,t]} \subseteq L^2(R)$ be a set of functions with support in $[s,t]$ and totalizing for $\mathscr{F}_{[s,t]}$. Let $\mathscr{B}_S$ and $\{P_{f,g} : f, g \in \mathscr{S}_{0,[s,t]}\}$ be as in Theorem 8. Then the following are equivalent:*

(i) *There exists a completely positive map $j_{s,t} : \mathscr{B}_S \to \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F}_{[s,t]})$ such that* (17) *and* (18) *hold.*

(ii) *Condition* (ii) *of Theorem* 8 *is satisfied for all $f, g \in \mathscr{S}_{0,[s,t]}$.*

Proof. This is obtained from the proof of Theorem 8 replacing everywhere $\mathscr{F}$ by $\mathscr{F}_{[s,t]}$. □

A known theorem by Schur states that, if $a = (a_{ij})$ and $b = (b_{ij})$ are positive definite matrices, then their pointwise product $c_{ij} = a_{ij} b_{ij}$ is positive definite. The following is a generalization of this result to completely positive kernels.

Lemma 10. *In the notations and assumptions of Theorem* 8, *let $(Q_{f,g})$ and $(P_{f,g})$ be completely positive kernels on $\mathscr{B}_S$ based on $\chi_{[r,s]}\mathscr{S}_0$ and $\chi_{[s,t]}\mathscr{S}_0$, respectively, we use the convention*

$$Q_{f,g} = Q_{\chi_{[r,s]}f, \chi_{[r,s]}g}, \qquad P_{f,g} = P_{\chi_{[s,t]}f, \chi_{[s,t]}g}.$$

Then their product

$$Q_{f,g} P_{f,g}, \quad f, g \in \mathscr{S}_0, \tag{24}$$

meant in the sense of composition of maps from $\mathscr{B}_S$ to $\mathscr{B}_S$ is also a completely positive kernel.

Proof. Fix $r, s, t \in R$, with $r < s < t$ and let $j_{r,s}$ denote the completely positive map from $\mathscr{B}_S$ to $\mathscr{B}(\mathscr{H}_S \otimes \mathscr{F}_{[r,s]})$ associated to the completely positive kernel $(Q_{f,g})$ with $f, g \in \chi_{[r,s]}\mathscr{S}_0$. Let $j_{s,t}$ denote the completely positive map from $\mathscr{B}_S$ to $\mathscr{B}(\mathscr{H}_S \otimes \mathscr{F}_{[s,t]})$ obtained in the same way from $(P_{f,g})$, with $f, g \in \chi_{[s,t]}\mathscr{S}_0$. Now we extend the map $j_{r,s}$ to a map from $\mathscr{B}_S \otimes \mathscr{B}(\mathscr{F}_{[s,t]})$ to $\mathscr{B}(\mathscr{H}_S \otimes \mathscr{F}_{[r,t]})$ by the prescription

$$j_{r,s}(x \otimes X_{s,t}) = j_{r,s}(x) \otimes X_{s,t} \equiv j_{r,s}(x) X_{s,t} \tag{25}$$

for any $x \in \mathscr{B}_S$ and $X_{s,t} \in \mathscr{B}(\mathscr{F}_{[s,t]})$. This extension is completely positive being identified to $j_{r,s} \otimes id_{s,t}$. Notice that (25) implies that

$$\langle \psi_f, j_{r,s}(A_{s,t}) \psi_g \rangle := \left\langle \psi_{f_{[r,s]}}, j_{r,s}\left(\left\langle \psi_{f_{[s,t]}}, A_{s,t} \psi_{g_{[s,t]}} \right\rangle \right) \psi_{g_{[r,s]}} \right\rangle \tag{26}$$

for any operator $A_{s,t} \in \mathscr{B}_S \otimes \mathscr{B}(\mathscr{F}_{[s,t]})$ and for any $f, g \in \chi_{[r,t]}\mathscr{S}_0$.

Using (25) and (26) we obtain, for any $x \in \mathscr{B}_S$ and $f, g \in \chi_{[r,t]}\mathscr{S}_0$

$$\langle \psi_f, j_{r,s} j_{s,t}(x)\psi_g \rangle = \left\langle \psi_{f_{[r,s]}}, j_{r,s}\left(\left\langle \psi_{f_{[s,t]}}, j_{s,t}(x)\psi_{g_{[s,t]}} \right\rangle\right)\psi_{g_{[r,s]}} \right\rangle = Q_{f,g}P_{f,g}(x). \tag{27}$$

But $j_{r,s} \circ j_{s,t} : \mathscr{B}_S \to \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F}_{[r,t]})$ is a completely positive map and therefore the left-hand side of (27) is a completely positive kernel on $\mathscr{B}_S$ based on $\chi_{[r,t]}\mathscr{S}_0$. This proves the lemma. □

Theorem 11. *Let $j_{s,t}$ be a Markov flow on C^*-algebra $\mathscr{A} = \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})$ with the localization $\mathscr{A}_{[s,t]} = \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F}_{[s,t]})$ and let $\mathscr{S}_0 \subseteq L^2(R)$ be a totalizing set. Then the family*

$$\left\{P^{s,t}_{f,g} : s, t \in R;\ s < t;\ f, g \in \mathscr{S}_0\right\} \tag{28}$$

defined by (7) *has the following properties*:

(i) *for any $f, g \in \mathscr{S}_0$ and for any $r, s, t \in R$ with $r < s < t$, $P^{s,t}_{f,g}$ is a linear map of $\mathscr{B}_S = \mathscr{B}(\mathscr{H}_S)$ into itself and*

$$P^{r,t}_{f,g} = P^{r,s}_{f,g}P^{s,t}_{f,g} \tag{29}$$

(ii) *for any $f, g \in \mathscr{S}_0$ and $s, t \in R$, $s < t$*

$$P^{s,t}_{f,g}(1) = \mathrm{e}^{\langle \chi_{[s,t]}f, \chi_{[s,t]}g\rangle} \tag{30}$$

(iii) *for each $s, t \in R$ with $s < t$, the family $\{P^{s,t}_{f,g} : f, g \in \mathscr{S}_0\}$ is a completely positive kernel on $\mathscr{B}_S$ based on $\mathscr{S}_0$.*

Conversely, given a family of the form (28) *satisfying* (i)–(iii), *there exists a conservative Markov flow $j_{s,t}$* on *$\mathscr{A}$ such that each $P^{s,t}_{f,g}$ is given by formula* (7).

Proof. *Necessity*. If $j_{s,t}$ is a Markov flow, then properties (i) and (ii) follow from Lemma 3 and property (iii) from Theorem 8.

