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Noncommutative Markov Chains Associated to a Preassigned Evolution: An Application to the Quantum Theory of Measurement*

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INTRODUCTION

It is well known that for each Markov probability measure μ on $(\Omega, \mathfrak{B}) = \Pi_{\mathbb{R}^+}(S, \mathscr{B})$ — $(S, \mathscr{B})$ being a standard Borel space, there is naturally associated a positive evolution, i.e., a family $(P(t, s))$, $s < t$, $s, t \geqslant 0$ of positivity preserving linear operators $P(t, s): L^\infty(S, \mathscr{B}, \mu_t) \to L^\infty(S, \mathscr{B}, \mu_s)$ (μ_t being the restriction of μ on the tth factor of the product $\Pi_{\mathbb{R}^+}(S, \mathscr{B})$) such that

$$P(s, r) \cdot P(t, s) = P(t, r), \qquad r < s < t, \tag{1}$$

$$P(t, s)(1_t) = 1_s, \tag{2}$$

(1_t being the identity function $L^\infty(S, \mathscr{B}, \mu_t)$). Conversely, if the maps $P(t, s)$ are assumed to be normal, such an evolution determines the measure μ up to the "initial measure" μ_0 by means of the equalities:

$$\mu_{0,t_1,\ldots,t_n}(f_0 \otimes f_{t_1} \otimes \cdots \otimes f_{t_n}) = \mu_0(f_0 \times P(t_1, 0)(f_{t_1} \times P(t_2, t_1)(f_{t_2} \times \cdots \times P(t_n, t_{n-1})(f_{t_n})\cdots))), \tag{3}$$

where

$$f_{t_j} \in L^\infty(S, \mathscr{B}, \mu_{t_j}), \qquad j = 1,\ldots, n;\ 0 < t_1 < \cdots < t_n;$$

$$\mu(F) = \int_\Omega F\, d\mu$$

and $\mu_{0,t_1,\ldots,t_n}$ is the restriction of μ on

$$\bigvee_{j=0}^{n} \Pi_{t_j}^{-1}(\mathscr{B}),\ (t_0 = 0);$$

Π_s being the projection onto the sth factor of the product $\Pi_{\mathbb{R}^+}(S, \mathscr{B})$. The natural translation of the above construction into a noncommutative context, obtained by letting the f_{t_j}'s in (3) vary in a C^*-algebra $\mathscr{A}_{t_j}$, leads, as it is well known, to unpleasant features such as complex mean values for positive operators.

Starting from an analysis of the concept of conditional expectation and of the Markov property in a noncommutative context [1, 2], it is possible to define a family of states on C^*-algebras of the form $\mathscr{A} = \otimes_{\mathbb{R}^+} \mathscr{A}_t$ ($\mathscr{A}_t$ an approximately finite C^*-algebra) whose structure is formally identical to that of classical Markov states and reduces to it when the algebras are commutative [3]. States of this class will be called noncommutative Markov chains. They can be defined on any C^*-algebra with a local structure (cf. [2]) but in this note we shall only consider C^*-algebras of the type mentioned above.

The main features of noncommutative Markov chains are that they are completely, and explicitly, determined by local characteristics—the transition expectations (cf. Theorem 1.1) and that to each of them a completely positive evolution is naturally associated. In [3] it has been shown that any *reversible* completely positive evolution determines, up to the initial state, a noncommutative Markov chain on $\mathscr{A} = \otimes_{\mathbb{R}^+} \mathscr{B}$, where $\mathscr{B}$ is the algebra of all bounded operators on a complex Hilbert space $\mathscr{H}$ and the cross norm on $\mathscr{A}$ is determined by the tensor products of finite sets of copies of $\mathscr{B}$. In general, however, unlike the commutative case, a completely positive evolution does not determine a noncommutative Markov chain up to the initial state (i.e., these data allow us to compute the expectations of observables at any fixed instant of time, but not, in general, the joint expectations of observables at different times). In Sections 1 and 2 it is shown that, under some continuity assumptions on the dependence of the local characteristics (i.e., the "transition expectations") on the parameter $t \in \mathbb{R}^+$ the noncommutative Markov chains associated to a preassigned evolution can be completely classified.

The resulting structure is very simple. To illustrate it, let us consider the equalities (3) in the commutative case, and denote by T_s the multiplication operator $T_s(f_s \otimes g_s) = f_s \cdot g_s; f_s, g_s \in L^\infty(S, \mathscr{B}, \mu_s)$. Defining $\mathscr{E}_{t,s}: L^\infty(S, \mathscr{B}, \mu_s) \otimes L^\infty(S, \mathscr{B}, \mu_t) \to L^\infty(S, \mathscr{B}, \mu_s)$ by

$$\mathscr{E}_{t,s}(f_s \otimes f_t) = T_s(f_s \otimes P(t, s) f_t) \tag{4}$$

the operator $\mathscr{E}_{t,s}$ is (completely) positive and (3) becomes:

$$\mu_{0,t_1,\ldots,t_n}(f_0 \otimes f_{t_1} \otimes \cdots \otimes f_{t_n}) = \mu_0(\mathscr{E}_{t_1,0}(f_0 \otimes \mathscr{E}_{t_2,t_1}(f_{t_1} \otimes \cdots \otimes \mathscr{E}_{t_n,t_{n-1}}(f_{t_{n-1}} \otimes f_{t_n})\cdots))). \tag{5}$$

In the noncommutative case the multiplication operator is no longer positive so that, even if the evolution $(P(t, s))$ is positive, the equalities (3) will not, in general, define a state.

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It is possible, however, to take the operators $\mathscr{E}_{t,s}$, the transition expectations, as basic objects of our analysis and impose conditions on them so that the family $(\mu_{0,t_1,\ldots,t_n})$ defined by (5) determines a unique state μ on $\mathscr{A} = \bigotimes_{t\in\mathbb{R}^+} \mathscr{A}_t$, i.e., it is a projective family of states. In the commutative case the structure (5) of the measure μ is deduced from a general, intrinsic, property of the conditional expectations associated to it—the Markov property. The same is true for noncommutative Markov chains (cf. Theorem 1.1). In Section (2) it is shown that, if the transition expectations $\mathscr{E}_{t,s}$ depend regularly enough on the parameters t, s, then they must be of the form (4) for some completely positive evolution $(P(t, s))$ and some completely positive linear map $T_s: \mathscr{A}_s \otimes \mathscr{A}_s \to \mathscr{A}_s$. In this case the projectivity conditions for the states defined by (5) are expressed in terms of two equations in the unknowns T_s (cf. Section 2, Theorem 1.1(iv), (v)) whose solutions parametrize, up to the initial state, the Markov chains associated to the evolution $(P(t, s))$ and with a regular dependence of the transition expectations on the parameters t, s.

In Section 3 one looks for the solutions of the above-mentioned equations which are "covariant" (cf. (29)) with respect to a given reversible evolution and such that the range of the operator T_0, of "instantaneous coupling at time 0", is a C^*-algebra. It is shown that there is only one class of such solutions which satisfy a continuity condition (cf. (32)). The evolution naturally associated to this class is shown to be a generalization of the evolution postulated in von Neumann's theory of the quantum measurement process. The joint expectations of the noncommutative Markov chains associated to this class of solutions are computed and shown to be the noncommutative analog of those associated to classical Markov chains describing a system which undergoes a deterministic evolution but whose initial state is subject to an indeterminacy.

The usual quantum systems correspond to a class of "weak" solutions (cf. [3] and Remark 1 after Theorem 1.1) of the compatibility equations. This class is uniquely characterized by the property that the range of the operator T_0 (of "instantaneous coupling" at time 0) is all of $\mathscr{B}(\mathscr{H})$—while for the unique class of strong solutions mentioned above, the range of T_0 is necessarily an Abelian algebra.

