

NOTE ON QUANTUM DYNAMICAL ENTROPIES

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Classical dynamical entropy is an important tool to analyse the efficiency of information transmission in communication processes. Quantum dynamical entropy was first studied by Connes, Størmer and Emch. Since then, there have been many attempts to formulate or compute the dynamical entropy for some models. Here we review four formulations due to (a) Connes, Narnhofer and Thirring, (b) Ohya, (c) Accardi, Ohya and Watanabe, (d) Alicki and Fannes. We consider mutual relations between these formulations and we show some concrete computations for a model.

Introduction

The classical dynamical (or Kolmogorov–Sinai) entropy $S(T)$ [13, 26] for a measure preserving transformation T was defined on a message space through finite partitions of the measurable space. The classical coding theorems of Shannon are important tools to analyse communication processes which have been formulated by the mean dynamical entropy and the mean dynamical mutual entropy. The mean dynamical entropy represents the amount of information per one letter of a signal sequence sent from an input source, and the mean dynamical mutual entropy does the amount of information per one letter of the signal received in an output system.

The quantum dynamical entropy (QDE) has been studied by Connes and Størmer [10], Emch [11], Connes, Narnhofer and Thirring [9], Alicki and Fannes [5], and others [7, 22]. Their dynamical entropies were defined in the observable spaces.

Recently, the quantum dynamical entropy and the quantum dynamical mutual entropy were studied by one of the present authors [14, 23]. They are formulated in the state spaces through the complexity of Information Dynamics [21, 23]. Furthermore, another formulation of the dynamical entropy through the quantum Markov chain (QMC) was done in [4].

In Section 1 we briefly review the formulation by Connes, Narnhofer and Thirring (CNT). In Section 2 we explain the formulation by complexity [23, 24]. In Section 3 we

review the formulation by QMC, and in Section 4 we briefly explain the formulation by Alicki and Fannes (AF). Mutual relations among these four formulations are discussed in Section 4, and in Section 6 we compute the mean dynamical entropies in quantum communication processes.

1. The CNT formulation

Let \mathcal{A} be a unital C^* -algebra, θ be an automorphism of \mathcal{A} , and φ be a stationary state over \mathcal{A} with respect to θ ; $\varphi \circ \theta = \varphi$. Let \mathcal{B} be a finite dimensional C^* -subalgebra of \mathcal{A} .

The CNT entropy [9] for a subalgebra \mathcal{B} is given by

$$H_\varphi(\mathcal{B}) = \sup \left\{ \sum_k \lambda_k S(\omega_k | \mathcal{B}, \varphi | \mathcal{B}); \quad \varphi = \sum_k \lambda_k \omega_k \text{ finite decomposition of } \varphi \right\},$$

where $\varphi | \mathcal{B}$ is the restriction of the state φ to \mathcal{B} and $S(\cdot, \cdot)$ is the relative entropy for C^* -algebra [6, 27, 28].

The CNT dynamical entropy with respect to θ and \mathcal{B} is given by

$$\tilde{H}_\varphi(\theta, \mathcal{B}) = \limsup_{N \rightarrow \infty} \frac{1}{N} H_\varphi(\mathcal{B} \vee \theta \mathcal{B} \vee \dots \vee \theta^{N-1} \mathcal{B}),$$

and the dynamical entropy for θ is defined by

$$\tilde{H}(\theta) = \sup_{\mathcal{B}} \tilde{H}_\varphi(\theta, \mathcal{B}).$$

2. Formulation by complexity

In this section, we first review the concepts of channel and complexity, which are the key concepts of ID (Information Dynamics) introduced by Ohya [21, 23].

Let $(\mathcal{A}, \Sigma(\mathcal{A}), \alpha(G))$, $(\bar{\mathcal{A}}, \bar{\Sigma}(\bar{\mathcal{A}}), \bar{\alpha}(\bar{G}))$ be an input (initial) and an output (final) C^* -systems, respectively, where \mathcal{A} (resp. $\bar{\mathcal{A}}$) is a unital C^* -algebra, $\Sigma(\mathcal{A})$ (resp. $\bar{\Sigma}(\bar{\mathcal{A}})$) is the set of all states on \mathcal{A} (resp. $\bar{\mathcal{A}}$) and $\alpha(G)$ (resp. $\bar{\alpha}(\bar{G})$) is the group of automorphisms of \mathcal{A} (resp. $\bar{\mathcal{A}}$) indexed by a group G (resp. \bar{G}).

The channel is a map Λ^* from $\Sigma(\mathcal{A})$ to $\bar{\Sigma}(\bar{\mathcal{A}})$. If the dual map Λ from $\bar{\mathcal{A}}$ to \mathcal{A} of Λ^* satisfies the complete positivity, the channel Λ^* is called a complete positive channel (CP channel).

For a weak $*$ compact convex subset \mathcal{S} of Σ , there exists a maximum measure μ with the barycenter φ such that

$$\varphi = \int_{\mathcal{S}} \omega d\mu.$$

The compound state, introduced in [17, 18], which exhibits the correlation between the initial φ and final $\Lambda^*\varphi$ states is given by

$$\mathcal{E}_\mu^* \varphi = \int_{\mathcal{S}} \omega \otimes \Lambda^* \omega d\mu.$$

In the sequel, we use a CP channel Λ^* and the compound state to formulate the dynamical entropy.

There are two types of complexities in ID. One of them is the complexity $C_T^{\mathcal{S}}(\varphi)$ of the system itself, the other is the transmitted complexity $T^{\mathcal{S}}(\varphi; \Lambda^*)$ from an initial system to a final system. These complexities should satisfy the following conditions:

(i) For any $\varphi \in \mathcal{S} \subset \Sigma$,

$$C^{\mathcal{S}}(\varphi) \geq 0, \quad T^{\mathcal{S}}(\varphi; \Lambda^*) \geq 0.$$

(ii) If there exists an orthogonal bijection $j : \text{ex}\Sigma \rightarrow \text{ex}\Sigma$; the set of all extremal points in Σ , then

$$\begin{aligned} C^{j(\mathcal{S})}(j(\varphi)) &= C^{\mathcal{S}}(\varphi), \\ T^{j(\mathcal{S})}(j(\varphi); \Lambda^*) &= T^{\mathcal{S}}(\varphi; \Lambda^*). \end{aligned}$$

(iii) For a state $\Psi = \varphi \otimes \psi \in \mathcal{S}_t$, with $\varphi \in \mathcal{S}$, $\psi \in \bar{\mathcal{S}}$,

$$C^{\mathcal{S}_t}(\Psi) = C^{\mathcal{S}}(\varphi) + C^{\bar{\mathcal{S}}}(\psi).$$

(iv) $0 \leq T^{\mathcal{S}}(\varphi; \Lambda^*) \leq C^{\mathcal{S}}(\varphi)$.