Sufficiency. Let $(P^{s,t}_{f,g})$ be a family satisfying (i)–(iii). Then, for any $f, g \in \mathscr{S}_0$ and $s, t \in R$, $s < t$, we know from Corollary 9 that there exists a linear, completely positive, identity preserving map

$$j_{s,t} : \mathscr{B}_S \to \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F}_{[s,t]})$$

characterized by

$$\langle \psi_f, j_{s,t}(x)\psi_g \rangle = P^{s,t}_{f,g}(x) \quad \forall x \in \mathscr{B}_S$$

for any $f, g \in \mathscr{S}_0$ with $\operatorname{supp} f$, $\operatorname{supp} g \subseteq [s,t]$. Each $j_{s,t}(x)$ $(x \in \mathscr{B}_S)$ is then uniquely extended to an operator in $\mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})$, still denoted with the same symbol, by the prescription

$$\langle \psi_f, j_{s,t}(x)\psi_g \rangle := \left\langle \psi_{\chi_{[s,t]^c}f}, \psi_{\chi_{[s,t]^c}g} \right\rangle P^{s,t}_{f,g}(x), \tag{31}$$

where $\chi_{[s,t]^c} = 1 - \chi_{[s,t]}$. We now extend the map

$$j_{s,t} : \mathscr{B}_S \to \mathscr{B}\big(\mathscr{H}_S \otimes \mathscr{F}_{[s,t]}\big)$$

to a map

$$j_{s,t} : \mathscr{B}_S \otimes \mathscr{B}(\mathscr{F}_{[t}) \to \mathscr{B}\big(\mathscr{H}_S \otimes \mathscr{F}_{[s}\big)$$

by the prescription

$$\langle \psi_f, j_{s,t}(X_{[t})\psi_f \rangle := \left\langle \psi_{f_{t]}}, j_{s,t}\left(\left\langle \psi_{f_{[t}}, X_{[t}\psi_{g_{[t}} \right\rangle\right)\psi_{g_{t]}} \right\rangle$$

for any $f, g \in \mathscr{D}_0$ and any operator $X_{[t}$ in $\mathscr{B}(\mathscr{H}_S \otimes \mathscr{F}_{[t})$. With this prescription it makes sense to speak of $j_{r,s}j_{s,t}(x)$, for $x \in \mathscr{B}_S$ and $r < s < t$. Moreover one has, for any $f, g \in \mathscr{D}_0$

$$\begin{aligned}\langle \psi_f, j_{r,s}j_{s,t}(x), \psi_g \rangle &= \left\langle \psi_{f_{s]}}, j_{r,s}\left(\left\langle \psi_{f[s,t]}, j_{s,t}(x)\psi_{g_{[s,t]}} \right\rangle\right)\psi_{g_{s]}} \right\rangle = \left\langle \psi_{f_{[r,s]}}, j_{r,s}\left(P^{s,t}_{f,g}(x)\right)\psi_{g_{[r,s]}} \right\rangle \left\langle \psi_{\chi_{[r}f}, \psi_{\chi_{[r}g} \right\rangle \\ &= P^{r,s}_{f,g}P^{s,t}_{f,g}(x)\left\langle \psi_{\chi^c_{[r,t]}f}, \psi_{\chi_{[r,t]^c}g} \right\rangle = P^{r,t}_{f,g}(x)\left\langle \psi_{\chi_{[r,t]^c}f}, \psi_{\chi_{[r,t]^c}g} \right\rangle = \langle \psi_f, j_{r,t}(x)\psi_g \rangle.\end{aligned}$$

Therefore, $j_{s,t}$ is a right multiplicative functional.

Because of (31) $j_{s,t}$ is localized in $[s,t]$, therefore $j_{s,t}$ is a Markov flow. □

4. Semigroups associated to Markovian cocycles

Lemma 12. *Suppose that $f, g \in L^2(R, \mathscr{K})$ assume a constant value, in the interval $[s,t]$, equal respectively to f_0 and g_0 (vectors in $\mathscr{K}$). Then if $j_{s,t}$ is a covariant Markovian cocycle, for any $x \in \mathscr{B}_S$, one has*

$$\left\langle \psi_{f_{[s,t]}}, j_{s,t}(x)\psi_{g_{[s,t]}} \right\rangle = P^{s,t}_{f,g}(x) = \left\langle \psi_{\chi_{[0,t-s]}f_0}, j_{0,t-s}(x)\psi_{\chi_{[0,t-s]}g_0} \right\rangle. \tag{32}$$

Proof. The covariance condition (4) implies that

$$\left\langle \psi_{f_{[s,t]}}, j_{s,t}(x)\psi_{g_{[s,t]}} \right\rangle = \left\langle \psi_{f_{[s,t]}}, u^0_s j_{0,t-s}(x)\psi_{g_{[s,t]}} \right\rangle$$

using the explicit form of u^0_t

$$u^0_s(x) = \Gamma(S_s)(x)\Gamma(S_s)^*,$$

where S_s is the shift in $L^2(R, \mathscr{K})$, defined by

$$S_s f(\tau) = f(\tau - s)$$

this becomes

$$\left\langle \psi_{f_{[s,t]}}, \Gamma(S_s) j_{0,t-s}(x)\Gamma(S_s)^*\psi_{g_{[s,t]}} \right\rangle = \left\langle \Gamma(S^*_s)\psi_{f_{[s,t]}}, j_{0,t-s}(x)\Gamma(S^*_s)\psi_{g_{[s,t]}} \right\rangle = \left\langle \psi_{S_{-s}f_{[s,t]}}, j_{0,t-s}(x)\psi_{S_{-s}g_{[s,t]}} \right\rangle. \tag{33}$$

Under our assumptions on f, one has, for $\tau \in [0, t-s]$

$$S_{-s}f_{[s,t]}(\tau) = S_{-s}\chi_{[s,t]}f(\tau) = \chi_{[s,t]}(\tau + s)f(\tau + s) = \chi_{[0,t-s]}(\tau_0)f_0.$$

Therefore, the right-hand side of (33) is equal to

$$\left\langle \psi_{f_0\chi_{[0,t-s]}}, j_{0,t-s}(x)\psi_{g_0\chi_{[0,t-s]}} \right\rangle$$

and this proves (32). □

Lemma 13. *Let f, g be as in Lemma 12 and define, for $\tau \in [0, b-a]$ and $x \in \mathscr{B}_S$*

$$P^\tau_{f,g}(x) := \left\langle \psi_{\chi_{[0,\tau]}f_0}, j_{0,\tau}(x)\psi_{\chi_{[0,\tau]}g_0} \right\rangle. \tag{34}$$

Then $P^\tau_{f,g} : \mathscr{B}_S \to \mathscr{B}_S$ is the restriction of a semigroup to the interval $[0, b-a]$, i.e., if $\rho, \sigma \in [0, b-a]$ are such that $\rho + \sigma \in [0, b-a]$, then

$$P^\rho_{f,g}P^\sigma_{f,g} = P^{\rho+\sigma}_{f,g}.$$

Remark 14. If a semigroup (P^t) is defined on an interval $[0, T]$ one can always extend it to $[0, 2T]$ by putting

$$P^{T+s} := P^T P^s$$

therefore, proceeding by induction, one can extend it to the whole of R_+. Clearly if P^t is strongly continuous in $[0, T]$ its extension will have the same property.

