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Stochastic Evolutions Driven by Nonlinear Quantum Noise. II

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Abstract. We consider the renormalized Itô table for higher powers of white noise and, assuming that there exists a Hilbert space in which these powers have an operator realization, we prove existence, uniqueness, and unitarity for stochastic equations driven by these powers.

1. INTRODUCTION

A general class of quantum stochastic differential processes was defined in [3], [4], and [5] by the relation

$$dB_{(m,n)}(t) = b^+(t)^m b(t)^n \, dt, \tag{1.1}$$

where $t \geqslant 0$, m, $n \in \{0, 1, \dots\}$, and the noise functionals b^+ and b are defined as follows.

Let $L^2_{\text{sym}}(\mathbb{R}^n)$ be the space of square-integrable functions on $\mathbb{R}^n$ that are symmetric under the permutations of their arguments, and let

$$F = \bigoplus_{n=0}^{\infty} L^2_{\text{sym}}(\mathbb{R}^n),$$

where, if $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in F$, then $\psi^{(0)} \in \mathbb{C}$, $\psi^{(n)} \in L^2_{\text{sym}}(\mathbb{R}^n)$ for $n \geqslant 1$, and

$$\|\psi\|^2 = |\psi(0)|^2 + \sum_{n=1}^{\infty} \int_{R^n} |\psi^{(n)}(s_1, \dots, s_n)|^2 \, ds_1 \dots ds_n.$$

Denote by $S \subset L^2(\mathbb{R}^n)$ the Schwartz space of smooth functions rapidly decreasing at infinity (faster than any polynomial) and let

$$D = \Big\{ \psi \in F | \psi^{(n)} \in S, \quad \sum_{n=1}^{\infty} n \|\psi^{(n)}\|^2 < \infty \Big\}.$$

For each $t \in \mathbb{R}$, we define the linear operator $b(t) \colon D \to F$ by

$$(b(t)\psi)^{(n)}(s_1, \dots, s_n) = \sqrt{n+1}\,\psi^{(n+1)}(t, s_1, \dots, s_n),$$

and the operator-valued distribution $b^+(t)$ by

$$(b^+(t)\psi)^{(n)}(s_1, \dots, s_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta(t - s_i)\psi^{(n-1)}(s_1, \dots, \hat{s}_i, \dots, s_n).$$

Since $L^2_{\text{sym}}(\mathbb{R}^n) = L^2_{\text{sym}}(\mathbb{R})^{\otimes n}$, we can identify F with the symmetric (boson) Fock space over S. For the case in which the elements of S are defined on $[0, +\infty)$, we denote the Fock space by $\Gamma(S_+)$.

For $\psi = \{(n!)^{-1/2} f^{\otimes n}\}$, we denote ψ by $\psi(f)$. For any $m, n \in \{0, 1, \dots\}$ we have

$$\langle \psi(g), b_t^{+^m} \psi(f) \rangle = \overline{g(t)^m} f(t)^n \langle \psi(g), \psi(f) \rangle, \tag{1.2}$$

where it is assumed that $\langle \cdot , \cdot \rangle$ is linear in the second argument and antilinear in the first.

We couple $\Gamma(S_+)$ with an *initial* Hilbert space H_0 and define an *adapted process* $A = \{A(t) \mid t \geqslant 0\}$ to be a family of operators on $H_0 \otimes \Gamma(S_+)$ such that $A(t) = A_t \otimes I$ for any t, where A_t acts on $H_0 \otimes \Gamma(S_+^{t]})$ and I is the identity operator on $\Gamma(S_+^{(t})$, and $S_+^{t]}$ and $S_+^{(t}$ are defined by

$$S_+^{t]} = \{f \cdot \chi_{[0,t]} \mid f \in S\} \quad \text{and} \quad S_+^{(t} = \{f \cdot \chi_{(t,+\infty)} \mid f \in S\},$$

respectively. If $A(t) = A \otimes I$ for each t, where A is an operator on H_0 and I is the identity operator on $\Gamma(S_+)$, then A is said to be a *constant process*. If $A(t)$ is a bounded operator for each t, then A is said to be a *bounded process* and, if $A^*(t)A(t) = A(t)A^*(t) = I$, then A is said to be a *unitary process*.

If $T \geqslant 0$ and the mapping $t \in [0, T] \to A(t)u \otimes \psi(f)$ is continuous for each $u \in H_0$ and $f \in S_+$, then A is said to be a *strongly continuous process* on $[0, T]$. If A is strongly continuous and the mapping $(u, f) \in H_0 \otimes S_+ \to A(t)u \otimes \psi(f)$ is continuous for all $t \in [0, T]$, then A is said to be *doubly continuous*.

In what follows we identify $B_{(m,n)}(t)$ with $I \otimes B_{(m,n)}(t)$, where I is the identity on H_0. For a constant adapted process $A = \{A(t) \mid t \geqslant 0\}$, we simply write A instead of $A(t)$.

Stochastic integrals with respect to time and to $B_{(m,n)}$ are defined in Lemma 2.1.

In [1, 2] we studied the existence and uniqueness problem for solutions of quantum stochastic evolutions of the form

$$dU(t) = \left[\sum_{\substack{m,n=0 \\ m+n \leqslant 2}}^{2} A_{(m,n)}(t) dB_{(m,n)}(t) \right] U(t), \qquad U(0) = U_0 , \qquad 0 \leqslant t \leqslant T < +\infty, \tag{1.3}$$

where $dB_{(0,0)}(t) = dt$, $dB_{(1,0)}(t) = dA^+(t)$, $dB_{(0,1)}(t) = dA(t)$, and A and A^+ are the annihilation and creation processes (see [7]). It was shown that, if the coefficient processes $A_{(m,n)}$ are adapted and such that

$$\sup_{0 \leqslant s \leqslant T} \|A_{(m,n)}(s)\|_{H_0 \otimes \Gamma(S_+)} < +\infty$$

for all m and n, and if U_0 is a bounded operator on $H_0 \otimes \Gamma(S_+)$, then (1.3) admits a unique adapted strongly continuous solution $U = \{U(t) \mid 0 \leqslant t \leqslant T < +\infty\}$ defined on the *span of the set* $\{u \otimes \psi(f)\}$, where $|f(s)| \leqslant 1$ for all $s \in [0, T]$ (cf. [6]).