(v) $T^{\mathcal{S}}(\varphi; \text{id}) = C^{\mathcal{S}}(\varphi)$.

Next we review the definitions of the three types of entropic complexities introduced in [23].

Let $(\mathcal{A}, \Sigma(\mathcal{A}), \alpha(G))$, $(\bar{\mathcal{A}}, \bar{\Sigma}(\bar{\mathcal{A}}), \bar{\alpha}(\bar{G}))$ and \mathcal{S} be as above. Let $M_{\varphi}(\mathcal{S})$ be the set of all maximal measures μ on \mathcal{S} with the fixed barycenter φ , and let $F_{\varphi}(\mathcal{S})$ be the set of all measures having finite support with the fixed barycenter φ . Then the three pairs of complexities are

$$\begin{aligned} T^{\mathcal{S}}(\varphi; \Lambda^*) &\equiv \sup \left\{ \int_{\mathcal{S}} S(\Lambda^* \omega, \Lambda^* \varphi) d\mu; \quad \mu \in M_{\varphi}(\mathcal{S}) \right\}, \\ C_T^{\mathcal{S}}(\varphi) &\equiv T^{\mathcal{S}}(\varphi; \text{id}), \\ I^{\mathcal{S}}(\varphi; \Lambda^*) &\equiv \sup \left\{ S \left(\int_{\mathcal{S}} \omega \otimes \Lambda^* \omega d\mu, \varphi \otimes \Lambda^* \varphi \right); \quad \mu \in M_{\varphi}(\mathcal{S}) \right\}, \\ C_I^{\mathcal{S}}(\varphi) &\equiv I^{\mathcal{S}}(\varphi; \text{id}), \\ J^{\mathcal{S}}(\varphi; \Lambda^*) &\equiv \sup \left\{ \int_{\mathcal{S}} S(\Lambda^* \omega, \Lambda^* \varphi) d\mu_f; \quad \mu_f \in F_{\varphi}(\mathcal{S}) \right\}, \\ C_J^{\mathcal{S}}(\varphi) &\equiv J^{\mathcal{S}}(\varphi; \text{id}). \end{aligned}$$

Based on the above complexities, we explain the quantum dynamical complexity (QDC) [17].

Let θ (resp. $\bar{\theta}$) be a stationary automorphism of \mathcal{A} (resp. $\bar{\mathcal{A}}$); $\varphi \circ \theta = \varphi$, and Λ be a covariant CP map (i.e., $\Lambda \circ \theta = \bar{\theta} \circ \Lambda$) from $\bar{\mathcal{A}}$ to \mathcal{A} . \mathcal{B}_k (resp. $\bar{\mathcal{B}}_k$) is a finite subalgebra

of \mathcal{A} (resp. $\bar{\mathcal{A}}$). Moreover, let α_k (resp. $\bar{\alpha}_k$) be a CP unital map from \mathcal{B}_k (resp. $\bar{\mathcal{B}}_k$) to \mathcal{A} (resp. $\bar{\mathcal{A}}$) and α^M and $\bar{\alpha}_\Lambda^N$ are given by

$$\begin{aligned}\alpha^M &= (\alpha_1, \alpha_2, \dots, \alpha_M), \\ \bar{\alpha}_\Lambda^N &= (\Lambda \circ \bar{\alpha}_1, \Lambda \circ \bar{\alpha}_2, \dots, \Lambda \circ \bar{\alpha}_N).\end{aligned}$$

Two compound states for α^M and $\bar{\alpha}_\Lambda^N$, with respect to $\mu \in M_\varphi(\mathcal{S})$, are defined as

$$\begin{aligned}\Phi_\mu^{\mathcal{S}}(\alpha^M) &= \int_{\mathcal{S}} \bigotimes_{m=1}^M \alpha_m^* \omega \, d\mu, \\ \Phi_\mu^{\mathcal{S}}(\alpha^M \cup \bar{\alpha}_\Lambda^N) &= \int_{\mathcal{S}} \bigotimes_{m=1}^M \alpha_m^* \omega \bigotimes_{n=1}^N \bar{\alpha}_n^* \Lambda^* \omega \, d\mu.\end{aligned}$$

Using the above compound states, the three transmitted complexities [23] are defined by

$$T_\varphi^{\mathcal{S}}(\alpha^M, \bar{\alpha}_\Lambda^N) \equiv \sup \left\{ \int_{\mathcal{S}} S \left(\bigotimes_{m=1}^M \alpha_m^* \omega \bigotimes_{n=1}^N \bar{\alpha}_n^* \Lambda^* \omega, \Phi_\mu^{\mathcal{S}}(\alpha^M) \otimes \Phi_\mu^{\mathcal{S}}(\bar{\alpha}_\Lambda^N) \right) d\mu; \mu \in M_\varphi(\mathcal{S}) \right\},$$

$$I_\varphi^{\mathcal{S}}(\alpha^M, \bar{\alpha}_\Lambda^N) \equiv \sup \left\{ S \left(\Phi_\mu^{\mathcal{S}}(\alpha^M \cup \bar{\alpha}_\Lambda^N), \Phi_\mu^{\mathcal{S}}(\alpha^M) \otimes \Phi_\mu^{\mathcal{S}}(\bar{\alpha}_\Lambda^N) \right); \mu \in M_\varphi(\mathcal{S}) \right\},$$

$$J_\varphi^{\mathcal{S}}(\alpha^M, \bar{\alpha}_\Lambda^N) \equiv \sup \left\{ \int_{\mathcal{S}} S \left(\bigotimes_{m=1}^M \alpha_m^* \omega \bigotimes_{n=1}^N \bar{\alpha}_n^* \Lambda^* \omega, \Phi_\mu^{\mathcal{S}}(\alpha^M) \otimes \Phi_\mu^{\mathcal{S}}(\bar{\alpha}_\Lambda^N) \right) d\mu_f; \mu_f \in F_\varphi(\mathcal{S}) \right\}.$$

When $\mathcal{B}_k = \bar{\mathcal{B}}_k = \mathcal{B}$, $\mathcal{A} = \bar{\mathcal{A}}$, $\theta = \bar{\theta}$, $\alpha_k = \theta^{k-1} \circ \alpha = \bar{\alpha}_k$, where α is a unital CP map from \mathcal{A}_0 to \mathcal{A} , the mean transmitted complexities are

$$\begin{aligned}\tilde{T}_\varphi^{\mathcal{S}}(\theta, \alpha, \Lambda^*) &\equiv \limsup_{N \rightarrow \infty} \frac{1}{N} T_\varphi^{\mathcal{S}}(\alpha^N, \bar{\alpha}_\Lambda^N), \\ \tilde{T}_\varphi^{\mathcal{S}}(\theta, \Lambda^*) &\equiv \sup_\alpha \tilde{T}_\varphi^{\mathcal{S}}(\theta, \alpha, \Lambda^*),\end{aligned}$$

and the same for $\tilde{I}_\varphi^{\mathcal{S}}$ and $\tilde{J}_\varphi^{\mathcal{S}}$. These quantities have properties similar to those of the CNT entropy [14, 23].

