Note that $V(P) = V((V(P)/(1-\alpha))\, M_\alpha + (1 - (V(P)/(1-\alpha))\, U)$ and that, moreover, $P \succcurlyeq$ the distribution within the outer brackets on the right. From this, and a similar remark about Q, it follows the observations, and the facts that $P * U = U$ and $V(p\,P + q\,U) = p\,V(P)$, that $W_1(\overline{P} * \overline{Q}) \geqq W_1(\overline{P})\, W_1(\overline{Q})$. By the remark preceding observation 1, this proves the lemma for absolutely continuous P.

Since a unimodal symmetric P is a mixture of an absolutely continuous distribution and a degenerate one concentrated at a single point, the proof of the lemma for T is completed.

The proof of $\mathbb{Z}/(n)$ is quite similar. The principal difference is perhaps the fact that the equality in observation 1 is replaced by an inequality. More precisely, if $\overline{M}_\alpha$ is the distribution assigning the probability $1/\alpha$ to each of α adjacent integers, and similarly for $\overline{M}_\beta$, then $1/4\alpha\beta \geqq W_a(\overline{M}_\alpha * \overline{M}_\beta) - W_a(\overline{M}_\alpha)\, W_a(\overline{M}_\beta) \geqq 0$.

Extension. – Since it was essentially the unimodality and symmetry of $\overline{P}$ that was used in the proof, it is easy to see that the lemma [for $\mathbb{Z}/(n)$] remains valid also if P is the reduction (mod n) of a unimodal symmetic distribution, on an arithmetic progression of integers.

3.1. *Proof of theorem* 1. – We may assume that P is unimodal symmetric about 0. Consider first absolutely continuous P. It is the limit of finite mixtures of rectangular distributions M_α (also symmetric about 0). Since $\hat{M}_\alpha(j) = (\sin \pi_j\, \alpha)/\pi_j\, \alpha$ it follows that $|\hat{P}(j)| \leqq \hat{P}(1)$ for $|j| > 1$ (equality occurring only when $P = U$).

By the Upper Bound Lemma of Diaconis-Shahshahani $|V(P^{*m})|^2 \leqq 1/4 \sum_{j \neq 0} |\hat{P}(j)|^{2m}$ and hence $\leqq ((1/2) + o(1))|\hat{P}(1)|^{2m}$ as $m \to \infty$. But, by our lemma, this could not happen if (1) is violated, even for a single m.

If P is not absolutely continuous then $P = p\,A + q\,P_r$ with A consisting of a single atom. Since V is a convex functional $(V(pP + qQ) \leqq pV(P) + qV(Q))$ it follows that $V(P^{*m}) \leqq \sum_{i=0}^{m} p^{m-i} q^i V(P_r^{*i}) \leqq (p + q|\hat{P}_r(1)|)^m = |\hat{P}(1)|^m$ and the proof is completed.

3.2. *Proof of theorem* 2. – Consider first the case $d = 1$. The proof follows, step by step, the one of theorem 1. In this case $\hat{M}_\alpha(j) = (\sin \pi\alpha j/n)/\alpha \sin \pi j/n$ and again $|\hat{P}(j)| \leqq |\hat{P}(1)|$ for $1 < j \leqq n-1$. The only point to worry about is if equality occurs for any $j \neq n-1$. But this can occur only if P is either the uniform distribution, or assigns the same probability to all but one of the integers $0, 1, \ldots, n-1$. In the latter case, the validity of the theorem follows from an easy direct calculation (*see* e.g. exercise 6 on page 25 of [1]).

The case $d > 1$ is treated in exactly the same way, via the extension to the lemma [$|\hat{P}(j)|$ for $0 < j < n$ is maximized for $j = r$].

3.3 *Proof of theorem* 3. – By the assumptions of the theorem $P_n = p_n M_{\alpha_n} + (1-p_n)\, Q_n$, where $\alpha_n = [\lambda\, n]$ and $p_n = a\, \alpha_n$. The result then follows directly from the lemma, the convexity of V mentioned above and the fact that $\hat{M}_\alpha(1) = (\sin \pi\alpha/n)/\alpha \sin \pi/n$.

