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Non–Commutative (Quantum) Probability, Master Fields and Stochastic Bosonization

L. Accardi **Y.G. Lu**(*) **I. Volovich**(**)

Centro V. Volterra
Università di Roma Tor Vergata
via di Tor Vergata, 00133 – Roma
e–Mail volterra@mat.utovrm.it

Abstract. In this report we discuss some results of non–commutative (quantum) probability theory relating the various notions of statistical independence and the associated quantum central limit theorems to different aspects of mathematics and physics including: q–deformed and free central limit theorems; the description of the master (i.e. central limit) field in matrix models along the recent Singer suggestion to relate it to Voiculescu's results on the freeness of the large N limit of random matrices; quantum stochastic differential equations for the gauge master field in QCD; the theory of stochastic limits of quantum fields and its applications to stochastic bosonization of Fermi fields in any dimensions; new structures in QED such as a nonlinear modification of the Wigner semicircle law and the interacting Fock space: a natural explicit example of a self–interacting quantum field which exhibits the non crossing diagrams of the Wigner semicircle law.

(*) Dipartimento di Matematica, Università di Bari.
(**) On leave absence from Steklov Mathem.Inst, Vavilov St.42,GSP–1,117966, Moscow.

(1.) Introduction

In this report we discuss some basic results of non-commutative (quantum) probability theory and their applications to large N limit and stochastic limit in quantum field theory. 1/N expansion is probably one of the most powerful methods in investigation of essential non-linear effects in quantum field theory and statistical physics. There are two main types of models admitting the large N expansion. These are so called vector and matrix models. Matrix models are more difficult for investigation. 1/N expansion was used to describe phase transition of N component spin models [St,BrZ]. It was shown [Ar76] that for vector models the using $1/N$ expansion permits to prove the renormalizability of 3 dim σ-model which is nonrenormalizable in the standard perturbation theory. The large N limit for vector models can be interpreted as a central limit theorem from the point of view of probability theory [AAV93]. For matrix model the large N colour expansion introduced by 't Hooft [tHo74] is one of most promising technique for arriving at an analytical understanding of long distance properties of non abelian gauge theories. The major obstacle to realization of this program is the fact that it has been impossible up to now to compute the leading term in this expansion, i.e. the sum of the planar graphs in the closed form unless one works in zero or one dimension [BIPZ]. Many matrix models simplify greatly in the limit of large N and it was suggested that the limiting behaviour of a model can be described in terms of so called the *master field* [Wit,MM, Ha80, Are81]. An equation for the Wilson loop functional in the large N limit was obtained by Makeenko and Migdal [MM].

In what follows we will explain that the master field for the simplest vector model is nothing but the operator of coordinate of the standard quantum mechanical harmonic oscillator. Recently Singer [Sin94], based on Voiculescu's works, has shown that the master field for matrix models can be described in terms of free random variables satisfying free commutation relations. The free commutation relations is a q-deformation of canonical commutation relations for $q = 0$. Non-commutative (quantum) probability theory considers central limit theorems for q-deformed commutation relations and provides an unified approach to both vector and matrix models.

The simplest classical central limit theorem for Gaussian random variables reads

$$\lim_{N\to\infty} \langle S_N^k \rangle_N = \int_R \lambda^k \varrho(\lambda) d\lambda \tag{1.1}$$

where $k = 1, 2, ...,$

$$S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i,$$

$$\langle f(x) \rangle_N = \frac{1}{Z_N} \int_{R^N} f(x) e^{-\frac{x^2}{2}} dx \tag{1.2}$$

$$Z_N = \int_{R^N} e^{-\frac{x^2}{2}} dx = (2\pi)^{\frac{N}{2}} \ , \qquad x = (x_1, ..., x_n)$$

The function $\varrho(\lambda)$ is the Gaussian density,

$$\varrho(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} \tag{1.3}$$

Below we will use also other notations for the expectation value,

$$\langle f \rangle = Ef = \varphi(f)$$

One can rewrite (1.1) as follows. Let us introduce an operator

$$Q = a^* + a \tag{1.4}$$

where a and a^* are usual annihilation and creation operators satisfying

$$[a, a^*] = 1 \tag{1.5}$$

Then one has

$$\int_R \lambda^k \varrho(\lambda) d\lambda = (\Phi, Q^k \Phi) \tag{1.6}$$

The vector Φ in (1.6) is the vacuum vector in the Fock space for the quantum harmonic oscillator, $a\Phi = 0$. As it is well known this Fock space is isomorphic to $L^2(R)$.

We can interpret the position operator Q as a *master field* (a *random variable* in the sense of quantum probability). So, the master field for simplest vector model is the operator of coordinate for harmonic oscillator,

$$\lim_{N \to \infty} \langle S_N^k \rangle_N = (\Phi, Q^k \Phi). \tag{1.7}$$

If the covariance of our random variables is not 1 but $\hbar$,

$$\langle x_i^2 \rangle_N = \hbar$$

then instead of (1.5) one gets

$$[a, a^*] = \hbar \quad (1.8).$$

Now, the position operator gives (roughly speaking) *one half* of the quantum observables: the other half being given by the momentum operator $P = (a - a^+)/i$. In this sense one can say that we can get *half* quantum mechanics (more exactly, an abelian subalgebra of the quantum mechanical algebra generated by the operators of momenta and position) as a limit in classical probability theory. In order to obtain the *other half* the classical central limit theorems are not sufficient and one must look at the quantum central limit theorems. This was done in the late seventies.

For a non-Gaussian family of independent random variables the central limit theorem can be formulated in a form similar to (1.1), (1.7) but now

$$S_N = \frac{1}{\sqrt{D_N}} \sum_{i=1}^{N} x_i \tag{1.9}$$

where the covariance is $D_N = \langle S_N^2 \rangle_N$ and the exponential function in (1.2) can have non-quadratic terms. If the limit in (1.7) is non-Gaussian, then the master field Q will be a non-linear function of a and a^*, $Q = Q(a, a^*)$.

Now let us discuss matrix models. Let $A = (A_{ij})_{i,j=1}^N$ be an ensemble of real symmetric $N \times N$ matrices with distribution given by

$$\langle f(A) \rangle_N = \frac{1}{Z_N} \int f(A) e^{-NS(A)} \prod_{i \le j} dA_{ij}, \tag{1.10}$$

$$Z_N = \int e^{-NS(A)} \prod_{i \le j} dA_{ij}$$

where the *action* $S(A)$ is quadratic,

$$S(A) = \frac{1}{2} tr A^2 \tag{1.11}$$

It follows from the works of Wigner [Wig] and Voiculescu [Voi92] that the limits

$$\lim_{N \to \infty} \langle \frac{1}{N} tr(A^k) \rangle_N = (\Phi, Q^k \Phi) \qquad ; \qquad k = 1, 2, \ldots \tag{1.12}$$

exists. The limiting object Q in (1.1) is a quantum random variable (in the sense of Section (2.) below) which is called *the master field*. It is defined as

$$Q = a^* + a \tag{1.13}$$

where a^* and a are *free* creation and annihilation operators, i.e. they do not satisfy the usual canonical commutation relations but the following

$$a a^* = 1 \tag{1.14}$$

The vector Φ in (1.1) is a *vacuum vector* in a *free* Fock space,

$$a \Phi = 0$$

The free Fock space is constructed starting from a^* and Φ, by the usual procedure but only by using the operators a and a^* satisfying (1.14) (cf. Section (4.) for more details). The expectation value in (1.12) is known to be

$$(\Phi, Q^k \Phi) = \int_R \lambda^k w(\lambda) d\lambda \tag{1.15}$$

where $w(\lambda)$ is the Wigner semicircle density,

$$w(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2} \qquad , \quad |\lambda| \le 2 \tag{1.16}$$

and $w(\lambda) = 0$ for $|\lambda| \geq 2$. The Wigner semicircle distribution in noncommutative probability plays the role of the Gaussian distribution in classical probability (cf. Section 7 for a substantiation of this statement).

Generalizations of the usual commutation (or anticommutation) relations have been studied, in relation to quantum groups (cfr., for example, [AV91, APVV]) and to quantum probability (cf. [Schü93]). One can consider the relation (1.14) as a limiting case ($q = 0$) of q-deformed commutation relations,

$$aa^* - qaa^* = \hbar \tag{1.17}$$

which reduce to the usual quantum mechanical oscillator and classical probability theory when $q = 1$. From this point of view a q–deformation ($q = 0$)of canonical commutation relations corresponds to a transition from the vector models to matrix models in the large N limit.

If one considers a more general action of the form

$$S(A) = S_0(A) + S_I(A)$$

where $S_0(A)$ has the form (1.11) and $S_I(A)$ is a more general nonlinear functional, then expanding the term $\exp S_I(A)$ in power series and exchanging formally the summation and the limit in (1.12), one still is lead to consider limits of the form (1.12). For this reason it is quite natural to conjecture, that the Voiculescu results should find a natural application in the large N expansion of quantum field models. In particular in QCD would be important to describe the large N behaviour. Recently Singer [Sin94] has suggested to use Voiculescu's results on free random variables in non-commutative probability theory to describe the master field and he considered the master field for 2 dimensional QCD. This was further discussed in [Do94,GG94].

Singer's conjecture is also supported by another result, obtained a couple of years ago [AcLu91], [AcLu93a], [AcLu93b]) which shows a surprising emergence of the semi–circle diagrams (not of the semicircle law but a nonlinear modification of it) in the stochastic limit of quantum electrodynamics. Since QCD is a generalization of QED it is natural to expect that these diagrams should play a role also in QCD (cf. Section (11.) below for an outline of the QED application and [AcLuVo94a] for a preliminary result concerning QCD: the latter paper, although written independently and based on completely different techniques goes in the same direction of the recent work [Wil94].

The master field as a limiting quantum random process there appears not only in the large N limit but also in the so colled stochastic limit of quantum field theory. In [AcLuVo93] we have found the following direct relation between quantum theory in real time and stochastic processes:

$$\lim_{\lambda\to 0} \lambda A\left(\frac{t}{\lambda^2}\,,\ k\right) = B(t,k) \tag{1.18}$$

where $A(t,k)$ is a free dynamical evolution of a usual (Boson or Fermion) annihilation operator $a(k)$ in a Fock space $\mathcal{F}$, i.e.

$$A(t,k) = e^{itE(k)}a(k)$$

k is a momentum variable and $B(t,k)$ is a Boson or Fermion quantum field acting in another Fock space $\mathcal{H}$ and satisfying the canonical commutation relations:

$$[B(t,k), B(t',k')]_{\pm} = 2\pi\delta(t-t')\delta(k-k')\delta(E(k)) \tag{1.19}$$

A (Boson or Fermion) Fock field satisfying commutation relations of the form (1.19) is called a *quantum white noise* [Ac Fri Lu1] and is a prototype example of quantum stochastic process. In the context of this our paper one can call it also the master field. In terms of the field operators $\phi(t,x)$ one can rewrite (1.18) as

$$\lim_{\lambda\to 0} \lambda\phi\left(\frac{t}{\lambda^2}\ ,\ x\right) = W(t,x) \tag{1.20}$$

and one can prove that the vacuum correlations of $W(t,x)$ coincide with those of a *classical Brownian motion*. The relation (1.18) can be interpreted as a quantum central limit theorem with λ playing the role of $\frac{1}{\sqrt{N}}$.

The scaling $t \mapsto t/\lambda^2$ has its origins in the early attempts by Pauli, Friedrichs and Van Hove to deduce a *master equation*.

In a series of papers [AcLu85],[AcLu93] it has been shown that the fact that, under the scaling $t \to t/\lambda^2$, quantum fields converge to quantum Brownian motions and the Heisenberg equation to a quantum Langevin equation, is a rather universal phenomenon as shown in a multiplicity of quantum systems, involving the basic physical interactions.

This kind of limit and the set of mathematical techniques developed to established it, was called in [AcLuVo93] *the stochastic limit of quantum field theory*. We call it also the $1+3$ asymptotical expansion to distinguish it from analogous scalings generalizing the present one in the natural direction of rescaling other variables beyond (or instead of) time: space, energy particle density,...

A generalization of (1.18) to a pair of *Fermi operators* A, A^+ has been obtained in [AcLuVo94] and it can be described by the formula

$$\lim_{\lambda\to 0} \lambda A\left(\frac{t}{\lambda^2}\ ,\ k_1\right) A\left(\frac{t}{\lambda^2}\ ,\ k_2\right) = B(t,k_1,k_2) \tag{1.21}$$

The remarkable property of the formula (1.21) is that while the $A(t,k)$ are **Fermion** annihilation operators the limit field $B(t,k_1,k_2)$ is a **Boson** annihilation operator which satisfies with its hermitian conjugate, the canonical bosonic commutation relations. For this reasons we call the formula (1.21): *stochastic bosonization*.

The bosonization of Fermions is well known in $1+1$ dimensions, in particular in the Thirring–Luttinger model see, for example [Wh, SV] and in string theory [GSW]. The stochastic bosonization (1.21) takes place in the real $1+3$ dimensional space–time and in fact in any dimension. For previous discussion of bosonization in higher dimensions cf. [Lut], [Hal].