3. Formulation by QMC

A construction of dynamical entropy is due to the quantum Markov chain [4].

Let \mathcal{A} be a von Neumann algebra acting on a Hilbert space \mathcal{H} and let φ be a state on \mathcal{A} and $\mathcal{A}_0 = M_d$ ($d \times d$ matrix algebra). Take the transition expectation $\mathcal{E}_\gamma : \mathcal{A}_0 \otimes \mathcal{A} \rightarrow \mathcal{A}$ of Accardi [1, 2] such that

$$\mathcal{E}_\gamma(\tilde{A}) = \sum_i \gamma_i A_{ii} \gamma_i,$$

where $\tilde{A} = \sum_{i,j} e_{ij} \otimes A_{ij} \in \mathcal{A}_0 \otimes \mathcal{A}$ and $\gamma = \{\gamma_j\}$ is a finite partition of unity $I \in \mathcal{A}$.

Quantum Markov chain is defined by $\psi \equiv \{\varphi, \mathcal{E}_{\gamma, \theta}\} \in \Sigma(\bigotimes_1^\infty \mathcal{A}_0)$ such that

$$\psi(j_1(A_1) \cdots j_n(A_n)) \equiv \varphi(\mathcal{E}_{\gamma, \theta}(A_1 \otimes \mathcal{E}_{\gamma, \theta}(A_2 \otimes \cdots \otimes A_{n-1} \mathcal{E}_{\gamma, \theta}(A_n \otimes I) \cdots))),$$

where $\mathcal{E}_{\gamma, \theta} = \theta \circ \mathcal{E}_{\gamma}$, $\theta \in \text{Aut}(\mathcal{A})$, and j_k is an embedding of \mathcal{A}_0 into $\bigotimes_1^{\infty} \mathcal{A}_0$ such that $j_k(A) = I \otimes \cdots \otimes I \otimes A \otimes I \cdots$.

Suppose that for φ there exists a unique density operator ρ such that $\varphi(A) = \text{Tr} \rho A$ for any $A \in \mathcal{A}$. Let us define a state ψ_n on $\bigotimes_1^n \mathcal{A}_0$ expressed as

$$\psi_n(A_1 \otimes \cdots \otimes A_n) = \psi(j_1(A_1) \cdots j_n(A_n)).$$

The density operator ξ_n for ψ_n is given by

$$\xi_n \equiv \sum_{i_1} \cdots \sum_{i_n} \text{Tr}_{\mathcal{A}}(\theta^n(\gamma_{i_n}) \cdots \gamma_{i_1} \rho \gamma_{i_1} \cdots \theta^n(\gamma_{i_n})) e_{i_1 i_1} \otimes \cdots \otimes e_{i_n i_n}.$$

Put

$$P_{i_n \cdots i_1} = \text{Tr}_{\mathcal{A}}(\theta^n(\gamma_{i_n}) \cdots \gamma_{i_1} \rho \gamma_{i_1} \cdots \theta^n(\gamma_{i_n})).$$

The dynamical entropy through QMC is defined by

$$\tilde{S}_{\varphi}(\theta; \gamma) \equiv \limsup_{n \rightarrow \infty} \frac{1}{n} \left(-\text{Tr} \xi_n \log \xi_n \right) = \limsup_{n \rightarrow \infty} \frac{1}{n} \left(- \sum_{i_1, \dots, i_n} P_{i_n \cdots i_1} \log P_{i_n \cdots i_1} \right).$$

If $P_{i_n \cdots i_1}$ satisfies the Markov property, then the above equality is written by

$$\tilde{S}_{\varphi}(\theta; \gamma) = - \sum_{i_1, i_2} P(i_2 | i_1) P(i_1) \log P(i_2 | i_1).$$

The dynamical entropy through QMC with respect to θ and a von Neumann subalgebra \mathcal{B} of \mathcal{A} is given by

$$\tilde{S}_{\varphi}(\theta; \mathcal{B}) \equiv \sup \left\{ \tilde{S}_{\varphi}(\theta; \gamma) : \gamma \subset \mathcal{B} \right\}.$$

4. Formulation by AF

Let \mathcal{A} be a C^* -algebra, θ be an automorphism on \mathcal{A} and φ be a stationary state with respect to θ and \mathcal{B} be a unital $*$ -subalgebra of \mathcal{A} . A set $\gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of elements of \mathcal{B} is called a finite operational partition of unity of size k if γ satisfies the following condition:

$$\sum_{i=1}^k \gamma_i^* \gamma_i = I. \tag{4.1}$$

The operation \circ is defined by

$$\gamma \circ \xi \equiv \{\gamma_i \xi_j ; \quad i = 1, 2, \dots, k, \quad j = 1, 2, \dots, l\}$$

for any partitions $\gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ and $\xi = \{\xi_1, \xi_2, \dots, \xi_l\}$. For any partition γ of size k , a $k \times k$ density matrix $\rho[\gamma] = (\rho[\gamma]_{i,j})$ is given by

$$\rho[\gamma]_{i,j} = \varphi(\gamma_j^* \gamma_i).$$

Then the dynamical entropy $\tilde{H}_\varphi(\theta, \mathcal{B}, \gamma)$ with respect to the partition γ and shift θ is defined by von Neumann entropy $S(\cdot)$;

$$\tilde{H}_\varphi(\theta, \mathcal{B}, \gamma) = \limsup_{n \rightarrow \infty} \frac{1}{n} S(\rho[\theta^{n-1}(\gamma) \circ \dots \circ \theta(\gamma) \circ \gamma]). \tag{4.2}$$

