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ON THE SPACE OF SQUARE ROOTS OF MEASURES

UDC 51

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1. In this note we establish a connection between the theory of infinite direct products of Hilbert spaces and the theory of measures on products of infinite families of measurable spaces. In particular, a classification of infinite product measures is obtained in terms of certain incomplete components of an infinite tensor product of Hilbert spaces. The assertion about the possibility of the application of the theory of infinite product measures was formulated by J. von Neumann in [1], Introduction, §6. It turns out that the natural space for this problem is the complex Hilbert space generated by "square roots" of real bounded measures on a given measurable space.

2. Let $(\Omega, \mathfrak{B})$ be a measurable space. We denote by $\mathfrak{M}_R(\Omega, \mathfrak{B})$ the real Banach space of bounded real countably additive measures on $(\Omega, \mathfrak{B})$. If $x, y \in \mathfrak{M}_R(\Omega, \mathfrak{B})$, then we will write $x \perp y$ if x is orthogonal to y (see [7]), we set $|x| = x^+ + x^-$, where $x = x^+ - x^-$ is the Jordan decomposition of x, and we will write $x \prec y$ if x is absolutely continuous with respect to y. The "Jordan decomposition of Ω" is the measurable decomposition $\{E^+, E^-\}$, of Ω such that $x^+(E^+) = x^+(\Omega)$, $x^-(E^-) = x^-(\Omega)$.

We denote by $\mathfrak{M}^+_R(\Omega, \mathfrak{B})$ the cone of positive measures in $\mathfrak{M}_R(\Omega, \mathfrak{B})$. If $m, n \in \mathfrak{M}^+_R(\Omega, \mathfrak{B})$, then there exists a unique positive measure defined by the expression

$$A \in \mathfrak{B} \mapsto \int_A \sqrt{\frac{dm}{dq}\frac{dn}{dq}}\, dq,$$

where q is any measure such that $m < q$, $n < q$. The above integral does not depend on the choice of q; therefore we may denote the measure defined by it by $\sqrt{mn}$.

Let $\mathcal{H}(\Omega, \mathfrak{B})$ be the complex vector space generated by the symbols $[x]$, $x \in \mathfrak{M}_R(\Omega, \mathfrak{B})$, with the following relations among the generators:

[r1]. $[x + x'] - [x] - [x'] = 0$, if $x \perp x'$;

[r2]. $[\xi \cdot x] - \sqrt{\xi} \cdot [x] = 0$, $\xi \in \mathbb{R}$, $i^2 = -1$;

[r3]. $[m] + [n] - [m + n + 2\sqrt{mn}] = 0$, $m, n \in \mathfrak{M}_R^+(\Omega, \mathfrak{B})$;

[r4]. $[m] - [n] - [\chi_+ \cdot (m + n - 2\sqrt{mn})] + [\chi_- \cdot (m + n - 2\sqrt{mn})] = 0$,

where $m, n \in \mathfrak{M}^+_R(\Omega, \mathfrak{B})$, and χ_+, χ_- are the characteristic functions of the Jordan decomposition of Ω corresponding to the measure $m - n$.

We denote by E the free complex vector space generated by the elements $[x]$, and by E_0 its subspace generated by the expressions $[r1], \dots, [r4]$.

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Lemma 1. *Each element $e \in E$ determines in a unique way two measures $x, y \in \mathfrak{M}_R(\Omega, \mathfrak{B})$ such that $e = [x^+] - [x^-] + i([y^+] - [y^-]) \pmod{E_0}$.*

We define a sesquilinear form on $E \times E$ by extending the equation

$$\beta([x], [y]) = \rho(x^+, y^+) - i\rho(x^+, y^-) + i\rho(x^-, y^+) + \rho(x^-, y^-), \qquad (1)$$

where $x, y \in \mathfrak{M}_R(\Omega, \mathfrak{B})$, and $\rho(m, n)$ denotes the Hellinger integral of $m, n \in \mathfrak{M}_R^+(\Omega, \mathfrak{B})$, defined by the equation $\rho(m, n) = \sqrt{mn}(\Omega)$.