In conclusion, the above results show that noncommutative Markov theory is a natural unifying context for the description of usual quantum mechanics and for the theory of the quantum measurement process. In both cases simple explicit formulas for the joint and the transition expectations of observables at different times are deduced as particular cases of a general formula valid also in the case of irreversible evolutions.

1. Markov States

Let $\mathscr{H}$ be a complex separable Hilbert space. In the following $\mathscr{B} = \mathscr{B}(\mathscr{H})$ shall denote the algebra of all bounded operators on $\mathscr{H}$, $\mathscr{A} = \bigotimes_{\mathbb{R}^+} \mathscr{B}$ the C^*-infinite tensor product of card $\mathbb{R}^+$-copies of $\mathscr{B}$; $J_t: \mathscr{B} \hookrightarrow \mathscr{A}$ the natural injection of $\mathscr{B}$ onto the tth factor of the product $\bigotimes_{\mathbb{R}^+} \mathscr{B}$; $\mathscr{A}_t = J_t(\mathscr{B})$, for $t \geqslant 0$; $\mathscr{A}_I = \bigvee_{t\in I} \mathscr{A}_t$ the C^*-subalgebra of $\mathscr{A}$ generated by the family $(\mathscr{A}_t)_{t\in I}$ where I is any subset of $\mathbb{R}^+$.

Let us recall from [3] that a quasi-conditional expectation with respect to the triple $\mathscr{A}_{[0,s[} \subseteq \mathscr{A}_{[0,s]} \subseteq \mathscr{A}_{[0,t]}$ is a positive linear map $E_{t,s}: \mathscr{A}_{[0,t]} \to \mathscr{A}_{[0,s]}$ such that, if $r < s$ then:

$$E_{t,s}(a \cdot b) = a \cdot E_{t,s}(b); \qquad a \in \mathscr{A}_{[0,s]}; \qquad b \in \mathscr{A}_{[0,t]} \tag{6}$$

for $s = 0$ a quasi-conditional expectation is defined as a completely positive (cf. [7]) linear map $E_{t,0}: \mathscr{A}_{[0,t]} \to \mathscr{A}_0$—and that a Markov state on $\mathscr{A}$ (with respect to the "localization" $(\mathscr{A}_{[0,t]})$) is a state φ on $\mathscr{A}$ such that there exists a family $(E_{t,s})$ of quasi-conditional expectations with respect to the triples $\mathscr{A}_{[0,s[} \subseteq \mathscr{A}_{[0,s]} \subseteq \mathscr{A}_{[0,t]}$ satisfying the equalities

$$\varphi_{[0,t]} = \varphi_{[0,s]} \cdot E_{t,s}, \qquad s < t, \tag{7}$$

where $\varphi_{[0,t]}$ denotes the restriction of φ on $\mathscr{A}_{[0,t]}$, and $\varphi_{[0,s]} \mapsto \varphi_{[0,s]} \cdot E_{t,s}$ denotes the adjoint action of $E_{t,s}$ from the dual of $\mathscr{A}_{[0,s]}$ to the dual of $\mathscr{A}_{[0,t]}$ and $\varphi_{[0,s]} \cdot E_{t,s}(a) = \varphi_{[0,s]}(E_{t,s}(a))$. In the following, by a quasi-conditional expectation we shall mean a normalized one, i.e., one for which:

$$E_{t,s}(1) = 1. \tag{8}$$

As shown in [3] the quasi-conditional expectation $E_{t,s}$ satisfies:

$$E_{t,s}(\mathscr{A}_{[s,t]}) \subseteq \mathscr{A}_s. \tag{9}$$

If the algebras $\mathscr{A}_t$ are commutative and $E_{t,s}$ is a conditional expectation as characterized by Moy [8], then (9) is equivalent to the usual Markov property and (7) implies that φ is a Markov measure in the usual sense.

In the general, noncommutative case repeated application of (6) and (7) leads to the equality

$$\varphi(a_0 \times a_{t_1} \times \cdots \times a_{t_n})$$
$$= \varphi_0(\bar{E}_{t_1,0}(a_0 \times \bar{E}_{t_2,t_1}(a_{t_1} \times \cdots \times \bar{E}_{t_n,t_{n-1}}(a_{t_{n-1}} \times a_{t_n})\cdots))), \tag{10}$$

where φ_0 denotes the restriction of φ on $\mathscr{A}_0$; $0 < t_1 < \cdots < t_n$; $a_v \in \mathscr{A}_v$, and $\bar{E}_{t,s}$ denotes the restriction of $E_{t,s}$ on $\mathscr{A}_s \vee \mathscr{A}_t = C^*$-algebra generated by $\mathscr{A}_s$ and $\mathscr{A}_t$. Because of (9) (Markov property) $\bar{E}_{t,s}(\mathscr{A}_s \vee \mathscr{A}_t) \subseteq \mathscr{A}_s$, hence the right-hand side of (10) is well defined. From (10) easily follows that

$$\bar{E}_{s,r}(a_r \cdot \bar{E}_{t,s}(1_s \cdot a_t)) = \bar{E}_{t,r}(a_r \cdot a_t); \qquad \mathrm{mod}\{\varphi_0; (\bar{E}_{t,s})\} \tag{11}$$

$$\bar{E}_{s,r}(a_r \cdot \bar{E}_{t,s}(a_s \cdot 1_t)) = \bar{E}_{s,r}(a_r \cdot a_s); \qquad \mathrm{mod}\{\varphi_0; (\bar{E}_{t,s})\} \tag{12}$$

where $r < s < t$; 1_s denotes the identity operator in $\mathscr{A}_s$, and equality $\mathrm{mod}\{\varphi_0; (\bar{E}_{t,s})\}$ means that in any expression of the form

$$\varphi(\bar{E}_{t_1,0}(a_0 \times \bar{E}_{t_2,t_1}(a_{t_1} \times \cdots \times \bar{E}_{t_n,t_{n-1}}(a_{t_{n-1}} \times a_{t_n})\cdots)))$$

the right-hand side of (11) or (12) can be substituted by the left-hand side leaving unaltered its value (cf. [3]). Clearly, if equalities (11), (12) are satisfied in the usual sense they are also satisfied $\mathrm{mod}\{\varphi_0, (\bar{E}_{t,s})\}$. Finally, if $s > 0$, the positivity of $E_{t,s}$, the relation $\mathscr{A}_{[0,t]} \cong \mathscr{A}_{[0,s[} \otimes \mathscr{A}_{[s,t]}$ and the fact that $\mathscr{A}_{[0,s[}$ contains matrix algebras of arbitrarily high order imply that $\bar{E}_{t,s}$ is a completely positive linear map. If $s = 0$, $\bar{E}_{t,0}$ is completely positive being the restriction of a completely positive map. Equality (10) shows that the state φ is completely determined by its restriction φ_0 on $\mathscr{A}_0$ and by the family $(\bar{E}_{t,s})$. More precisely:

THEOREM 1.1. *Every noncommutative Markov state on $\mathscr{A}$ determines a pair $\{\varphi_0, (\bar{E}_{t,s})_{0\leqslant s<t}\}$ with the following properties:*

(i) φ_0 *is a state on* $\mathscr{A}_0$ ($\cong \mathscr{B}$),

(ii) $\bar{E}_{t,s}: \mathscr{A}_s \vee \mathscr{A}_t \to \mathscr{A}_s$ *is a completely positive linear map* ($s < t$),

(iii) $\bar{E}_{t,s}(1_s \cdot 1_t) = 1_s$,

(iv) $\bar{E}_{s,r}(a_r \cdot \bar{E}_{t,s}(1_s \cdot a_t)) = \bar{E}_{t,r}(a_r \cdot a_t)$; $r < s < t$,

(v) $\bar{E}_{s,r}(a_r \cdot \bar{E}_{t,s}(a_s \cdot 1_t)) = \bar{E}_{s,r}(a_r \cdot a_s)$; $r < s < t$ ($a_v \in \mathscr{A}_v$).

where equalities (iv), (v) *hold* $\mathrm{mod}\{\varphi_0, (\bar{E}_{t,s})\}$. Conversely, every pair $\{\varphi_0, (\bar{E}_{t,s})\}$. *Conversely, every pair* $\{\varphi_0, (\bar{E}_{t,s})\}$ *satisfying* (v) $\mathrm{mod}\{\varphi_0, (\bar{E}_{t,s})\}$ *and* (i)–(iv) *determines a unique Markov state on* $\mathscr{A}$.