In particular in quantum chromodynamics one can think of the Fermi–operators $A(t,k)$ as corresponding to quarks then the bosonic operator $B(t,k_1,k_2)$ can be considered as describing a meson (cf. [AcLuVo94b] for a preliminary approach to QCD in this spirit).

An extension of the notion of the stochastic limit is the notion of *anisotropic asymptotics*[AV94]. Anisotropic asymptotics describe behaviour of correlation functions when some components of coordinates are large as compare with others components. It is occurred that (2+2) anisotropic asymptotics for 4-points functions are related with the well known Regge regime of the scattering amplitudes. Let us explane what we mean by anisotropic asymptotics. If x^μ are space-time coordinates, one denotes $x^\mu = (y^\alpha, z^i)$, where $\alpha = 0, 1,\ i = 2, 3$ for 2+2 decomposition and $\alpha = 0,\ i = 1, 2, 3$ for 1+3 decomposition. Anisotropic asymptotics mean the evaluation of asymptotics of correlation functions for a field $\Phi(\lambda y, z)$

$$< \Phi_{i_1}(\lambda y_1, z_1)\Phi_{i_2}(\lambda y_2, z_2)...\Phi_{i_n}(\lambda y_n, z_n) >$$

when $\lambda \to 0$ (or $\lambda \to \infty$), i.e. when some of variables are much larger than others.

In the following section we shall describe some results of quantum probability relevant for the present exposition (for a more detailed review cf. [Ac90]) and then we consider a generalization of the relations of (1.7) and (1.12) to the case of several matrix random variables, to the non–Gaussian case and to the case of q–deformed algebras of the CCR.

(2.) Algebraic (quantum, non–commutative) probability

A positive linear $*$-functional φ on a $*$–algebra $\mathcal{A}$ (i.e. a complex algebra with an involution denoted $*$ and a unit denoted 1) such that $\varphi(1) = 1$ is called a **state**. An **algebraic probability space** is a pair $\{\mathcal{A},\ \varphi\}$ where $\mathcal{A}$ is a *–algebra and φ is a state on $\mathcal{A}$.

If $\mathcal{A}$ is a commutative algebra we speak of a **classical probability space**; if $\mathcal{A}$ is noncommutative of a **quantum probability space**.

If $\mathcal{A}$ is a commutative separable von Neumann algebra, then it is isomorphic to the algebra $L^\infty(\Omega, \mathcal{F}, P)$ where $(\Omega, \mathcal{F}, P)$ is a probability space in the usual sense (Ω is a set, $\mathcal{F}$ a σ–algebra of subsets of Ω, P a positive measure with $P(\Omega) = 1$). Moreover, in this isomorphism, the state φ becomes the usual integral with respect to P, i.e. $\int_\Omega (\cdot)\ dP$. algebra, then, with the Gelfand–Naimark–Segal (GNS) construction with respect to φ. Thus classical probability is naturally included in algebraic probability. In fact one can prove that there is an isomorphism, in the sense of categories, between classical probability spaces in the algebraic sense and probability spaces in the sense of classical probability theory, which form a category with morphisms defined by measure preserving measurable transformations defined up to sets of zero probability.

An investigation of such and more general relations as (1.1) and (1.5) is one of subjects of non-commutative (quantum) probability theory.

The oldest example of quantum probability space is the pair $\{\mathcal{B}(\mathcal{H}), \varphi\}$ where $\mathcal{H}$ is a Hilbert space and φ a state on the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. Typical examples of states are $\varphi(a) =< \Phi, a\Phi >$ where Φ is a vector in $\mathcal{H}$ (e.g. Fock vacuum) or $\varphi(a) = Tr(wa)$, where w is a density matrix (e.g. an equilibrium finite temperature state).

A *random variable* with probability space $\{\mathcal{A}, \varphi\}$ is an element of $\mathcal{A}$.

A *stochastic process* is family (X_t), of random variables, indexed by an arbitrary set T. Given a stochastic process $X \equiv (X_t)$, one can form the polynomial $*$– algebra $\mathcal{P}(X)$, which is a $*$–subalgebra of $\mathcal{A}$. The restriction of the state φ on this algebra gives a state φ_X on $\mathcal{P}(X)$ which is called the **distribution** of the process X. If $T = \{1, \ldots, n\}$ is a finite set, the state φ_X is called the **joint distribution** of the random variables $\{X_1, \ldots, X_n\}$. If T consists of a single element, then one speaks of the **distribution** of the random variable X.

It is known that the free $*$–algebra $\mathcal{C} < \tilde{X}_t \ : \ t \in T >$ generated by the (noncommuting) indeterminates $\tilde{X}_t$ $(t \in T)$ has the following universal property: for any $*$–algebra $\mathcal{P}$ and for any family $\{X_t \ : \ t \in T\}$ of its elements, there exists a unique homomorphism $\alpha : \mathcal{C} < \tilde{X}_t \ : \ t \in T > \to \mathcal{P}$, characterized by the property:

$$\alpha(\tilde{X}_t) = X_t \qquad ; \ \forall t \in T$$

Using this α one can *pull back* the state φ on $\mathcal{A}$ to the state $\tilde{\varphi} := \varphi \circ \alpha$ on the free algebra and, because of the universal property, the algebraic probability space

$$\{\mathcal{C} < \tilde{X}_t \ : \ t \in T >, \tilde{\varphi}\} \tag{2.1}$$

contains all the informations on the stochastic process X. Summing up: **to give an algebraic stochastic process indexed by a set T, is the same as giving a probability space of the form (2.1)**.

This *pull back* is useful because *it depends only on the index set T*. This implies that, given a sequence (or, more generally, a net) of stochastic processes

$$X^{(n)} \equiv \{\mathcal{A}^{(n)}, X_t^{(n)}, t \in T, \ \varphi^{(n)}\} \ ; \quad n = 1, 2, \ldots \tag{2.2}$$

one can define a sequence (net) $(\tilde{\varphi}^{(n)})$ of states **on the same algebra** $\mathcal{C}\langle \tilde{X}_t : t \in T\rangle$ and therefore it makes sense to speak of convergence of $(\varphi^{(n)})$ to a limiting state $\varphi^{(\infty)}$ on $\mathcal{C}\langle \tilde{X}_t : t \in T\rangle$. By the above remark, the algebraic probability space $\{\mathcal{C}\langle \tilde{X}_t : t \in T\rangle, \varphi^{(\infty)}\}$ uniquely defines an algebraic stochastic process which is called **the limit in law** (or in distribution) of the sequence of processes $X^{(n)}$ given by (2.2). This notion of convergence in law was pioneered by von Waldenfels [vWI88] and used by Voiculescu [Voi92]. For results such as the law of large numbers and the central limit theorem (cf. Section 8.) below) this notion is sufficient, but for more interesting applications to physics, a more sophisticated notion (which turns out to be nearer to the spirit of traditional quantum theory) is required (cf. Section (12.) below).

The notion of convergence in law plays a crucial role in the stochastic limit of quantum field theory (cf. Section () below) and it is likely to play a similar role in any constructive developement of quantum field theory.

In fact it is known (von Neumann–Friedrichs–Haag phenomenon) that an interacting field cannot live on the same Hilbert space of its associated free field. Therefore, if we want to approximate, in some sense, the interacting correlations by local perturbations of free correlations, then we cannot use the standard operator topologies, which do not allow to *go out* from the original space.

Now let $\mathcal{F}$ be the (GNS) space of this probability space, Φ_F the cyclic vector and π the Gelfand–Naimark–Segal GNS representation of $\mathcal{C} < \tilde{X}_t \ : \ t \in T >$ in the linear operators on $\mathcal{F}$. Denoting $\mathcal{H}_1$ the Hilbert subspace of $\mathcal{F}$ generated by the vectors $\{\pi(\tilde{X}_t)\cdot\Phi_F \,:\, t\in T\}$ (the 1–particle space), we can identify $\mathcal{F}$ to the full Fock space $\Gamma(\mathcal{H}_1)$ over the *one particle space* $\mathcal{H}_1$ with a unitary isomorphism that maps Φ_F to the vacuum vector Φ and intertwines the GNS representation of $\mathcal{C} < \tilde{X}_t \,:\, t \in T >$ with tensor multiplication (these two properties uniquely determine the unitary).

The *free (or full) Fock space* $\mathcal{F}(H)$ over a pre–Hilbert space H is just the tensor algebra over H, i.e.

$$\mathcal{F}(H) := \oplus_{n\geq 0}(\otimes H)^n = \mathbf{C}\cdot\Phi \oplus H \oplus H\otimes H \oplus \dots$$

where $\Phi = 1$ is the vacuum vector. The Boson (resp. Fermion) Fock space differs from it because the tensor product in the n–particle space is symmetrized (resp. antisymmetrized).

The creation and annihilation operators in the full Fock space are defined by

$$A(f)\Phi = 0$$

$$A(f)g_1\otimes g_2\otimes\dots\otimes g_n = \langle f, g_1\rangle g_2\otimes\dots\otimes g_n$$

$$A^+(f)g_1\otimes\dots\otimes g_n = f\otimes g_1\otimes\dots\otimes g_n$$

Here f, $g_1,\dots,g_n$ are elements from H. From these definitions it follows that

$$A(f)A^+(g) = \langle f, g\rangle \tag{2.3}$$

Operators $A(f)$ and $A^+(f)$ are bounded operators in the Hilbert space $\mathcal{F}(H)$.

Notice that, in the above mentioned identification between the GNS space of the free algebra with the full Fock space, the GNS representation $\pi(X_t)$ of a random variable X_t becomes the creation operator $a^+(X_t)$.

Summing up: since the full Fock algebra is a universal object for stochastic processes (in the sense explained above), it follows that it is a natural programme to study subalgebras of this algebra and states on them. The symmetric (Boson) and antisymmetric (Fermion) algebra and corresponding classes of states have been widely studied. Another interesting subalgebra is the so–called *Toeplitz algebra*, which is the norm closure of the algebra generated by the creation and annihilation operators, plus the identity (in the identification of $\Gamma(\mathcal{H}_1)$ with $\mathcal{H}_F$ this is simply the norm closure of the GNS representation of $\mathcal{C} < \tilde{X}_t \,:\, t \in T >$). For a single random variable this algebra is isomorphic to the one–sided shift; for a finite stochastic process (a random variable with values in a finite dimensional vector space) it is isomorphic to an extension, by the compact operators, of the *Cuntz algebra* O_n (the C^*–algebra generated by n isometries with orthogonal ranges); for an infinite stochastic process ($card(T) = \infty$) the Toeplitz and the Cuntz algebra coincide.

The definition of random variable given above shall be sufficient for our purposes, but it should be remarked that, it only includes those random variables which have finite moments of all orders. Thus, for example, the Cauchy (Lorentz) distribution, ubiquitous in exponential decays, or the stable laws, which receive more and more attention in the study of chaotic systems, are excluded from this approach.

If one wants to include all random variables one needs the following more subtle defintion (cf. [AFL82]). An **algebraic random variable** is a triple

$$\{\{\mathcal{A},\ \varphi\},\ \mathcal{B},\ j\} \tag{2.4}$$

where $\{\mathcal{A},\ \varphi\}$ is an algebraic probability space (**sample algebra**); $\mathcal{B}$ is a *–algebra (**state algebra**) and $j : \mathcal{B} \to \mathcal{A}$ is an embedding. In the following, unless explicitly stated, we shall assume that j is a *-homomorphism, i.e. $j(1_{\mathcal{B}}) = 1_{\mathcal{A}}$. When this is not the case, we shall speak of a **non conservative** algebraic random variable. If $\mathcal{B}$ is commutative we speak of a **classical** algebraic random variable. If $\mathcal{B}$ is non-commutative, of a **quantum random variable**. The state φ_j, on $\mathcal{B}$, is defined by:

$$\varphi_j := \varphi \circ j \tag{2.5}$$

is called the **distribution** of the random variable j. As usual a stochastic process is a family of random variables (j_t) indexed by a set T. The process is called *classical* if $\mathcal{B}$ is commutative and the algebras $j_t(\mathcal{B})$ commute for different t. If x_j $(j \in F)$ is a set of generators of the algebra $\mathcal{B}$, we define the new index set $T' := F \times T$ and the set $\{X_{(j,t)} = j_t(x_j)\ :\ (j,t) \in T' = F \times T\}$ is a stochastic process in the sense of the previous definition.

The finite dimensional joint correlations of a stochastic process are the quantities

$$\varphi(j_{t_1}(b_1)\dots j_{t_n}(b_n)) \tag{2.6}$$

for all $n \in \mathbf{N}$, $b_1 \dots, b_n \in \mathcal{B}$ and $t_1 \dots, t_n \in T$. It can be shown that they uniquely characterize the stochastic process up to isomorphism (cf. [AFL82]): this a generalization of the Kolmogorov reconstruction theorem of a stochastic process in terms of the finite dimensional joint probabilities as well as of the Wightman reconstruction theorem.

Moreover one can prove that the convergence of the correlation kernels reduces, when one fixes a set of generators, to the notion of convergence in distribution, as introduced above.