Proof. It is convenient to write

$$\rho = s - r, \qquad \sigma = t - s,$$

with $r, s, t \in [a, b]$. Then we have

$$P^{s-r}_{f,g}P^{t-s}_{f,g}(x) = \left\langle \psi_{\chi_{[0,s-r]}f_0}, j_{0,s-r}\left(\left\langle \psi_{\chi_{[0,t-s]}f_0}, j_{0,t-s}(x)\psi_{\chi_{[0,t-s]}g_0} \right\rangle\right)\psi_{\chi_{[0,s-r]}g_0} \right\rangle. \tag{35}$$

Using identity (32) the right-hand side of (35) becomes

$$\left\langle \psi_{\chi_{[a+r,a+s]}f_0}, j_{a+r,a+s}\left(\left\langle \psi_{\chi_{[a+s,a+t]}f_0}, j_{a+s,a+t}(x)\psi_{\chi_{[a+s,a+t]}g_0} \right\rangle\right)\psi_{\chi_{[a+r,a+s]}g_0} \right\rangle$$

and, because of the factorization properties of the exponential vectors, this is equal to

$$\left\langle \psi_{\chi_{[a+r,a+t]}f_0}, j_{a+r,a+s}j_{a+s,a+t}(x)\psi_{\chi_{[a+r,a+t]}g_0} \right\rangle = \left\langle \psi_{\chi_{[a+r,a+t]}f_0}, j_{a+r,a+t}(x)\psi_{\chi_{[a+r,a+t]}g_0} \right\rangle.$$

Using again identity (32), this is equal to

$$\left\langle \psi_{\chi_{[0,t-r]}f_0}, j_{0,t-r}(x)\psi_{\chi_{[0,t-r]}g_0} \right\rangle = P^{t-r}_{f,g}(x)$$

and this ends the proof. □

Lemma 15. *In the notations and assumptions of Lemma* 12, *if the cocycle* $(j_{s,t})$ *is strongly continuous, then the semigroups* $P^{\tau}_{f,g}$, *defined by* (34) *are strongly continuous and one has, denoting* $|f|$ *the norm in* $\mathscr{K}$:

$$\|P^t_{f,g}(x)\| \leqslant e^{(t/2)(|f_0|^2+|g_0|^2)}\|x\|. \tag{36}$$

Proof. From (34) we deduce

$$\|P^t_{f,g}(x)\| \leqslant \|\psi_{\chi_{[0,t]}f_0}\| \cdot \|x\| \cdot \|\psi_{\chi_{[0,t]}g_0}\| = e^{(-t/2)(|f_0|^2+|g_0|^2)}\|x\|.$$

The strong continuity is clear. □

Theorem 16. *Let* $\mathscr{F} = \Gamma(L^2(R))$ *and let* $(j_{s,t})$ *be a Markov flow from* $\mathscr{B}_S$ *to* $\mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})$. *Define, for any* $t \geqslant 0$ *and any pair of vectors* $f_0, g_0 \in \mathscr{K}$ *the map*

$$P^t_{f_0,g_0}(x) := \left\langle \psi_{f_0\chi_{[0,t]}}, j_{0,t}(x)\psi_{g_0\chi_{[0,t]}} \right\rangle, \quad x \in \mathscr{B}_S. \tag{37}$$

Then:

(i) *for any* $t \geqslant 0$, $f_0, g_0 \in \mathscr{K}$, $P^t_{f_0,g_0}$ *is a* C_0*-semigroup from* $\mathscr{B}_S$ *into itself*

(ii) *for* t, f_0, g_0 *as in* (i)

$$P^t_{f_0,g_0}(1) = e^{t\langle f_0,g_0\rangle_{\mathscr{K}}} \tag{38}$$

(iii) *for each* $t \geqslant 0$, *the family*

$$\left\{ P^t_{f_0,g_0} : f_0, g_0 \in \mathscr{K} \right\} \tag{39}$$

is a completely positive kernel from $\mathscr{B}_S$ *to* $\mathscr{B}_S$ *based on* $\mathscr{K}$.

Conversely, given a family of the form (39) *satisfying* (i)–(iii), *there exists a conservative Markov flow from* $\mathscr{B}_S$ *to* $\mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})$ *such that each* $P^t_{f_0,g_0}$ *is given by* (37).

Proof. It is a known and elementary fact that the family $\mathscr{S}_0$ of functions of the form $f_0\chi_{[0,t]}$, with $f_0 \in \mathscr{K}$ and $\chi_{[0,t]}$ a characteristic function of a bounded interval in R, is totalizing for $\mathscr{F}$. With this choice of $\mathscr{S}_0$ we can apply Theorem 11 and conclude that the assignment of a Markov flow is equivalent to the assignment of a family $(P^{s,t}_{f,g})$ with $s, t \in R$, $s \leqslant t$, $f, g \in \mathscr{S}_0$. Now fix, s, t, f, g as above. Then there exists a partition

$$s = t_0 < t_1 < \cdots < t_n < t = t_{n+1}$$

of the interval $[s, t]$ such that f (resp. g) has the constant value $f_j \in C$ (resp. $g_j \in C$) in the interval $[t_j, t_{j+1})$ $(j = 0, \ldots, n)$. Correspondingly, we have

$$P^{s,t}_{f,g} = P^{s,t_1}_{f,g}P^{t_1,t_2}_{f,g} \cdot \ldots \cdot P^{t_n,t}_{f,g} = P^{t_1-s}_{f_0,g_0}P^{t_2-t_1}_{f_1,g_1} \ldots P^{t-t_n}_{f_{n+1},g_{n+1}}. \tag{40}$$

Therefore, the assignment of the family $(P^{s,t}_{f,g})$, or equivalently of the flow, is equivalent to the assignment of the family of semigroups $(P^t_{f_0,g_0})$ $(t \geqslant 0,\ f_0, g_0 \in C)$. Moreover, by the noncommutative Schur Lemma 10, if for each $t \in R_t$ the maps $\{P^t_{f_0,g_0} : f_0, g_0 \in C\}$ are a completely positive kernel, then for each $s \leqslant t$ the maps $\{P^{s,t}_{f,g} : f, g \in \mathscr{S}_0\}$ have the same property. Since the converse is clear because the semigroups $P^t_{f_0,g_0}$ are a subset of the maps $P^{s,t}_{f,g}$, the theorem is proved. □

5. The extended semigroup of a flow

In this section, restrict our considerations to the case $\mathscr{K} = C$, so that $L^2(R, \mathscr{K}) = L^2(R)$. We shall use the following theorem.