Moreover, U is a unitary process if and only if $U_0 = I$ and

$$\begin{aligned}
&A_{(0,0)} + A^*_{(0,0)} + \gamma A_{(0,1)} A^*_{(0,1)} = 0, && A_{(0,1)} + A^*_{(1,0)} + 2\gamma A_{(0,2)} A^*_{(0,1)} + \gamma A_{(0,1)} A^*_{(1,1)} = 0,\\
&A_{(0,2)} + A^*_{(2,0)} + 2\gamma A_{(0,2)} A^*_{(1,1)} = 0, && A_{(1,1)} + A^*_{(1,1)} + 4\gamma A_{(0,2)} A^*_{(0,2)} + \gamma A_{(1,1)} A^*_{(1,1)} = 0,\\
&A^*_{(0,0)} + A_{(0,0)} + \gamma A^*_{(1,0)} A_{(1,0)} = 0, && A^*_{(1,0)} + A_{(0,1)} + \gamma A^*_{(1,0)} A_{(1,1)} + 2\gamma A^*_{(2,0)} A_{(1,0)} = 0,\\
&A^*_{(2,0)} + A_{(0,2)} + 2\gamma A^*_{(2,0)} A_{(1,1)} = 0,
\end{aligned} \tag{1.4}$$

where the quantity $\gamma > 0$ defined by

$$[b(t), b^+(s)] = \gamma \cdot \delta(t - s) \tag{1.5}$$

is the *variance of the quantum Brownian motion* defined by $B_{(0,1)}$ and $B_{(1,0)}$, δ is the delta function (cf. [3, 4]), and * denotes the dual operator. These results extended those in [7] referred to stochastic evolutions driven by *linear quantum noise* $B_{(1,0)}$ and $B_{(0,1)}$.

The proof made use of Itô's table

$$
\begin{array}{l|cccccc}
 & dt & dB_{(0,1)} & dB_{(1,0)} & dB_{(0,2)} & dB_{(2,0)} & dB_{(1,1)} \\
\hline
dt & 0 & 0 & 0 & 0 & 0 & 0 \\
dB_{(0,1)} & 0 & 0 & \gamma dt & 0 & 2\gamma dB_{(1,0)} & \gamma dB_{(0,1)} \\
dB_{(1,0)} & 0 & 0 & 0 & 0 & 0 & 0 \\
dB_{(0,2)} & 0 & 0 & 2\gamma dB_{(0,1)} & 0 & 4\gamma dB_{(1,1)} & 2\gamma dB_{(0,2)} \\
dB_{(2,0)} & 0 & 0 & 0 & 0 & 0 & 0 \\
dB_{(1,1)} & 0 & 0 & \gamma dB_{(1,0)} & 0 & 2\gamma dB_{(2,0)} & \gamma dB_{(1,1)}
\end{array}
\tag{1.6}
$$

It was also shown in [1] that the stochastic differentials dt, $dB_{(0,1)}$, $dB_{(1,0)}$, $dB_{(0,2)}$, $dB_{(2,0)}$, and $dB_{(1,1)}$ are *linearly independent*, which means that

$$\sum_{\substack{m,n=0\\ m+n\leqslant 2}}^{2} A_{(m,n)}(t)\, dB_{(m,n)}(t) = 0 \quad \text{for all} \quad t \in [0,T] \quad \Longrightarrow \quad A_{(m,n)} \equiv 0 \quad \text{for all} \quad m,n. \tag{1.7}$$

In what follows we take $\gamma = 1$ and use the generalized Itô table

$$dB_{(m,n)}\, dB_{(k,l)} = nk\, dB_{(m+k-1,n+l-1)}, \tag{1.8}$$

where m, n, k, and l are any natural numbers such that $m+k-1 \geqslant 0$ and $n+l-1 \geqslant 0$ (cf. [4]) to extend the results of [1] and [2] to quantum stochastic evolutions of the form

$$dU(t) = \left[\sum_{m,n=0}^{\infty} A_{(m,n)}(t)\, dB_{(m,n)}(t)\right] U(t), \qquad U(0) = U_0\,, \qquad 0 \leqslant t \leqslant T < +\infty. \tag{1.9}$$

For an adapted process $X = \{X(t) \mid t \geqslant 0\}$, its stochastic differential $dX = \{dX(t) \mid t \geqslant 0\}$ is defined by

$$dX(t) = X(t+dt) - X(t). \tag{1.10}$$

For two adapted processes X and Y, we have

$$d(X \cdot Y)(t) = dX(t) \cdot Y(t) + X(t) \cdot dY(t) + dX(t) \cdot dY(t). \tag{1.11}$$

We also note that adapted processes commute with the stochastic differentials $dB_{(m,n)}$.

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF STOCHASTIC EVOLUTIONS DRIVEN BY NONLINEAR QUANTUM NOISE

Definition 2.1. Let $\tau \geqslant 0$. We say that a family $\{A_{(n,k)} \mid n,k = 0,1,\dots\}$ of adapted processes is of $L^2[0,T]$*-type* if

$$\|A_{(n,k)}\|_{u,f}^{2,T} \stackrel{\text{def}}{=} \sum_{n,k=0}^{\infty} n \left[\int_0^t \|A_{(n,k)}(s)u \otimes \psi(f)\|^2 ds\right]^{1/2} < +\infty \tag{2.1}$$

for all $t \in [0,T]$, $u \in H_0$, and $f \in S_+$ with $|f(s)| \leqslant 1$ for $s \in [0,t]$.

Lemma 2.1. *Let $t \geqslant 0$, u, $\nu \in H_0$, f, $g \in S_+$ with $|f(s)| \leqslant 1$, $|g(s)| \leqslant 1$ for all $s \in [0,t]$, let $\{A_{(n,k)} \mid n,k = 0,1,\dots\}$ and $\{C_{(n,k)} \mid n,k = 0,1,\dots\}$ be families of adapted processes of $L^2[0,t]$-type and let*

$$\Pi_1(t) = \sum_{n,k=0}^{\infty} \int_0^t A_{(n,k)}(s)\,dB_{(n,k)}(s) \tag{2.2}$$

and

$$\Pi_2(t) = \sum_{n,k=0}^{\infty} \int_0^t C_{(n,k)}(s)\,dB_{(n,k)}(s). \tag{2.3}$$

Then:

(i) $$\langle \Pi_1(t)u\otimes\psi(f), \nu\otimes\psi(g)\rangle = \sum_{n,k=0}^{\infty}\int_0^t g(s)^n\overline{f(s)^k}\langle A_{(n,k)}(s)u\otimes\psi(f),\nu\otimes\psi(g)\rangle\,ds, \tag{2.4}$$