Remark. The choice of a set of generators corresponds, in probability, to the use of local coordinates in geometry: it is useful to do calculations, but it is not intrinsic. The more general, purely algebraic, point of view, described at the end of the present section, provides a more intrinsic invariant approach to stochastic processes.

(3.) Statistical Independence

Mathematically the notion of *statistical independence* is related to that of product: classical (Boson) independence – to the usual tensor product; Fermion independence – to the $\mathbf{Z}_2$–graded tensor product; quantum group independence – to more general forms of twisted tensor products (cf. [Schü93]; free independence – to the free product.

Statistical independence means, in physical terms, *absence of interaction*. Notice that also in absense of interaction one can have *kinematical relations*, which are reflected by algebraic constraints. For example two Fermion operators $F(f_1)$, $F(f_2)$ corresponding to orthogonal test functions, are independent in the vacuum state, but related by the CAR: $F(f_1)F(f_2) = -F(f_2)F(f_1)$.

In classical probability one can *encode* all the algebraic relations into the state (thus making than statistical relations). The analogous result for quantum probability is the **reconstruction theorem** of [AFL82]. Concrete examples of such encodings are the deduction of the CCR (resp. the CAR from a boson (resp. Fermion) gaussian state [GivWa78], [vWa78] (cf. Section (3.) below) deduction of the Cuntz algebra from a free–gaussian state [Fag93].

In view of this *we conjecture that the natural generalizations of the Heisenberg commutation relations shall be deduced from the GNS representation of stochastic processes in the sense of quantum probability* (independent processes, as generalizations of free systems, Markov processes as generalizations of interacting systems). More precisely *the various possible notions of free physical system*. For this reason an investigation of the notion of statistical independence is relevant for physics.

The following definition includes all the notions of independence considered up to now:

Definition ( ). Let $\{\mathcal{A}, \varphi\}$ be an algebraic probability space; T a set; $(\mathcal{A}_t)_{t\in T}$ a family of subalgebras of $\mathcal{A}$; $\mathcal{F}$ a subset of $\bigcup_{n=0}^{\infty} T^n$ i.e. $\mathcal{F}$ is a family of ordered n–tuples of elements of T; and, for each $(t_1, \ldots, t_n) \in \mathcal{F}$, let be given a family of subsets of the set $\{t_1, \ldots, t_n\}$.

The algebras $(\mathcal{A}_t)_{t\in T}$ are called $(\mathcal{F}\ , \varphi)$-**independent** if for any $(t_1, \ldots, t_n) \in \mathcal{F}$ and for any choice of $a_{t_j} \in \mathcal{A}_{t_j}(X) \quad (j = 1, \ldots, n)$ such that, for some $\{t'_1, \ldots, t'_m\} \in \mathcal{F}$ one has:

$$\varphi\left(a_{t'_j}\right) = 0 \tag{3.0}$$

for all $j = 1, \ldots, m$, then one also has

$$\varphi\left(a_{t_1} \cdot \ldots \cdot a_{t_n}\right) = 0 \tag{3.1}$$

It is moreover required that for each n–tuple $(t_1, \ldots, t_n)$, the family $F(t_1, \ldots, t_n)$ includes all the *singletons* of the set $\{t_1, \ldots, t_n\}$ (i.e. those subsets consisting of a single element $t_1 \in \{t_1, \ldots, t_n\}$. The classical, Boson and Fermion independence correspond to the case in which:

i) $\mathcal{F}$ is the set of n–tuples $(t_1, \ldots, t_n) \in T^n (n \in \mathbf{N})$ such that $t_i \neq t_j$ for all $i \neq j \quad (i, j = 1, \ldots, n)$

ii) for each n–tuple $(t_1, \ldots, t_n)$ the family $F(t_1, \ldots, t_n)$ consists of all the *singletons* of the set $\{t_1, \ldots, t_n\}$.

If in a general product $a_{t_1} \cdot \ldots \cdot a_{t_n}$ one adds and subtracts from a_{t_1} its φ–expectation, condition (3.1) implies, that

$$\varphi\left(a_{t_1} \cdot \ldots \cdot a_{t_n}\right) = \varphi\left(a_{t_2}\right) \cdot \varphi\left(a_{t_2} \cdot \ldots \cdot a_{t_i} \cdot \ldots \cdot a_{t_n}\right) \tag{3.2}$$

Notice that, as far as indepndence is concerned, there is no difference between the three cases. However, in the classical case the algebras $\mathcal{A}_t$ are commutative and commute for different t; in the Boson case they are noncommutative but still commute for different t; in the Fermi case they are noncommutative and anticommute for different t.

The *free independence*, introduced by Voiculescu, corresponds to the case in which:

i) $\mathcal{F}$ is the set of n–tuples $(t_1, \ldots, t_n) \in T^n$ $(n \in \mathbf{N})$ such that $t_i \neq t_{i+1}$, $i = 1, 2, \ldots, n$.

ii) The subset F of $\{t_1, \ldots, t_n\}$ coincides with $\{t_1, \ldots, t_n\}$ itself (this means that, for the expectation (3.1) to be zero, all the a_{t_i} must have mean zero).

Notice that the factorization property of the classical independent random variables:

$$\varphi(a_{t_1} \cdot \ldots \cdot a_{t_n}) = \varphi(a_{t_1}) \cdot \ldots \cdot \varphi(a_{t_n})$$

whenever $\varphi(a_{t_j}) = 0$ and $t_j \neq t_k$ for $j \neq k$, follows by induction for all the notions of independence discussed here.

Intermediate cases have not yet been studied and in fact it is not clear, up to now, if other cases are possible (cf. [Schü94] for a general analysis of the motion of independence); for other generalizations of the notion of independence cf [BoSp91], [SpevW92].

Notice that, in general, the notion of independence alone is not sufficient to determine the form of all the correlation functons, i.e. the equivalence class of the process. In fact it gives no information about the expectations of products of the form $a_{t_1} \cdot \ldots \cdot a_{t_n}$ with $(t_1, \ldots, t_n) \notin \mathcal{F}$.

In the usual cases however this reconstruction is possible due to commutation relations or to stronger conditions (as in the free case).

Comparing the formulation of condition (i) in the usual independence and in the free one, one sees that the notion of free independence is *essentially non commutative*.

For example the condition $\varphi(a_s a_t a_s a_t) = 0$ if $\varphi(a_s) = \varphi(a_t) = 0$ and $s \neq t$, would imply, if the algebras $\mathcal{A}_s$ and $\mathcal{A}_t$ commute (or anticommute), that $\varphi(a_s^2 a_t^2) = 0$ which implies $a_s = a_t = 0$, if both a_s and a_t are self–adjoint and the state φ faithful (strictly positive on positive non zero elements).

Given a stochastic process $(X_t)_{(t \in T)}$ in a probability space $\{\mathcal{A}, \varphi\}$, one can associate to each random variable X_t the $*$–algebra $\mathcal{A}_t$, generated by X_t and the identity.

The random variables $(X_t)(t \in T)$ are called free–independent (or simply free) if the corresponding algebras $\mathcal{A}_t$ are free independent in the sense of the above definition.

(4.) Free Fock space and q–deformations

Let us now consider the Boson (B), Fermi (F) and free (Fr) vacuum expectations

$$\langle \Phi, A^{\varepsilon(1)}(f_1)A^{\varepsilon(2)}(f_2)\dots A^{\varepsilon(N)}(f_N)\Phi\rangle_V \tag{4.1}$$

where $N \in \mathbf{N}$, $\varepsilon \in \{0,1\}^N$,

$$A^\varepsilon = \begin{cases} A & \text{if } \varepsilon = 0 \\ A^+ & \text{if } \varepsilon = 1 \end{cases}$$

$V \in \{B, F, Fr\}$, and $\langle , \rangle_B$ (resp. $\langle , \rangle_F$, $\langle , \rangle_{Fr}$) denotes the Boson (resp. Fermi, Free)–vacuum expectation on the Boson (resp. Fermi, Free) Fock space over a certain Hilbert space H.

A common feature of the three cases in that (4.1) is equal zero if $N = 2n+1$ is odd.

In the following we shall investigate (4.1) for $N = 2n$. Another common feature of the three cases is that (4.1) is equal to zero if either $\exists\, r \le 2n$, such that, denoting $|\cdot|$ the cardinality of a set

$$|\{k : k \le r,\ \varepsilon(k) = 0\}| < |\{k : k \le r,\ \varepsilon(k) = 1\}| \tag{4.2a}$$

or

$$|\varepsilon(k) : k \le 2n,\ \varepsilon(k) = 0\}| \ne |\{\varepsilon(k) : k \le 2n,\ \varepsilon(k) = 1\}| \tag{4.2b}$$

For each $\varepsilon \in \{0,1\}^{2n}$ such that (4.2a) and (4.2b) are not true. We connect with a line a vertex $k \in \{i : \varepsilon(i) = 0\}$ and a vertex $m \in \{i : \varepsilon(i) = 1\}$. Thus we need n–lines to connect the $2n$ vertices and in the following we call this a *connected diagram*.

Let us denote by $1 = p_1 < p_2 < \dots < p_{2n-1} < p_{2n} < 2n$, the vertices such that $\varepsilon(p_k) = 0$, $k = 1, 2, \dots, n$ (i.e. the creator vertices). Then by the CCR or the CAR, (.1) is equal to

$$\sum_{\substack{1<q_1,\dots,q_n=2n \\ q_h>p_h, h=1,\dots,n \\ \{q_h\}_{h=1}^n=\{1,\dots,2n\}\setminus\{p_h\}_{h=1}^n}} \kappa_\varepsilon \prod_{h=1}^n \langle f_{p_h}, f_{q_h}\rangle \tag{4.3}$$

where, for $V \in \{F, Fr\}$, $\kappa_\varepsilon \in \{0, \pm 1\}$ is determined uniquely by $\{p_h, q_h\}$ and it is always equal to 1 in the case $V = B$.

In (4.3), for each choice of $\{q_h\}_{h=1}^n$ we have a diagram with lines $\{(p_h, q_h)\}_{h=1}^n$. In other words, we have a pair–partition $\{(p_h, q_h)\}_{h=1}^n$ of $\{1, 2, \dots, 2n\}$. For fixed $\{p_n\}_{h=1}^n$, in the cases B and F, we have many choices of $\{q_h\}_{h=1}^n$'s and each choice determines a diagram. But for each fixed $\{p_h\}_{h=1}^n$ there is at most one choice of the $\{q_h\}_{h=1}^n$ such that the associated diagram is non–crossing, i.e. for all $1 \le k \ne h \le n$,

$$p_k < p_h \Rightarrow q_k < p_h \tag{4.4}$$

Thus, for $V = Fr$, (4.1) differs from zero only if ε allows a non–crossing diagram $\{(p_h, q_h)\}_{h=1}^n$ and if so (4.1) is given by

$$\prod_{h=1}^n \langle f_{p_h}, f_{q_h}\rangle \tag{4.5}$$

In this case the distribution of the field operators $A(f)+A^+(f)$, $f \in H$ is a joint semi–circle distribution (in the sense that the marginals are semi–circle laws with variance proportional to the square norm of the test functions). In Section (12.) we shall see that the stochastic limit of quantum electrodynamics sugggests a nonlinear modification of the expression (4.5) which no longer gives the semi–circle law for the field operators, but still keeps the structure of the non–crossing diagrams.

The q–deformed commutation relations in the full Fock space $\mathcal{F}(H)$ (cf. for example [APVV] and references therein

$$c(f)^*(g) - qc^*(g)c(f) = \langle f, g\rangle$$

interpolate between Bose relations ($q=1$), Fermi ($q=-1$) and free ($q=0$).

For $-1 < q < 1$ one gets the relation [BoSp91]

$$\langle \Phi, (c(f)+c^*(f))^{2n+1}\Phi >= 0$$

$$\langle \Phi, (c(f)+c^*(f))^{2n}\Phi = \int_{2/\sqrt{1-q}} x^{2n}\nu_q(x)dx$$

where

$$\nu_q(x) = \frac{\sqrt{1-q}}{\pi} \sin\ \text{Arccos}\ \frac{\sqrt{1-q}}{2} \cdot \prod_{k=1}^{\infty}(1-q^k)\cdot$$
$$\cdot \left|1 - q^k \exp\left\{-2i\ \text{Arccos}\ \frac{\sqrt{1-q}}{2}\, x\right\}\right|^2$$

In particular for $q=0$ one has the Wigner semicircle distribution, $\nu_0 = w_{1/2}$ and

$$\nu_q(x) \to \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right),\ q \to 1(\text{Bose})$$

$$\nu_q(x) \to \frac{1}{2}\left(\delta(x-1)+\delta(x+1)\right),\ q \to -1\ (\text{Fermi})$$

For the free algebra (4.1) one has

$$\langle \Phi, (a(f)+a^*(f))^{2n+1}\Phi\rangle = 0$$

$$\langle \Phi, (a(f)+a^*(f))^{2n}\Phi\rangle = x_{2n}\cdot \|f\|^{2n}$$

where x_{2n} is the number of non–crossing partitions of $\{1,\dots,2n\}$, the so–called Catalan number,

$$x_{2n}\cdot\|f\|^{2n} = \int_{\mathbf{R}^N} x^{2n} W_{\|f\|}(x)dx$$

(5.) Gaussianity (classical, Boson, Fermion, Free, ...)