Theorem 17. *Let $\mathscr{S}_0 \subseteq L^2(R)$ denote the set of finite sums of characteristic functions over bounded disjoint intervals, i.e.,*

$$\mathscr{S}_0 = \left\{ \sum_{j=1}^{n} \chi_{[a_j,b_j]};\; n \in N,\; a_i, b_j \in R,\; a_j < b_j,\; (a_j, b_j) \cap (a_k, b_k) = \emptyset,\; if \quad j \neq k \right\}. \tag{41}$$

Then the set of exponential vectors with test functions in $\mathscr{S}_0$ are total in $L^2(R)$.

Proof. An elementary proof of this theorem is in [16]. More elaborated proofs are in [8,15]. In this section, the family $\mathscr{S}_0$ will be fixed and given by (41). □

Theorem 18. *Let $\mathscr{F} = \Gamma(L^2(R))$ and let $(j_{s,t})$ be a Markov flow from $\mathscr{B}_S$ to $\mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})$. Let ψ_0 denote the vacuum vector in $\mathscr{F}^0$ (we use the same notation for the vacuum vector in any $\mathscr{F}_{[s,t]}$). Define, for $x \in \mathscr{B}_S$ and $t \geqslant 0$*

$$P^t_{00}(x) := \langle \psi_0, j_{0,t}(x)\psi_0 \rangle, \tag{42}$$

$$P^t_{01}(x) := \left\langle \psi_0, j_{0,t}(x)\psi_{\chi_{[0,t]}} \right\rangle, \tag{43}$$

$$P^t_{10}(x) := \left\langle \psi_{\chi_{[0,t]}}, j_{0,t}(x)\psi_0 \right\rangle, \tag{44}$$

$$P^t_{11}(x) := \left\langle \psi_{\chi_{[0,t]}}, j_{0,t}(x)\psi_{\chi_{[0,t]}} \right\rangle. \tag{45}$$

Then:

(i) *for each $\varepsilon, \varepsilon' \in \{0, 1\}$, $(P^t_{\varepsilon,\varepsilon'})$ is a C_0-semigroup on $\mathscr{B}_S$,*

(ii) *the semigroups $(P^t_{\varepsilon,\varepsilon'})$ satisfy*

$$P^t_{00}(1) = P^t_{10}(1) = P^t_{01}(1) = 1, \tag{46}$$

$$P^t_{11}(1) = e^t, \tag{47}$$

(iii) *the map*

$$x \in \mathscr{B}_S \mapsto \begin{pmatrix} P^t_{00}(x) & P^t_{01}(x) \\ P^t_{10}(x) & P^t_{11}(x) \end{pmatrix} \in M(2, C) \otimes \mathscr{B}_S, \tag{48}$$

where $M(2, C)$ denotes the algebra of 2×2 complex matrices, is completely positive.

Conversely, given four C_0-semigroups $(P^t_{\varepsilon,\varepsilon'})$ $(\varepsilon, \varepsilon' \in \{0, 1\})$ satisfying condition (46)–(48) *then there exists a unique conservative Markov flow on $\mathscr{B}_S \otimes \mathscr{B}(\mathscr{F})$ satisfying conditions* (42)–(45).

Proof. Let $f, g \in \mathscr{S}_0$ and $s, t \in R$, $s < t$. Then there exists a partition

$$s = t_0 < t_1 < \cdots < t_n < t_{n+1} = t \tag{49}$$

such that both f and g are constant in each interval of this partition. In our assumptions this constant value can only be 0 or 1 so, in notation (37), we have only four possibilities:

$$P^t_{00}, P^t_{01}, P^t_{10}, P^t_{11}. \tag{50}$$

In these notations

$$P^{s,t}_{f,g} = P^{t_1,s}_{\varepsilon_0,\delta_0} P^{t_2-t_1}_{\varepsilon_1,\delta_1} \ldots P^{t-t_n}_{\varepsilon_{n+1},\delta_{n+1}}, \tag{51}$$

where $\varepsilon_j, \delta_j \in \{0, 1\}$. Thus $(P^{s,t}_{f,g})$ is uniquely determined by the four semigroups (42)–(45).

According to Definition 7 the complete positivity of the kernel $(P^t_{f_0,g_0})$ means that, for any $n \in N$, $b_1, \ldots, b_n \in \mathscr{B}_S$, $f_1, \ldots, f_n \in C$ the map

$$x \in \mathscr{B}_S \mapsto \sum_{j,k=1}^{n} b_j^* P^t_{f_j,f_k}(x) b_k \tag{52}$$

is completely positive. If the f_j can only take values 0 and 1 we can assume, up to a relabeling of the indices, that

$$f_j = 0 \text{ for } j = 1, \ldots, n_1; \quad f_j = 1 \text{ for } j = n_1 + 1, \ldots, n. \tag{53}$$

With this notation the right-hand side of (52) becomes

$$\sum_{j,k=1}^{n_1} b_j^* P_{00}^t(x) b_k + \sum_{j=1}^{n_1} \sum_{k=n_1+1}^{n} b_j^* P_{01}^t(x) b_k + \sum_{j=n_1+1}^{n} \sum_{k=1}^{n_1} b_j^* P_{10}^t(x) b_k + \sum_{j,k=n_1+1}^{n} b_j^* P_{11}^t(x) b_k$$
$$= c_0^* P_{00}^t(x) c_0 + c_0^* P_{01}^t(x) c_1 + c_1^* P_{10}^t(x) c_0 + c_1^* P_{11}^t(x) c_1,$$

where we have put

$$c_0 := \sum_{j=1}^{n_1} b_j; \qquad c_1 := \sum_{j=n_1+1}^{n} b_j.$$

Since the b_j are arbitrary in $\mathscr{B}_S$, so are c_0, c_1. So, under our assumptions, the complete positivity of the map (52) is equivalent to the complete positivity of the map (48).

Finally from condition (ii) of Theorem 16 one immediately deduces (46) and (47).