(ii) $$\begin{aligned}&\langle \Pi_1(t)u\otimes\psi(f), \Pi_2(t)\nu\otimes(g)\rangle\\ &= \sum_{n,k,N,K=0}^{\infty}\Big[\int_0^t \overline{g(s)^n}f(s)^k\int_0^s f(z)^N\overline{g(z)}^K\langle A_{(n,k)}(s)u\otimes\psi(f), C_{(N,K)}(z)\nu\otimes\psi(g)\rangle\,dzds\\ &+\int_0^t f(s)^N\overline{g(s)^K}\int_0^s f(z)^k\overline{g(z)^n}\langle A_{(n,k)}(z)u\otimes\psi(f),\ C_{(N,K)}(s)\nu\otimes\psi(g)\rangle\,dzds\\ &+nN\int_0^t f(s)^{N+k-1}\overline{g(s)}^{K+n-1}\langle A_{(n,k)}(s)u\otimes\psi(f),\ C_{(N,K)}(s)\nu\otimes\psi(g)\rangle ds\Big],\end{aligned} \tag{2.5}$$

(iii) $$|\langle \Pi_1(t)u\otimes\psi(f),\nu\otimes\psi(g)\rangle| \leqslant t^{1/2}\|A_{(n,k)}\|_{u,f}^{2,t}\|\nu\otimes\psi(g)\|, \tag{2.6}$$

(iv) $$\|\Pi_1(t)u\otimes\psi(f)\|^2 \leqslant (2t+1)\big[\|A_{(n,k)}\|_{u,f}^{2,t}\big]^2. \tag{2.7}$$

Proof. (i) The proof immediately follows from (1.1) and (1.2). (ii) By (1.11), if n, k, N, and $K \in \{0,1,\dots\}$, then

$$\begin{aligned}&d\Big[\Big\langle \int_0^\tau A_{(n,k)}(s)dB_{(n,k)}(s)u\otimes\psi(f), \int_0^\tau C_{(N,K)}(s)dB_{(N,K)}(s)\nu\otimes\psi(g)\Big\rangle\Big]\\ &= \Big\langle A_{(n,k)}(\tau)u\otimes\psi(f), \int_0^\tau C_{(N,K)}(s)dB_{(N,K)}(s)\nu\otimes\psi(g)\Big\rangle\\ &+\Big\langle \int_0^\tau A_{(n,k)}(s)dB_{(n,k)}(s)u\otimes\psi(f), C_{(N,K)}(\tau)dB_{(N,K)}(\tau)\otimes\psi(g)\Big\rangle\\ &+\Big\langle A_{(n,k)}(\tau)dB_{(n,k)}(\tau)u\otimes\psi(f),\ C_{(N,K)}(\tau)dB_{(N,K)}(\tau)\nu\otimes\psi(g)\Big\rangle.\end{aligned}$$

By (1.1), (1.2), $dB^*_{(N,K)}\,dB_{(n,k)} = dB_{(K,N)}dB_{(n,k)}$, and (1.8), the right-hand side is equal to

$$\begin{aligned}&\overline{g(\tau)^n}f(\tau)^k\int_0^\tau f(s)^N\overline{g(s)^K}\langle A_{(n,k)}(\tau)u\otimes\psi(f), C_{(N,K)}(s)\nu\otimes\psi(g)\rangle\,dsd\tau\\ &+f(\tau)^N\overline{g(\tau)^K}\int_0^\tau f(s)^k\overline{g(s)^n}\langle A_{(n,k)}(s)\otimes\psi(f), C_{(N,K)}(\tau)\nu\otimes\psi(g)\rangle\,dsd\tau\\ &+nNf(\tau)^{N+k-1}\overline{g(\tau)}^{K+n-1}\langle A_{(n,k)}(\tau)u\otimes\psi(f),\ C_{(N,K)}(\tau)\nu\otimes\psi(g)\rangle\,d\tau\end{aligned}$$

from which the result follows by integrating from 0 to t and by summing over n, k, N, and K.

(iii) The proof immediately follows from (2.4) and (2.1), and from Hölder's inequality.
(iv) By (2.5), for $u = \nu$, $f = g$, and $A_{(n,k)} = C_{(n,k)}$, we have

$$\begin{aligned}
\|\Pi_1(t)u \otimes \psi(f)\|^2 \leqslant & \sum_{n,k,N,K=0}^{\infty} \bigg[\int_0^t\int_0^s \|A_{(n,k)}(s)u\otimes\psi(f)\| \cdot \|A_{(N,K)}(z)u\otimes\psi(f)\|\,dzds \\
&+ \int_0^t\int_0^s \|A_{(n,k)}(z)u\otimes\psi(f)\| \cdot \|A_{(N,K)}(s)u\otimes\psi(f)\|\,dzds \\
&+ nN\int_0^t \|A_{(n,k)}(s)u\otimes\psi(f)\| \cdot \|A_{(N,K)}(s)u\otimes\psi(f)\|ds \bigg]\\
\leqslant & \, 2\sum_{n,k=0}^{\infty}\int_0^t \|A_{(n,k)}(s)u\otimes\psi(f)\|ds \sum_{N,K=0}^{\infty}\int_0^t \|A_{(N,K)}(s)u\otimes\psi(f)\|ds \\
&+ \sum_{n,k,N,K=0}^{\infty} nN\left[\int_0^t \|A_{(n,k)}(s)u\otimes\psi(f)\|^2ds\right]^{1/2}\left[\int_0^t \|A_{(N,K)}(s)u\otimes\psi(f)\|^2ds\right]^{1/2} \\
\leqslant & \, 2\left[\sum_{n,k=0}^{\infty}\int_0^t \|A_{(n,k)}(s)u\otimes\psi(f)\|ds\right]^2 + \left[\sum_{n,k=0}^{\infty} n\left[\int_0^t \|A_{(n,k)}(s)u\otimes\psi(f)\|^2ds\right]^{1/2}\right]^2 \\
\leqslant & \, 2\left[\sum_{n,k=0}^{\infty} t^{1/2}\left[\int_0^t \|A_{(n,k)}(s)u\otimes\psi(f)\|^2ds\right]^{1/2}\right]^2 + \left[\sum_{n,k=0}^{\infty} n\left[\int_0^t \|A_{(n,k)}(s)u\otimes\psi(f)\|^2ds\right]^{1/2}\right]^2 \\
\leqslant & \, 2t\left[\sum_{n,k=0}^{\infty} n\left[\int_0^t \|A_{(n,k)}(s)u\otimes\psi(f)\|^2ds\right]^{1/2}\right]^2 + \left[\sum_{n,k=0}^{\infty} n\left[\int_0^t \|A_{(n,k)}(s)u\otimes\psi(f)\|^2ds\right]^{1/2}\right]^2 \\
\leqslant & \, (2t+1)[\|A_{(n,k)}\|_{u,f}^{2,t}]^2. \qquad \square
\end{aligned}$$