The results of the central limit theorems suggest the independent study of the class of maps which canonically arise from these theorems, these are the so–called *Gaussian maps.*

Let $\mathcal{B}$ and $\mathcal{C}$ be *–algebras and let $E : \mathcal{B} \longrightarrow \mathcal{C}$ be a linear $*$–map. A family $B \subseteq \mathcal{B}$ is called a **mean zero generalized Gaussian family** with respect to E, if for each $n \in \mathbf{N}$ and for each sequence $b_1, \ldots, b_n$ of elements of B, not necessarily different among themselves, one has:

$$E(b_1 \cdot \ldots \cdot b_n) = 0 \quad \text{if } n \text{ is odd} \tag{5.1}$$

and for each even $n = 2p$, there exists a subset $\mathcal{PP}_o(n)$, of the set $\mathcal{PP}(n)$ of all the ordered pair partitions of the set $\{1, \ldots, n = 2p\}$ such that

$$E(b_1 \cdot \ldots \cdot b_{2p}) = \frac{1}{p!} \sum_{(i_1,j_1,\ldots,i_p,j_p)\in\mathcal{PP}_o(n)} \epsilon(i_1, j_1;\, i_2, j_2;\, \ldots ;i_p, j_p) E(b_{i_1} b_{j_1}) \cdot \ldots \cdot E(b_{i_p} \cdot b_{j_p}) \tag{5.2}$$

where the pair partition $(i_1, j_1;\, i_2, j_2;\, \ldots ;i_p, j_p)$ is called ordered if a permutation of the pairs $(i_\alpha,\, j_\alpha)$ changes the partition while

$$i_\alpha < j_\alpha \qquad ; \qquad \alpha = 1, \ldots, p$$

and for each natural integer n, ϵ_n is a complex valued character of the permutation group over $2n$ elements, e.g. $\epsilon_n(j_1, \ldots, j_{2n}) = +1$ for all permutation (Boson case) or $\epsilon_n(j_1, \ldots, j_{2n}) = sgn(j_1, \ldots, j_{2n})$ (Fermion case).

Notice that positivity is not required a priori, but the most intresting examples are obtained in correspondence of positive (even completely positive) maps. It is not known for which subsets $\mathcal{PP}_o(n)$ of pair partitions these prescriptions define a positive functional. The standard Gaussian (Boson or Fermion) case corresponds to the choice of all pair partitions; the free case corresponds to the so–called *non crossing* (or rainbow) partitions, which are characterized by the condition:

$$i_k < i_h < j_k \leftrightarrow i_k < i_h < j_h < j_k \tag{5.3}$$

in other terms: *the intervals corresponding to two pairs* (i_k, j_h) and (i_h, j_k) *are ither disjoint or one is contained in the other*.

Other subsets which give positivity can be constructed by hands, but it is not known if they come from some central limit theorem.

One easily checks that generalized gaussianity is preserved under linear combinations in the sense that, if $B \subseteq \mathcal{B}$ is an E–Gaussian family then the linear span $[B]$ of B is also an E–Gaussian family. The sesquilinear map on $[B]$

$$q(b_1, b_2) := E(b_1^* \cdot b_2)$$

is called the **covariance** of the Gaussian map E.

If $\mathcal{C} = \mathbf{C}$ = the complex numbers, E is positive and $E(1) = 1$, then E is called a **Gaussian state**. In the physical literature, Boson and Fermion Gaussian states are called **quasi-free** states.

In case of Boson Gaussian states (which include the classical probability measures) one easily checks the well known identity for Gaussian measures i.e., in case of a single self–adjoint random variable b:

$$E(e^{zb}) = e^{\frac{1}{2}z^2 E(b^2)} \tag{5.4}$$

In case of a single random variable, if $\mathcal{PP}_0(n)$ is the set of non crossing partitions and ε is identically equal to 1, then condition (5.12) gives the momenta of the **Wigner semicircle law**.

The following theorem, essentially due to von Waldenfels, shows that the canonical commutation and anticommutation relations have their deep root in the notion of Gaussianity.

Theorem 5.1. *Let $\{\mathcal{A}, \varphi\}$ be an algebraic probability state. Let B be a family of algebraic generators of $\mathcal{A}$ and suppose that B is a mean zero (Boson or Fermion) Gaussian family with respect to φ in the sense of the above definition.*

Denote, for $a, b \in \mathcal{A}$, for any natural integer n and $j_1, \ldots, j_n \leq n$ natural integers :

$$[a, b]_\epsilon = \begin{cases} ab - ba, & \text{if } \varphi \text{ is Boson-Gaussian} \\ ab + ba, & \text{if } \varphi \text{ is Fermion-Gaussian} \end{cases} \tag{5.5}$$

and let $\{\mathcal{H}, \pi, \Phi\}$ be the GNS representation of the pair $\{\mathcal{A}, \varphi\}$. Then for any pair $b_1, b_2 \in B$ one has

$$[\pi(b_1), \pi(b_2)]_\epsilon = \varphi([b_1, b_2]_\epsilon) \tag{5.6}$$

The above theorem suggests to look at the GNS representation of Gaussian (or even more general) states as a natural source of generalized commutaion relations. In other terms: *up to now the commutation or anticommutation relations have been related to the kinematics (e.g group representations)*. The **locality condition** itself should find its natural formulation as an expression of some form of *statistical independence* rather than in terms of commutation relations: the former can be formulated uniquely in terms of observable quantities; the latter is strongly model dependent. *The above theorem suggests that their deeper root is statistic*. The statistical root is related to a universal phenomenon such as the central limit theorem and the theory of unitary group representations should enter the picture as a theory of *statistical symmetries*.

In particular one might ask oneself which are the commutation relations coming from free gaussianity. This problem has been solved by Fagnola [Fag93] who proved that essentially they are the Cuntz relations. More precisely, in full analogy with the Boson and the Fermion case, in the case of a single non self–adjoint random variable a, a^+, if the rank of the covariance matrix is one, the GNS representation is unitarily equivalent to the free (full) Fock representation described in Section (4.); if it is two, then it is unitarily isomorphic to the free product of a free Fock and a free anti–Fock representation. The analogy with the Boson and the Fermion case suggests to interpret this latter type of states as *the free analogue of the finite temperature states.* A strong confirmation to this conjecture would come if one could prove that, in the stochastic limit of QED without dipole approximation, if the field is taken in a finite temperature representation, rather than in the Fock one, as in [AcLu92], one obtains not the interacting Fock module but the corresponding finite temperature module (cf. Section (12.) below for the terminology).

(6.) Algebraic laws of large numbers and central limit theorems

Let $\{\mathcal{A}, \varphi\}$ be an algebraic probability space and let $j_n : \mathcal{B} \to \mathcal{A}$ ($n \in \mathbf{N}$) be an algebraic stochastic process. The laws of large numbers (LLN) study the behaviour of the sums

$$\frac{1}{N} S_N(b) := \frac{1}{N} \sum_{n=1}^{N} j_n(b) \tag{6.1}$$

for some $b \in \mathcal{B}$, and the central limit theorems (CLT), the behaviour of the centered sums

$$\frac{1}{\sqrt{N}} \bar{S}_N(b) := \frac{1}{\sqrt{N}} \sum_{n=1}^{N} [j_n(b) - \varphi(j_n(b))] \tag{6.2}$$

In case of a vector or matrix valued (k–dimensional) random variables, one studies the sums (6.1), (6.2) not for a single $b \in \mathcal{B}$, but for a k–tuple of elements

$$b^{(1)}, \ldots, b^{(k)} \in \mathcal{B}$$

Thus introducing, for $q = 1, \ldots, k$, the notation

$$b_n^{(q)} := \begin{cases} j_n(b^{(q)}) & \text{LLN–case} \\ [j_n(b(q)) - \varphi(j_n(b^{(q)}))] & \text{CLT–case} \end{cases} \tag{6.3}$$

one can reduce both the laws of large numbers and the central limit theorems, to the study the behaviour of the sums

$$\frac{1}{N^\alpha} S_N(b^{(q)}) := \frac{1}{N^\alpha} \sum_{n=1}^{N} (b_n^{(q)}) \qquad ; \ q = 1, \ldots, k \tag{6.4}$$

More precisely, one is interested in the limit, as $N \to \infty$ of expressions of the form

$$\varphi \left(P \left[\frac{S_N(b^{(1)})}{N^\alpha}, \frac{S_N(b^{(2)})}{N^\alpha}, \ldots, \frac{S_n(b^{(k)})}{N^\alpha} \right] \right) \tag{6.5}$$

where $\alpha > 0$ is a scalar, $b^{(1)}, \ldots, b^{(k)} \in B$ and P is a polynomial in the k noncommuting indeterminates $X_1, \ldots, X_k$. The law of large numbers is obtained when $\alpha = 1$ and the central limit theorem when $\alpha = 1/2$.

This convergence is called **convergence in the sense of moments**. In the classical (commutative) case, one replaces the polynomial P in (6.5) by a continuous bounded function, and obtains the (strictly stronger) notion of **convergence in law**; however if the $b_n^{(q)}$) do not commute there is no natural meaning for the expression (6.5) if P is not a polynomial (or an holomorphic function, if a topology is given on $\mathcal{A}$).

By linearity one can replace the polynomial P in (6.5) by a noncommutative monomial and the study of expressions of the type ( ) is reduced to the study of expressions of the type

$$\varphi\left(\frac{S_n(b^{(1)})}{N^\alpha}\cdot\frac{S_n(b^{(2)})}{N^\alpha}\cdot\dots\cdot\frac{S_n(b^{(k)})}{N^\alpha}\right) \tag{6.6}$$

where $b_1,\dots,b_k\in\mathcal{B}$ and it is not required that $b_n^{(i)}\neq b_n^{(h)}$ if $h\neq i$.

The stochastic process $j_n:\mathcal{B}\to\mathcal{A}$ is called *homogeneous* (or stationary) if

$$\varphi\circ j_k=\varphi\circ j_o=:\varphi_o \qquad ; \quad \forall k\in\mathbf{N} \tag{6.7}$$

and this notion can be extended to the case when φ is a general linear map from $\mathcal{A}$ to a subalgebra $\mathcal{C}$ rather than a state on $\mathcal{A}$. In fact almost all the results of this Section extend to maps

As in classical probability, also in algebraic probability the laws of large numbers and the central limit theorem are proved under independence assumptions. Here however one has to distinguish between the various types of independence which are possible.

In the above notations, a linear map $\varphi:\mathcal{A}\to\mathcal{C}$ is called **factorizable** or simply a **product map**, if for each integer n and $b_1,\dots,b_n\in\mathcal{B}$ one has

$$\varphi(j_1(b_1)\cdot\dots\cdot j_n(b_n))=\varphi(j_1(b_1))\cdot\dots\cdot\varphi(j_n(b_n)) \tag{6.8}$$

Notice that factorization is required only in the case when the indices are in **strictly increasing** order. If moreover the stationarity condition ( ) is satisfied then φ is called a **homogeneous product map** with marginal φ_o.

Theorem 6.1. *Let $\mathcal{C}\subseteq$ Center($\mathcal{A}$) and $\varphi:\mathcal{A}\to\mathcal{C}$ be a $\mathcal{C}$-linear product map and suppose that, for $j\neq k$, $j_h(\mathcal{B})$ and $j_k(\mathcal{B})$ commute (Boson case) and that $\mathcal{A}$ is generated by*

$$\{\mathcal{C}\vee j_k(\mathcal{B})\ :\ k=1,2,\dots\}$$

Then, if the limit:

$$\lim_{N\to\infty}\varphi\left(\frac{S_N(b)}{N}\right)=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\varphi(j_k(b))=:\varphi_o(b) \tag{6.9}$$

exists in the norm topology of $\mathcal{C}$ for each $b\in B$, it follows that for any natural integer k, for any $b_1,\dots,b_k\in\mathcal{B}$, and for any polynomial P in k non commuting indeterminates, one has

$$\lim_{N\to\infty}\varphi\left(P\left(\frac{S_N(b_1)}{N},\dots,\frac{S_N(b_k)}{N}\right)\right)=P\left(\varphi_o(b_1),\dots,\varphi_o(b_k)\right) \tag{6.10}$$

The following is the simplest algebraic central limit theorem. It extends the classical central limit theorem for independent identically distributed random variables to the Boson independent case.