Conversely, let be given the four semigroups (42)–(45). Then, for $f, g \in \mathscr{S}_0$ with associated partition (49), we can define $P_{f,g}^{s,t}$ by (11). The evolution property (33) follow immediately from the semigroup property and (51).

Because of complete positivity of (48), $\{P_{f,g}^{s,t} : f, g \in \mathscr{S}_0\}$, defined by (51), is a completely positive kernel on $\mathscr{B}_S$ based on $\mathscr{S}_0$. This finishes the proof of the theorem. □

Formula (48) naturally suggests to study the following semigroup.

Definition 19. The one-parameter semigroup

$$\begin{pmatrix} x_{00} & x_{11} \\ x_{10} & x_{11} \end{pmatrix} \in M(2, C) \otimes \mathscr{B}_S \mapsto \begin{pmatrix} P_{00}^t(x_{00}) & P_{01}^t(x_{01}) \\ P_{10}^t(x_{10}) & P_{11}^t(x_{11}) \end{pmatrix} \in M(2, C) \otimes \mathscr{B}_S \tag{54}$$

will be called *the extended semigroup* of the flow $(j_{s,t})$ and denoted $\tilde{P}^t$.

It is clear that the map (48) is obtained by restriction of the extended semigroup $\tilde{P}^t$ to the subspace of $M_2 \otimes \mathscr{B}_S$ formed by the matrices of the form

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix}, \quad x \in \mathscr{B}_S.$$

Lemma 20. *Let $\mathscr{H}_S$ and $\mathscr{F}$ be Hilbert spaces and for $n \in N$, let be given n vectors $\psi_1, \ldots, \psi_n \in \mathscr{F}$. Then the map*

$$X = (x_{jk}) \in M(n, \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})) \mapsto (\langle \psi_j, x_{jk} \psi_k \rangle) \in M(n, \mathscr{B}(\mathscr{H}_S))$$

is completely positive.

Proof. It is sufficient to prove the positivity of the above map because, since $\mathscr{H}_S$ is arbitrary, we can always replace it by $\mathscr{H}_S \otimes C^k$ $(k \in N)$. Let $x = (x_{jk}) \in M(n, \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F}))$ be positive. We want to show that, for any $c_1, \ldots, c_n \in \mathscr{B}(\mathscr{H}_S)$ the operator

$$\sum_{jk} c_j^* \langle \psi_j, x_{jk} \psi_k \rangle c_k \in \mathscr{B}(\mathscr{H}_S) \tag{55}$$

is positive. Clearly (55) is a self-adjoint element of $\mathscr{B}(\mathscr{H}_S)$ and, if $\xi \in \mathscr{H}_S$ is any vector, then

$$\left\langle \xi, \sum_{jk} c_j^* \langle \psi_j, x_{jk} \psi_k \rangle c_k \xi \right\rangle = \sum_{jk} \left\langle c_j \xi, \langle \psi_j, x_{jk} \psi_k \rangle c_k \xi \right\rangle = \sum_{jk} \left\langle c_j \xi \otimes \psi_j, x_{jk} (c_k \xi \otimes \psi_k) \right\rangle = \langle \varphi, x \varphi \rangle \geqslant 0,$$

where $\varphi := \otimes_{j=1}^n c_j \xi \otimes \psi_j$. □

Theorem 21. *The extended semigroup* $\tilde{P}^t : M(2,\mathscr{B}_S) \to M(2,\mathscr{B}_S)$ *is a completely positive semigroup satisfying the condition*

$$\tilde{P}^t\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & e^t\end{pmatrix}. \tag{56}$$

Conversely any completely positive semigroup $\tilde{P}^t$ *on* $M(2,\mathscr{B}_S)$ *satisfying condition* (56) *is the extended semigroup of a unique Markov flow on* $\mathscr{B}_S$ *which is completely determined by the coefficients of* $\tilde{P}^t$ *via Theorem* 18.

Proof. *Necessity*. The complete positivity of $(j_{s,t})$ implies that the flow

$$j^{(2)}_{s,t} : (x_{ij}) \in M(2,\mathscr{B}_S) \to (j_{s,t}(x_{ij})) \in M(2,\mathscr{B}(\mathscr{H}_S \otimes \mathscr{F}))$$

is completely positive. Since $\tilde{P}^t$ is obtained by composing this flow with the map

$$\begin{pmatrix} j_{0,t}(x_{00}) & j_{0,t}(x_{01})\\ j_{0,t}(x_{10}) & j_{0,t}(x_{11})\end{pmatrix} \mapsto \begin{pmatrix} \langle\psi_0, j_{0,t}(x_{00})\psi_0\rangle & \left\langle\psi_0, j_{0,t}(x_{01})\psi_{\chi_{[0,t]}}\right\rangle\\ \left\langle\psi_{\chi_{[0,t]}}, j_{0,t}(x_{10})\psi_0\right\rangle & \left\langle\psi_{\chi_{[0,t]}}, j_{0,t}(x_{11})\psi_{\chi_{[0,t]}}\right\rangle\end{pmatrix} = \tilde{P}^t((x_{ij}))$$

its complete positivity follows from Lemma 20. The normalization condition (56) follows from conditions (46) and (47) of Theorem 18.

Sufficiency. Let $\tilde{P}^t$ be as in the statement of the theorem and let $(e_{\varepsilon\varepsilon'})(\varepsilon,\varepsilon' = 0,1)$ be a system of matrix units for $M(2,C)$. Define for any $\varepsilon,\varepsilon' = 0,1$ and $x \in \mathscr{B}_S$

$$P^t_{\varepsilon,\varepsilon'}(x) := \tilde{P}^t(x \otimes e_{\varepsilon,\varepsilon'}).$$

Then each $P^t_{\varepsilon,\varepsilon'}$ is a C_0-semigroup on $\mathscr{B}_S$. The complete positivity of $\tilde{P}^t$ and condition (56) imply that all the conditions of Theorem 18 are satisfied. Therefore, the result follows from Theorem 18. □

Remark 22. The above theorem naturally suggests the following interpretation of the extended semigroup as matrix elements of a very weak stochastic equation for the extended flow.