Theorem 1. *The sesquilinear form β induces a scalar product in the space $\mathcal{H}(\Omega, \mathfrak{B}) = E/E_0$. This scalar product defines on $\mathcal{H}(\Omega, \mathfrak{B})$ the structure of a complex Hilbert space.*

The proof of this theorem is based on the fact that $E_0 = \{c \in E: \beta(c, c) = 0\}$ and uses the following assertion.

Lemma 2. *For every $\epsilon > 0$ we define in $\mathfrak{M}_R(\Omega, \mathfrak{B}) \times \mathfrak{M}_R(\Omega, \mathfrak{B})$ the subset $V(\epsilon) = \{(x, y): \|x\| + \|y\| - 2[\rho(x^+, y^+) + \rho(x^-, y^-)] \le \epsilon\}$.*

Then the sets $V(\epsilon)$ represent neighborhoods of the diagonal in a base of a uniform structure which is isomorphic to the uniform structure induced by the norm on each subset of $\mathfrak{M}_R(\Omega, \mathfrak{B})$ that is bounded in norm.

Let $\pi: E \to E/E_0 = \mathcal{H}(\Omega, \mathfrak{B})$ be the canonical projection. We define the mapping

$$\varkappa:\ x \in \mathfrak{M}_R(\Omega, \mathfrak{B}) \to \pi[x] \in \mathcal{H}(\Omega, \mathfrak{B}). \qquad (2)$$

Theorem 2. *The mapping κ is a homeomorphism of $\mathfrak{M}_R(\Omega, \mathfrak{B})$ onto its image, and it has the following properties:*

[i1]. $\varkappa(x) = \varkappa(x^+) + i\varkappa(x^-), \quad i^2 = -1$;

[i2]. $\varkappa(\xi \cdot x) = \sqrt{\xi} \cdot \varkappa(x), \quad \xi \in \mathbf{R}$;

[i3]. $\|\varkappa(x)\|^2_{\mathcal{H}(\Omega,\mathfrak{B})} = \|x\|_{\mathfrak{M}_R(\Omega,\mathfrak{B})}$;

[i4]. $\varkappa(m) + \varkappa(n) = \varkappa(m + n + 2\sqrt{mn}),\ m, n \in \mathfrak{M}_R^+(\Omega, \mathfrak{B})$;

[i5]. $\varkappa(m) - \varkappa(n) = \varkappa[\chi_+ \cdot (m + n - 2\sqrt{mn})] - \varkappa[\chi_- \cdot (m + n - 2\sqrt{mn})]$,

where $m, n \in \mathfrak{M}_R^+(\Omega, \mathfrak{B})$, and χ_+, χ_- are the characteristic functions of the Jordan decomposition corresponding to the measure $m - n$.

An enumeration of the above properties justifies an interpretation of the element $\kappa(x)$ as the "square root" of the measure $x \in \mathfrak{M}_R(\Omega, \mathfrak{B})$. For this reason we will frequently use the notation $\kappa(x) = \sqrt{x}$ in what follows. Any vector in $\mathcal{H}(\Omega, \mathfrak{B})$ has the form

$$h = (\sqrt{x^+} - \sqrt{x^-}) + i(\sqrt{y^+} - \sqrt{y^-}).$$

Consequently, we may define on $\mathcal{H}(\Omega, \mathfrak{B})$ a canonical involution $J(h) = \sqrt{x^+} - \sqrt{x^-} - i(\sqrt{y^+} - \sqrt{y^-})$, and the subspace of fixed elements for J is the real space generated by the cone $\kappa\{\mathfrak{M}_R^+(\Omega, \mathfrak{B})\}$. We denote this space by $\mathcal{H}_r(\Omega, \mathfrak{B})$.