Definition 2.2. Let $T \geqslant 0$. We say that a family $\{A_{(n,k)} \mid n, k = 0, 1, \ldots\}$ of bounded adapted processes is of $L_\infty[0,T]$-*type* if

$$\|A_{(n,k)}\|^{\infty,T} \stackrel{\text{def}}{=} \sum_{n,k=0}^{\infty} n \sup_{s\in[0,T]} \|A_{(n,k)}(s)\| < +\infty. \tag{2.8}$$

Remark. An $L_\infty[0,T]$-type family is also of $L^2[0,T]$-type.

Lemma 2.2. *In the notation of Lemma 2.1, suppose that* $A_{(n,k)}(s) = \alpha_{(n,k)}(s)\beta(s)$, *where* $\{\alpha_{(n,k)} \mid n, k = 0, 1, \ldots\}$ *is a family of bounded adapted processes of* $L_\infty[0,t]$-*type, and*

$$\sup_{s\in[0,t]} \|\beta(s)u\otimes\psi(f)\| < +\infty.$$

Then

$$\|\Pi_1(t)u\otimes\psi(f)\|^2 \leqslant t(2t+1)[\|\alpha_{(n,k)}\|^{\infty,t} \sup_{s\in[0,t]} \|\beta(s)u\otimes\psi(f)\|^2. \tag{2.9}$$

Proof. By Lemma 2.1 (iv) we have

$$\|\Pi_1(t)u\otimes\psi(f)\|^2 \leqslant (2t+1)\left[\sum_{n,k=0}^{\infty} n\left[\int_0^t \|A_{(n,k)}(s)u\otimes\psi(f)\|^2ds\right]^{1/2}\right]^2$$

$$\leqslant (2t+1)\left[t^{1/2}\sup_{s\in[0,t]}\|\beta(s)u\otimes\psi(f)\|\sum_{n,k=0}^{\infty}n\sup_{s\in[0,t]}\|\alpha_{(n,k)}(s)\|\right]^2$$

$$\leqslant (2t+1)t\sup_{s\in[0,t]}\|\beta(s)u\otimes\psi(f)\|^2[\|\alpha_{(n,k)}\|^{\infty,t}]^2. \qquad \square$$

Lemma 2.3. *In the notation of Lemma 2.1, suppose that $A_{(n,k)}(s) = \alpha_{(n,k)}(s)\beta(s)$, where $\{\alpha_{(n,k)} \mid n,k=0,1,\dots\}$ is a family of bounded adapted processes of $L_\infty[0,t]$-type and*

$$\int_0^t \|\beta(s)u\otimes\psi(f)\|^2 ds < +\infty.$$

Then

$$\|\Pi_1(t)u\otimes\psi(f)\|^2 \leqslant (2t+1)[\|\alpha_{(n,k)}\|^{\infty,t}]^2\int_0^t\|\beta(s)u\otimes\psi(f)\|^2\,ds.$$

Proof. By Lemma 2.1 (iv),

$$\|\Pi_1(t)u\otimes\psi(f)\|^2 \leqslant (2t+1)\left[\sum_{n,k=0}^{\infty}n\left[\int_0^t\|\alpha_{(n,k)}(s)\beta(s)u\otimes\psi(f)\|^2ds\right]^{1/2}\right]^2$$

$$\leqslant (2t+1)\left[\sum_{n,k=0}^{\infty}n\sup_{s\in[0,t]}\|\alpha_{(n,k)}(s)\|\left[\int_0^t\|\beta(s)u\otimes\psi(f)\|^2ds\right]^{1/2}\right]^2$$

$$\leqslant (2t+1)[\|\alpha_{(n,k)}\|^{\infty,t}]^2\int_0^t\|\beta(s)u\otimes\psi(f)\|^2ds. \qquad \square$$

Theorem 2.1. *Let $T\geqslant 0$, and let $\{A_{(n,k)} \mid n,k=0,1,\dots\}$ be a family of bounded adapted processes of $L_\infty[0,T]$-type. If U_0 is a bounded operator on $H_0\otimes\Gamma(S_+)$, then the quantum stochastic differential equation*

$$dU(t) = \left[\sum_{n,k=0}^{\infty}A_{(n,k)}(t)\,dB_{(n,k)}(t)\right]U(t), \qquad U(0)=U_0, \tag{2.10}$$

or, in integral form,

$$U(t) = U_0 + \sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)U(s)\,dB_{(n,k)}(s), \tag{2.11}$$

admits a unique adapted strongly continuous solution $U=\{U(t)\mid 0\leqslant t\leqslant T<+\infty\}$ defined on the span of the set $\{u\otimes\psi(f)\}$, where $|f(s)|\leqslant 1$ for all $s\in[0,T]$.

Proof. Define a sequence $\{U_N\}_{N=0}^{\infty}$ of adapted processes by setting $U_0(t)=U_0$ and

$$U_N(t) = U_0 + \sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)U_{N-1}(s)dB_{(n,k)}(s) \tag{2.12}$$

for $N\geqslant 1$. Then, for $0\leqslant t\leqslant t'\leqslant T<+\infty$, it follows from Lemma 2.2 that

$$\|[U_N(t)-U_N(t')]u\otimes\psi(f)\|^2 = \left\|\sum_{n,k=0}^{\infty}\int_{t'}^{t}A_{(n,k)}(s)U_{N-1}(s)dB_{(n,k)}(s)u\otimes\psi(f)\right\|^2$$

$$\leqslant (2(t'-t)^2+(t'-t))[\|A_{(n,k)}\|^{\infty,T}]^2\sup_{0\leqslant s\leqslant T}\|U_{N-1}(s)u\otimes\psi(f)\|^2,$$

from which it follows that $\{U_N\}_{N=0}^{\infty}$ is a sequence of strongly continuous adapted processes. Moreover, using Lemma 2.3 repeatedly, we obtain