We may say that a Markov flow $j_{s,t} : \mathscr{B}_S \to \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})$ satisfies a *weak stochastic equation* in the time interval $[S,T]$ if, for any subinterval $[s,t] \subseteq [S,T]$ and for any $\varepsilon,\varepsilon' \in \{0,1\}$, there exist dense subspaces $\mathscr{D}_{\varepsilon,\varepsilon'} \subseteq \mathscr{B}_S$ and linear maps

$$\kappa_{\varepsilon,\varepsilon'} : \mathscr{D}_{\varepsilon,\varepsilon'} \to \mathscr{B}_S \tag{57}$$

such that, for any matrix $(x_{\varepsilon,\varepsilon'})_{\varepsilon,\varepsilon'\in\{0,1\}} \in M(2,\mathscr{B}_S)$ with $x_{\varepsilon,\varepsilon'} \in \mathscr{D}_{\varepsilon,\varepsilon'}$, one has

$$\begin{pmatrix} j_{s,t}(x_{00}) & j_{s,t}(x_{01})\\ j_{s,t}(x_{10}) & j_{s,t}(x_{11})\end{pmatrix} = \begin{pmatrix} x_{00} & x_{01}\\ x_{10} & x_{12}\end{pmatrix} + \int_s^t \begin{pmatrix} j_{s,\tau}(\kappa_{00}(x_{00}))\,d\tau & j_{s,\tau}(\kappa_{01}(x_{01}))\,dA_\tau\\ j_{s,\tau}(\kappa_{10}(x_{10}))\,dA^+_\tau & j_{s,\tau}(\kappa_{11}(x_{11}))\,dN_\tau\end{pmatrix}, \tag{58}$$

where the stochastic integrals in (58) are meant componentwise and the identity means that it holds after taking expectation with respect to the $M(2,\mathscr{B}_S)$-valued map

$$\begin{pmatrix} \langle\psi_0, \cdot\psi_0\rangle & \left\langle\psi_0, \cdot\psi_{\chi_{[s,t]}}\right\rangle\\ \left\langle\psi_{\chi_{[s,t]}}, \cdot\psi_0\right\rangle & \left\langle\psi_{\chi_{[s,t]}}, \cdot\psi_{\chi_{[s,t]}}\right\rangle\end{pmatrix}, \tag{59}$$

which is completely positive by Lemma 20.

In this sense, we may say that every covariant Markov flow $j_{s,t} : \mathscr{B}_S \to \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})$ satisfies a weak stochastic equation and the map

$$\kappa := \begin{pmatrix} \kappa_{00} & \kappa_{01}\\ \kappa_{10} & \kappa_{11}\end{pmatrix}$$

is the generator of the extended semigroup associated to the flow.

Conversely if κ is the generator of a completely positive semigroup in $M(2,\mathscr{B}_S)$ then a Markov flow $j_{s,t} : \mathscr{B}_S \to \mathscr{B}(\mathscr{H}_S \otimes \mathscr{F})$ satisfying Eq. (58) exists if and only if the element

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

is in the domain of κ and

$$\kappa\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

6. Existence of dynamics

In Section 5, we have shown that the theory of (nonnecessarily homomorphic) Markov flows can be reduced to the theory of completely positive semigroups on the algebra $M(2,\mathscr{B}_S)$ with an additional normalization condition. In this section, we apply this result to the question of the existence and uniqueness of a homomorphic Markov flow as solution j_t of a quantum stochastic differential equation (QSDE) of the form

$$\mathrm{d}j_t(x) = j_t \circ \sum_\alpha \theta_\alpha(x)\,\mathrm{d}M^\alpha(t) \tag{60}$$

with initial condition

$$j_0(x) = x, \tag{61}$$

where $x \in \mathscr{B}_S$ and the $\mathrm{d}M_\alpha(t)$ are the three standard boson Fock stochastic diffferentials,

$$\mathrm{d}M^{-1}(t) = \mathrm{d}A(t), \quad \mathrm{d}M^1(t) = \mathrm{d}A^\dagger(t), \quad \mathrm{d}M^0(t) = \mathrm{d}t$$

with the Ito table and conjugation rules

$$\mathrm{d}M^{-1}(t)\,\mathrm{d}M^1(t) = \mathrm{d}M^0(t); \quad M^{0*} = M^0, \ M^{1*} = M^{-1}.$$

The inclusion of the number process can be dealt with in a similar techinque, but for the application we have in mind (cf. Section 7 below) it is not necessary. We write the above relations among the stochastic differentials in the compact notations

$$M^{\alpha*} = M^{\bar{\alpha}}, \tag{62}$$

$$\mathrm{d}M^\beta(t)\,\mathrm{d}M^\gamma(t) = \sum_\alpha c_\alpha^{\beta\gamma}\,\mathrm{d}M^\alpha(t), \tag{63}$$

where the $c_\alpha^{\beta\gamma}$ are complex numbers (the structure constants). Since we are interested in homomorphic flows, we assume that the maps θ_α are unital

$$\theta_\alpha(1) = 0 \quad \forall\alpha, \tag{64}$$

symmetric (the conjugation rules for the α's are the same as for stochastic differentials)

$$\theta_\alpha(x^*) = \theta_{\bar{\alpha}}(x)^* \quad \forall\alpha, \ \forall x \in \mathscr{B}_S \tag{65}$$

and satisfy the stochastic Leibnitz rule

$$\theta_\alpha(xy) = \theta_\alpha(x)y + x\theta_\alpha(y) + \sum_{\beta,\gamma} c_\alpha^{\beta\gamma}\theta_\beta(x)\theta_\gamma(y), \tag{66}$$

where the structure constants $c_\alpha^{\beta\gamma}$ are the same as in the Ito table.

Our idea is the following: we express the generator of the extended semigroup $\tilde{P}^t$ in terms of maps θ_α. Then we take advantage of the specific properties of these maps to prove that we are in the conditions to apply the Hille–Yosida theorem giving the existence and uniqueness of the extended semigroup $\tilde{P}^t$ and therefore, because of our results, of the flow itself.

To express the generator of the extended semigroup in terms of the structure maps let us take the expectation of Eq. (60) with respect to the functionals ψ_i, ψ_j, where $\psi_1 = \psi_{\chi[0,T]}$ and the equation is considered in the interval $[0,T]$. This means that we derive the evolution equations for $P_{ij}^{\prime t} = P_{ij}^t$ for $(i,j) = (0,0),(0,1),(1,0)$ and $P_{11}^{\prime t} = P_{11}^t \mathrm{e}^{T-t}$.

Using the factorization property for exponential vectors $\psi_f = \psi_{f_{t]}} \otimes \psi_{f_{[t}}$ we get

$$\left\langle j_t \circ \sum_\alpha \theta_\alpha(x)\,\mathrm{d}M_\alpha(t) \right\rangle_{ij} = \sum_\alpha \langle j_t \circ \theta_\alpha(x)\rangle_{ij} \langle \mathrm{d}M^\alpha(t)\rangle_{ij}, \quad \langle\cdot\rangle_{ij} = \langle \psi_i, \cdot\psi_j\rangle.$$

Finally for P''^t_{ij} we get

$$\frac{\mathrm{d}}{\mathrm{d}t}P''^t_{ij}(x) = P''^t_{ij} \circ \sum_\alpha \mu^\alpha_{ij}(t)\theta_\alpha(x), \quad \mu^\alpha_{ij}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle M^\alpha\rangle_{ij}. \tag{67}$$

The generator L of the semigroup $\tilde{P}^t$ is given by application, to the RHS of Eq. (60), of the matrix expectation $(\langle\cdot\rangle_{ij})$ (where $i,j = 0,1$ $\langle\cdot\rangle_{ij} = \langle\psi_i, \cdot\psi_j\rangle$, and ψ_0, ψ_1 are as above).