Theorem 6.2. *Let $\varphi_o : \mathcal{B} \longrightarrow \mathcal{C}$ be a map and let φ be an homogeneous product map on $\mathcal{A}$ with marginal φ_o . Suppose moreover that, for $j \neq k$, $j_h(\mathcal{B})$ and $j_k(\mathcal{B})$ commute Then, for any $k \in \mathbf{N}$ and any elements $b_1, \ldots, b_k \in \mathcal{B}$ such that*

$$\varphi_o(b_j) = 0 \qquad ; \qquad j = 1, \ldots, k \tag{6.11}$$

one has

$$\lim_{N\to\infty} \varphi\left(\frac{S_N(b_1)}{N^{1/2}} \cdot \frac{S_N(b_2)}{N^{1/2}} \cdot \ldots \cdot \frac{S_N(b_k)}{N^{1/2}}\right) = 0 \tag{6.12}$$

if k is odd and, if $k = 2p$ for some integer p then

$$\lim_{N\to\infty} \varphi\left(\frac{S_N(b_1)}{\sqrt{N}} \cdot \frac{S_N(b_2)}{\sqrt{N}} \cdot \ldots \cdot \frac{S_N(b_k)}{\sqrt{N}}\right) =$$

$$= \sum_{(S_1,\ldots,S_p)\in\mathcal{P}_{p,2p}} \frac{1}{p!} \sum_{\pi\in\mathcal{S}_p} \sigma(\pi, S_1, \ldots, S_p, b_1, \ldots, b_k)\varphi_o(b_{S_{\pi(1)}}) \cdot \ldots \cdot \varphi_o(b_{S_{\pi(p)}}) \tag{6.13}$$

where $\sigma(\pi, S_1, \ldots, S_p, b_1, \ldots, b_k)$ is a constant depending only on the permutation π, on the partition $(S_1, \ldots, S_p)$ of $(1, \ldots, k)$ and on $b_1, \ldots, b_k$.

Remark. A partition $(S_1, \ldots, S_p)$ of the set $\{1, \ldots, k\}$ will be said to contain a **singleton** if, for some j=1,. . . .,p, S_j consists of a single element. The basic idea in the proof of the law of large numbers and of the central limit theorems for product maps, with the method of moments, is that the partitions are divided into two classes: those with a singleton and those without. Those with a singleton give zero contribution because of (10.1) and the product map assumption. For a partition $(S_1, \ldots, S_p)$, with no singleton, p must be $\leq k/2$ and the number of terms in the sum is of order N^p, hence it is balanced by the normalizing factor $N^{\alpha k}$. Thus, in the mean zero case, a finite nontrivial limit can only take place if $\alpha = 1/2$ and in this case p must be equal to $k/2$, i.e. $(S_1, \ldots, S_p)$ must be a pair partition.

The *invariance principles* (or functional central limit theorems) study the sums $S_{[Nt]}(b)/N^{1/2}$, where t is a real number and $[Nt]$ denotes the integer part of t, and show that, under natural conditions they converge to some (classical or quantum) Brownian motion. Recalling that

$$\frac{1}{N^{1/2}} S_{[Nt]}(b) = \frac{1}{N^{1/2}} \sum_{k=1}^{[Nt]} j_k(b) = \frac{1}{N^{1/2}} \sum_{k=1}^{[Nt]} \int_{k-1}^{k} j_k(b) ds =$$

$$= \frac{1}{N^{1/2}} \sum_{k=1}^{[Nt]} \int_{k-1}^{k} \chi_{[k-1,k]}(s) j_k(b) ds$$

where $\chi_{[k-1,k]}(s) = 0$ if $s \in [k-1,k]$ and $= 1$ if $s \notin [k-1,k]$, one can generalize the above sums by considering limits, as $\lambda \to 0$ of expressions of the form

$$\lambda \int_{S^2/\lambda^2}^{T^2/\lambda^2} X(s) ds = S_\lambda([S,T], X)$$

where $X(s)$ is an operator valued function. These are precisely the kind of limits that one meets in the stochastic limit of quantum field theory (cf. Section (10.) below).

In their full generality, the first Boson and Fermion central limit theorems were proved by von Waldenfels [GivWa78], [vWa78] (cf. also pioneering work by Hudson [Hu71], [Hu73]. Voiculescu [Voi91] proved the first free central limit theorem. Speicher [Sp90] showed that the same techniques used in the proof of the usual quantum central limit theorems allow to prove also the free central limit theorem. The first quantum invariance principle was proved by Accardi and Bach [AcBa87] in the Boson case and by Lu [Lu89] in the Fermion case.

As already said, independence corresponds to free systems, whose physics is usually not very interesting. Interaction corresponds to statistical dependence. However, as in classical statistical mechanics an Euclidean field theory, the locality properties of the interactions allow to express nontrivial informations in terms of local perturbations of the free fields. In probabilistic language, this leads to the notion of *weak dependence* (manifested by a decay of correlations). Under weak dependence assumptions, the CLT still holds (cf. [Verb89] for the CCR case and [AcLu90b] for general q–deformed commutation relations.

(7.) Large N limits of matrix models and asymptotic freeness

We try to explain here why it is natural to expect, as suggested by Singer [Sin94], that some results of Voiculescu on random matrices might help clarifying the $1/N$ expansion of matrix models and its extensions.

The basic idea can be summarized in the following statement: a set $A_1^{(N)}, \dots, A_k^{(N)}$ (k can be infinite) of $N \times N$ symmetric random matrices with independent (modulo symmetry) gaussian entries, with mean zero and variances of order $1/N$, **tend to become free random variables as N becomes large** ($N \to \infty$).

The physical interpretation of this phenomenon is best understood in terms of chaos: **freenes is an indication of chaotic behaviour**. In fact, just as statistical independence means *absence of statistical relations*, so freeness means *absence of algebraic relations*. Therefore, in some sense, *free independence* denotes a maximum of chaoticity.

In the $1/N$–expansion one is precisely concerned with the limit, as $N \to \infty$, of $N \times N$ matrix models. Truly, the large matrices are not always hermitean (e.g. they can be unitary) and their distributions are not indpendent gaussian because of the interaction and of the algebraic constraints.

However intuitively one expects that if the deviation from gaussianity is not too large (cf. Theorem 5.2 below) the general picture should not change too–much.

In the following we review some definitions and theorems that make this picutre more precise.

Let $\{\mathcal{A}_N, \varphi_N\}$ be a sequence of algebraic probability spaces and, for each N let $A^{(N)} = (A_i^{(N)})_{i\in I}$ be a family of random variables in $\mathcal{A}_N$. The sequence of joint distributions $\varphi_{A^{(N)}}$ converge as $N \to \infty$, if there exists a (state) $\tilde{\varphi}$ such that

$$\varphi_{A^{(N)}}(P) \to \tilde{\varphi}(P)$$

as $n \to \infty$ for every P in $\mathbf{C}\langle \tilde{X}_i | i \in I\rangle$ (algebra of polynomials on noncommuting indeterminants X_i) $\tilde{\varphi}$ is called the limit distribution.

Let $I = \bigcup_{j\in J} I_j$ be a partition of I.

A sequence of stochastic processes $(\{A_i^{(N)} | i \in I_j\}_{j\in J}$ is *asymptotically free* as $N \to \infty$ if it has a limit distribution $\tilde{\varphi}$ and if $(\{X_i | i \in I_j\})_{j\in J}$ is a free–independent stochastic process in the probability space $\{\mathbf{C}\langle X_i | i \in I\rangle, \tilde{\varphi}\}$.

Let $(\Omega, \mathcal{F}, P)$ be a classical probability space denote, L the algebra $\bigcap_{1\le p<\infty} L^p(\Omega)$ equipped with the expectation functional E given by the integration with respect to P.

Let $\{\mathcal{M}_N, \varphi_N\} = \{L \otimes M_N(\mathbf{C}), E \otimes \tau_N\}$ denote the noncommutative probability space of $N \times N$ random matrices with entries in L. Here τ_N is the normalized matrix trace, $\tau_N(A) = \frac{1}{N}\, trA$. We will consider a sequence of random matrices $A_i^{(N)} = (A_{i,\alpha\beta}^{(N)})_{\alpha,\beta=1}^N \in \mathcal{M}_N$ where $A_{i,\alpha\beta}^{(N)}$ denotes matrix elements, $\alpha, \beta = 1, 2, \ldots, N$ and $i = 1, 2, \ldots$.

Theorem 7.1. *[Voi92] . Let $A_i^{(N)} \in \mathcal{M}_N$, $i = 1, 2, \ldots$ be hermitian random matrices such that $A_i^{(N)*} = A_i^{(N)}$ and both Re $A_{i,\alpha\beta}^{(N)}$ and Im $A_{i,\alpha\beta}^{(N)}$ are independent Gaussian random variables such that for $i = 1, 2, \ldots$ one has*

$$E(A_{i,\alpha\beta}^{(N)}) = 0\ , \quad 1 \le \alpha \le \beta \le N$$

$$E((Re\ A_{i,\alpha\beta}^{(N)})^2) = E((Im\ A_{i,\alpha\beta}^{(N)})^2) = \frac{1}{2N}\ ; \quad 1 \le \alpha < \beta \le N$$

$$E((A_{i,\alpha\alpha}^{(N)})^2) = \frac{1}{N}\ , \quad 1 \le \alpha \le N$$

Then the family $(\{A_1^{(N)}\},\ \{A_2^{(N)}\}, \ldots)$ of matrix random variables is asymptotically free as $N \to \infty$ and moreover the limit distribution of each $A_i^{(N)}$ is a semicircle law.

One can rewrite this theorem as follows

$$\lim_{N\to\infty} \varphi_N\big(A_{i_1}^{(N)} \ldots A_{i_k}^{(N)}\big) = \tilde{\varphi}(\tilde{X}_{i_1} \ldots \tilde{X}_{i_k}) \tag{7.1}$$

where $\tilde{\varphi}$ is a limiting state on $\mathcal{C}\langle \tilde{X}_i : i \in \mathbf{N}\rangle$ and the A_i are noncommutative indeterminates.

If the family $(A_i^{(N)})$ consist only of one sequence $A^{(N)}$ of random matrices, in the same notations of the introduction, one can write (7.1) as

$$\lim_{N\to\infty} \langle \frac{1}{N} tr(A^{(N)m}) \rangle_N = \varphi(\tilde{X}^m) = \int_{\mathbf{R}} x^m w(x)dx \qquad ; \; m = 1, 2, \ldots \tag{7.2}$$

where $w(x)$ is the Wigner semicircle distribution with unit variance. By the GNS representation one gets an operator realization in the free Fock space of the free algebra $\mathbf{C}\langle A_i^{\infty}\rangle$ In particular in the simplest case of only one sequence $A^{(N)}$ of random matrices one gets the formulae (1.1) and (1.5) from the introduction where $\Phi = \pi_\varphi(X)$.

There are extensions of Theorem 7.1 in different directions. In particular there is an analogous result for a *standard family of independent unitary $N \times N$ random matrices*. In this case a limiting distribution is the Haar measure on the unite circle. There are extensions also to other groups of matrices, to the fermionic case and to the *non–Gaussian* case.

Theorem 7.2. *[Voi92] Let $A_i^{(N)} \in \mathcal{M}_N$, $i = 1, 2, \ldots$ be Hermitian $A_i^{(N)*} = A_i^{(N)}$ and $A_{i,\alpha\beta}^{(N)}$ is an independent set of random variables such that*

$$E(A_{i,\alpha,\beta}^{(N)}) = 0$$

$$E(|A_{i,\alpha\beta}^{(N)}|^2) = \frac{1}{N}$$

$$\sup E(|A_{i,\alpha\beta}^{(N)}|^m) = O(N^{-m/2}),\; m \geq 1\,,\; i = 1, 2\ldots, 1 \leq \alpha \leq \beta \leq N$$

The the family $(\{A_1^{(N)}\}, \{A_2^{(N)}\}, \ldots)$ of sets of random variables is asymptotically free as $N \to \infty$ and the limit distribution of each $A_i^{(N)}$ is a semicircle law.

(8.) Quantum stochastic differential equations

Let $X(f)$ be an operator depending on a test function f (e.g. a field or creation or annihilation or number operator). Suppose that f depends on space–time (or time–momentum) variables, then if we denote by χ_I the characteristic function of the time interval I, the operator $X(\chi_I f)$ is localized (in time) in the interval I and, moreover, the map $I \mapsto X(\chi_I f)$ is an operator valued measure on the real line $\mathbf{R}$. Under mild regularity conditions such a measure is called a *semi–martingale* (or a stochastic integrator) and integration with respect to measures of this kind is called **quantum stochastic integration**. Classical stochastic integration is recovered when the algebraic random variables $X(\chi_I f)$ commute for $I \subseteq \mathbf{R}$ and f varying among the test functions. In classical probability a random variable valued measure on the real line is also called an *additive process* (referring to the property $X(\chi_{[r,s]}f) + X(\chi_{[s,t]}f) = X(\chi_{[r,t]}f)$) and often in the following we shall use this terminology. Often a time origin, say $t = 0$ is fixed and one uses the notations

$$X(\chi_{[o,t]} f) =: X_t(f)$$

in these notations $X(\chi_{[s,t]} f) = X_t(f) - X_s(f)$ and the notation on the right hand side is more frequently used in classical probability. In the following we shall include the function f in definition of X and drop it from the notations.