$$\begin{aligned}
&\|[U_N(t)-U_{N-1}(t)]u\otimes\psi(f)\|^2\\
&=\left\|\sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)[U_{N-1}(t_1)-U_{N-2}(t_1)]dB_{(n,k)}(s)u\otimes\psi(f)\right\|^2\\
&\leqslant(2T+1)[\|A_{(n,k)}\|^{\infty,T}]^2\int_0^t\|[U_{N-1}(t_1)-U_{N-2}(t_1)]u\otimes\psi(f)\|^2\,dt_1\\
&\leqslant(2T+1)^2[\|A_{(n,k)}\|^{\infty,T}]^4\int_0^t\int_0^{t_1}\|[U_{N-2}(t_2)-U_{N-3}(t_2)]u\otimes\psi(f)\|^2\,dt_2dt_1\\
&\quad\vdots\\
&\leqslant(2T+1)^{N-1}[\|A_{(n,k)}\|^{\infty,T}]^{2(N-1)}\\
&\quad\times\int_0^t\int_0^{t_1}\cdots\int_0^{t_{N-2}}\left\|\sum_{n,k=0}^{\infty}\int_0^{t_{N-1}}A_{(n,k)}(s)U_0dB_{(n,k)}(s)u\otimes\psi(f)\right\|^2dt_{N-1}\dots dt_1\\
&\leqslant(2T+1)^N[\|A_{(n,k)}\|^{\infty,T}]^{2N}\|U_0\|^2\|u\otimes\psi(f)\|^2\int_0^t\int_0^{t_1}\cdots\int_0^{t_{N-1}}dsdt_{N-1}\dots dt_1\\
&\leqslant(2T+1)^N[\|A_{(n,k)}\|^{\infty,T}]^{2N}\|U_0\|^2\|u\otimes\psi(f)\|^2\frac{t^N}{n!},
\end{aligned}$$

which implies that

$$\sum_{N=1}^{\infty}\|[U_N(t)-U_{N-1}(t)]u\otimes\psi(f)\|\leqslant+\infty,$$

and thus we can define a process $U=\{U(t)\mid 0\leqslant t\leqslant T<+\infty\}$ by

$$U(t)u\otimes\psi(f)=\lim_N U_N(t)u\otimes\psi(f). \tag{2.13}$$

As a uniform limit of adapted processes, U is a strongly continuous adapted process. Moreover,

$$\begin{aligned}
&\left\|\left[U(t)-U_0-\sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)U(s)dB_{(n,k)}(s)\right]u\otimes\psi(f)\right\|\\
&=\left\|\left[U(t)-U_{N+1}(t)+U_{N+1}(t)-U_0-\sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)U_N(s)dB_{(n,k)}(s)\right.\right.\\
&\quad\left.\left.+\sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)U_N(s)dB_{(n,k)}(s)-\sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)U(s)dB_{(n,k)}(s)\right]u\otimes\psi(f)\right\|\\
&\leqslant\|[U(t)-U_{N+1}(t)]u\otimes\psi(f)\|+\left\|\left[U_{N+1}-U_0-\sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)U_N(s)dB_{(n,k)}(s)\right]u\otimes\psi(f)\right\|\\
&\quad+\left\|\sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)[U_N(s)-U(s)]dB_{(n,k)}(s)u\otimes\psi(f)\right\|.
\end{aligned}\tag{2.14}$$

The first term of (2.14) tends to zero as $n\to\infty$ by (2.13).

The second term of (2.14) is equal to zero by (2.12).

By Lemma 2.2, the third term of (2.14) can be estimated, namely, it is

$$\leqslant [T(2T+1)]^{1/2}\|A_{(n,k)}\|^{\infty,T} \sup_{0\leqslant s\leqslant T} \|[U_n(s)-U(s)]u\otimes\psi(f)\|,$$

which tends to zero as $n\to\infty$ because the convergence in (2.13) is uniform.

Thus,

$$U(t) = U_0 + \sum_{n,k=0}^{6} \int_0^t A_{(n,k)}(s)U(s)dB_{(n,k)}(s), \tag{2.15}$$

i.e., U is a solution of (2.11).

If $\{V(t) \mid 0\leqslant t\leqslant T\}$ is another process such that

$$V(t) = U_0 + \sum_{n,k=0}^{\infty} \int_0^t A_{(n,k)}(s)V(s)\,dB_{(n,k)}(s), \tag{2.16}$$

then

$$\|[U(t)-V(t)]u\otimes\psi(f)\|^2 \leqslant (2T+1)[\|A_{(n,k)}\|^{\infty,T}]^2 \int_0^t \|[U(s)-V(s)]u\otimes\psi(f)\|^2\,ds$$

by Lemma 2.3, and we conclude by Gronwall's inequality that

$$\|[U(t)-V(t)]u\otimes\psi(f)\| = 0.$$

Hence, $U=V$ on the span of $\{u\otimes\psi(f)\}$, which proves uniqueness. □

3. LINEAR INDEPENDENCE OF STOCHASTIC DIFFERENTIALS

Proposition 3.1. *Let $T\geqslant 0$, let N and K be any two nonnegative integers, and let*

$$\{A_{(n,k)} \mid n=0,1,\dots,N,\ k=0,1,\dots,K\}$$

be a family of doubly continuous adapted processes on $[0,T]$ such that

$$\sum_{n=0}^{N}\sum_{k=0}^{K} A_{(n,k)}(t)dB_{(n,k)}(t) = 0 \tag{3.1}$$

weakly on the linear span of

$$\{u\otimes\psi(f) \mid u\in H_0,\ \textit{where}\ f\in S_+\ \textit{and}\ |f(t)|\leqslant 1\ \textit{on}\ [0,T]\}.$$

Then

$$A_{(n,k)}(t)=0 \quad \textit{for all} \quad n=0,1,\dots,N\,,\quad k=0,1,\dots,K,\quad \textit{and}\quad t\in[0,T].$$

Proof. We use mathematical induction on N and K. For $N=0,1$ and $K=0,1$, the claim is true (cf. [1]).

Suppose that the assertion is true for some N and K. Let us prove that it is true for $N+1$ and $K+1$. Thus, assume that

$$\sum_{n=0}^{N+1}\sum_{k=0}^{K+1} A_{(n,k)}(t)dB_{(n,k)}(t) = 0 \tag{3.2}$$

for any $t \in [0,T]$. Then it follows from (1.1) and (1.2) that

$$\sum_{n=0}^{N+1}\sum_{k=0}^{K+1} \overline{f(t)}^n g(t)^k \langle u\otimes\psi(f),\ A_{(n,k)}(t)\nu\otimes\psi(g)\rangle = 0 \tag{3.3}$$

for any u, $\nu, \in H_0$ and f, $g \in S^+$ such that $|f(t)| \leqslant 1$ and $|g(t)| \leqslant 1$ on $[0,T]$.