Using the known properties of matrix elements of stochastic differentials between exponential vectors one can express L as a function of the structure maps θ_α as follows:

$$L = \sum_\alpha L^\alpha\theta_\alpha, \quad \alpha = -1, 0, 1,$$

where

$$L^0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad L^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad L^1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

More explicitly

$$L = \begin{pmatrix} \theta_0 & \theta_0 + \theta_{-1} \\ \theta_0 + \theta_1 & \theta_0 + \theta_1 + \theta_{-1} \end{pmatrix}, \tag{68}$$

where the matrix L acts elementwise on $M(2, \mathscr{B}_S)$: $(L_{ij})(x_{ij}) = (L_{ij}(x_{ij}))$. For the calculations in the following lemma it is convenient to introduce the following notations: for the action of L^α on a generic 2×2-matrix $x = (x_{ij})$ we use the convenction

$$L^0(x) = x, \quad L^1(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}x, \quad L^{-1}(x) = x\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where the products on the right-hand sides are interpreted as usual matrix multiplications.

The following lemma is important to reduce the problem of existence, uniqueness and conservativity of a flow to an application of the Hille–Yosida theorem.

Lemma 23. *If the structure maps θ_α in* (60) *have a common core $\mathscr{B}_0$ which is a $*$-subalgebra invariant under the square root, then the operator L is symmetric and generates a completely positive semigroup whenever there exists a $\lambda > 0$ such that the range of* 1–λL *is dense.*

Proof. The symmetry is obvious because the complete positivity of the semigroup $\tilde{P}^t$ implies that $\tilde{P}^t(x^*) = \tilde{P}^t(x)^*$.

To prove the existence of the semigroup with the generator L we will use the Hille–Yosida theorem (cf. [7]). To use this theorem we have to prove the dissipativity of the generator.

To prove this we will use that $\theta_{\pm 1}$ are mutually adjoint derivations and θ_0 is a dissipation satisfying the property

$$\theta_0(AB) = \theta_0(A)B + A\theta_0(B) + \theta_{-1}(A)\theta_1(B)$$

that is exactly formula (66). Applying the generator L to 2×2-matrix with the entries in $\mathscr{B}_S$ we get

$$\begin{aligned} L(x^*x) &= (\theta_{-1}(x^*)x + x^*\theta_{-1}(x))\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(\theta_1(x^*)x + x^*\theta_1(x)) + \theta_0(x^*)x + x^*\theta_0(x) + \theta_{-1}(x^*)\theta_1(x) \\ &= L(x^*)x + x^*L(x) + \theta_{-1}(x^*)\theta_1(x) + \theta_{-1}(x^*)\left[x, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, x^*\right]\theta_1(x) \\ &= L(x^*)x + x^*L(x) + \left|\theta_1(x) + \left[x, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]\right|^2 - \left|\left[x, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]\right|^2, \end{aligned} \tag{69}$$

where we use the notation $|A|^2 = A^*A$. The third term in (69) is nonnegative, therefore

$$L(x^*x) \geqslant L(x^*)x + x^*L(x) - |\delta(x)|^2 \tag{70}$$

with δ given by

$$\delta(x) = \mathrm{i}\left[x, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

From the identity

$$\delta^2(x^*x) = \delta(\delta(x^*)x + x^*\delta(x)) = \delta^2(x^*)x + 2\delta(x^*)\delta(x) + x^*\delta^2(x)$$

and from the fact that δ is a symmetric derivation, it follows that

$$|\delta(x)|^2 = \frac{1}{2}\left(\delta^2(x^*x) - \delta^2(x^*)x - x^*\delta^2(x)\right). \tag{71}$$

Given (71), (70) is equivalent to

$$\left(L + \frac{1}{2}\delta^2\right)(x^*x) \geqslant \left(L + \frac{1}{2}\delta^2\right)(x^*)x + x^*\left(L + \frac{1}{2}\delta^2\right)(x). \tag{72}$$

Because all maps θ^α have a common core $\mathscr{B}_0$, that is a $*$-subalgebra, invariant under the square root, according to [7, (3.2.22)] it follows that $L + \frac{1}{2}\delta^2$ is a closable dissipation. By similar arguments one can prove complete dissipativity.

By the Hille–Yosida theorem (cf. [7]) the dissipative operator

$$S = L + \frac{1}{2}\delta^2 = \begin{pmatrix} \theta_0 & \theta_0 + \theta_{-1} - \frac{1}{2} \\ \theta_0 + \theta_1 - \frac{1}{2} & \theta_0 + \theta_{+1} + \theta_{-1} \end{pmatrix}$$

generates a semigroup of contractions P^t on $M(2, \mathscr{B}_S)$.

The proof of this theorem is by the resolvent approximation. In this approximation the semigroup e^{tS} is the strong limit of the semigroups $\mathrm{e}^{tS_\varepsilon}$ with the bounded generator

$$S_\varepsilon = S(1 - \varepsilon S)^{-1} = -\varepsilon^{-1}(1 - (1 - \varepsilon S)^{-1}).$$

But by [11] a bounded symmetric completely dissipative operator on a W^*-algebra generates a completely positive (contractive) semigroup. Since $\mathrm{e}^{tS_\varepsilon}$ converges strongly to e^{tS}, also the semigroup e^{tS} is completely positive.

The operators L and $\frac{1}{2}\delta^2$ commute. Now $\frac{1}{2}\delta^2$ is the generator of the C^0-semigroup

$$\mathrm{e}^{\frac{t}{2}\delta^2}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} & \mathrm{e}^{-\frac{t}{2}}x_{12} \\ \mathrm{e}^{-\frac{t}{2}}x_{21} & x_{22} \end{pmatrix}.$$

We get that

$$\mathrm{e}^{t\left(L+\frac{1}{2}\delta^2\right)}\mathrm{e}^{-\frac{t}{2}\delta^2}$$

is a completely positive semigroup with generator L.