Proof. Sufficiency. Let $\{\varphi_0, (\bar{E}_{t,s})\}$ be a pair satisfying (i)–(iv). For each $s < t$ denote by $\mathscr{F}_{s,t}$ the family of ordered finite sets $G = \{s < s_1 < \cdots < s_n < t\}$. $\mathscr{F}_{s,t}$, ordered by inclusion, is an increasing net. For each $G = \{s < s_1 < \cdots < s_n < t\} \in \mathscr{F}_{s,t}$ define:

$$\begin{aligned} &E_{G,s}(a_s \cdot a_{s_1} \times \cdots \times a_{s_n} \times a_t) \\ &\quad = \bar{E}_{s_1,s}(a_s \cdot \bar{E}_{s_2,s_1}(a_{s_1} \times \cdots \times \bar{E}_{t,s_n}(a_{s_n} \cdot a_t)\cdots)). \end{aligned} \tag{13}$$

Because of (ii) and (iii), $E_{G,s}$ extends to a completely positive linear map, still denoted $E_{G,s}$, from $\mathscr{A}_G$ into $\mathscr{A}_s$; moreover $E_{G,s}(1_G) = 1_s$. Property (iv) implies that the family of completely positive linear maps $\{E_{G,s}: G \in \mathscr{F}_{s,t}\}$ is projective, that is,

$$F \subseteq G \Rightarrow E_{G,s} \upharpoonright \mathscr{A}_F = E_{F,s}, \qquad F, G \in \mathscr{F}_{s,t}.$$

Therefore there exists a unique linear map $E'_{t,s}: \mathscr{A}_{[s,t]} \to \mathscr{A}_s$ such that:

$$E'_{t,s} \upharpoonright \mathscr{A}_G = E_{G,s}, \qquad \forall G \in \mathscr{F}_{s,t};$$

$E'_{t,s}$ is completely positive; and $E'_{t,s}(1_{[s,t]}) = 1_s$ since it is a pointwise norm limit of maps with these properties. Since $\mathscr{A}_{[0,t]} \cong \mathscr{A}_{[0,s[} \otimes \mathscr{A}_{[s,t]}$ the map $E_{t,s} = \mathrm{id}_{[0,s[} \otimes E'_{t,s}: \mathscr{A}_{[0,t]} \to \mathscr{A}_{[0,s]}$ (where $\mathrm{id}_I: \mathscr{A}_I \to \mathscr{A}_I$ is the identity map) is a quasi-conditional expectation with respect to the triple $\mathscr{A}_{[0,s[} \subseteq \mathscr{A}_{[0,s]} \subseteq \mathscr{A}_{[0,t]}$. By construction (cf. (13)) it follows that

$$E_{s,r} \cdot E_{t,s} = E_{t,r}; \qquad r < s < t. \tag{14}$$

Property (v) implies that the family $(E_{t,s})$ of quasi-conditional expectations is projective, that is:

$$E_{t,s} \upharpoonright \mathscr{A}_{[0,t_0]} = E_{t_0,s}, \qquad 0 \leqslant s < t_0 < t; \tag{15}$$

therefore, the family $(\varphi_{[0,t]})$ of states on $(\mathscr{A}_{[0,t]})$ defined by

$$\varphi_{[0,t]} = \varphi_0 \cdot E_{t,0} \tag{16}$$

is projective, hence it defines a unique state φ on $\mathscr{A}$. If condition (v) is verified only $\mathrm{mod}\{\varphi_0, (\bar{E}_{t,s})\}$ then (16) still defines a projective family of states.

Because of (14) and (16) the state φ, uniquely defined by the family $(\varphi_{[0,t]})$, satisfies:

$$\varphi_{[0,t]} = \varphi_{[0,s]} \cdot E_{t,s}, \qquad s < t; \tag{17}$$

hence it is a Markov state. The necessity of conditions (i)–(v) has been proved in the discussion before the formulation of the theorem.

Remark 1. A simple example of pair $\{\varphi_0, (\bar{E}_{t,s})\}$ satisfying (i)–(iv) but for which (v) is satisfied only $\mathrm{mod}\{\varphi_0, (\bar{E}_{t,s})\}$ is the following: Let φ_0 be a state on $\mathscr{B}$ and $(Z(t,s))$ a *quantum dynamical evolution*, i.e., a family of completely positive maps $Z(t,s): \mathscr{B} \to \mathscr{B}$; $s < t$, such that:

$$Z(t,s)[1] = 1, \tag{18}$$

$$Z(s,r) \cdot Z(t,s) = Z(t,r), \qquad r < s < t, \tag{19}$$

the map $s \in [r, t[\to Z(t,s)$ has a pointwise strongly continuous extension from $[r,t]$ to $\mathscr{B}$. (20)

$Z(t,s)$ is strongly continuous for every $s < t$ (21)

(the strong topology on $\mathscr{B}$ is defined by the seminorms $x \in \mathscr{B} \mapsto \varphi(x^*x)^{1/2}$, where φ is a positive element in $\mathscr{B}_*$—the predual of $\mathscr{B}$; cf. [9, p. 20]).

Define $\varphi_s = \varphi_0 \cdot Z(s,0)$ and

$$\bar{E}_{t,s}(\bar{a}_s \cdot \bar{a}_t) = \varphi_s(a_s) \cdot \bar{Z}(t,s)[\bar{a}_t]; \qquad \bar{a}_v = J_v(a_v); \quad a_v \in \mathscr{B};$$

where $\bar{Z}(t,s) = J_s \cdot Z(t,s) \cdot J_t^*$. Then clearly the family $(\bar{E}_{t,s})$ satisfies (ii), (iii), (iv), but if $a_t \in \mathscr{B}$, $\bar{a}_t = J_t(a_t)$: $\bar{E}_{s,r}(\bar{a}_r \cdot \bar{E}_{t,s}(\bar{a}_s \cdot 1)) = \varphi_r(a_r) \cdot \varphi_s(a_s) \cdot 1 \neq$

$\varphi_r(a_r) \cdot \bar{Z}(s, r)[\bar{a}_s] = \bar{E}_{s,r}(a_r \cdot a_s)$. However, for any $0 < r_1 < \cdots < r < s < t$ one has:

$$\begin{aligned}\varphi_0(\bar{E}_{r_1,0}(\bar{a}_0 \times \bar{E}_{r_2,r_1}(\bar{a}_{r_1} \times \cdots \times \bar{E}_{s,r}(\bar{a}_r \cdot \bar{a}_s)\cdots))) \\ = \varphi_0(a_0) \cdot \varphi_{r_1}(a_{r_1}) \times \cdots \times \varphi_r(a_r) \times \varphi_s(a_s) \\ = \varphi_0(\bar{E}_{r_1,0}(\bar{a}_0 \times \bar{E}_{r_2,r_1}(\bar{a}_{r_1} \times \cdots \times \bar{E}_{s,r}(\bar{a}_r \times \bar{E}_{t,s}(a_s \times 1_t))\cdots))),\end{aligned} \tag{22}$$

that is, in this case, (v) takes place mod$\{\varphi_0, (\bar{E}_{t,s})\}$. As shown in [3] if the evolution $(Z(t, s))$ is defined by inner automorphisms of $\mathscr{B}$ then the Markov states defined by (22) are the only ones whose canonical evolution is $(Z(t, s))$. In particular, a Schrödinger evolution uniquely determines, up to an initial state, the Markov state naturally associated to it.