The dynamical entropy $\tilde{H}_\varphi(\theta, \mathcal{B})$ is given by taking the supremum over operational partition of unity in \mathcal{B} as

$$\tilde{H}_\varphi(\theta, \mathcal{B}) = \sup \left\{ \tilde{H}_\varphi(\theta, \mathcal{B}, \gamma); \gamma \subset \mathcal{B} \right\}. \tag{4.3}$$

5. Relations among the four formulations

In this section we discuss relations among the above four formulations. The \mathcal{S} -mixing entropy in GQS introduced in [20] is

$$S^\mathcal{S}(\varphi) = \inf \{ H(\mu); \mu \in M_\varphi(\mathcal{S}) \},$$

where $H(\mu)$ is given by

$$H(\mu) = \sup \left\{ - \sum_{A_k \in \tilde{\mathcal{A}}} \mu(A_k) \log \mu(A_k) : \tilde{\mathcal{A}} \in P(\mathcal{S}) \right\},$$

and $P(\mathcal{S})$ is the set of all finite partitions of \mathcal{S} .

The following theorem [14, 23] shows the relation between the formulation by CNT and that by complexity.

THEOREM 5.1. *Under the above settings, we have the following relations:*

- (1) $0 \leq I^\mathcal{S}(\varphi; \Lambda^*) \leq T^\mathcal{S}(\varphi; \Lambda^*) \leq J^\mathcal{S}(\varphi; \Lambda^*)$,
- (2) $C_I^\Sigma(\varphi) = C_T^\Sigma(\varphi) = C_J^\Sigma(\varphi) = S^\Sigma(\varphi) = H_\varphi(\mathcal{A})$,
- (3) $\mathcal{A} = \tilde{\mathcal{A}} = B(\mathcal{H})$, for any density operator ρ , and

$$0 \leq I^\mathcal{S}(\rho; \Lambda^*) = T^\mathcal{S}(\rho; \Lambda^*) \leq J^\mathcal{S}(\rho; \Lambda^*).$$

Since there exists a model showing that $S^{I(\alpha)}(\varphi) \geq H_\varphi(\mathcal{A}_\alpha)$, $S^\mathcal{S}(\varphi)$ distinguishes states more sharply than $H_\varphi(\mathcal{A})$, where $\mathcal{A}_\alpha = \{A \in \mathcal{A}; \alpha(A) = A\}$.

Furthermore, we have the following results [24].

- (1) When $\mathcal{A}_n, \mathcal{A}$ are abelian C^* -algebras and α_k is an embedding map, then

$$\begin{aligned} T^\Sigma(\mu; \alpha^M) &= S_\mu^{\text{classical}} \left(\bigvee_{m=1}^M \tilde{A}_m \right), \\ I^\Sigma(\mu; \alpha^M, \bar{\alpha}^N) &= I_\mu^{\text{classical}} \left(\bigvee_{m=1}^M \tilde{A}_m, \bigvee_{n=1}^N \tilde{B}_n \right) \end{aligned}$$

are satisfied for any finite partitions \tilde{A}_m, \tilde{B}_n on the probability space $(\Omega = \text{spec}(\mathcal{A}), \mathcal{F}, \mu)$.

(2) When Λ is the restriction of \mathcal{A} to a subalgebra \mathcal{M} of \mathcal{A} ; $\Lambda = |\mathcal{M}$,

$$H_\varphi(\mathcal{M}) = J^\Sigma(\varphi; |\mathcal{M}) = J_\varphi^\Sigma(\text{id}; |\mathcal{M}).$$

Moreover, when

$$\begin{aligned} \mathcal{N} \subset \mathcal{A}_0, \quad \mathcal{A} &= \bigotimes_{\mathbb{N}} \mathcal{A}_0, \quad \theta \in \text{Aut}(\mathcal{A}); \\ \alpha^N &\equiv (\alpha, \theta \circ \alpha, \dots; \theta^{N-1} \circ \alpha); \\ \alpha &= \bar{\alpha}; \quad \mathcal{A}_0 \rightarrow \mathcal{A} \quad \text{an embedding}; \\ \mathcal{N}_N &\equiv \bigotimes_1^N \mathcal{N}, \end{aligned}$$

we have

$$\tilde{H}_\varphi(\theta; \mathcal{N}) = \tilde{J}_\varphi^\Sigma(\theta; \mathcal{N}) = \limsup_{N \rightarrow \infty} \frac{1}{N} J_\varphi^\Sigma(\alpha^N; |\mathcal{N}_N).$$

We show the relation between the formulation by complexity and that by QMC. Under the same settings in Section 3, we define a map $\mathcal{E}_{(n,\gamma)}^*$ from $\Sigma(\mathcal{H})$, the set of all density operators in \mathcal{H} , to $\Sigma(\bigotimes_1^n \mathbb{C}^d \otimes \mathcal{H})$ by

$$\begin{aligned} \mathcal{E}_{(n,\gamma)}^*(\rho) &= \sum_{i_1} \cdots \sum_{i_{n-1}} \sum_{i_n} e_{i_1 i_1} \otimes \cdots \otimes e_{i_{n-1} i_{n-1}} \otimes e_{i_n i_n} \otimes \\ &\quad \otimes \theta^{n-1}(\gamma_{i_n}) \theta^{(n-2)}(\gamma_{i_{n-1}}) \cdots \gamma_{i_1} \rho \gamma_{i_1} \cdots \theta^{(n-2)}(\gamma_{i_{n-1}}) \theta^{n-1}(\gamma_{i_n}), \end{aligned}$$

for any density operator $\rho \in \Sigma(\mathcal{H})$. Let us take a map $E_{(n)}^*$ from $\Sigma(\bigotimes_1^n \mathbb{C}^d \otimes \mathcal{H})$ to $\Sigma(\bigotimes_1^n \mathbb{C}^d)$ such that

$$E_{(n)}^*(\sigma) = \text{Tr}_{\mathcal{H}} \sigma, \quad \forall \sigma \in \Sigma\left(\left(\bigotimes_1^n \mathbb{C}^d\right) \otimes \mathcal{H}\right).$$

Then a map $\Gamma_{(n,\gamma)}^*$ from $\Sigma(\mathcal{H})$ to $\Sigma(\bigotimes_1^n \mathbb{C}^d)$ is given by

$$\Gamma_{(n)}^*(\rho) \equiv E_{(n)}^*\left(\mathcal{E}_{(n,\gamma)}^*(\rho)\right), \quad \forall \rho \in \Sigma(\mathcal{H}),$$

so that $\Gamma_{(n,\gamma)}^*(\rho) = \xi_n$ and

$$\tilde{S}_\varphi(\theta; \gamma) = \limsup_{n \rightarrow \infty} \frac{1}{n} S(\Gamma_{(n,\gamma)}^*(\rho)).$$

From Theorem 5.1, we have $C_I^\Sigma(\Gamma_{(n,\gamma)}^*(\rho)) = S(\Gamma_{(n,\gamma)}^*(\rho))$. Hence

$$\tilde{S}_\varphi(\theta; \gamma) = \tilde{C}_I^\Sigma(\Gamma_{(\gamma)}^*(\rho)) \left(\equiv \limsup_{n \rightarrow \infty} \frac{1}{n} C_I^\Sigma(\Gamma_{(n,\gamma)}^*(\rho)) \right).$$

Now we briefly show the relation between the formulation by complexity and that by AF.