This finishes the proof of the lemma. □

Lemma 24. *The semigroup* $\tilde{P}^t = \mathrm{e}^{tL}$ *satisfies*

$$\mathrm{e}^{tL}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \quad \mathrm{e}^{tL}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \mathrm{e}^t \end{pmatrix}.$$

Proof. The semigroup $\mathrm{e}^{tL_\varepsilon}$ is given by the iterated series

$$\mathrm{e}^{tL_\varepsilon} = \sum_{k=0}^{\infty} \frac{t^k}{k!} L_\varepsilon^k,$$

where the regularization L_ε have the form

$$L_\varepsilon = L(1 - \varepsilon L)^{-1} = -\varepsilon^{-1}(1 - (1 - \varepsilon L)^{-1}).$$

From formula (68) follows that the generator L and therefore its regularization L_ε kills

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(because each of the θ_α kills 1). Therefore from the form of the iterated series it follows that the semigroup e^{tL_ε} is conservative ($e^{tL_\varepsilon}(1) = 1$). Applying to

$$e^{tL_\varepsilon}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

the limit $\varepsilon \to 0$ and using that e^{tL} is the strong limit of e^{tL_ε} we get the conservativity of e^{tL}. Using the rescaling $P'^{t}_{11} = P^{t}_{11}e^{T-t}$ we get the statement of the lemma. □

Theorem 25. *In the assumptions of Lemma* 23, *there exists a unique flow* $j_t(x)$ *whose extended semigroup is* e^{tL}. *Moreover* $j_t(x)$ *is the unique solution of the Cauchy problem* (60) *and* (61).

Proof. Because of Lemma 23, L is the generator of a semigroup e^{tL} satisfying the conditions of Theorem 21. Hence there exists a unique flow $j_t(x)$, whose extended semigroup is e^{tL}. Therefore, if we prove that this flow satisfies Eq. (60) with initial condition (61) then, by the uniqueness of the matrix elements in the exponential vectors, the uniqueness of the solution follows.

First we prove that the flow $j_t(x)$ satisfies (60) and (61). To this goal it is sufficient to prove that for a dense set of vectors ψ, ψ' the matrix element $\langle\psi, j_t(x)\psi'\rangle$ satisfies the equation obtained by taking $\langle\psi, \cdot\psi'\rangle$-expectation of Eq. (60) and moreover $\langle\psi, j_t(1)\psi'\rangle = 1$. But this is true because of (67). This finishes the proof of the theorem. □

7. Application to quantum Glauber dynamics

In this section, we apply our technique to prove the existence of the flow for a quantum system of spins on a lattice. Starting with the work of Glauber [17] the dynamics of infinite classical lattice systems has been considered by many authors and has led us to study the ergodic and equlibrium properties of a new class of classical Markov semigroups (cf. [18] for a general survey and for further references). Quantum analog of these semigroups have also been considered by several authors (e.g., [19–21,23,24]). However the problem of deriving these Markovian semigroups, and more generally the stochastic flows, as limits of Hamiltonian systems, was open both in the classical and in the quantum case. This problem was solved in the paper [5], where a QSDE describing the dynamics of a infinite volume spin system interacting with bosonic white noise was derived. In the simplest case (one-dimensional lattice with nearest neighbor, translation invariant interaction) we have the following picture: the spins (or more generally two state systems) are enumerated by integer numbers and the dynamics of the system (the flow) is given by the following QSDE (Langevin equation)

$$dj_t(X) = \sum_{\alpha=-1,0,1} j_t \circ \theta_\alpha(X)\, dM^\alpha(t), \tag{73}$$

where

$$dM^{-1}(t) = dB(t), \quad dM^{1}(t) = dB^*(t),$$

$$\theta_{-1}(X) = -i\left[X, F_\Lambda^{(++)*}\right], \quad \theta_1(X) = -i\left[X, F_\Lambda^{(++)}\right],$$

$$\theta_0(X) = \left(\theta_0^{(0,-1)} + \theta_0^{(0,1)} + \theta_0^{(-1)} + \theta_0^{(1)}\right)(X)\left(\sum_{\varepsilon,\mu}\left(-i\mathrm{Im}(g|g)^-_{(\varepsilon\mu)}[X, F_\Lambda^{\varepsilon\mu*}F_\Lambda^{\varepsilon\mu}] + i\mathrm{Im}(g|g)^+_{(\varepsilon\mu)}[X, F_\Lambda^{\varepsilon\mu}F_\Lambda^{\varepsilon\mu*}]\right)\right.$$

$$\left. + \mathrm{Re}(g|g)^-_{(++)}\left(2F_\Lambda^{(++)*}XF_\Lambda^{(++)} - \left\{X, F_\Lambda^{(++)*}F_\Lambda^{(++)}\right\}\right) + \mathrm{Re}(g|g)^+_{(++)}\left(2F_\Lambda^{(++)}XF_\Lambda^{(++)*} - \left\{X, F_\Lambda^{(++)}F_\Lambda^{(++)*}\right\}\right)\right),$$

where $(g|g)^\pm_{(\varepsilon\mu)}$ are constants with nonnegative real part (whose explicit form is given in [5]) and the stochastic differentials satisfy the following Ito table

$$dB(t)\, dB^*(t) = \mathrm{Re}(g|g)^+_{(++)}\, dt,$$

$$dB^*(t)\, dB(t) = \mathrm{Re}(g|g)^-_{(++)}\, dt.$$

The operators $F_\Lambda^{(\varepsilon,\mu)}$ acting on the spin degrees of freedom have the form

$$F_\Lambda^{(\varepsilon\mu)} = \sum_{r\in\Lambda} F_r^{(\varepsilon,\mu)},$$

where Λ is a subset of the lattice of integer numbers and

$$F_r^{(++)} = |1_{r-1}\rangle\langle 1_{r-1}\|1_r\rangle\langle -1_r\|1_{r+1}\rangle\langle 1_{r+1}| + |-1_{r-1}\rangle\langle -1_{r-1}\|-1_r\rangle\langle 1_r\|-1_{r+1}\rangle\langle -1_{r+1}|$$

and the other operators $F_r^{(\varepsilon,\mu)}$ are defined correspondingly. For fixed spin number r the operators $F_r^{(\varepsilon\mu)}$ are indexed by configurations of nearest neighbors of the spin at r (for every r we have four configurations). We denote these configurations $++$, $+-$, $-+$ and $--$ (the first symbol is the orientation of the spin on the left of r and the second on the right).

One can prove that the structure maps θ_α in (73) satisfy the conditions of Lemma 23 and conditions (64)–(66). Therefore, we can apply to the stochastic dynamics of the considered system of spins the approach developed in the present paper and prove the existence of the flow which solves the QSDE (73). These considerations continue to hold also for multidimensional lattices, but in this case the Langevin equation is more complex [5].

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