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Probabilités/*Probability Theory*

The Ornstein-Uhlenbeck process and the Dirichlet form associated to the Lévy Laplacian

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Abstract – We prove the existence of an Ornstein-Uhlenbeck type process associated to the Lévy Laplacian. The corresponding semigroup is explicity described. The Dirichlet form associated to this process gives a new type of "gradient" (or classical) Dirichlet forms of essentially infinite dimensional character because it vanishes identically on the cylindrical functions.

Le processus d'Ornstein-Uhlenbeck et la forme de Dirichlet associées au laplacien de Lévy

Résumé – *On montre l'existence d'un processus du type d'Ornstein-Uhlenbeck associé au laplacien de Lévy. Le semigroupe correspondant est construit explicitement. La forme de Dirichlet associée à ce processus engendre un type nouveau de forme de Dirichlet du type gradient qui est essentiellement de dimension infinie puisqu'elle s'annule identiquement sur les fonctions cylindriques.*

Version française abrégée – Soient E un espace localement convexe et $H \subset E^*$ un espace de Hilbert dense dans E^*, tel que l'inclusion naturelle soit continue. Alors on obtient le triplet $E \subset H^* = H \subset E^*$. Soit $\{e_n\}_{-\infty}^{\infty}$ une base orthogonale de H telle que $\{e_n\} \subset E$. Supposons que $S : E^* \to E^*$ soit un homéomorphisme linéaire topologique tel que $S(H) = S$ et $Se_j = e_{j+1}$. Désignons par $\mathcal{M}$ la classe des mesures de Radon réelles μ sur E^* invariantes par rapport à S, telles que

$$\int_{E^*} z_j^2(z)\, |\mu|(dz) < \infty, \qquad \forall j,$$

où $z_j(z) = \langle z, e_j \rangle$. D'après le théorème ergodique, pour toute mesure $\mu \in \mathcal{M}$ la limite

$$g(z) = \lim_{n\to\infty} \frac{\sum_{j=-n}^{n} z_j^2}{2n+1} \quad \text{existe } \mu\text{-p.p.}$$

Désignons par $\mathcal{F}$ la classe des transformées de Fourier

$$\hat{\mu} : \quad E \to C, \qquad \hat{\mu}(x) = \int \exp(i\langle z, x \rangle)\, \mu(dz), \qquad \mu \in \mathcal{M}.$$

Le complété $\mathcal{C}_E$ de $\mathcal{F}$ par rapport à la norme uniforme est une C^*-algèbre isomorphe à la C^*-algèbre $C(Q)$, Q étant le spectre de $\mathcal{C}_E$ qui est un espace compact. Il existe une injection naturelle continue de E dans Q. Nous étudions le processus $Z(t)$ à générateur $A_L/2$, où

$$A_L f(x) := \Delta_L f(x) - \langle f'(x), x \rangle := \lim_{n\to\infty} \left(\frac{\sum_{i=-n}^{n} \partial_{e_i}^2 f}{2n+1} - \sum_{i=-n}^{n} x_i\, \partial_{e_i} f(x) \right),$$

$$f \in \mathcal{F}.$$

Note présentée par Paul Malliavin.

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Soit $\{ w_n(t) \}$ une suite des processus de Wiener réels indépendants. Introduisons les processus à valeurs dans E :

$$S_n(t) = \frac{\sum_{i=-n}^{n} w_i(t)\, e_i}{\sqrt{2n+1}}, \qquad dZ_n(t) = dS_n(t) - \frac{1}{2} Z_n(t)\, dt, \qquad Z_n(0) = 0.$$

Notons $T_t^{(n)}$ et $P_t^{(n)}$ les semigroupes correspondants.