Allowing f and g to vary over the compactly supported continuous functions on $[0,t)$, we see from (3.3) that $\langle u\otimes\psi(f),\ A_{(0,0)}(t)\nu\otimes\psi(g)\rangle = 0$. Since $A_{(0,0)}$ is adapted and the set

$$\{u\otimes\psi(f) \mid f \text{ is continuous with compact support in } [0,t)\}$$

is total, it follows that $A_{(0,0)} \equiv 0$ on $[0,T]$.

Letting f and g vary over continuous functions with $f(t) \neq 0$ and $\sup pg \subset [0,t)$, we get from (3.3) that

$$\sum_{n=1}^{N+1} \overline{f(t)}^n \langle u\otimes\psi(f),\ A_{(n,0)}(t)\nu\otimes\psi(g)\rangle = 0, \tag{3.4}$$

which implies the relation

$$\sum_{n=0}^{N} \overline{f(t)}^n \langle u\otimes\psi(f),\ A_{(n,0)}(t)\nu\otimes\psi(g)\rangle = 0 \tag{3.5}$$

because $f(t) \neq 0$, and hence

$$\Big\langle u\otimes\psi(f),\ \sum_{n=0}^{N} A_{(n,0)}(t)dB_{(n,0)}(t)\nu\otimes\psi(g)\Big\rangle = 0. \tag{3.6}$$

Since the value t is arbitrary, it now follows from the totality argument that

$$\sum_{n=0}^{N} A_{(n,0)}(t)dB_{(n,0)}(t) = 0 \tag{3.7}$$

and thus $A_{(n,0)} \equiv 0$ on $[0,T]$ for all $n = 0,1,\dots,N$ by the induction hypothesis. Similarly, $A_{(0,k)} \equiv 0$ on $[0,T]$ for any $k = 0,1,\dots,K$.

Thus, (3.3) reduces to

$$\sum_{n=1}^{N+1}\sum_{k=1}^{K+1} \overline{f(t)}^n g(t)^k \langle u\otimes\psi(f),\ A_{(n,k)}(t)\nu\otimes\psi(g)\rangle = 0. \tag{3.8}$$

Considering, as before, f and g such that $\overline{f(t)}g(t) \neq 0$, we conclude that

$$\sum_{n=0}^{N}\sum_{k=0}^{K} \overline{f(t)}^n g(t)^k \langle u\otimes\psi(f),\ A_{(n,k)}(t)\nu\otimes\psi(g)\rangle = 0, \tag{3.9}$$

i.e.,

$$\sum_{n=0}^{N}\sum_{k=0}^{K} A_{(n,k)}(t)dB_{(n,k)}(t) = 0, \tag{3.10}$$

from which it follows by the induction assumption that $A_{(n,k)} \equiv 0$ on $[0,T]$ for any $n = 0,1,\dots,N$ and $k = 0,1,\dots,K$ with $n+k \geqslant 1$. Thus, by (3.3) we have

$$f(t)^{N+1}g(t)^{K+1}\langle u\otimes\psi(f),\ A_{(N+1,K+1)}(t)\nu\otimes\psi(g)\rangle = 0, \tag{3.11}$$

and, since f, g, u, and ν are arbitrary, it follows that $A_{(N+1,K+1)} \equiv 0$ on $[0,T]$. □

Proposition 3.2. *Let $T \geqslant 0$ and let $\{A_{(n,k)} \mid n,k=0,1,\dots\}$ be a family of constant adapted processes on $[0,T]$ such that*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{(n,k)} dB_{(n,k)}(t) = 0 \tag{3.12}$$

weakly on the span of

$$\{u \otimes \psi(f) \mid u \in H_0, f \in S_+, |f(t)| \leqslant 1 \ \ on \ \ [0,T]\}.$$

Then

$$A_{(n,k)} = 0 \quad for\ all \quad n,k = 0,1,\dots \tag{3.13}$$

Proof. By (3.12), (1.1), and (1.2) we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \overline{f(t)}^n g(t)^k \langle u \otimes \psi(f),\ A_{(n,k)}\nu \otimes \psi(g)\rangle = 0 \tag{3.14}$$

for all u, $\nu \in H_0$, and $f, g \in S_+$ with $|f(t)| \leqslant 1$ and $|g(t)| \leqslant 1$ on $[0,T]$.

Since $A_{(n,k)} = \widetilde{A}_{(n,k)} \otimes I$, where $\widetilde{A}_{(n,k)}$ acts on H_0 and I is the identity operator on $\Gamma(S_+)$, relation (3.14) implies that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \overline{f(t)}^n g(t)^k \langle u, \widetilde{A}_{(n,k)}\nu\rangle \langle \psi(f), \psi(g)\rangle = 0 \tag{3.15}$$

on $[0,T]$, and, since the inner product of two exponential vectors is never zero, it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \overline{f(t)}^n g(t)^k \langle u, \widetilde{A}_{(n,k)}\nu\rangle = 0 \tag{3.16}$$

for all $t \in [0,T]$. Fix some $t \in [0,T]$. Then, as f, g vary, the left-hand side of (3.16) is a double power series in $\overline{f(t)}$ and $g(t)$ which is identically equal to zero. Thus, $\langle u, \widetilde{A}_{(n,k)}\rangle\nu = 0$, and hence $\widetilde{A}_{(n,k)} = 0$ because u and ν are arbitrary, and thus $A_{(n,k)} = 0$ for any $n, k = 0, 1, \dots$ □