Under the same settings in Section 4, for any partitions $\gamma = \{\gamma_1, \dots, \gamma_k\}$ of size k , the $k \times k$ density matrix $\rho[\gamma] = (\rho[\gamma]_{i,j})$ is defined by

$$\rho[\gamma]_{i,j} = \varphi(\gamma_j^* \gamma_i)$$

which is acting on the k -dimensional Hilbert space \mathcal{H}_k . We define a map $\Xi_{(m,\gamma)}^*$ from $\Sigma(\mathcal{H}_k)$ to $\Sigma(\mathcal{H}_{k^m})$ by

$$\Xi_{(m,\gamma)}^*(\rho[\gamma]) = \rho[\theta^{m-1}(\gamma) \circ \dots \circ \theta(\gamma) \circ \gamma]$$

for any partitions $\gamma = \{\gamma_1, \dots, \gamma_k\}$ of size k and any density matrices $\rho[\gamma] \in \Sigma(\mathcal{H}_k)$. The dynamical entropy by AF is given by

$$\tilde{H}_\varphi(\theta, \mathcal{B}, \gamma) = \limsup_{m \rightarrow \infty} \frac{1}{m} S(\Xi_{(m,\gamma)}^*(\rho[\gamma])).$$

Since $C_I^\mathcal{F}(\Xi_{(m,\gamma)}^*(\rho[\gamma])) = S(\Xi_{(m,\gamma)}^*(\rho[\gamma]))$, we have

$$\tilde{H}_\varphi(\theta, \mathcal{B}, \gamma) = \tilde{C}_I^\mathcal{F}(\Xi_{(\gamma)}^*(\rho[\gamma])).$$

In any case, the formulation by the entropic complexities contains other formulations. Moreover it opens other possibilities to classify dynamical systems more fine [5].

6. Computation of quantum dynamical complexity for a model

Let $X = \{a_1, \dots, a_M\}$ be an alphabet used to construct the input signals and let $\mathcal{S} = \{E_1, \dots, E_M\}$ be a set of one-dimensional projections on a Hilbert space \mathcal{H}_0 satisfying (1) $E_n \perp E_m$ ($n \neq m$) and (2) E_n corresponds to the character a_n .

By Σ_0 we denote the set of density operators generated by \mathcal{S}

$$\Sigma_0 \equiv \left\{ \sigma = \sum_{n=1}^M \lambda_n E_n ; \sum_{n=1}^M \lambda_n = 1, \lambda \geq 0 \right\}.$$

Suppose that the input quantum state is an element of Σ_0 . To send the information effectively, the state is first transmitted through a quantum modulator; the transmitted state is called the quantum modulated state. Let $\gamma_{(\mathcal{M})}^*$ be a map from Σ_0 to $\Sigma_0^{(\mathcal{M})}$ such that $\gamma_{(\mathcal{M})}$ is a completely positive unital map from \mathcal{A} to \mathcal{A} induced by the modulator (\mathcal{M}) . For any $E_n \in \mathcal{S}$, the modulated state $E_n^{(\mathcal{M})}$ is given by

$$E_n^{(\mathcal{M})} = \gamma_{(\mathcal{M})}^*(E_n).$$

By $\Sigma_0^{(\mathcal{M})}$ we denote the set of modulated states

$$\Sigma_0^{(\mathcal{M})} \equiv \left\{ \sigma^{(\mathcal{M})} = \sum_{n=1}^M \lambda_n E_n^{(\mathcal{M})} ; \sum_{n=1}^M \lambda_n = 1, \lambda \geq 0 \right\}.$$

In this paper, we consider the modulated states constructed by the photon number states.

- (1) For any $E_n \in \Sigma_0$, the modulated state $E_n^{(PAM)}$ for PAM (Pulse Amplitude Modulator) is defined by

$$E_n^{(PAM)} = \gamma_{(PAM)}^*(E_n) = |n\rangle\langle n|,$$

where $|n\rangle\langle n|$ is the n photon number state on \mathcal{H} .

- (2) For any $E_n \in \Sigma_0$, the modulated state $E_n^{(PPM)}$ for PPM (Pulse Position Modulator) is defined by

$$\begin{aligned} E_n^{(PPM)} &= \gamma_{(PPM)}^*(E_n) \\ &= \underbrace{E_0^{(PAM)} \otimes \dots \otimes E_0^{(PAM)} \otimes \overbrace{E_d^{(PAM)}^{n\text{-th}}} \otimes E_0^{(PAM)} \otimes \dots \otimes E_0^{(PAM)}}_M, \end{aligned}$$

where $E_0^{(PAM)}$ is the vacuum state and $E_d^{(PAM)} = |d\rangle\langle d|$ (d is fixed).

The transmission efficiency using the MER (mutual entropy — entropy ratio) [29] is calculated for some modulators.

Now we compute the mean dynamical mutual entropy for PAM and PPM expressed by the photon number state (as above).