THÉORÈME 1. – (i) *Les semigroupes $P_t^{(n)}$ et $T_t^{(n)}$ sur $\mathcal{F}$ convergent fortement vers des semigroupes markoviens P_t et T_t qui possèdent des extensions fortement continues sur $\mathcal{C}_E$ (désignées par les mêmes symboles). Pour toute $f = \hat{\nu} \in \mathcal{F}$ on a :*

$$T_t f(x) = \int_{E^*} \exp(i\langle z, e^{-t/2} x\rangle) \exp\left(-\frac{1-e^{-t}}{2} g(z)\right) \nu(dz).$$

De plus, le générateur de T_t sur $\mathcal{F}$ coïncide avec A_L.

(ii) *Il existe des processus markoviens $W(t)$ et $Z(t)$ à valeurs dans Q tel que les semigroupes associés sur $C(Q)$ soient P_t et T_t respectivement.*

(iii) *La loi Λ_1 de $W(1)$ est une mesure invariante pour $Z(t)$ et*

$$\Lambda_1(f) = \int_{E^*} \exp(-g(z)/2)\nu(dz), \qquad \forall f = \hat{\nu} \in \mathcal{F}$$

THÉORÈME 2. – *La suite des formes de Dirichlet $\mathcal{E}_n$ associées aux processus $Z_n(t)$ converge sur $\mathcal{F}$ vers la forme de Dirichlet $\mathcal{E}_L$ liée à $Z(t)$ par $\mathcal{E}_L(f, f) = -\Lambda_1(f A_L f)$.*

THÉORÈME 3. – *Le semigroupe T_t est hypercontractif et pour tout $f \in \mathcal{F}$ on a :*

$$\Lambda_1(|f|^2 \log |f|) \leqq \Lambda_1((\nabla_L f, \nabla_L f)_K) + 2^{-1} \Lambda_1(|f|^2) \log \Lambda_1(|f|^2).$$

THE ORNSTEIN-UHLENBECK SEMIGROUP ASSOCIATED TO THE LÉVY BROWNIAN MOTION. – Let E be a locally convex space, E^* its dual, H a separable Hilbert space continuously and densely embedded into E^*. Denote by $\langle \cdot, \cdot \rangle$ the duality between E^* and E. For any $v \in E$ the functional $h \mapsto \langle h, v\rangle$ is continuous on H, hence there is a vector $j(v) \in H$ such that $\langle h, v\rangle = (h, j(v))_H$, $\forall h \in H$. Thus, we obtain the embedding $j : E \to H$. We shall identify E with $j(E)$.

Let $\{ e_n \}_{n=-\infty}^{\infty}$ be an orthonormal basis in H such that $\{ e_n \} \in j(E)$. We shall assume that there is a continuous linear homeomorphism $S : E^* \to E^*$ such that $S(H) \subset H$ and $Se_j = e_{j+1}$. For any n put $z_n : E^* \to R^1$, $z \mapsto \langle z, e_n\rangle$, $x_n : E \to R^1$, $x \mapsto \langle e_n, x\rangle$.

We study the stochastic processes with generators $\Delta_L/2$ and $A_L/2$, where

(1) $$\Delta_L := \lim_{n\to\infty} \frac{\sum_{i=-n}^{n} \partial_{e_i}^2}{2n+1},$$

(2) $$A_L := \Delta_L - x\nabla := \lim_{n\to\infty} \left(\frac{\sum_{i=-n}^{n} \partial_{e_i}^2}{2n+1} - \sum_{i=-n}^{n} x_i\, \partial_{e_i} \right).$$

Solutions of the classical Yang-Mills equations have been shown in [1] to be harmonic functions of the Lévy Laplacian Δ_L. Unlike the case of the Volterra-Gross Laplacian, in our setting there are no processes in E^* with the generators of the form (1), (2).