4. UNITARITY OF SOLUTIONS OF STOCHASTIC EVOLUTIONS DRIVEN BY NONLINEAR QUANTUM NOISE

Theorem 4.1 (Necessary unitarity conditions). *If a unique solution of the evolution*

$$\begin{gathered} dU(t) = \left[\sum_{n,k=0}^{\infty} A_{(n,k)} dB_{(n,k)}(t)\right] U(t), \\ U(0) = I, \qquad 0 \leqslant t \leqslant T < +\infty, \end{gathered} \tag{4.1}$$

is unitary, then

$$A^*_{(k,n)} + A_{(n,k)} + \sum_{\substack{\alpha,\beta,\lambda,\mu \\ \alpha+\mu=n+1 \\ \beta+\lambda=k+1}} \beta\mu A_{(\alpha,\beta)} A^*_{(\lambda,\mu)} = 0 \tag{4.2}$$

for any $n, k = 0, 1, 2, \dots$, and

$$A^*_{(k,n)} + A_{(n,k)} + \sum_{\substack{\alpha,\beta,\lambda,\mu \\ \alpha+\mu=k+1 \\ \beta+\lambda=n+1}} a\lambda A^*_{(\alpha,\beta)} A_{(\lambda,\mu)} = 0, \tag{4.3}$$

where α, β, λ, $\mu = 0, 1, 2, \dots$

Proof. Since U is unitary, it follows that

$$U(t)U(t)^* = I, \tag{4.4}$$
$$U(t)^*U(t) = I \tag{4.5}$$

for each $t \in [0,T]$. By (1.10) and (1.11), it follows from relation (4.4) upon taking the differential that

$$dU(t)U(t)^* + U(t)dU(t)^* + dU(t)dU(t)^* = 0. \tag{4.6}$$

Since $dB^*_{(n,k)} = dB_{(k,n)}$, it follows from (4.1) that

$$dU(t)^* = U(t)^*\left[\sum_{n,k=0}^{\infty} A^*_{(n,k)}dB_{(k,n)}(t)\right] \tag{4.7}$$
$$U(0)^* = I, \qquad 0 \leqslant t \leqslant T < +\infty,$$

and hence, by Itô's table (1.8), relation (4.6) yields

$$\sum_{n,k=0}^{\infty} A^*_{(n,k)}dB_{(k,n)}(t) + \sum_{n,k=0}^{\infty} A_{(n,k)}dB_{(n,k)}(t) + \sum_{n,k,N,K=0}^{\infty} A_{(n,k)}A^*_{(N,K)}kK\, dB_{(n+K,k+N-1)}(t) = 0,$$

i.e.,

$$\sum_{n,k=0}^{\infty} [A^*_{(k,n)} + A_{(n,k)}]dB_{(n,k)}(t) + \sum_{\alpha,\beta,\lambda,\mu=0}^{\infty} \beta\mu A_{(\alpha\beta)}A^*_{(\lambda,\mu)}dB_{(\alpha+\mu-1,\beta+\lambda-1)}(t) = 0.$$

This, together with Proposition 3.2, implies (4.2).

Similarly, (4.5) implies (4.3). □

Lemma 4.1. *Let $t \geqslant 0$, u, $\nu \in H_0$, f, $g \in S_+$ with $|f(s)| \leqslant 1$, $|g(s)| \leqslant 1$ for all $s \in [0,t]$, let $\{A_{(n,k)} \mid n,k = 0,1,\dots\}$ and $\{C_{(n,k)} \mid n,k = 0,1,\dots\}$ be families of adapted processes of $L^2[0,t]$-type, and let*

$$\Pi_1(t) = \sum_{n,k=0}^{\infty} \int_0^t A_{(n,k)}(s)dB_{(n,k)}(s), \qquad \Pi_2(t) = \sum_{n,k=0}^{\infty} \int_0^t C_{(n,k)}(s)dB_{(n,k)}(s).$$

Then

$$\langle \Pi_1(t)u\otimes\psi(f), \Pi_2(t)\nu\otimes\psi(g)\rangle = \int_0^t \Bigg[\sum_{n,k=0}^{\infty} \overline{f(s)}^k g(s)^n \langle A_{(n,k)}(s)u\otimes\psi(f), \Pi_2(s)\nu\otimes\psi(g)\rangle$$
$$+ \sum_{n,k=0}^{\infty} \overline{f(s)}^n g(s)^k \langle \Pi_1(s)u\otimes\psi(f),\ C_{(n,k)}(s)\nu\otimes\psi(g)\rangle \tag{4.8}$$
$$+ \sum_{n,k=0}^{\infty}\sum_{N,K=0}^{\infty} nN\overline{f(s)}^{k+N+1} g(s)^{n+K-1}\langle A_{(n,k)}(s)u\otimes\psi(f),\ C_{(N,K)}(s)\nu\otimes\psi(g)\rangle\Bigg]\, ds.$$

Proof. As in the proof of Lemma 2.1 (ii), it follows from (1.2), (1.8), and (1.11) that

$$\begin{aligned} d\langle\Pi_1(t)u\otimes\psi(f),\Pi_2(t)\nu\otimes\psi(g)\rangle &= \langle d\Pi_1(t)u\otimes\psi(f),\Pi_2(t)\nu\otimes\psi(g)\rangle \\ &+ \langle\Pi_1(t)u\otimes\psi(f),d\Pi_2(t)\nu\otimes\psi(g)\rangle + \langle d\Pi_1(t)u\otimes\psi(f),d\Pi_2(t)\nu\otimes\psi(g)\rangle \\ &= \sum_{n,k=0}^{\infty}\overline{f(k)}^k g(t)^n\langle A_{(n,k)}(t)u\otimes\psi(f),\Pi_2(t)\nu\otimes\psi(g)\rangle dt \\ &+ \sum_{n,k=0}^{\infty}\overline{f(t)}^n g(t)^k\langle\Pi_1(t)u\otimes\psi(f),\ C_{(n,k)}(t)\nu\otimes\psi(g)\rangle dt \\ &+ \sum_{n,k=0}^{\infty}\sum_{N,K=0}^{\infty} nN\overline{f(t)}^{k+N+1} g(t)^{n+K+1}\langle A_{(n,k)}(t)u\otimes\psi(f),C_{(N,K)}(t)\nu\otimes\psi(g)\rangle dt \end{aligned}$$

from which the result follows by integrating from 0 to t and making use of the relations $\Pi_1(0) = \Pi_2(0) = 0$. □

Theorem 4.2 (Sufficient unitarity conditions). *Suppose that the coefficient processes $A_{(n,k)}$, $n, k = 0, 1, \dots$, are such that the evolution*

$$dU(t) = \left[\sum_{n,k=0}^{\infty} A_{(n,k)}(t)dB_{(n,k)}(t)\right]U(t), \qquad U(0) = I, \qquad 0 \leqslant t \leqslant T < +\infty, \tag{4.9}$$

admits a unique solution $U = \{U(t) \mid 0 \leqslant t \leqslant T < +\infty\}$ and satisfies the identities

$$A^*_{(k,n)}(t) + A_{(n,k)}(t) + \sum_{\substack{\alpha,\beta,\lambda,\mu \\ \alpha+\mu=n+1 \\ \beta+\lambda=k+1}} \beta\mu A_{(\alpha,\beta)}(t)A^*_{(\lambda,\mu)}(t) = 0, \tag{4.10}$$

$$A^*_{(k,n)}(t) + A_{(n,k)}(t) + \sum_{\substack{\alpha,\beta,\lambda,\mu \\ \alpha+\mu=k+1 \\ \beta+\lambda=n+1}} \alpha\lambda A^*_{(\alpha,\beta)}(t)A_{(\lambda,\mu)}(t) = 0 \tag{4.11}$$

for any $t \in [0, T]$, where α, β, $\mu = 0, 1, 2, \dots$ Then U is a unitary process.