Recently various kinds of stochastic calculi have been introduced in quantum probability, corresponding to different choices of the operator valued measures $I \mapsto X(\chi_I f)$, the first one of these being due to [HuPa84a], [HuPa84b]. The common features of these calculi are:

– One starts from a representation of some commutation relations (not necessarily the usual CCR or CAR) over a space of the form $L^2(I, dt; \mathcal{K})$ where $\mathcal{K}$ is a given Hilbert space and $I \subseteq \mathbf{R}$ is an interval.
– One introduces a family of operator valued measures on $\mathbf{R}_+$, related to this representation, e.g. expressed in terms of creation or annihilation or number or field operators.
– One shows that it is possible to develop a theory of stochastic integration with respect to these operator valued measures sufficiently rich to allow to solve some nontrivial stochastic differential equations.

The basic application of such a theory is the construction of unitary wave operators (Markovian cocycles in the quantum probabilistic terminology) as solutions of quantum stochastic differential equations generalizing the usual Schrödinger equation (in interaction representation). In order to achieve this goal it is necessary to prove an operator form of the Itô formula (whose content is that the product of two stochastic integrals is a sum of stochastic integrals).

The exposition that follows is aimed at giving a quick idea of the basic constructions of quantum stochastic calculus and by no means should be considered a complete exposition (for an approach to stochastic integration which unifies the classical as well as the several different quantum theories, cf. [AcFaQu90] and [Fag90] ; for a more concrete approach, based on the Fock space over $L^2(I, dt; \mathcal{K})$, cf. [Par93]).

Let be given a complex separable Hilbert space $\mathcal{H}$ and a filtration $(\mathcal{H}_{t]})$ in $\mathcal{H}$ (i.e. a family of subspaces $\mathcal{H}_{t]}$ of $\mathcal{H}$ such that, for $s \le t$ one has $\mathcal{H}_{s]} \subseteq \mathcal{H}_{t]}$: intuitively $\mathcal{H}_{t]}$ represents the history of the system up to time t. We write $\mathcal{B}(\mathcal{H})$ to denote the vector space of all bounded operators on $\mathcal{H}$.

Let $\mathcal{D}$ be a total subset of $\mathcal{H}$ (the linear conbinations of elements of $\mathcal{D}$ are dense).

When dealing with stochastic integrals one has to do with unbounded random variables. In particular one cannot single out, in the quantum case, some natural and easily verifiable conditions that assure that all the stochastic integrals considered leave invariant a sufficiently large domain of vectors. For this domain reasons even the general algebraic context of $*$–algebras is too strong to deal with stochastic integrals and one needs a weaker definition of random variable.

A *random variable* is a pair (F, F^+) of linear operators on $\mathcal{H}$ with domain containing $\mathcal{D}$ and such that for all elements η , ξ in this domain

$$< \eta, F\xi >=< F^+\eta, \xi >$$

The (linear) space of random variables shall be denoted $\mathcal{L}(\mathcal{D};\mathcal{H})$ and a random variable (F, F^+) shall be simply denoted F. A **stochastic process** is a family $(F_t)_{\ t\geq 0}$ of random variables; it is called **adapted** to the filtration $(\mathcal{H}_{t]})$ if, for all $t \geq 0$ one has $F_t\mathcal{H}_{t]} \subseteq \mathcal{H}_{t]}$. This is a *causality condition* it intuitively expresses the fact that the random variable F_t does not depend on the future (with respect to t) history of the system. A *step* (or elementary) process is an adapted process of the form

$$F(t) = \sum_{k=1}^{n} \chi_{(t_k,t_{k+1}]}(t) F_{t_k}$$

with $0 \leq t_o < t_1 < ... < t_n < \infty$.

An *additive process* on $\mathcal{H}$ is a family $(X(s,t))$ $(0 \leq s < t)$ of elements of $\mathcal{L}(\mathcal{D};\mathcal{H})$ such that, for each s the process $t \mapsto X(s,t)$ is adapted and for all $r < s < t$ we have

$$X(r,t) = X(r,s) + X(s,t)$$

Given an additive process X and a step process F, one can define the **left stochastic integral**

$$\int dX_s F_s = \sum_{k=1}^{n} X(t_k, t_{k+1}) F_{t_k} \tag{8.1}$$

and the right stochastic integral is defined similarly. Stochastic integrals of step process are called *simple* stochastic integrals.

An additive process X is called a *regular semimartingale* (or simply a stochastic integrator) for the set $\mathcal{D}$ if for all elements $\xi \in \mathcal{D}$ there exist two positive funtions $g_\xi^\epsilon \in L^1_{loc}(\mathbf{R}_+)$ $(\epsilon = 0,1)$ such that, for any step process F and all $t \geq 0$ we have:

$$\| \int_0^t dX_s F_s^\epsilon \xi \|^2 \leq c_{t,\xi} \cdot \int_0^t \| F_s^\epsilon \xi \|^2 \, g_\xi^\epsilon(s) ds \tag{8.2}$$

where F^ϵ stands for F or F^+ (to include free stochastic integration one should replace F^ϵ on the right hand side of (8.2) by $\mathcal{T}_X(F^\epsilon)$ where $\mathcal{T}_X$ is a linear map depending on the integrator X (cf. [Fag92]).

The inequalities (8.2) depend on the domain on which the basic integrators are considered. The known examples of basic integrators are creation, annihilation and number processes with respect to a given quasi-free (gaussian) representation of the CCR or of the CAR. In the Boson case they verify these inequalities on the domain of coherent vectors but for more general domains (such as the invariant domain which combines both the previous domains), or to include the Fermi and the free case (for which the natural domain is the linear span of the n-particle vectors), we need the inequalities of [AcFaQu90].

Using the inequality (8.2) one can complete the simple stochastic integrals in the natural seminorms given by the right hand side of (8.2) and obtain the (vector) space of stochastic integrals (with respect to X).

The basic example of the situation described above is the following: let

– $\mathcal{H} = \Gamma(L^2(\mathbf{R}_+)$ the Fock space over the one-particle space $L^2(\mathbf{R}_+)$
– $\mathcal{D} = \{\psi(f) = \sum_{n\geq 0} \frac{\otimes^n f}{n!} : \ f \in L^2(\mathbf{R}_+)\}$ the set of exponential vectors in $\mathcal{H}$.
– $\Phi = \psi(0)$ the vacuum state in $\mathcal{H}$
– $\mathcal{H}_{t]} = \Gamma(\chi_{[0,t]})\mathcal{H} = \Gamma(L^2([0,t]) \otimes \Phi_{[t}$ where $t \geq 0$ and $\Gamma(\chi_{[0,t]})$ is the orthogonal projector defined by

$$\Gamma(\chi_{[0,t]})\psi(f) = \psi(\chi_{[0,t]}f)$$

– A the annihilation field defined, on $\mathcal{D}$ by the relation:

$$A(f)\psi(g) =< f,g > \psi(g)$$

– A^+ the corresponding creation field, the adjoint of the annihilation.

For all $f \in L^2_{\text{loc}}(\mathbf{R}_+)$, the additive processes $A(\chi_{(s,t]}f)$, $A^+(\chi_{(s,t]}f)$ are regular semimartingales for the set $\mathcal{D}$ and, on this domain, they satisfy the commutation relations

$$[A(f), A^+(g)] =< f,g > \tag{8.3}$$

The process $(A_t, A_t^+) = (A(\chi_{(0,t]})$, $A^+(\chi_{(0,t]})$ is called *the standard Boson Fock Brownian motion* . In physics one uses more often the notation:

$$A(f) = \int_{\mathbf{R}} f(s)a_s ds \tag{8.4}$$

where a_s is an operator valued distribution (annihilation density) satisfying the commutation relations

$$[a_s, a_t^+] = \delta(t-s) \tag{8.5}$$

The (distribution valued) process (a_s, a_t^+) is called *quantum white noise.*

The classical Brownian motion is the process $Q_t = A_t + A_t^+$ (notice the continuum analogue of the master field (1.9a)). The process

$$P_t := \frac{1}{2i}\{A(\chi_{[0,t]}) - A^+(\chi_{[0,t]})\} \tag{8.6}$$

is also a classical Brownian motion and one has

$$[P_s, Q_t] = i \min\{s,t\}$$

in this sense one says that a quantum BM is a pair of non commuting classical BM.

The space $H = \Gamma(L^2(R))$ or $(\Gamma(L^2([0,+\infty))))$ is the simplest model for the state space of a quantum noise in the theory of dissipative quantum phenomena (cf. [Ac90]). It is canonically isomorphic to the L^2-space of the increments of the Wiener process (the white noise space) and its factorizing property

$$\Gamma(L^2(R)) \cong \Gamma(L^2((-\infty,t])) \otimes \Gamma(L^2((t,+\infty))) \tag{8.7}$$

corresponds to the independence of the increments of the Wiener process.

The essence of Ito's formula is contained in the commutation relations (6.6) and in the algebraic identity ( )

To show this, consider the increments of the creation and annihilation operators processes:

$$A_t - A_s = A(\chi_{[s,t]}) \qquad ; \qquad A^+_t - A^+_s = A^+(\chi_{[s,t]})$$

where $\chi_{[s,t]}(r)$ denotes the characteristic function of the interval $[s,t] \subseteq \mathbf{R}$ i.e.

$$\chi_{[s,t]} = 0 \qquad \text{if } r \notin [s,t] \qquad ; \qquad = 1 \ \text{ otherwise}$$

Now choose a small interval $[t, t+dt]$ and apply formula (8.6) above to the increment of the annihilation process over this interval, i.e.

$$dA_t = A_{t+dt} - A_t = A(\chi_{[t,t+dt]})$$

Then for any continuous function g one finds the approximation:

$$dA_t\psi(g) = \langle \chi_{[t,t+dt]}, g\rangle \psi(g) = \left(\int_t^{t+dt} g(s)ds \right) \psi(g) \equiv g(t)dt \cdot \psi(g)$$

where for a **numerical** function $F(t)$ we write

$$F(t+dt) - F(t) \equiv 0 \Leftrightarrow \lim_{dt\to 0} \big(sup_{t\in[o,T]}(F(t+dt) - F(t)\big)/dt \to 0 \ ; \ \forall T < +\infty$$

in this case we say that the difference $F(t+dt) - F(t)$ *is of order* $o(dt)$. This suggests that the topology of weak convergence on the set of the exponential vectors is a natural one to give a meaning to a quantum analogue of the Ito table. In fact the relation

$$dA_t\psi(g) \equiv g(t)dt \cdot \psi(g)$$

implies that the two-parameter families

$$A^2(\chi_{[s,t]}) \qquad , \qquad A^{+2}(\chi_{[s,t]}) \qquad , \qquad A^+(\chi_{[s,t]}) \cdot A(\chi_{[s,t]})$$

are of order $o(dt)$ for the topology of weak convergence on the exponential vectors $\psi(f)$ with f continuous and square integrable. In fact, for any pair $\psi(f)$, $\psi(g)$ of such vectors one has,

$$< \psi(f), A^2(\chi_{[t,t+dt]})\psi(g) >= \left(\int_t^{t+dt} gds \right)^2 < \psi(f), \psi(g) > \qquad (8.8a)$$

$$< \psi(f), A^{+2}(\chi_{[t,t+dt]})\psi(g) >= \left(\int_t^{t+dt} gds \right)^2 < \psi(f), \psi(g) > \qquad (8.8b)$$

$$\langle \psi_f, dA^+_t dA_t \psi_g\rangle = \langle dA_t\psi_f, dA_t\psi_g\rangle \equiv (f(t)dt)^*(g(t)dt) \cdot \langle \psi_f, \psi_g\rangle \equiv O(dt^2)\langle \psi_f, \psi_g\rangle$$

which are all of order $o(dt)$. Moreover

$$< \psi(f), A^+(\chi_{[t,t+dt]}) \cdot A(\chi_{[t,t+dt]})\psi(g) >= \left(\int_t^{t+dt} \overline{f} g ds\right) \cdot < \psi(f), \psi(g) > \qquad (8.8c)$$

which is of order dt. Hence, in our notations:

$$dA_t^+ dA_t \equiv 0 \qquad ; \qquad dA_t^+ dA_t^+ \equiv dt$$

Similarly one proves that:

$$(dA_t)^2 \equiv (dA_t^+)^2 \equiv 0$$

However $A(\chi_{[t,t+dt]}) \cdot A^+(\chi_{[t,t+dt]})$ is not of order $o(dt)$ in the same topology since, from the Heisenberg commutation relation:

$$[A(f), A^+(g)] =< f, g > \qquad (8.9)$$

and from (8.8c), one deduces that

$$< \psi(f), A(\chi_{[t,t+dt]}) \cdot A^+(\chi_{[t,t+dt]})\psi(g) >=$$

$$= \left(\int_t^{t+dt} \overline{f}(s) g(s) ds\right) \cdot < \psi(f), \psi(g) > + o(dt) \equiv \overline{f}(t) g(t) dt \cdot < \psi(f), \psi(g) >$$

Hence, in our notations:

$$dA_t dA_t^+ \equiv dt$$

The classical Ito table, deduced as an application of the above, is the set of equations ( ), ( ), ( ) plus the obvious equation $dtdt \equiv 0$.