Denote by $\mathcal{M}$ the space of bounded signed Radon measures μ on E^* satisfying the following two conditions:

$$\mu \circ S^{-1} = \mu, \quad \int_{E^*} x_i^2 \,|\mu|(dx) < \infty, \qquad \forall i.$$

According to the ergodic theorem for any measure $\mu \in \mathcal{M}$ the limit

$$g(x) = \lim_{n\to\infty} \frac{\sum_{i=-n}^{n} x_i^2}{2n+1} \quad \text{exists } \mu\text{-a.e. and in } L^2(\mu).$$

The Fourier transform of a measure μ on E^* is defined by

$$\hat{\mu} : \ E \to C^1, \qquad \hat{\mu}(v) = \int_{E^*} \exp(i\langle x, v\rangle)\, \mu(dx).$$

Denote by $\mathcal{F}$ the space of the Fourier transforms of measures in $\mathcal{M}$. The completion $\mathcal{C}_E$ of $\mathcal{F}$ with the sup-norm is an abelian C^*-algebra, so it is algebraically and topologically isomorphic to the algebra $C(Q)$ of all continuous functions on its spectrum Q. Q is a compact space and we may assume that E is continuously embedded in Q.

Now let $\{ w_n(t) \}$ be a sequence of independent real standard Wiener processes on a probability space (Ω, P). Denote by E_n the linear span of the vectors $e_{-n}, \ldots, e_n$ and define the $E_n \subseteq E$-valued (hence Q-valued) processes $S_n(t)$ and $Z_n(t)$ by

$$S_n(t) = \frac{\sum_{i=-n}^{n} w_i(t)\, e_i}{\sqrt{2n+1}}, \qquad dZ_n(t) = dS_n(t) - \frac{1}{2} Z_n(t)\, dt, \qquad Z_n(0) = 0.$$

and denote by $P_t^{(n)}$ and $T_t^{(n)}$ the corresponding semigroups. The processes $S_n(t) + x$ and $Z_n(t) + e^{-t/2} x$ start from the generic point in $x \in E$.

1. THEOREM. – (i) *For any $t \geqq 0$ the operators $T_t^{(n)}$, $P_t^{(n)}$ converge strongly to bounded operators T_t, P_t. The families $\{ T_t \}_{t\geqq 0}$ and $\{ P_t \}_{t\geqq 0}$ are strongly continuous semigroups on $\mathcal{F}$ hence they extend to strongly continuous semigroups on $\mathcal{C}_E$ (thus on $C(Q)$). For any $f = \hat{\nu} \in \mathcal{F}$ one has:*

$$T_t f(x) = \int_{E^*} \exp(i\langle z, x\rangle) \exp\left[-\frac{1-e^{-t}}{2} g(z)\right] \nu(dz).$$

The generator of T_t coincides with A_L on $\mathcal{F}$.

(ii) *The semigroups T_t and P_t give rise to two Markov processes $Z(t)$ and $W(t)$, with state space Q, which are the limits in distribution of the finite-dimensional processes $Z_n(t)$ and $S_n(t)$.*

(iii) *The law Λ_1 of $W(1)$ is an invariant probability of the process $Z(t)$. For any $f = \hat{\nu} \in \mathcal{F}$ one has:*

$$\Lambda_1(f) = \int_{E^*} \exp(-g(z)/2)\nu(dz).$$

Proof. – The semigroups $P_t^{(n)}$ and $T_t^{(n)}$ have the following representation, where $f = \hat{\nu}$:

$$P_t^{(n)} f(x) = \int \hat{\nu}(x-y)\, \mu_{n,t}(dy), \qquad T_t^{(n)} f(x) = \int \hat{\nu}(y)\, \lambda_{n,t,x}(dy),$$

where $\mu_{n,t}$ is the image of the standard Gaussian measure on R^{2n+1} under the mapping $(t_{-n}, \dots, t_n) \mapsto (t/(2n+1))^{1/2} \sum_{i=-n}^{n} t_i e_i$ and $\lambda_{n,t,x}$ is the image of the measure $\mu_{n,1}$ under the map $y \mapsto e^{-t/2} x + \sqrt{1-e^{-t}}\, y$. For any measure λ on E and any measure ν on E^* the following Parseval identity holds:

$$\int_{E^*} \hat{\lambda}(y)\, \nu(dy) = \int_E \hat{\nu}(x)\, \lambda(dx).$$

Therefore, we have:

$$T_t^{(n)} f(x) = \int_{E^*} \hat{\lambda}_{n,t,x}(z)\, \nu(dz)$$
$$= \int_{E^*} \exp(i\langle z, e^{-t/2} x\rangle) \exp\left[-\frac{1-e^{-t}}{4n+2} \sum_{i=-n}^{n} z_i^2\right] \nu(dz),$$
$$P_t^{(n)} f(x) = \int_{E^*} \exp(i\langle z, x\rangle) \exp\left(-\frac{t}{4n+2} \sum_{i=-n}^{n} z_i^2\right) \nu(dz).$$

Now the first assertion can be easily deduced from the Lebesgue theorem. Both semigroups $P_t^{(n)}$ and $T_t^{(n)}$ are Markovian. Hence, P_t and T_t are Markovian as well. Therefore, they possess unique extensions to $\mathcal{C}_E$. These semigroups give rise to two Markov processes $W(t)$ and $Z(t)$ with state space Q. By our construction for any fixed t the laws of $S_n(t)$ and $Z_n(t)$ converge weakly to the laws of $W(t)$ and $Z(t)$ respectively. The law of $W(1)$ is an invariant probability for the process $Z(t)$, since the invariant probabilities μ_n of the processes $Z_n(t)$ converge weakly to the measure Λ_1.

2. Remark. – For $f \in \mathcal{F}$ the function $A_L f$ typically is not the Fourier transform of a bounded measure. For example, let ν be the Gaussian measure on E^* with Fourier transform $f(x) = \exp(-(x, x)_H/2)$. Then $(\Delta_L - A_L) f(x) = (x\nabla) f(x) = -(x, x) \exp(-(x, x)_H/2)$, which is in $\mathcal{C}_E$ but is not the Fourier transform of any bounded measure on E^*.

The Lévy gradient and the associated Dirichlet form. – Note that the process $Z_n(t)$ is the diffusion corresponding to the Dirichlet form

$$\mathcal{E}_n(f, f) = (2n+1)^{-1} \int \sum_{i=-n}^{n} (\partial_{e_i} f(x))^2\, \mu_n(dx).$$

3. Theorem. – *The sequence of the Dirichlet forms $\mathcal{E}_n$ converge pointwise on $\mathcal{F}$. We denote its limit by $\mathcal{E}_L$. For any $f \in \mathcal{F} \cap D(A_L)$ one has: $\mathcal{E}_L(f, f) = -\Lambda_1(f A_L f)$.*

We shall construct a gradient (which will be called the Lévy gradient) in such a way that

$$\mathcal{E}_L(f, f) = \Lambda_1((\nabla_L f, \nabla_L f)_{\mathcal{K}}). \tag{3}$$

Let X be the space of all real sequences (x_n) satisfying the condition

$$\limsup_n \frac{\sum_{i=-n}^{n} x_i^2}{n} < \infty.$$

As shown in [3], there exists a pre-scalar product $(\cdot, \cdot)_0$ on X with the following property: if the limit $\lim_{n\to\infty} (1/(2n+1)) \sum_{i=-n}^{n} h_i k_i$ exists for two elements $h = (h_n)$ and $k = (k_n)$ in X then this limit coincides with $(h, k)_0$. The completion of the quotient space $X/(\cdot, \cdot)_0$ with respect to this inner product is denoted by $\mathcal{K}$.

For any $f = \hat{\nu} \in \mathcal{F}$ let $\nabla_L f(x) = (\partial_{e_i} f(x))$. Since the sequence $\sum_{i=-n}^{n} z_i^2/(2n+1)$ converges in $L^1(\mu)$, it follows that $\nabla_L f(x) \in \mathcal{K}$.