Proof. For $u, \nu \in H_0$ and for $f, g \in S_+$ such that $|f(s)| \leqslant 1$ and $|g(s)| \leqslant 1$ on $[0, T]$, it follows from the integral form of (4.9) that

$$\begin{aligned} &\langle U(t)u\otimes\psi(f),U(t)\nu\otimes\psi(g)\rangle - \langle u\otimes\psi(f),\ \nu\otimes\psi(g)\rangle \\ &= \left\langle u\otimes\psi(f),\ \sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)U(s)dB_{(n,k)}(s)\nu\otimes\psi(g)\right\rangle \\ &+ \left\langle \sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)U(s)dB_{(n,k)}(s)u\otimes\psi(f),\nu\otimes\psi(g)\right\rangle \\ &+ \left\langle \sum_{n,k=0}^{\infty}\int_0^t A_{(n,k)}(s)U(s)dB_{(n,k)}(s)u\otimes\psi(f),\ \sum_{N,K=0}^{\infty}\int_0^t A_{(N,K)}(s)U(s)dB_{(N,K)}(s)\nu\otimes\psi(g)\right\rangle \end{aligned}$$

which, by Lemma 2.1 (i) and Lemma 4.1, is equal to

$$\int_0^t \sum_{n,k=0}^{\infty} \overline{f(s)}^n g(s)^k \langle u \otimes \psi(f), A_{(n,k)}(s)U(s)\nu \otimes \psi(g)\rangle ds$$
$$+ \int_0^t \sum_{n,k=0}^{\infty} \overline{f(s)}^k g(s)^n \langle A_{(n,k)}(s)U(s)u \otimes \psi(f), \nu \otimes \psi(g)\rangle ds$$
$$+ \int_0^t \Bigg[\sum_{n,k=0}^{\infty} \overline{f(s)}^k g(s)^n \langle A_{(n,k)}(s)U(s)u \otimes \psi(f), (U(s)-I)\nu \otimes \psi(g)\rangle$$
$$+ \sum_{n,k=0}^{\infty} \overline{f(s)}^n g(s)^k \langle (U(s)-I)u \otimes \psi(f), A_{(n,k)}(s)U(s)\nu \otimes \psi(g)\rangle$$
$$+ \sum_{n,k=0}^{\infty} \sum_{N,K=0}^{\infty} nN\overline{f(s)}^{k+N-1} g(s)^{n+K-1} \langle A_{(n,k)}(s)U(s)u \otimes \psi(f), A_{(N,K)}(s)U(s)\nu \otimes \psi(g)\rangle \Bigg] ds$$
$$= \int_0^t \Bigg\langle U(s)u \otimes \psi(f), \Bigg[\sum_{n,k=0}^{\infty} \overline{f(s)}^k g(s)^n A^*_{(n,k)}(s) + \sum_{n,k=0}^{\infty} \overline{f(s)}^n g(s)^k A_{(n,k)}(s)$$
$$+ \sum_{n,k=0}^{\infty} \sum_{N,K=0}^{\infty} nN\overline{f(s)}^{k+N-1} g(s)^{n+K-} A^*_{(n,k)}(s)A_{(N,K)}(s) \Bigg] U(s)\nu \otimes \psi(g) \Bigg\rangle ds. \qquad (4.12)$$

By (4.11), the coefficient of $\overline{f(s)}^n g(s)^k$ in (4.12) is equal to zero. Thus,

$$\langle U(t)u \otimes \psi(f), U(t)\nu \otimes \psi(g)\rangle - \langle u \otimes \psi(f), \nu \otimes \psi(g)\rangle = 0,$$

and hence, by Proposition 7.2 of [8], the operator $U(t)$ extends to a unique linear isometry on $H_0 \otimes \Gamma(S_+)$.

Similarly, by considering the dual equation of (4.9) and using (4.10), we can conclude that $U(t)$ extends to a coisometry in $H_0 \otimes \Gamma(S_+)$. Thus, U is a unitary process. □

Remark. We can readily see that the unitarity conditions (1.4) are contained in (4.10) and (4.11). For example, if $n = 0$ and $k = 1$, then (4.10) implies $\alpha = 0$, $\mu = 1$ and either $\beta = 1$, $\lambda = 1$ or $\beta = 2$, $\lambda = 0$, thus yielding the second condition of (1.4), etc.

REFERENCES

1. Accardi, L. and Boukas, A., Unitarity Conditions for Stochastic Differential Equations Driven by Non-Linear Quantum Noise, submitted.
2. Accardi, L. and Boukas, A., Stochastic Evolutions Driven by Non-Linear Quantum Noise, submitted.
3. Accardi, L., Lu, Y. G., and Obata, N., Towards a Non-Linear Extension of Stochastic Calculus, *Publications of the Research Institute for Mathematical Sciences, Kyoto, RIMS Kokyuroku*, 1996, vol. 957, pp. 1–15.
4. Accardi, L., Lu, Y. G., and Volovich, I., White Noise Approach to Stochastic Calculus and Nonlinear Itô Tables, submitted for publication in *Nagoya Journal of Mathematics*.
5. Accardi, L. and Volovich, I., Quantum White Noise with Singular Non-Linear Interaction, submitted.
6. Fagnola, F., On Quantum Stochastic Differential Equations with Unbounded Coefficients, *Probab. Th. Rel. Fields*, 1990, vol. 86, pp. 501–516.
7. Hudson, R. L. and Parthasarathy, K. R., Quantum Itô's Formula and Stochastic Evolutions, *Comm. Math. Phys.*, 1984, vol. 93, pp. 301–323.
8. Parthasarathy, K. R., *An Introduction to Quantum Stochastic Calculus*, Boston: Birkhauser, 1992.