Define $W_t = A_t + A_t^+$. If $s < t$

$$[W_s, W_t] = [W_s, W_s] + [W_s, W_{[s,t]}] = 0$$

where, in the last identity, we have used the property that *the future commutes with the past*. We conclude that (W_t) is a commutative family and therefore $\{(W_t), \Phi\}$ is a classical stochastic process. Moreover (W_t) is mean 0 and Gaussian since both A_t, A_t^+ are. Finally

$$dW_t^2 = (dA_t + dA_t^+)^2 = dA_t^2 + dA_t dA_t^+ + dA_t^+ dA_t + (dA_t^+)^2 \equiv dt$$

Therefore (W_t) is a classical Wiener process.

Another important example of stochastic process is the number (or gauge) process.

Let be given a pre-Hilbert space H; let $\Gamma(H)$ be the Fock space over H, Φ the Fock vacuum. For each $X \in B(H)$ with the norm less than 1, define a bounded operator $\Gamma(X)$ on $\Gamma(H)$ by the relation

$$\Gamma(X)\psi(f) := \psi(Xf) \qquad (8.10)$$

A special case of (8.10) is $X = e^{ith}$ with the operator h being self-adjoint. By the Stone-von Neumann Theorem and (8.10), (e^{ith}), as well as $(\Gamma(e^{ith}))$, are unitary group on H, and $\Gamma(H)$ respectively. Denote $N(h)$ the generator of $(\Gamma(e^{ith}))$, i.e.

$$e^{itN(h)}\psi(f) := \psi(e^{ith}f) = \Gamma(e^{ith})\psi(f) \tag{8.11}$$

and moreover for an arbitrary operator k, we define $N(k)$ by complex linearity:

$$N(k) = N\left(\left[\frac{k+k^*}{2}\right] + i\left[\frac{k-k^*}{2i}\right]\right) := N\left(\frac{k+k^*}{2}\right) + iN\left(\frac{k-k^*}{2i}\right) \tag{8.12}$$

The distribution of $N(h)$ in a coherent state (described by normalized exponential vectors–coherent vectors) is (a linear combination of) Poisson, i.e., if $h = h^*$

$$\begin{aligned}
&\langle e^{-\frac{1}{2}\|f\|}\psi(f),\ e^{itN(h)} \cdot e^{-\frac{1}{2}\|f\|}\psi(f)\rangle = \\
&= e^{-\|f\|^2}\langle\psi(f),\ \Gamma(e^{ith})\psi(f)\rangle = e^{-\|f\|^2}\langle\psi(f), \psi(e^{ith}f)\rangle \\
&= \exp\{-\|f\|^2 + \langle f, e^{ith}f\rangle\} = \exp\langle f, (e^{ith} - 1)f\rangle
\end{aligned} \tag{8.13}$$

Example Let $h = h^2 = h^*$ be an orthogonal projection. Then:

$$e^{ith} - 1 = e^{it} \cdot h + h^{\perp} - 1 = (e^{it} - 1)h \tag{8.14}$$

therefore the characteristic function of $N(h)$ is $\exp(e^{it} - 1)\|hf\|^2$ which corresponds to a Poisson distribution with intensity

$$\|hf\|^2$$

(9.) Stochastic limits and anisotropic asymptotics

Let us study the behaviour of correlation functions when only some components of coordinates in a preferable frame are suppose large or small. One of the most important examples corresponds to the (2+2)-decomposition and it describes the case when all longitudinal components in the central mass frame are assumed much smaller then transersal ones. Let x^{μ} be coordinates in the 4-dimensional Minkowski space-time and denote $x^{\mu} = (y^{\alpha}, z^i)$, $\alpha = 0, 1$, $i = 2, 3$. 2+2 Anisotropic asymptotics of Green functions of scalar selfinteracting theory describe the behaviour of the following correlation functions

$$G_n(\{\lambda y_i, z_i\}) =< \phi(\lambda y_1, z_1)\phi(\lambda y_2, z_2)...\phi(\lambda y_n, z_n) > \tag{9.1}$$

for $\lambda \to 0$.

For the scalar selfinteracting theory (9.1) is given by

$$< \phi(\lambda y_1, z_1)\phi(\lambda y_2, z_2)...\phi(\lambda y_n, z_n) >= \int \phi(\lambda y_1, z_1)\phi(\lambda y_2, z_2)...\phi(\lambda y_n, z_n)\cdot \tag{9.2}$$

$$\exp\{\int d^4x[\frac{1}{2}(\partial_\alpha \phi)^2 + (\partial_i \phi)^2 + V(\phi)]\}d\phi.$$

Performing in (9.2) the rescaling

$$\phi(\lambda y, z) = \tilde{\phi}(y, z)$$

and the change of variables in the action $y \to \lambda y$ one gets

$$G_n(\{\lambda y_i, z_i\}) = \int \tilde{\phi}(y_1, z_1)\tilde{\phi}(y_2, z_2)...\tilde{\phi}(y_n, z_n)\cdot$$

$$\exp\{\int d^4x \frac{1}{2}(\partial_\alpha \tilde{\phi})^2 + \lambda^2(\partial_i \tilde{\phi})^2 + \lambda^2 V(\tilde{\phi})\}d\tilde{\phi},$$

i.e. the theory with the effective action

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_\alpha \tilde{\phi})^2 + \frac{\lambda^2}{2}(\partial_i \tilde{\phi})^2 + \lambda^2 V(\tilde{\phi}) \tag{9.3}$$

Now we consider the asymptotics of the Green functions for the theory with the action (9.3) when $\lambda \to 0$. Let us discuss the free action. The free propagator for the action (9.3) has the form

$$G_\lambda(y, z) = \frac{1}{(2\pi)^4} \int \frac{e^{ikx}}{k_\alpha^2 + \lambda^2 k_i^2} dk = \frac{1}{(2\pi)^2} \frac{1}{\lambda^2 y^2 + z^2} \tag{9.4}$$

Let us examine the limit of (9.4) for $\lambda \to 0$ in the sense of theory of distributions, i.e. the asymptotic behaviour of the integral

$$(G_\lambda, f) = \frac{1}{(2\pi)^2} \int d^2y d^2z \frac{f(y, z)}{\lambda^2 y^2 + z^2}$$

when $\lambda \to 0$. Here $f(y, z)$ is a test function. One gets

$$G_\lambda(y, z) = \frac{1}{4\pi}\delta^{(2)}(z) \ln \frac{1}{\lambda^2} + \frac{1}{4\pi^2}\frac{1}{z^2} + \frac{1}{4\pi}\delta^{(2)}(z) \ln \frac{1}{y^2} + o(1)]$$

Here

$$\frac{1}{z^2} = Reg \frac{1}{z^2},$$

$$(\frac{1}{z^2}, f) = \int_{|z| \leq 1} d^2z \frac{f(z) - f(0)}{z^2} + \int d^2y \int_{|z|>1} d^2z \frac{f(z)}{z^2}$$

For a consideration of anisotropic asymptotics in scalar and gauge theories see [AV94].

(10.) Stochastic bosonization

Let $\Gamma_-(\mathcal{H}_1)$ denote the Fermi Fock space on the 1–particle space $\mathcal{H}_1$ and let, for $f \in \mathcal{H}_1$, $A(f)$, $A^+(f)$ denote the creation and annihilation operators on $\Gamma_-(\mathcal{H}_1)$ which satisfy the usual canonical anticommutation relations (CAR):

$$A(f)A^+(g) + A^+(g)A(f) = \langle f, g\rangle$$

The main idea of stochastic bosonization is that, in a limit to be specified below, two Fermion operators give rise to a Boson operator. To substantiate the idea let us introduce the operators

$$\mathcal{A}(f,g) := A(g)A(f), \qquad \mathcal{A}^+(f,g) := \big(\mathcal{A}(f,g)\big)^*$$

then by the CAR we have that

$$\mathcal{A}(f,g)\mathcal{A}^+(f',g') = A(g)A(f)A^+(f')A^+(g') =$$

$$= < g,g' >< f,f' > - < f,f' >< g,g' > + \mathcal{A}^+(f',g')\mathcal{A}(f,g) + R(f,g;f',g')$$

i.e.

$$[\mathcal{A}(f,g), \mathcal{A}^+(f',g')] = < (f\otimes g)(f'\otimes g') > + R(f,g;f',g')$$

where we introduce the notations

$$R(f,g;g',f') := < f,g' > A^+(f')A(g) - < f,f' > A^+(g')A(g) - < g,g' > A^+(f')A(f)$$

$$< (f\otimes g)(f'\otimes g') > := < f,f' >< g,g' > - < f,g' >< g,f' >$$

Moreover,

$$\mathcal{A}(f,g)\mathcal{A}(f',g') = \mathcal{A}(f',g')\mathcal{A}(f,g)$$

One can prove that, in the stochastic limit, the remainder term R tends to zero so that, in this limit, the *quasi CCR* become *bona fide* CCR.

The first step in the stochastic limit of quantum field theory is to introduce the *collective operators*. In our case we associate to the quadratic Fermion operator the collective creation operator defined by:

$$A^+_\lambda(S,T;f_0,f_1) := \lambda \int_{S/\lambda^2}^{T/\lambda^2} e^{i\omega t} A^+(S_t f_0)A^+(S_t f_1)dt \tag{10.1}$$

where ω is a real number, $S \leq T$, $f_0, f_1 \in \mathcal{K}$ and $S_t : \mathcal{H}_1 \to \mathcal{H}_1$ is the one–particle dynamical evolution whose second quantization gives the free evolution of the Fermi fields, i.e.

$$A(f) \mapsto A(S_t f) \ ; \quad A^+(f) \mapsto A^+(S_t f)$$

The collective annihilation operators are defined as the conjugate of the collective creators.

The introduction of operators such as the right hand side of (10.1) is a standard technique in the stochastic limit of QFT, we refer to [AcAlFriLu] for a survey and a detailed discussion. Here we limit ourselves to state that the choice (10.1) is dictated by the application of first order perturbation theory to an interaction Hamiltonian.

Having introduced the collective operators, the next step of the stochastic approximation is to compute the 2–point function

$$< \Phi, A_\lambda(S,T;f_0,f_1)A^+_\lambda(S',T';f_0',f_1')\Phi >=$$

$$= \lambda^2 \int_{S/\lambda^2}^{T/\lambda^2} dt \int_{S'/\lambda^2}^{T'/\lambda^2} ds e^{i\omega(t-s)} < \Phi, A(S_t f_0)A(S_t f_1)A^+(S_s f_0')A^+(S_s f_1')\Phi > \tag{10.2}$$

By the CAR, the scalar product in the right hand side of (10.2) is equal to

$$< S_t f_0, S_s f_1' >< S_t f_1, S_s f_0' > - < S_t f_0, S_s f_0' >< S_t f_1, S_s f_1' >$$

By standard arguments [AcLu1] one proves that the limit, as $\lambda \to 0$, of (10.2) is

$$< \chi_{[S,T]}, \chi_{[S',T']} >_{L^2(\mathbf{R})} \cdot \int_{-\infty}^{\infty} ds e^{i\omega s} \big[< f_0, S_s f_1' >< f_1, S_s f_0' > - < f_0, S_s f_0' >< f_1, S_s f_1' > \big]$$

Let us introduce on the algebraic tensor product $\mathcal{K} \odot \mathcal{K}$ the pre–scalar product $(\cdot|\cdot)$ defined by:

$$(f_0 \otimes f_1 | f_0' \otimes f_1') :=$$

$$= \int_{-\infty}^{\infty} ds e^{i\omega s} \big[< f_0, S_s f_1' >< f_1, S_s f_0' > - < f_0, S_s f_0' >< f_1, S_s f_1' > \big]$$

and denote by $\mathcal{K} \otimes_{\mathrm{FB}} \mathcal{K}$ the Hilbert space obtained by completing $\mathcal{K} \odot \mathcal{K}$ with this scalar product. Now let us consider the correlator

$$< \Phi, \prod_{k=1}^{n} A_\lambda^{\varepsilon(k)}(S_k, T_k; f_{0,k}, f_{1,k})\Phi > \tag{10.3}$$

where,

$$A^\varepsilon := \begin{cases} A^+, & \text{if } \varepsilon = 1, \\ A, & \text{if } \varepsilon = 0 \end{cases}$$

By the CAR, it is easy to see that (10.3) is equal to zero if the number of creators is different from the number of annihilators, i.e.

$$\Big| \{k;\ \varepsilon(k) = 1\} \Big| \neq \Big| \{k;\ \varepsilon(k) = 0\} \Big|$$

or if there exsits a $j = 1, \cdots, n$ such that the number of creators on the left of j is greater than the number of annihilators with the same property, i.e.

$$\Big| \{k \leq j;\ \varepsilon(k) = 1\} \Big| > \Big| \{k \leq j;\ \varepsilon(k) = 0\} \Big|$$

THEOREM (10.1) The limit, as $\lambda \to 0$, of (10.3) is equal to

$$< \Psi, \prod_{k=1}^{n} a^{\varepsilon(k)}(\chi_{[S_k,T_k]} \otimes f_{0,k} \otimes_{\mathrm{FB}} f_{1,k})\Psi >$$

where, a, a^+ and Ψ are (Boson) annihilation, creation operators on the Boson Fock space $\Gamma_+(L^2(\mathbf{R}) \otimes \mathcal{K} \otimes_{\mathrm{FB}} \mathcal{K})$ respectively and Ψ is the vacuum vector.