4. Theorem. – *Equality* (3) *holds for any* $f \in \mathcal{F}$.

It is possible to show that the Dirichlet form $\mathcal{E}_L$ is in a certain sense a diffusion Dirichlet form.

Hypercontractivity.

5. Theorem. – *The Ornstein-Uhlenbeck semigroup T_t associated with the Lévy Laplacian is hypercontractive,* i.e. (cf. [8]) *$\|T_t f\|_q \leqq \|f\|_p$ holds for any $t > 0$, $p > 1$, $q > 1$, such that $e^{2t} \geqq (q-1)/(p-1)$. Equivalently, the following logarithmic Sobolev inequality holds:*

$$\int f^2 \log f\, \Lambda_1 \leqq \int (\nabla_L f, \nabla_L f)_{\mathcal{K}}\, \Lambda_1 + \|f\|_2^2 \log \|f\|_2.$$

Let us compare the process $Z(t)$ with the standard Ornstein-Uhlenbeck process $\xi(t)$ in E^* associated with H (which exists if, for example, E is a nuclear Fréchet space). Using the basis $\{e_n\}$ we can define the process $\xi(t)$ by $\xi(t) = \sum_{n=-\infty}^{\infty} \xi(t)\, e_n$, where $\{\xi_n(t)\}$ is a sequence of independent standard real Ornstein-Uhlenbeck processes. If we try to obtain the process $\xi(t)$ by the method applied above for constructing $Z(t)$, we have the finite dimensional processes $\eta_n(t) = \sum_{i=-n}^{n} \xi_i(t)\, e_i$ and the corresponding semigroup $T_t^{\eta,n}$ given by

$$T_t^{\eta,n} f(x) = \int f(e^{-t/2} x + \sqrt{1-e^{-t}}\, y)\, \gamma_n(dy),$$

where γ_n is the Gaussian measure on E_n with Fourier transform $\exp\left(-\sum_{i=-n}^{n} x_i^2/2\right)$.

By the Parseval formula (with $f = \hat{\nu}$) we have:

$$T_t^{\eta,n} f(x) = \int \exp(i\langle z, e^{-t/2} x\rangle) \exp\left(-\frac{1-e^{-t}}{2} \sum_{i=-n}^{n} z_i^2\right) \nu(dz).$$

Thus, if ν is the Dirac measure δ the limit equals 1, otherwise it is zero. However, in this situation we get a nontrivial limit semigroup if we choose a bigger function space. Let $\mathcal{F}_{\text{fin}}$ be the collection of the Fourier transforms of measures concentrated on the finite dimensional spaces E_n and let $\mathcal{F}_0 = \mathcal{F} + \mathcal{F}_{\text{fin}}$. $\mathcal{F} \cap \mathcal{F}_{\text{fin}}$ contains only the constants by S-invariance of the measures in $\mathcal{M}$. On this bigger functional space $\mathcal{F}_0$ the limit semigroup T_t^{ξ} (which is the transition semigroup of the process $\xi(t)$ on E^*) acts by

$$T_t^{\xi} f(x) = \int_{E^*} f(e^{-t/2} x + \sqrt{1-e^{-t}}\, y)\, \gamma(dy),$$

where γ is the Gaussian measure on E^* with the Fourier transform $\exp(-(x, x)_H/2)$ (this is the invariant measure of the Ornstein-Uhlenbeck process $\xi(t)$). Our new function space contains all trigonometric functions of the form $\exp\left(i \sum_{j=-n}^{n} c_j x_j\right)$ since these are the Fourier transforms of the atomic measures. The linear span of such functions is a core of the Dirichlet form corresponding to $\xi(t)$ defined by the formula

$$\mathcal{E}^{\xi}(f, f) = \int_{E^*} (\nabla_H f(z), \nabla_H f(z))_H \, \gamma(dz),$$

where

$$(\nabla_H f(z), h)_H = \partial_h f(z), \qquad \forall h \in H.$$

At this point the parallel between the processes $Z(t)$ and $\xi(t)$ is broken. Though the operators A_L and ∇_L are well-defined on $\mathcal{F}_{\text{fin}}$, this subspace is far from being a core for $\mathcal{E}_L$. $\mathcal{E}_L$ leads to a new class of Dirichlet forms of "gradient type" which principally differs from the examples studied in [4], [6], [7], [9].