Remark 2. Since, for each $s \in \mathbb{R}^+$, $\mathscr{A}_s \cong \mathscr{B}$, then for any $t_0 < t_1 < \cdots < t_n$ ($t_j \in \mathbb{R}^+$) the C^*-algebra $\mathscr{A}_{(t_1,\ldots,t_n)}$ can be identified with a dense sub-algebra of $\mathscr{B}(\mathscr{H} \otimes \cdots \otimes \mathscr{H})$ (n-fold tensor product). A state φ on $\mathscr{A}$ will be called *locally normal* if for any $t_0 < t_1 < \cdots < t_n$ the restriction of φ on $\mathscr{A}_{(t_0,\ldots,t_n)}$ has a normal extension to $\mathscr{B}(\mathscr{H} \otimes \cdots \otimes \mathscr{H})$.

Equality (10) implies that if for each $s < t$, $\bar{E}_{t,s}$ has a normal extension to $\mathscr{B}(\mathscr{H} \otimes \mathscr{H})$ and φ_0 is a normal state on $\mathscr{B}(\mathscr{H})$, then φ is locally normal.

2. Continuity Conditions

In the following, unless explicitly stated, we shall study the equalities (ii)–(v) of Theorem 1.1 without the weakening condition mod$\{\varphi_0, (\bar{E}_{t,s})\}$.

A family $(\bar{E}_{t,s})$ satisfying (ii)–(v) defines, by means of the equality:

$$\bar{E}_{t,s} \cdot (J_s \otimes J_t) = J_s \cdot \mathscr{E}_{t,s}; \qquad s < t; \tag{23}$$

a family $(\mathscr{E}_{t,s})$ of maps with the following properties:

(ii′) $\mathscr{E}_{t,s}: \mathscr{B} \otimes \mathscr{B} \to \mathscr{B}$ is a completely positive linear map,

(iii′) $\mathscr{E}_{t,s}(1 \otimes 1) = 1$,

(iv′) $\mathscr{E}_{s,r}(a \otimes \mathscr{E}_{t,s}(1 \otimes b)) = \mathscr{E}_{t,r}(a \otimes b)$, $r < s < t$,

(v′) $\mathscr{E}_{s,r}(a \otimes \mathscr{E}_{t,s}(b \otimes 1)) = \mathscr{E}_{s,r}(a \otimes b)$, $r < s < t$.

A family $(\mathscr{E}_{t,s})$ satisfying conditions (ii′)–(v′) will be called a family of *transition expectations*. Define, for $s < t$, $b \in \mathscr{B}$:

$$Z(t, s)[b] = \mathscr{E}_{t,s}(1 \otimes b). \tag{24}$$

Let us assume that the family $(\mathscr{E}_{t,s})$ satisfies the following conditions: For each $r < t \in \mathbb{R}^+$ and $b \in \mathscr{B}$, the map

$$s \in [r, t[\;\mapsto Z(t, s)[b] \tag{25}$$

extends to a strongly continuous map from $[r, t]$ to $\mathscr{B}$.

$$s\text{-}\lim_{s \downarrow r} \mathscr{E}_{s,r}(x) = T_r(x), \qquad x \in \mathscr{B} \otimes \mathscr{B} \tag{26}$$

exists uniformly on strongly compact sets of $\mathscr{B} \otimes \mathscr{B}$. Moreover we shall assume that, for each $s < t$ the map $\mathscr{E}_{t,s}$ has a normal extension to $\mathscr{B}(\mathscr{H} \otimes \mathscr{H})$ (i.e., we restrict our analysis to locally normal states). The above assumptions ((26) is not necessary, for this) imply that $(Z(t, s))$ is a quantum dynamical evolution. Thus, as shown in [1], the complete positivity of the $Z(t, s)$ is, in this context, a *consequence of the Markov property*.

Under the above assumptions, if $a, b \in \mathscr{B}$ and $r < t$ then, for every $s, r < s < t$:

$$\begin{aligned}&\mathscr{E}_{t,r}(a \otimes b) - T_r(a \otimes Z(t, r)[b]) \\ &= \{\mathscr{E}_{s,r}(a \otimes Z(t, r)[b]) - T_r(a \otimes Z(t, r)[b])\} \\ &\quad + \{\mathscr{E}_{s,r} - T_r\}(a \otimes [Z(t, s) - Z(t, r)][b]) \\ &\quad + T_r(a \otimes [Z(t, s) - Z(t, r)][b]).\end{aligned}$$

Because of (26) given a strong neighborhood W of the origin in $\mathscr{B}$ there is a $\delta_1 > 0$ such that

$$\mathscr{E}_{s,r}(a \otimes Z(t, r)[b]) - T_r(a \otimes Z(t, r)[b]) \in W$$

for $s \in [r, r + \delta_1]$. Because of (25) and (26) there is a δ_2 such that if $s \in [r, r + \delta_2]$, $\{\mathscr{E}_{s,r} - T_r\}(a \otimes y) \in W$ for every $y \in \{[Z(t, s) - Z(t, r)][b]: s \in [r, r + \delta_1]\}$. Since normality and complete positivity imply strong continuity (cf. [4, p. 53]), from (26) it follows that T_r is strongly continuous on compact sets. Hence, by (25) there is a $\delta_3 > 0$ such that, if $s \in [r, r + \delta_3]$ then

$$T_r(a \otimes [Z(t, s) - Z(t, r)][b]) \in W.$$

Therefore, if $s \in [r, r + \delta]$ with $\delta = \min\{\delta_1, \delta_2, \delta_3\}$

$$\mathscr{E}_{t,r}(a \otimes b) - T_r(a \otimes Z(t, r)[b]) \in W + W + W.$$

From the arbitrariness of W it follows that

$$\mathscr{E}_{t,r}(a \otimes b) = T_r(a \otimes Z(t, r)[b]). \tag{27}$$

Using (27) one can express the properties characterizing the family of transition expectations $(\mathscr{E}_{t,s})$ in terms of the one-parameter family (T_r) and the associated quantum dynamical evolution, namely,

(ii″) $T_r: \mathscr{B} \otimes \mathscr{B} \to \mathscr{B}$ is a completely positive linear map,

(iii″) $T_r(1 \otimes 1) = 1$,

(iv″) $T_r(a \otimes Z(t, r)[b]) = T_r(a \otimes Z(s, r) \cdot T_s(1 \otimes Z(t, s)[b])$,

(v″) $T_r(a \otimes Z(s, r)[b]) = T_r(a \otimes Z(s, r) \cdot T_s(b \otimes 1))$.

On account of (27), (23), and Theorem 1.1, the properties (ii″)–(v″) allow to associate a class of Markov states to a preassigned quantum dynamical evolution $(Z(t, s))$, by solving the equations (iv″), (v″) in the unknowns (T_r) subject to the conditions (ii″), (iii″).