For a consideration of the stochastic bosonization in a model with an interaction see [AcLuVo94a].

(11.) The interacting Fock module and QED

Here we briefly review how the stochastic limit for quantum electrodynamics without dipole approximation naturally leads to the non–crossing diagrams, corresponding to a nonlinear deformation of the Wigner semicircle law, as well as to a *generalization of the free algebra* and to the introduction of the so–called *interacting Fock space* [AcLu93a] . For simplicity we shall consider the simplest of this model: a **polaron type particle**, interacting via the minimal coupling with a quantum EM field described by the Hamiltonian

$$H = H_S + H_R + H_I = H_0 + H_I \tag{11.1}$$

where denoting $p = (p_1, p_2, p_3)$ the momentum operator, the particle (system) Hamiltonian is

$$H_S = \frac{p^2}{2} \otimes 1 \tag{11.2}$$

where the reservoir Hamiltonian is

$$H_R = 1 \otimes \int_{\mathbf{R}^3} |k| a_k^* a_k dk \tag{11.3}$$

a_k and a_k^* are bosonic annihilation and creation operators and the interaction Hamiltonian is

$$\lambda H_I = \lambda \int_{\mathbf{R}^3} g(k) e^{ikq} p \otimes \frac{a_k}{\sqrt{|k|}} dk + \text{ h.c.} \tag{11.4}$$

where $g(k)$ (11.4) is a cutoff test function, λ is a coupling constant and p and q satisfy the commutation relations $[q, p] = i$.

The Hamiltonian (11.1) is an operator in the tensor product of Hilbert spaces of the particle and the field, $\mathcal{H} \otimes \Gamma(\mathcal{H})$, where $\mathcal{H} = L^2(\mathbf{R}^3)$, $\Gamma(\mathcal{H})$ is the bosonic Fock space.

In the weak coupling limit (stochastic limit) one considers the asymptotic behaviour for $\lambda \to 0$, of the *time scaled* evolution operator in the interaction representation, $U(t/\lambda^2)$, where:

$$U(t) = e^{itH_0} e^{-itH}$$

satisfies the equation

$$\frac{d}{dt} U(t) = -i\lambda H_I(t) U(t)$$

and $H_I(t)$ is the interaction representation Hamiltonian:

$$H_I(t) = e^{itH_0} H_I e^{-itH_0} = A^*(S_t g)(-ip) + \text{ h.c.}$$

$$A^+(S_t g) = \int_{\mathbf{R}^3} dk e^{-ikq} e^{itkp} e^{-it|k|^2/2} (S_t g)(k) \otimes a_k$$

The factor $\exp -it|k|^2/2$ shall be incorporated into the free one particle evolution S_t giving rise to the new evolution $\exp -it|k|^2/2$ still denoted, for simplicity, with the same symbol S_t. One defines the *collective annihilator process* by

$$A_\lambda(S_1, T_1, g) = \lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt \int_{\mathbf{R}^3} dk e^{-itkp} e^{ikq} e^{-itk^2_2} \otimes \overline{S_t g}(k) a_k \tag{11.5}$$

and its conjugate, $A^*_\lambda(t)$, and consider the limit when $\lambda \to 0$. If one puts $\lambda = 1$ $(N \to \infty)$ and replaces $T/\lambda^2 = NT$ by its integer part, the analogy between (12.5) and (11.6a) below) and the functional central limit theorem of Section (7.) becomes apparent. Let $K \subset L^2(\mathbf{R}^3)$ be the subspace of the Schwartz functions such that for any pair $f, g \in K$ the condition

$$\int |\langle f, S_t g\rangle| dt < \infty$$

is satisfied.

Theorem 11.1. *For any $n \in \mathbf{N}$ and test functions $g_1, \ldots, g_n$, from K the limit of the collective scalar product*

$$\lim_{\lambda \to 0} < A^*_\lambda(S_1, T_1; g_1) \ldots A^*_\lambda(S_n, T_n; g_n) \xi \otimes \Omega, \eta \otimes \Omega > \tag{11.6a}$$

exists and is equal to

$$\langle A^*(\chi_1 \times g_1) \ldots A^*(\chi_n \times g_n) \xi \otimes \psi\,,\ \eta \otimes \psi \rangle \tag{11.6b}$$

where $\chi_j = \chi_{[S_j, T_j]}$ is the characteristic function of the interval $[S_j, T_j](\chi_j(t) = 1$ if $t \in [S_j, T_j]$; $= 0$ otherwise) and the right hand side of (11.5) is an inner product in a limit Hilbert spaces which is the tensor product of a Hilbert module with the system Hilbert space, endowed with a new type of scalar product, which is described in the following.

We shall now introduce the following notations.
For each $f \in K$ define

$$\tilde{f}(t) := \int_{\mathbf{R}^d} e^{-ik\cdot q} e^{ik\cdot p} (S_t f)(k) dk$$

then $\tilde{f} : \mathbf{R} \to \mathcal{B}(L^2(\mathbf{R}^d))$. Denote by $\mathcal{P}$ the momentum algebra of the particle (system) i.e. the von Neumann–algebra generated by $\{e^{ik\cdot p} : k \in \mathbf{R}^d\}$, and by $\mathcal{F}$ the $\mathcal{P}$–right–linear span of $\{\tilde{f} : f \in K\}$. Then the tensor product $L^2(\mathbf{R}) \otimes \mathcal{F}$ is a $\mathcal{P}$ (in fact $1 \otimes \mathcal{P}$)–right module, on which we introduce the inner product

$$(\alpha \otimes \tilde{f}|\beta \otimes \tilde{g}) := \langle \alpha, \beta \rangle_{L^2(\mathbf{R})} \int_{\mathbf{R}} du \int_{\mathbf{R}} d_k e^{-iuk \cdot p} \overline{f}(k)(S_u g)(k) \tag{11.7}$$

which is a positive $\mathcal{P}$–right–sesquilinear form.

Starting from $L^2(\mathbf{R}) \otimes \mathcal{F}$ and taking the quotienting by the elements of zero norm with respect to the inner product $(\cdot|\cdot)$ given by (11.7), one obtains a (pre–)Hilbert module, still denoted by $L^2(\mathbf{R}) \otimes \mathcal{F}$. By $(L^2(\mathbf{R}) \otimes \mathcal{F})^{\otimes n}$ we shall denote the n–th algebraic tensor power of $L^2(\mathbf{R}) \otimes \mathcal{F}$, on which we introduce the inner product

$$((\alpha_1 \otimes \tilde{f}_1) \otimes \ldots \otimes (\alpha_n \otimes \tilde{f}_n)|(\beta_1 \otimes \tilde{g}_1) \otimes \ldots \otimes (\beta_n \otimes \tilde{g}_n)) :=$$

$$:= \prod_{h=1}^{n} \langle \alpha_n, \beta_n \rangle_{L^2(\mathbf{R})} \cdot \int_{\mathbf{R}^n} du_1 \ldots du_n \int_{\mathbf{R}^{nd}} dk_1 \ldots dk_n$$

$$\prod_{h=1}^{n} [e^{-iu_h k_h \cdot p} \overline{f}_h(k_h)(S_{u_h} g_h)(k_h)] \cdot \exp\left(i \sum_{1 \le r \le h \le n-1} u_r k_r k_{h+1} \right) \tag{11.8}$$

Notice that because of the exponential factor, even if the p in the exponential were a scalar, and not, as it is an operator on the particle space, the left hand side of (11.8) would not be equal to

$$(\alpha_1 \otimes \tilde{f}_1|\beta_1 \otimes \tilde{g}_1) \cdot \ldots \cdot (\alpha_n \otimes \tilde{f}_n|\beta_n \otimes \tilde{g}_n) \tag{11.9}$$

i.e. to the usual scalar product in the full Fock space.

Thus, with respect to the usual Fock (or full Fock) space, in the expression (11.8) there are two new features:

i) the inner product is not scalar valued but takes values in the momentum algebra of the system (i.e. particle) space.

ii) If we interpret $(L^2(\mathbf{R}) \otimes \mathcal{F})^{\otimes n}$ as a kind of n–particle space, then the exponential factor in (11.8) indicates that **these n–particle interact**.

The feature (i) tells us that we are in presence of an Hilbert module (over the momentum space of the particle). The feature (ii) is a remnant of the interaction: before the limit the different modes of the field interacted among themselves only through the mediation of the particle. After the limit **there is a true self–interaction**. Although complex, formula (11.8) seems to be the first one to describe in an explicit way a real self–interaction between all the modes of a quantum field.

It would be therefore very interesting to obtain a **relativistic generalization** of formula (11.8), which should be an achievable goal because the interaction among the modes occurs via the scalar product, which is an invariant expression.

In analogy with the usual Fock space, we introduce the direct sum

$$\Gamma(L^2(\mathbf{R}) \otimes \mathcal{F}) := \mathbf{C} \cdot \Psi \oplus \bigoplus_{n=1}^{\infty} (L^2(\mathbf{R}) \otimes \mathcal{F})^{\otimes n} \tag{11.10}$$

where the unit vector Ψ is called *the vacuum*. If we endow each *n–particle space* in (11.10) with the inner product (11.8), we do not obtain the usual full Fock space, but a new object that, because of the reasons explained above, has been called **the interacting Fock module**.

In order to pursue the analogy with the usual (full) Fock space, define the creator by

$$A^+(\alpha\otimes\tilde f)[(\alpha_1\otimes\tilde f_1)\otimes\ldots\otimes(\alpha_n\otimes\tilde f_n)] := (\alpha\otimes\tilde f)\otimes(\alpha_1\otimes\tilde f_1)\otimes\ldots\otimes(\alpha_n\otimes\tilde f_n)$$

and the annihilator by

$$A(\alpha\otimes f) := [A^+(\alpha\otimes f)]^+$$

One easily shows that the action of the annihilator is the one suggested by the obvious analogy with the Fock case, although now the inner product is an operator

Theorem 11.2.

$$A(\alpha\otimes\tilde f)[A^+(\alpha_1\otimes\tilde f_1)\ldots A^+(\alpha_n\otimes\tilde f_n)\Psi]$$
$$= (\alpha\otimes f)|\alpha_1\otimes\tilde f_1)A^+(\alpha_2\otimes\tilde f_2)\ldots A^+(\alpha_n\otimes\tilde f_n)\Psi$$

where Ψ is the vacuum of $\Gamma(L^2(\mathbf{R})\otimes\mathcal{F})$.

The explicit calculation of the quantities of physical interest in made possible by the following:

Theorem. $\forall n\in\mathbf{N}\ \varepsilon\in\{0,1\}^n$, $A^0 := A$, $A^1 := A^+$

$$\langle\Psi, A^{\varepsilon(1)}(\alpha_1\otimes\tilde f_1)\ldots A^{\varepsilon(n)}(\alpha_n\otimes\tilde f_n)\Psi\rangle \tag{11.11}$$

is 0 if $\{\varepsilon(1),\ldots,\varepsilon(n)\}$ does not allow a non–crossing pair partitions; if $\{\varepsilon(1),\ldots,\varepsilon(n)\}$ allows non–crossing pair partition, then it is unique and $n=2j$ for some j. In this case if we denote by $1<m_1<m_2<\ldots<m_j=2j=n$ the position of the creators and by $\{m'_h\}_{h=1}^j$ the position of the corresponding annihilators, in the sense that the non–crossing pair partition is given by:

$$(m'_1,m_1),\ (m'_2,m_2),\ldots,(m'_j,m_j)$$

then (12.11) is given by

$$\prod_{h=1}^j\langle\alpha_{m_h},\alpha_{m'_h}\rangle_{L^2(\mathbf{R})}\int_{\mathbf{R}}du_1\ldots\int_{\mathbf{R}}du_j\int_{\mathbf{R}^{jd}}dk_1\ldots dk_j$$

$$\left[\prod_{h=1}^j(Su_hfm_h)(k_h)\overline{f}_{m'_h}(k_h)\cdot e^{iu_hk_h\cdot p}\right]\cdot$$

$$\cdot\exp\left(i\sum_{h=1}^{j-1}\sum_{r=h+1}^{j}u_hk_h\cdot h_r\chi_{(m'_r,m_r)}(m_h)\right)$$

It can also be proved that the limit (in the sense of quantum convergence in law) of the rescaled wave operator U_{t/λ^2} exists and satisfies a quantum stochastic differential equation on $\Gamma(L^2(\mathbf{R}) \otimes \mathcal{F}) \otimes \mathcal{H}_0$ (cf. the equation (11.2) of [AcLu]).

We shall not discuss here the quantum stochastic differential equation, associated to this model, because it requires the notion of *stochastic integration over Hilbert modules*, which introduces several new features with respect to the usual quantum stochastic integration. For this we refer to the paper [AcLu91], [AcLu92], [AcLu93a], [AcLu93b], for the model we are discussing here and to [Lu92a], [Lu92b], [Lu94], for the general theory.

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