Let $B(\mathcal{H}_0)$ (resp. $B(\overline{\mathcal{H}}_0)$) be the set of all bounded linear operators on a Hilbert space \mathcal{H}_0 (resp. $\overline{\mathcal{H}}_0$), and let \mathcal{B}_0 (resp. $\overline{\mathcal{B}}_0$) be a finite subset in $B(\mathcal{H}_0)$ (resp. $B(\overline{\mathcal{H}}_0)$). Let \mathcal{A} (resp. $\overline{\mathcal{A}}$) be an infinite tensor product space of $B(\mathcal{H}_0)$ (resp. $B(\overline{\mathcal{H}}_0)$) denoted by

$$\mathcal{A} \equiv \bigotimes_{i=-\infty}^{\infty} B(\mathcal{H}_0), \quad \overline{\mathcal{A}} \equiv \bigotimes_{i=-\infty}^{\infty} B(\overline{\mathcal{H}}_0).$$

Moreover, let θ (resp. $\overline{\theta}$) be a shift transformations on \mathcal{A} (resp. $\overline{\mathcal{A}}$) defined by

$$\begin{aligned} \theta\left(\bigotimes_{i=-\infty}^{\infty} A_i\right) &= \bigotimes_{i'=-\infty}^{\infty} A_{i'} \quad (i' = i - 1) \quad \text{for any} \quad \bigotimes_{i=-\infty}^{\infty} A_i \in \mathcal{A}, \\ \overline{\theta}\left(\bigotimes_{j=-\infty}^{\infty} \overline{A}_j\right) &= \bigotimes_{j'=-\infty}^{\infty} \overline{A}_{j'}, \quad (j' = j - 1) \quad \text{for any} \quad \bigotimes_{j=-\infty}^{\infty} \overline{A}_j \in \overline{\mathcal{A}}. \end{aligned}$$

Let α (resp. $\overline{\alpha}$) be the embedding map from \mathcal{B}_0 into \mathcal{A} (resp. $\overline{\mathcal{B}}_0$ to $\overline{\mathcal{A}}$) given by

$$\begin{aligned} \alpha(A) &= \dots I \otimes I \otimes A \otimes I \otimes \dots \in \mathcal{A}, \quad \text{for any} \quad A \in \mathcal{B}_0, \\ \overline{\alpha}(\overline{A}) &= \dots I \otimes I \otimes \overline{A} \otimes I \otimes \dots \in \overline{\mathcal{A}}, \quad \text{for any} \quad \overline{A} \in \overline{\mathcal{B}}_0. \end{aligned}$$

The set of all density operators on \mathcal{H}_0 (resp. $\overline{\mathcal{H}}_0$) we denote by Σ_0 (resp. $\overline{\Sigma}_0$), and let Σ (resp. $\overline{\Sigma}$) be the set of all density operators on \mathcal{A} (resp. $\overline{\mathcal{A}}$).

The maps $\alpha_{(\mathcal{M})}^N$ and $\bar{\alpha}_{\tilde{\Lambda}(\mathcal{M})}^N$ are given by

$$\begin{aligned} \alpha_{(\mathcal{M})}^N &\equiv (\alpha \circ \tilde{\gamma}_{(\mathcal{M})}, \theta \circ \alpha \circ \tilde{\gamma}_{(\mathcal{M})}, \dots, \theta^{N-1} \circ \alpha \circ \tilde{\gamma}_{(\mathcal{M})}), \\ \bar{\alpha}_{\tilde{\Lambda}(\mathcal{M})}^N &\equiv (\tilde{\gamma}_{(\mathcal{M})} \circ \tilde{\Lambda} \circ \bar{\alpha}, \tilde{\gamma}_{(\mathcal{M})} \circ \tilde{\Lambda} \circ \bar{\theta} \circ \bar{\alpha}, \dots, \tilde{\gamma}_{(\mathcal{M})} \circ \tilde{\Lambda} \circ \bar{\theta}^{N-1} \circ \bar{\alpha}), \end{aligned}$$

where we took a special channel and modulator such that $\tilde{\Lambda} \equiv \bigotimes_{i=-\infty}^{\infty} \Lambda$ and $\tilde{\gamma}_{(\mathcal{M})} \equiv \bigotimes_{i=-\infty}^{\infty} \gamma_{(\mathcal{M})}$.

(I) PAM. Let us take a stationary initial state $\rho \in \Sigma$

$$\rho = \sum_m \mu_m \bigotimes_{i=-\infty}^{\infty} \rho_m^{(i)} \quad \left(\sum_m \mu_m = 1, \quad \mu_m \neq \mu_{m'}, \quad (m \neq m') \right) \quad (6.1)$$

with the unique Schatten decomposition $\rho_m^{(i)} = \sum_{n_i=1}^M \lambda_{n_i}^{(m)} E_{n_i} \in \Sigma_0$ ($\sum_{n_i=1}^M \lambda_{n_i}^{(m)} = 1$, $\lambda_{n_i}^{(m)} \neq \lambda_{k_i}^{(m)}$, ($n_i \neq k_i$)) of $\rho_m^{(i)}$. Then we have

$$\Phi_E(\alpha_{(PAM)}^N) = \sum_{n_0=1}^M \cdots \sum_{n_{N-1}=1}^M \left(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)} \right) \left(\bigotimes_{i=0}^{N-1} E_{n_i}^{(PAM)} \right), \quad (6.2)$$

$$\Phi_E(\bar{\alpha}_{\tilde{\Lambda}(PAM)}^N) = \sum_{n_0=1}^M \cdots \sum_{n_{N-1}=1}^M \left(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)} \right) \left(\bigotimes_{i=0}^{N-1} \Lambda^* E_{n_i}^{(PAM)} \right). \quad (6.3)$$

When Λ^* is an attenuation channel, we obtain [17]

$$\Lambda^* E_{n_i}^{(PAM)} = \sum_{j_i=0}^{n_i} |C_{j_i}^{n_i}|^2 F_{j_i}^{(PAM)},$$

where $F_{j_i}^{(PAM)} = |j_i\rangle\langle j_i|$ is the j_i -photon number state in the output space $\bar{\Sigma}_0$ and

$$|C_{j_i}^{n_i}|^2 = \frac{n_i!}{j_i!(n_i - j_i)!} \eta^{j_i} (1 - \eta)^{(n_i - j_i)},$$

where η is the transmission rate of the channel [17, 19]. The compound states related to the channel $\tilde{\Lambda}^*$ become

$$\begin{aligned} &\Phi_E(\alpha_{(PAM)}^N) \cup \bar{\alpha}_{\tilde{\Lambda}(PAM)}^N \\ &= \sum_{n_0=1}^M \cdots \sum_{n_{N-1}=1}^M \left(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)} \right) \sum_{j_0=0}^{n_0} \cdots \sum_{j_{N-1}=0}^{n_{N-1}} \left(\prod_{k'=0}^{N-1} |C_{j_{k'}}^{n_{k'}}|^2 \right) \times \\ &\quad \times \left(\bigotimes_{i=0}^{N-1} E_{n_i}^{(PAM)} \right) \left(\bigotimes_{i'=0}^{N-1} F_{j_{i'}}^{(PAM)} \right), \end{aligned} \quad (6.4)$$