For stationary Markov chains (cf. [3, Eq. 4]), $T_r = T_0$ for every r, and (iv″), (v″), respectively become

$$T_0(a \otimes Z(t+s)[b]) = T_0(a \otimes Z(s) \cdot T_0(1 \otimes Z(t)[b])),$$
$$T_0(a \otimes Z(s)[b]) = T_0(a \otimes Z(s) \cdot T_0(b \otimes 1)).$$

The evolution canonically associated to the Markov states defined by a pair $\{(T_r), (Z(t, s))\}$ satisfying (ii″)–(v″), *will not be*, in general, $Z(t, s)$. This will be the case if and only if:

$$T_s(1 \otimes Z(t, s)[b]) = Z(t, s)[b], \qquad b \in \mathscr{B}. \tag{28}$$

Condition (28) clearly implies (iv″). Summing up:

THEOREM 2.1. *To every family* $(\mathscr{E}_{t,s})$ *of normal transition expectations satisfying the continuity conditions* (25), (26) *there is associated a couple* $\{(T_r), (Z(t, s))\}$ *such that* $(Z(t, s))$ *is a quantum dynamical evolution and* (T_r) *satisfies* (ii″)–(v″). *Every couple* $\{(T_r), (Z(t, s))\}$ *satisfying* (ii″)–(v″) *defines, through* (27) *a family of transition expectations* $(\mathscr{E}_{t,s})$. *If* $\{(T_r), (Z(t, s))\}$ *satisfies* (28), *the evolution naturally associated to* $(\mathscr{E}_{t,s})$ *coincides with* $(Z(t, s))$.

3. The Quantum Measurement Process

Let $(Z(t, s))$ be a quantum dynamical evolution. If the evolution $(Z(t, s))$ is *reversible* (meaning by this that for each t, s the adjoint action of $Z(t, s)$ on the predual of $\mathscr{B}$ is one to one from the set of normal states onto itself) then a theorem of Kadison [6] implies that each $Z(t, s)$ is an inner automorphism of $\mathscr{B}$ and, as shown in [3], the requirement that $(Z(t, s))$ be the evolution canonically associated to a Markov chain, uniquely determines this chain up to the initial state.

However, as shown in the preceding section, by solving the Equations (iv″), (v″) under the conditions (ii″), (iii″), it is possible to associate to a given evolution a Markov chain whose natural evolution is not the initial one. In particular, starting from a reversible evolution, one might obtain an irreversible one. In the present section we illustrate this circumstance by solving these equations for a given *reversible* evolution $(Z(t, s))$. We shall look for the solutions (T_r) of equations (iv″)–(v″) which satisfy the condition:

$$Z(r, 0) \cdot T_r = T_0 \cdot (Z(r, 0) \otimes Z(r, 0)), \tag{29}$$

which will be called $(Z(t, s))$-covariance.

If $T_0(1 \otimes \mathscr{B}) = \mathscr{B}$ letting $a = 1$ in (iv″) and multiplying both sides of the equality by $Z(r, 0)$ one finds:

$$T_0(1 \otimes Z(t, r)[b]) = T_0(1 \otimes T_0(Z(t, r)[b]))$$

since $Z(t, r)$ is invertible this is equivalent to:

$$\hat{T}_0(1 \otimes b) = \hat{T}_0^2(1 \otimes b), \qquad b \in \mathscr{B},$$

where

$$\hat{T}_0(x) = 1 \otimes T_0(x), \qquad x \in \mathscr{B} \otimes \mathscr{B}.$$

Thus $\hat{T}_0 \colon \mathscr{B} \otimes \mathscr{B} \to 1 \otimes \mathscr{B}$ is a norm one projection onto its range which is $1 \otimes \mathscr{B}$. Tomijama's theorem [12] implies that $\hat{T}_0$ is a conditional expectation, hence T_0 must be of the form

$$T_0(a \otimes b) = \varphi_0(a) \cdot b, \qquad a, b \in \mathscr{B} \tag{30}$$

for some state φ_0 on $\mathscr{B}$. Conversely, each operator T_0 of the form (30) gives rise, through (29), to a class of operators (T_r) satisfying (ii″), (iii″), (iv″). Condition (v″) is satisfied mod$(\varphi_0, (\overline{E}_{t,s}))$ where $\overline{E}_{t,s}$ is defined by (29), (27), (23).

Therefore, under assumption (29), if $T_0(1 \otimes \mathscr{B}) = \mathscr{B}$ there is exactly one class of operators (T_r) satisfying (ii″), (iii″), (iv″) and the corresponding class of noncommutative Markov chains is the one described in Remark 1 after Theorem 1.1 (i.e., the class of usual quantum system). The operator T_0 and the transition expectations $\mathscr{E}_{t,s}$ associated to the couple $\{T_0; (Z(t, s))\}$ through (29) and (27) satisfy the relations:

$$\mathfrak{A}_0 = T_0(\mathscr{B} \otimes \mathscr{B}) \text{ is a } C^*\text{-algebra}, \tag{31}$$

$$s\text{-}\lim_{s \downarrow r} \mathscr{E}_{s,r}(1 \otimes \mathscr{E}_{t,s}(x)) = \mathscr{E}_{t,r}(x); \qquad x \in \mathscr{B} \otimes \mathscr{B}. \tag{32}$$

LEMMA 3.1. *Assume that* (31), (32) *are satisfied by a linear strongly continuous operator* $T_0 \colon \mathscr{B} \otimes \mathscr{B} \to \mathscr{B}$. *Then* T_0 *gives rise through* (29) *and* (27) *to a solution of the equations* (iv″), (v″) *if and only if*:

$$\mathfrak{A}_0 = T_0(\mathscr{B} \otimes \mathscr{B}) \text{ is an Abelian } C^*\text{-algebra} \tag{33}$$

$$T_0(a \otimes b) = F_0(a) \cdot F_0(b), \qquad a, b \in \mathscr{B}, \tag{34}$$

where $F_0 \colon \mathscr{B} \to \mathfrak{A}_0$ *is a normal conditional expectation.*

Proof. *Necessity.* Condition (32) is equivalent to

$$\lim_{s \downarrow r} Z(r, 0)^{-1} \cdot T_0(1 \otimes T_0(Z(s, 0) \otimes Z(t, 0)))$$
$$= Z(r, 0)^{-1} \cdot T_0(Z(r, 0) \otimes Z(t, 0))$$

pointwise strongly. Since the $Z(t, s)$'s are invertible, strongly continuous, and $Z(s, 0) \to Z(r, 0)$ pointwise strongly, as $s \downarrow r$, this is equivalent to:

$$T_0 \cdot (1 \otimes T_0) = T_0$$

or denoting as above $\hat{T}_0 = 1 \otimes T_0$

$$\hat{T}_0^2 = \hat{T}_0$$

which, because of (iii″) means that $\hat{T}_0$ is a norm one projection onto $1 \otimes \mathfrak{A}_0$. Assumption (31) and Tomijama's theorem imply thus that $\hat{T}_0$ is a conditional expectation.

Condition (29) and the reversibility of $(Z(t, s))$ imply that (iv″) and (v″), respectively are equivalent to:

$$T_0(a \otimes b) = T_0(a \otimes T_0(1 \otimes b)), \tag{35}$$

$$T_0(a \otimes b) = T_0(a \otimes T_0(b \otimes 1)), \tag{36}$$

which, since $\hat{T}_0$ is a conditional expectation, are equivalent to:

$$T_0(a \otimes b) = T_0(a \otimes 1) \cdot T_0(1 \otimes b), \tag{37}$$

$$T_0(a \otimes b) = T_0(a \otimes 1) \cdot T_0(b \otimes 1). \tag{38}$$

Conditions (38) and (ii″) imply that for Hermitian (hence all) $a, b \in \mathscr{B}$

$$T_0(a \otimes b) = T_0(b \otimes a). \tag{39}$$

Thus, if $a_0, b_0 \in \mathfrak{A}_0$ one has

$$\begin{aligned} 1 \otimes a_0 b_0 &= \hat{T}_0(1 \otimes a_0) \cdot \hat{T}_0(1 \otimes b_0) \\ &= \hat{T}_0(a_0 \otimes 1) \cdot \hat{T}_0(1 \otimes b_0) \\ &= \hat{T}_0(a_0 \otimes b_0) = \hat{T}_0(b_0 \otimes a_0) \\ &= 1 \otimes b_0 a_0 \end{aligned}$$

thus $\mathfrak{A}_0$ must be Abelian. Denoting:

$$F_0(b) = T_0(1 \otimes b), \qquad b \in \mathscr{B}, \tag{40}$$

$F_0 \colon \mathscr{B} \to \mathfrak{A}_0$ is a conditional expectation which, because of (37), (39) satisfies (34).