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Probabilités/*Probability Theory*

Courbures de l'espace de probabilités d'un mouvement brownien riemannien

Ana-Bela Cruzeiro et Paul Malliavin

Résumé – Renormalisation hypoelliptique. Entrelacement des processus tangents. La connexion markovienne et la dérivation d'intégrales stochastiques. Géométrie différentielle en indices continus : torsion et courbure de la connexion markovienne. Tenseur de Ricci markovien. Formule de Weitzenböck et identité d'énergie des intégrales stochastiques anticipantes. Explosion de l'identité de Weitzenböck pour la connexion de Levi-Civita. Renormalisation infinitésimale des géodésiques.

Curvatures of the probability space associated to a Riemannian Brownian Motion

Abstract – *Hypoelliptic Renormalization. Intertwinning of Tangent processes. Markovian connection and derivation of stochastic integrals. Differential Geometry with continuous indices: Torsion and Curvature of the Markovian connection. Markovian Ricci tensor. Weitzenböck formula and energy identity for anticipative stochastic integrals. Explosion of the Weitzenböck identity for the Levi-Civita Connection. Infinitesimal renormalization of the geodesics.*

Le Calcul des Variations stochastiques a rendu naturel le concept d'*espace tangent à un espace de probabilités* sur lequel s'est trouvé ainsi développée une Géométrie différentielle du premier ordre. On se propose dans cette Note d'introduire une Géométrie différentielle du second ordre en calculant les *courbures de l'espace de probabilité* d'un mouvement brownien riemannien. Les éléments de calcul différentiel d'ordre 2 se trouvent contenus en germe dans la formule de Itô de changement de variables pour les semimartingales ; de ce fait il n'est pas complètement inattendu que la détermination de l'identité d'énergie pour les intégrales stochastiques anticipantes intrinsèques nécessite le calcul des courbures de l'espace de Probabilités ; un nouveau point de rencontre entre Géométrie et Analyse stochastique se trouve ainsi mis en évidence.

La Note précédente [1] a introduit un repère mobile canonique sur $P_{m_0}(M)$ l'espace des chemins d'une variété riemannienne compacte M ; ses fonctionnelles de structure ont été calculées ; la connexion de Levi-Civita de $P_{m_0}(M)$ a été exprimée dans ce repère mobile; toutes les renormalisations de cette géométrie différentielle de dimension infinie ont été effectuées de façon intrinsèque par des intégrales stochastiques. Dans cette Note nous utiliserons librement les notations de la Note précédente.

1. Renormalisation hypoelliptique. – Le transport parallèle de Itô définit une isométrie $t^p_{\tau_1 \leftarrow \tau_2} : T_{p(\tau_2)}(M) \to T_{p(\tau_1)}(M)$. Les *champs de vecteurs constants* sont les champs de vecteurs de la forme $Z_\tau = t^p_{\tau \leftarrow 0}(z(\tau))$ où z ne dépend pas de p ; les champs de vecteurs constants constituent le système de coordonnées intrinsèques que définit le repère mobile ; ce système de coordonnées permettra de transférer en dimension infinie les calculs en coordonnées locales de la dimension finie. Dans ce contexte l'équation de structure du repère mobile s'écrit [1] :

$$d_\tau([u, v]) = \Omega(u, v) \circ dx + (Q_u \dot{v} - Q_v \dot{u})\, d\tau$$

Note présentée par Paul Malliavin.

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