$$\begin{aligned} & \bar{\Phi}_E(\alpha_{(PAM)}^N) \otimes \bar{\Phi}_E(\bar{\alpha}_{\bar{A}(PAM)}^N) \\ &= \sum_{n_0=1}^M \cdots \sum_{n_{N-1}=1}^M \left(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)} \right) \sum_{n'_0=1}^M \cdots \sum_{n'_{N-1}=1}^M \left(\sum_{m'} \prod_{k'=0}^{N-1} \lambda_{n'_{k'}}^{(m')} \right) \times \\ & \times \sum_{j'_0=0}^{n'_0} \cdots \sum_{j'_{N-1}=0}^{n'_{N-1}} \left(\prod_{k''=0}^{N-1} |C_{j'_{k''}}^{n'_{k''}}|^2 \right) \left(\bigotimes_{i=0}^{N-1} E_{n_i}^{(PAM)} \right) \left(\bigotimes_{i'=0}^{N-1} F_{j'_{i'}}^{(PAM)} \right). \end{aligned} \quad (6.5)$$

For an initial state ρ in (6.1), we have

$$\begin{aligned} & \mathbf{I}_E(\rho; \alpha_{(PAM)}^N, \bar{\alpha}_{\bar{A}(PAM)}^N) \\ &= \sum_{j_0=0}^M \cdots \sum_{j_{N-1}=0}^M \sum_{n_0=J_0}^M \cdots \sum_{n_{N-1}=J_{N-1}}^M \left(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)} \right) \left(\prod_{k'=0}^{N-1} |C_{j_{k'}}^{n_{k'}}|^2 \right) \times \\ & \times \log \frac{\prod_{k=0}^{N-1} |C_{j_k}^{n_k}|^2}{\sum_{n'_0=J_0}^M \cdots \sum_{n'_{N-1}=J_{N-1}}^M \left(\sum_{m'} \mu_{m'} \prod_{k'=0}^{N-1} \lambda_{n'_{k'}}^{(m')} \right) \left(\prod_{k''=0}^{N-1} |C_{j_{k''}}^{n'_{k''}}|^2 \right)}. \end{aligned}$$

Having the above equality, we obtain the following theorem.

THEOREM 6.1. (1) For an initial state ρ in (6.1), we have the lower bound of $\tilde{C}_\rho(\theta, \alpha_{(PAM)})$ such that

$$\tilde{C}_\rho(\theta, \alpha_{(PAM)}) \geq \sum_m \mu_m S(\rho_m^{(0)}).$$

(2) Let $\mu_0 = 1, \mu_k = 0 (\forall k \geq 1)$ and $\lambda_n^{(0)} = \lambda_n$ in (6.1). When $\mathcal{A} = \bar{\mathcal{A}}, \theta = \bar{\theta}$ and $\alpha = \bar{\alpha}$, we obtain the following equalities:

$$\begin{aligned} \tilde{C}_\rho(\theta, \alpha_{(PAM)}) &= - \sum_{n=1}^M \lambda_n \log \lambda_n, \\ \tilde{I}_\rho^\Sigma(\theta, \alpha_{(PAM)}, \Lambda^*) &= \sum_{j=0}^M \sum_{n=J}^M \lambda_n |C_j^n|^2 \log \frac{|C_j^n|^2}{\sum_{n'=J}^M \lambda_{n'} |C_{j'}^{n'}|^2}. \end{aligned}$$

(II) PPM. For an initial state ρ in (6.1), we obtain the following compound states:

$$\bar{\Phi}_E(\alpha_{(PPM)}^N) = \sum_{n_0=1}^M \cdots \sum_{n_{N-1}=1}^M \left(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)} \right) \left(\bigotimes_{i=0}^{N-1} \left(\bigotimes_{j=0}^{N-1} E_{d\delta_{j,n_i}}^{(PAM)} \right) \right). \quad (6.6)$$

$$\begin{aligned} & \bar{\Phi}_E(\bar{\alpha}_{\bar{A}(PPM)}^N) = \sum_{n_0=1}^M \cdots \sum_{n_{N-1}=1}^M \left(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)} \right) \times \\ & \times \sum_{\ell_0=0}^{d\delta_{1,n_0}} \cdots \sum_{\ell_{N-1}=0}^{d\delta_{M,n_{N-1}}} |C_{\ell_0}^{d\delta_{1,n_0}}|^2 \cdots |C_{\ell_{N-1}}^{d\delta_{M,n_{N-1}}}|^2 \left(\bigotimes_{i=0}^{N-1} \left(\bigotimes_{j=1}^M F_{\ell_{iM+j}}^{(PAM)} \right) \right). \end{aligned} \quad (6.7)$$

$$\begin{aligned}
 & \Phi_E(\alpha_{(PPM)}^N) \cup \bar{\alpha}_{\bar{A}(PPM)}^N \\
 &= \sum_{n_0=1}^M \cdots \sum_{n_{N-1}=1}^M \left(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)} \right) \sum_{\ell_1=0}^{d\delta_{1,n_0}} \cdots \sum_{\ell_{N-1}=0}^{d\delta_{M,n_{N-1}}} |C_{\ell_1}^{d\delta_{1,n_0}}|^2 \cdots |C_{\ell_{N-1}}^{d\delta_{M,n_{N-1}}}|^2 \times \\
 & \quad \times \left(\bigotimes_{i=0}^{N-1} \left(\bigotimes_{j=1}^M E_{d\delta_{j,n_i}}^{(PAM)} \right) \right) \left(\bigotimes_{i'=0}^{N-1} \left(\bigotimes_{j'=0}^M F_{\ell_{i'M+j'}}^{(PAM)} \right) \right). \tag{6.8}
 \end{aligned}$$

$$\begin{aligned}
 & \Phi_E(\alpha_{(PPM)}^N) \otimes \Phi_E(\bar{\alpha}_{\bar{A}(PPM)}^N) \\
 &= \sum_{n_0=1}^M \cdots \sum_{n_{N-1}=1}^M \sum_{n'_0=1}^M \cdots \sum_{n'_{N-1}=1}^M \left(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)} \right) \times \\
 & \quad \times \sum_{\ell_1=0}^{d\delta_{1,n_0}} \cdots \sum_{\ell_{N-1}=0}^{d\delta_{M,n_{N-1}}} |C_{\ell_1}^{d\delta_{1,n_0}}|^2 \cdots |C_{\ell_{N-1}}^{d\delta_{M,n_{N-1}}}|^2 \sum_{n'_0=1}^M \cdots \sum_{n'_{N-1}=1}^M \left(\sum_{m'} \mu_{m'} \prod_{k'=0}^{N-1} \lambda_{n'_{k'}} \right) \times \\
 & \quad \times \left(\bigotimes_{i=0}^{N-1} \left(\bigotimes_{j=1}^M E_{d\delta_{j,n_i}}^{(PAM)} \right) \right) \left(\bigotimes_{i'=0}^{N-1} \left(\bigotimes_{j'=1}^M F_{\ell_{i'M+j'}}^{(PAM)} \right) \right). \tag{6.9}
 \end{aligned}$$