Sufficiency. If $\mathfrak{A}_0$ is Abelian and $F_0 \colon \mathscr{B} \to \mathfrak{A}_0$ is a conditional expectation, the equality $T_0(a \otimes b) = F_0(a) \cdot F_0(b)$ clearly defines a completely positive map and $T_0(1 \otimes 1) = 1$, (in fact $x \in \mathscr{B} \otimes \mathscr{B} \mapsto 1 \otimes T_0(x) \in 1 \otimes \mathfrak{A}_0$ is a conditional expectation). The identities (35), (36) are immediately verified. Therefore, the family (T_r) defined by T_0 through (29) satisfies (ii″)–(v″). And this proves the lemma.

Lemma 3.1 means that under assumptions (31) and (32) the couples $\{\mathfrak{A}_0, F_0\}$, where $\mathfrak{A}_0$ is an *Abelian* C^*-sub-algebra of $\mathscr{B}$ and $F_0 \colon \mathscr{B} \to \mathfrak{A}_0$ a conditional expectation, parametrize all the solutions of the equations (ii″)–(v″). If $\{\mathfrak{A}_0, F_0\}$ is such a couple, the corresponding transition expectations are defined, through (29) and (27) by:

$$\begin{aligned} \mathscr{E}_{t,s}(a \otimes b) &= Z(s, 0)^{-1} \cdot T_0(Z(s, 0)[a] \otimes Z(t, 0)[b]) \\ &= \{Z(s, 0)^{-1} \cdot F_0(Z(s, 0)[a])\} \cdot \{Z(s, 0)^{-1} \cdot F_0(Z(t, 0)[b])\} \end{aligned}$$

(where we have used the fact that $Z(s, 0)$ is an inner automorphism of $\mathscr{B}$). Defining

$$F_{t,s}(a) = Z(s, 0)^{-1} \cdot F_0(Z(t, 0)[a]), \tag{41}$$

$$F_s(a) = F_{s,s}(a), \tag{42}$$

one obtains:

$$\mathscr{E}_{t,s}(a \otimes b) = F_s(a) \cdot F_{t,s}(b)$$

hence, computing through formulas (10) and (23), the joint expectations of the Markov state defined by the family of transition expectations $(\mathscr{E}_{t,s})$, for an arbitrary initial state φ_0, one obtains: ($\bar{a}_v = J_t(a_v)$; $a_v \in \mathscr{B}$)

$$\varphi(\bar{a}_0 \times \bar{a}_{t_1} \times \cdots \times \bar{a}_{t_n}) = \varphi_0(F_0(a_0) \times F_{t_1,0}(a_{t_1}) \times \cdots \times F_{t_n,0}(a_{t_n})). \tag{43}$$

The meaning of the identity (43) is best understood by looking at its classical analog: Let S be a space and $T_{s,t} \colon S \to S$ a reversible evolution:

$$T_{r,s} \cdot T_{s,t} = T_{r,t}, \qquad T_{t,s} = T_{s,t}^{-1}, \qquad T_{0,0} = \mathrm{id},$$

and let w_0 be a probability measure on S. The joint probabilities naturally associated to the system are given by:

$$w(B_0 \times B_{t_1} \times \cdots \times B_{t_n}) = w_0(B_0 \cap T_{0,t_1}^{-1} B_{t_1} \cap \cdots \cap T_{0,t_n}^{-1} B_{t_n});$$

thus, if $a_0, a_{t_1}, \ldots, a_{t_n}$ are any measurable functions on S, their joint expectations, for w are given by

$$\begin{aligned} & w(a_0 \otimes a_{t_1} \otimes \cdots \otimes a_{t_n}) \\ &\quad = w_0(a_0 \times \hat{T}_{t_1,0}(a_{t_1}) \times \cdots \times \hat{T}_{t_n,0}(a_{t_n})), \end{aligned} \tag{44}$$

where $\hat{T}_{t,s}(a) = a_0 \cdot T_{t,s}$.

Thus the expression at the right-hand side of (43) gives the noncommutative analog for the joint expectations of a classical Markov chain with reversible evolution and with an initial distribution w_0. In the classical case, if w_0 is concentrated on a point, the joint expectations (44) factorize. In the noncommutative case, the factor determining the nontriviality of the joint expectations (43), i.e., that they do not reduce to the product of the expectations at different times—*is not* the density matrix of φ_0, but the operator of "instantaneous coupling" T_0.

Let now $\mathfrak{A}_0$ be an Abelian von Neumann algebra. Then the requirement that $F_0: \mathscr{B} \to \mathfrak{A}_0$ is normal faithful implies, as a consequence of a general result of Størmer [11], that $\mathfrak{A}_0$ is spanned by a family (l_j) of minimal, mutually orthogonal projections. In such case F_0 has the form:

$$F_0(a) = \sum_j \varphi_0{}^j(a) \cdot l_j,$$

where $\varphi_0{}^j$ is a normal state on $\mathscr{B}$ and $\varphi_0{}^j(l_k) = 0$ for $j \neq k$, $= 1$ for $j = k$. For any reversible evolution $(Z(t, s))$ on $\mathscr{B}$ the operator T_0, defined by (34) with the above choice of F_0, will determine, according to Lemma 3.1, (29), and (27), a family of transition expectations:

$$\mathscr{E}_{t,s}(a \otimes b) = \sum_j \varphi_0{}^j(Z(t, 0)[b]) \cdot \varphi_0{}^j(Z(s, 0)[a]) \cdot Z(s, 0)^{-1}[l_j]$$

and an evolution:

$$Z_m(t, s)[b] = \sum_j \varphi_0{}^j(Z(t, 0)[b]) \cdot Z(s, 0)^{-1}[l_j].$$

The adjoint action of $Z_m(t, s)$ on the set of density matrices: $W \mapsto WZ_m(t, s)$, is given by:

$$WZ_m(t, s) = \sum_j \tau(W \cdot Z(s, 0)^{-1}[l_j]) \cdot Z(t, 0)^{-1}[W_0{}^j],$$

where $W_0{}^j$ is the density matrix of $\varphi_0{}^j$. In particular, if the l_j are rank one projections in $\mathscr{B} = \mathscr{B}(\mathscr{H})$:

$$l_j = P_{\varphi_j}, \qquad P_{\varphi_j}(\psi) = \langle \varphi_j, \psi \rangle \cdot \varphi_j, \qquad \psi \in \mathscr{H}$$

with $\varphi_j \in \mathscr{H}$, $\langle \varphi_j, \varphi_k \rangle = 0$ for $j \neq k$, 1 for $j = k$ then $W_0{}^j = P_{\varphi_j}$. Therefore, if

$$Z(t, 0)[b] = e^{itH} \cdot b \cdot e^{-itH}, \qquad b \in \mathscr{B}$$

for some self-adjoint operator H on $\mathscr{H}$, then:

$$W \cdot Z_m(t, s) = \sum_j \tau(W \cdot P_{e^{-isH}\varphi_j}) \cdot P_{e^{-itH}\varphi_j}.$$

The above formula is a generalization of the one prescribed by von Neumann's theory of the quantum measurement process (cf. [13, Chap. V]) to which it reduces for $s = t = 0$. It can be shown that a modification of von Neumann's probabilistic argument, which keeps into account both the time in which the operation of measure is performed (i.e., s) and the time in which the a priori probability is required (i.e., t), yields exactly the above result. Moreover dropping the requirement of evolution-covariance one can easily construct examples of Markov processes of the quantum measurement type which allow different operators T of "instantaneous coupling" at different times.