For an initial state ρ in (6.1), we have

$$\begin{aligned}
 & I_E(\rho; \alpha_{(PPM)}^N, \bar{\alpha}_{\bar{A}(PPM)}^N) \\
 &= - \sum_{n_0=1}^M \cdots \sum_{n_{N-1}=1}^M \left(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)} \right) \sum_{p=1}^N \sum_{\{q_1, \dots, q_p\} \subset \{1, 2, \dots, N\}} \times \\
 & \quad \times \sum_{\ell_1=1}^d \cdots \sum_{\ell_p=1}^d |C_{\ell_1}^d|^2 \cdots |C_{\ell_p}^d|^2 (1-\eta)^{N-p} \eta^p \log \left(\sum_{m'} \mu_{m'} \prod_{k'=0}^p \lambda_{n_{q_{k'}}}^{(m')} \right).
 \end{aligned}$$

THEOREM 6.2. (1) For an initial state ρ in (6.1), we have the lower bound of $\tilde{C}_\rho(\theta, \alpha_{(PPM)})$ such that

$$\tilde{C}_\rho(\theta, \alpha_{(PPM)}) \geq \sum_m \mu_m S(\rho_m^{(0)}).$$

(2) Let $\mu_0 = 1, \mu_k = 0$ ($\forall k \geq 1$) and $\lambda_n^{(0)} = \lambda_n$ in (6.1). When $\mathcal{A} = \bar{\mathcal{A}}, \theta = \bar{\theta}$ and $\alpha = \bar{\alpha}$, we obtain the following equalities:

$$\begin{aligned}
 \tilde{C}_\rho(\theta, \alpha_{(PPM)}) &= - \sum_{n=1}^M \lambda_n \log \lambda_n, \\
 \tilde{I}_\rho^\Sigma(\theta, \alpha_{(PPM)}, \Lambda^*) &= (1 - (1-\eta)^d) \tilde{C}_\rho(\theta, \alpha_{(PPM)}).
 \end{aligned}$$

(3) If $\mathcal{A} = \bar{\mathcal{A}}, \theta = \bar{\theta}, \alpha = \bar{\alpha}$ and $d \geq N$, then

$$\tilde{I}_\rho^\Sigma(\theta, \alpha_{(PPM)}, \Lambda^*) \geq \tilde{I}_\rho^\Sigma(\theta, \alpha_{(PAM)}, \Lambda^*).$$

REFERENCES

- [1] L. Accardi: *Noncommutative Markov chains*, Internatinal School of Mathematical Physics, Camerino (1974), 268.
- [2] L. Accardi, A. Frigerio and J. Lewis: *Quantum stochastic processes*, Publications of the Research institute for Mathematical Sciences Kyoto University, **18** (1982), 97.
- [3] L. Accardi and M. Ohya: Compound channels, transition expectations and liftings, to appear in *J. Multivariate Analysis*.
- [4] L. Accardi, M. Ohya and N. Watanabe: Dynamical entropy through quantum Markov chain, to appear in *Open System and Information Dynamics*.
- [5] R. Alicki and M. Fannes: *Lett. Math. Phys.* **32** (1994), 75.
- [6] H. Araki: *Relative entropy for states of von Neumann algebras*, *Publications of the Research institute for Mathematical Sciences Kyoto University* **11** (1976), 809.
- [7] F. Benatti: *Deterministic Chaos in Infinite Quantum Systems*, Springer, Berlin 1993.
- [8] L. Bilingsley: *Ergodic Theory and Information*, Wiley, New York 1965.
- [9] A. Connes, H. Narnhoffer and W. Thirring: *Commun. Math. Phys.* **112** (1987), 691.
- [10] A. Connes and E. Størmer: *Acta Math.* **134** (1975), 289.
- [11] G. G. Emch: *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **29** (1974), 241.
- [12] L. Feinstein: *Foundations of Information Theory*, MacGrow-Hill, 1965.
- [13] A. N. Kolmogorov: *Dokl. Akad. Nauk SSSR* **119** (1958), 861.
- [14] N. Muraki and M. Ohya: *Lett. Math. Phys.* **36** (1996), 327.
- [15] J. von Neumann: *Die Mathematischen Grundlagen der Quantenmechanik*, Springer, Berlin 1932.
- [16] M. Ohya: *J. Math. Anal. Appl.* **84** (1981), 318.
- [17] M. Ohya: *IEEE Trans. Information Theory* **29** (1983), 770.
- [18] M. Ohya: *L. Nuovo Cimento* **38** (1983), 402.
- [19] M. Ohya: *J. Math. Anal. Appl.* **100** (1984), 222.
- [20] M. Ohya: *Rep. Math. Phys.* **27** (1989), 19.
- [21] M. Ohya: *Information dynamics and its application to optical communication processes*, Springer Lecture Notes in Physics, **378**, p. 81, Springer, Berlin 1991.
- [22] M. Ohya and D. Petz: *Quantum Entropy and its Use*, Springer, Berlin 1993.
- [23] M. Ohya: *Quantum Communications and Measurement* **2** (1995), 309.
- [24] M. Ohya: Foundation of entropy, complexity and fractal in quantum systems, to appear in International Congress of Probability Towards 2000, 1996.
- [25] M. Ohya and N. Watanabe: Note on irreversible dynamics and quantum information, to appear in the *Alberto Frigerio Conference Proceedings*.
- [26] J. G. Sinai: *Dokl. Akad. Nauk SSSR* **124** (1959), 768.
- [27] A. Uhlmann: *Commun. Math. Phys.* **54** (1977), 21.
- [28] H. Umegaki: *Kodai Math. Sem. Rep.* **14** (1962), 59.
- [29] N. Watanabe: *Quantum Probability and Related Topics* **6** (1991), 489.