4. Feynman–Kac Formula

Theorem 1.1 shows how to construct a Markov state from a family $(\bar{E}_{t,s})$ of transition expectations which satisfies a set of compatibility conditions. In the present section we show how, starting from a family $(\bar{E}_{t,s})$ which does not satisfy the projectivity conditions of Theorem 1.1 it is possible, in some cases, to construct a new family $(\tilde{E}_{t,s})$ satisfying these conditions. The study of non-projective families of transition expectations arises naturally when one considers "perturbations" of projective families $(\bar{E}^0_{t,s})$ of the form:

$$\bar{E}_{t,s}(\cdot) = \bar{E}^0_{t,s}(K^*_{t,s} \cdot K_{t,s}), \tag{45}$$

where $K_{t,s}$ is an operator affiliated to $\mathscr{A}_s \vee \mathscr{A}_t$ which, from a physical point of view, represents a local interaction.

Some examples (and a more detailed analysis which will appear elsewhere) show that the study of such perturbations is a natural context for the non-commutative formulation of the Feynman–Kac formula.

Lemma 4.1. *In the notations of Section 1 let $(\bar{E}_{t,s})_{s<t}$ be a family of completely positive linear maps $\bar{E}_{t,s}: \mathscr{A}_s \vee \mathscr{A}_t \to \mathscr{A}_s$ such that $\bar{E}_{t,s}(1) = 1$. Assume that for every $r < t$, $a_r \in \mathscr{A}_r$, $b_t \in \mathscr{A}_t$, the limit:*

$$\tilde{E}_{t,r}(a_r \times b_t) = \lim_{\mathscr{F}_{r,t}} \bar{E}_{t_1,r}(a_r \times \bar{E}_{t_2,t_1}(1_{t_1} \times \cdots \times \bar{E}_{t,t_n}(1_{t_n} \times 1_t)\cdots)) \tag{46}$$

exists in the strong topology for $\mathscr{A}_r(\simeq \mathscr{B})$ and that, for each $a_r \in \mathscr{A}_r$ the map

$$b_t \in \mathscr{A}_t \mapsto \tilde{E}_{t,r}(a_r \cdot b_t) \tag{47}$$

is strongly continuous.

Then the family $(\tilde{E}_{t,s})$ satisfies conditions (ii)–(iv) *of Theorem* 1.1.

Proof. Since conditions (ii), (iii) are clearly satisfied by $(\tilde{E}_{t,s})$ one has only to prove (iv), i.e.,

$$\tilde{E}_{s,r}(a_r \cdot \tilde{E}_{t,s}(1_s \cdot b_t)) = \tilde{E}_{t,r}(a_r \cdot b_t) \tag{48}$$

for $r < s < t$, $a_r \in \mathscr{A}_r$, $b_t \in \mathscr{A}_t$. To this aim denote, for

$$\{s < t_1 < \cdots < t_n < t\} \in \mathscr{F}_{s,t}:$$

$$b_{(s<t_1<\cdots<t_n<t)} = \bar{E}_{t_1,s}(1_s \times \bar{E}_{t_2,t_1}(1_{t_1} \times \cdots \times \bar{E}_{t,t_n}(1_{t_n} \times b_t)\cdots)),$$

Because of (46) and (47), given a strong neighborhood W of 0 in $\mathscr{A}_r$, there is an $F_0 \in \mathscr{A}_{s,t}$ such that, for every $\{s < t_1 < \cdots < t_n < t\} \supseteq F_0$ one has:

$$\tilde{E}_{s,r}(a_r \cdot \tilde{E}_{t,s}(1_s \cdot b_t)) - \tilde{E}_{s,r}(a_r \cdot b_{(s<t_1<\cdots<t_n<t)}) \in W.$$

For fixed $\{s < t_1 < \cdots < t_n < t\} \supseteq F_0$, there is a $G_0 \in \mathscr{F}_{r,s}$ such that, if $\{r < s_1 < \cdots < s_m < s\} \supseteq G_0$:

$$\tilde{E}_{s,r}(a_r \times b_{(s<t_1<\cdots<t_n<t)})$$
$$- \bar{E}_{s_1,r}(a_r \times \bar{E}_{s_2,s_1}(1_s \times \cdots \times \bar{E}_{s,s_m}(1_{s_m} \times b_{(s<\cdots<t)})\cdots)) \in W,$$

and there are $F_1 \in \mathscr{F}_{s,t}$, $G_1 \in \mathscr{F}_{r,s}$ such that if

$$\{r < s_1 < \cdots < s < t_1 < t\} \supseteq F_1 \cup G_1$$

$$\bar{E}_{s_1,r}(a_r \times \bar{E}_{s_2,s_1}(1_{s_1} \times \cdots \times \bar{E}_{t,t_n}(1_{t_n} \times b_t)\cdots)) - \tilde{E}_{t,r}(a_r \times b_t) \in W.$$

Since one can always assume that $G_0 \supseteq G_1$ and $F_0 \supseteq F_1$

$$\tilde{E}_{s,r}(a_r \cdot \tilde{E}_{t,s}(1_s \cdot b_t)) - \tilde{E}_{t,r}(a_r b_t) \in W + W + W,$$

Since W is arbitrary, this proves (48).

Remark. Since $\tilde{E}_{t,s}(1_s \cdot 1_t) = 1_s$ one has:

$$\tilde{E}_{s,r}(a_r \cdot \tilde{E}_{t,s}(1_s \cdot 1_t)) = \tilde{E}_{s,r}(a_r \cdot 1_t).$$

Hence, because of (48): $\tilde{E}_{t,r}(a_r \cdot 1_t) = \tilde{T}_r(a_r)$ independently of $t > r$. Therefore, condition (v) of Theorem 1.1 is equivalent, in this case to:

$$\tilde{E}_{s,r}(a_r \cdot \tilde{T}_s(a_s)) = \tilde{E}_{s,r}(a_r \cdot a_s) \tag{49}$$

so that in order to associate a Markov chain to the couple $\{\varphi_0, (\bar{E}_{t,s})\}$ φ_0 being a state on $\mathscr{A}_0$, it is sufficient that condition (49) be satisfied in the weak sense specified in Theorem (1.1).

Consider, as an example, the Markov chain $\{\varphi_0, (E^0_{t,s})\}$ associated to a quantum mechanical system with time-independent Hamiltonian H_0 and initial state φ_0. According to [3] the transition expectations of such a system have the form:

$$\bar{E}^0_{t,s}(J_s(a_s) \cdot J_t(a_t)) = \varphi_s(a_s) \cdot J_s[e^{-i(t-s)H_0} \cdot a_t \cdot e^{i(t-s)H_0}].$$

Let V_0 be a self-adjoint operator on $\mathscr{B}$ such that $H_0 + V_0$ is self-adjoint on $D(H_0) \cap D(V_0)$, and consider the perturbations of $\bar{E}^0_{t,s}$ defined by:

$$\bar{E}_{t,s}(a_s \cdot b_t) = \bar{E}^0_{t,s}(e^{-i(t-s)V_t} \cdot a_s b_t \cdot e^{i(t-s)V_t}),$$

where $e^{isV_t} = J_t(e^{isV_0})$.

From the equality: $(a_s, b_t \in \mathscr{B})$

$$\bar{E}_{t_1,s}(J_s(a_s) \times \bar{E}_{t_2,t_1}(1_{t_1} \times \cdots \times \bar{E}_{t,t_n}(1_{t_n} \times J_t(b_t))\cdots)))$$
$$= \varphi_s(a_s) \times J_s\{[(e^{i(t-t_n)V_0}e^{i(t-t_n)H_0}) \times \cdots \times (e^{i(t_1-s)V_0}e^{i(t_1-s)H_0})]^*$$
$$\times b_t \times [(e^{i(t-t_n)V_0}e^{i(t-t_n)H_0}) \times \cdots \times (e^{i(t_1-s)V_0}e^{i(t_1-s)H_0})]\}$$

and Lemma 4.2 below it follows that the limit

$$\lim_{\mathscr{F}_{s,t}} \bar{E}_{t_1,s}(a_s \times \bar{E}_{t_2,t_1}(1_{t_1} \times \cdots \times \bar{E}_{t,t_n}(1_{t_n} \times b_t)\cdots)) = \tilde{E}_{t,s}(a_s \times b_t)$$

exists in the strong operator topology for $\mathscr{A}_s$ identified to $\mathscr{B}$, for every $s < t$, and moreover

$$\tilde{E}_{t,s}(J_s(a_s) \times J_t(b_t)) = \varphi_s(a_s) \times J_s\{e^{-i(t-s)(H_0+V_0)} \times b_t \times e^{i(t-s)(H_0+V_0)}\}$$

Clearly, the $\tilde{E}_{t,s}$ satisfy condition (47). Moreover, since the transition expectations $\tilde{E}_{t,s}$ are of the the same type of those discussed in Remark 1 after Theorem 1.1, condition (v) is satisfied mod$\{\varphi_0, (\tilde{E}_{t,s})\}$ for every state φ_0 on $\mathscr{A}_0$ $(\simeq\mathscr{B})$. Hence, for any such a state the couple $\{\varphi_0, (\tilde{E}_{t,s})\}$ defines a unique Markov chain which, according to [3], corresponds to a quantum system with Hamiltonian $H_0 + V_0$.

Lemma 4.2. *Let A, B be self-adjoint operators on an Hilbert space $\mathscr{H}$ and suppose that $A + B$ is self-adjoint on $D = D(A) \cap D(B)$. Then $\forall\, \psi \in \mathscr{H}$*

$$\lim_{\mathscr{F}_{t_0,t}} (e^{i(t_1-t_0)A}e^{i(t_1-t_0)B}) \times \cdots \times (e^{i(t_n-t_{n-1})A}e^{i(t_n-t_{n-1})B})\psi$$
$$= e^{i(t-t_0)(A+B)}\psi, \tag{50}$$

where $t_0 < t_1 < \cdots < t_n = t$, and the limit is taken with respect to the net $\mathscr{F}_{t_0,t}$ of the partitions $\{t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[t_0, t]$.

Proof. We know (Simon [10, p. 296])[1] that: if $\psi \in D$ then for s belonging to any bounded interval:

$$\tau^{-1}[e^{i\tau A}e^{i\tau B} - e^{i\tau(A+B)}]\, e^{is(A+B)}\psi \to 0 \tag{51}$$

[1] The proof of the lemma is a simple modification of the result quoted from [10] which we have written down for completeness.

for $\tau \to 0$, uniformly in s. Denote:

$$\prod_{0 \leqslant j \leqslant K-1} e^{i(t_{j+1}-t_j)A} e^{i(t_{j+1}-t_j)B}$$

$$= (e^{i(t_1-t_0)A} e^{i(t_1-t_0)B}) \times \cdots \times (e^{i(t_K-t_{K-1})A} e^{i(t_K-t_{K-1})B})$$

and similarly for $e^{i(t_{K+1}-t_K)(A+B)}$. Remark that:

$$\sum_{K=0}^{n} \prod_{0 \leqslant j \leqslant K-1} [e^{i(t_{j+1}-t_j)A} e^{i(t_{j+1}-t_j)B}]$$

$$\cdot [e^{i(t_{K+1}-t_K)A} e^{i(t_{K+1}-t_K)B} - e^{i(t_{K+1}-t_K)(A+B)}]$$

$$\cdot \prod_{K<j\leqslant n-1} e^{i(t_{j+1}-t_j)(A+B)}$$

$$= \prod_{0\leqslant j \leqslant n-1} e^{i(t_{j+1}-t_j)A} e^{i(t_{j+1}-t_j)B} - e^{i(t-t_0)(A+B)}.$$

Hence, for every $\psi \in D$:

$$\left\| \left\{ \prod_{0\leqslant j\leqslant n-1} e^{i(t_{j+1}-t_j)A} e^{i(t_{j+1}-t_j)B} - e^{i(t-t_0)(A+B)} \right\} \psi \right\|$$

$$\leqslant \sum_{K=0}^{n-1} \| \{ e^{i(t_{K+1}-t_K)A} \cdot e^{i(t_{K+1}-t_K)B} - e^{i(t_{K+1}-t_K)(A+B)} \} e^{i(t-t_{K+1})(A+B)} \psi \|.$$

But from (51) we know that, given $\epsilon > 0$ and $\psi \in D$ there is a $\delta > 0$ such that if $\tau < \delta$ then

$$\| \tau^{-1} [e^{i\tau A} e^{i\tau B} - e^{i\tau(A+B)}] e^{is(A+B)} \psi \| \leqslant \epsilon$$

uniformly in $s \in [0, t]$.

Thus, if ϵ, ψ, δ are as above and $\{t_0 < \cdots < t_n = t\}$ is any partition of $[t_0, t]$ satisfying $\max_j (t_{j+1} - t_j) < \delta$ we deduce

$$\| \{ (e^{i(t_1-t_0)A} e^{i(t_1-t_0)B}) \times \cdots \times (e^{i(t-t_{n-1})A} e^{i(t-t_{n-1})B}) - e^{i(t-t_0)(A+B)} \} \psi \|$$

$$\leqslant \sum_{K=0}^{n-1} (t_{K+1} - t_K) \epsilon = (t - t_0) \epsilon$$

Therefore the equality (50) takes place for each $\psi \in D$. Since such ψ are norm dense in $\mathscr{H}$ and the family $\{ \prod_{0 \leqslant j \leqslant n-1} e^{i(t_{j+1}-t_j)A} e^{i(t_{j+1}-t_j)B} \}$ is bounded in norm by 1, the equality takes place everywhere in $\mathscr{H}$. Thus the lemma is proved.

References

1. L. Accardi, Non-commutative Markov chains, *in* "Proceedings of the International Summer School of Mathematical Physics, Sept. 30–Oct. 12, 1974," Università di Camerino.
2. L. Accardi, On the non-commutative Markovian property, *Functional Anal. Appl.* (in Russian) 9 (1975).
3. L. Accardi, Non relativistic Quantum Mechanics as a non-commutative Markof process, *Advances in Math.* 20 (1976), 329–366.
4. J. Dixmier, "Les algèbres d'opérateurs dans l'espace Hilbertien," Gauthier–Villars, Paris, 1969.
5. J. L. Doob, "Stochastic Processes," Wiley, New York, 1967.
6. R. Kadison, Transformations of states in operator theory and dynamics, *Topology* 3, Suppl. 2 (1965), 177–198.
7. C. Lance, Tensor Products of C^*-Algebras in "C^*-Algebras and Their Applications to Statistical Mechanics and Quantum Field Theory," North-Holland, Amsterdam, 1976.
8. S. C. Moy, Characterizations of Conditional expectation as a transformation on function space, *Pacific J. Math.* 4 (1954), 47–64.
9. S. Sakai, "C^*-Algebras and W^*-Algebras," Ergebnisse der Mathematik and ihrer Grenzgebiete Band 60, Springer, Berlin/New York, 1971.
10. B. Simon, "Quantum Mechanics for Hamiltonians Defined as Quadratic Forms," Princeton Univ. Press, Princeton, N.J., 1971.
11. E. Størmer, On projection maps of von Neumann algebras, *Math. Scand.* 30 (1972), 46–50.
12. J. Tomijama, On the projection of norm one in W^*-algebras, *Proc. Japan Acad.* 33 (1957), 608–612.
13. J. von Neumann, "Mathematical Foundations of Quantum Mechanics," Princeton Univ. Press, Princeton, N.J., 1955.

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