Theorem 3. *There exists a homeomorphism $\alpha: \mathfrak{M}_R(\Omega, \mathfrak{B}) \to \mathcal{H}_r(\Omega, \mathfrak{B})$ which has the* *properties* [i3], [i4], [i5] *stated in Theorem 2, and, in addition,*

$$[i1'].\ \alpha(x)=\alpha(x^+)-\alpha(x^-);$$
$$[i2'].\ \alpha(\xi x)=\sqrt{\xi}\cdot\alpha(x),\quad \xi\in R^+.$$

These properties characterize the space $\mathcal{H}_r(\Omega, \mathfrak{B})$ *to within a unitary isomorphism.*

Since $\mathcal{H}(\Omega, \mathfrak{B}) = \mathcal{H}_r(\Omega, \mathfrak{B}) \oplus i\mathcal{H}_r(\Omega, \mathfrak{B})$, the space $\mathcal{H}(\Omega, \mathfrak{B})$ also is characterized by these properties to within a unitary isomorphism. It may be shown that the space $\mathcal{H}(\Omega, \mathfrak{B})$ is isomorphic to the inductive limit of the family $L^2_C(\Omega, \mathfrak{B}, x)$ for $x \in \mathfrak{M}_R(\Omega, \mathfrak{B})$. These properties characterize the space $\mathcal{H}_r(\Omega, \mathfrak{B})$ to within unitary isomorphism.

3. Let $(\Omega_\iota, \mathfrak{B}_\iota)_{\iota\in I}$ be a family of measurable spaces.

Definition 1. Two product measures $\Pi_{\iota\in I}x_\iota$ and $\Pi_{\iota\in I}y_\iota$ are called *asymptotically equivalent* if there exists a finite subset $F \subset I$ such that the two product measures $\Pi_{\iota\in I-F}x_\iota$ and $\Pi_{\iota\in I-F}y_\iota$ are not orthogonal.

Lemma 3. *The relation of asymptotic equivalence is an equivalence relation on the set of product measures.*

In this way, the set of all product measures is partitioned into asymptotic equivalence classes. For each such class $\overline{\mathfrak{G}}$ we define the closed subspace $\mathcal{K}(\overline{\mathfrak{G}})$ of the space $\mathcal{H}_r(\Pi_{\iota\in I}\Omega_\iota, \Pi_{\iota\in I}\mathfrak{B}_\iota)$ which is generated by the images of the product measures in the equivalence class $\overline{\mathfrak{G}}$ under the mapping α defined in Theorem 3. We form the infinite direct product $\bigotimes_{\iota\in I}\mathcal{H}_r(\Omega_\iota, \mathfrak{B}_\iota)$. For each $\iota \in I$ let α_ι be the mapping defined in Theorem 3, and let $(\alpha_\iota(x_\iota))_{\iota\in I}$ be a C_0-family (see [1], Definition 3.31).

Definition 2. We will say that the C_0-family $(\alpha_\iota(x_\iota))_{\iota\in I}$ is *essentially positive* if there exists a finite set $F \subset I$ such that x_ι is positive for all $\iota \in I - F$. We will call the equivalence class $\mathfrak{G}$ of C_0-families (in the sense of [1]) *essentially positive* if it contains only a positive C_0-family.

Theorem 4. *There exists a one-to-one correspondence between the essentially positive equivalence classes of C_0-families and the asymptotic equivalence classes of product measures on* $\Pi_{\iota\in I}(\Omega_\iota, \mathfrak{B}_\iota)$. *If* $\overline{\mathfrak{G}}^+$ *is the class of product measures corresponding to* $\mathfrak{G}^+$, *then there exists a unitary isomorphism* $U_{\mathfrak{G}^+}$ *between the incomplete tensor product* $\bigotimes^+_{\iota\in I}\mathcal{H}_r(\Omega_\iota, \mathfrak{B}_\iota)$ *and the space* $\mathcal{H}(\overline{\mathfrak{G}}^+)$, *characterized by the property* $U_{\mathfrak{G}^+}((\alpha_\iota(x_\iota))) = \alpha(\Pi_{\iota\in I}x_\iota)$ *for every essentially positive C_0-family* $(\alpha_\iota(x_\iota))_{\iota\in I}$ *in* $\mathfrak{G}^+$.

The proof of this theorem uses Kakutani's theorem [2], some results from [1], and the following assertion.

Lemma 4. *Let* $(\alpha_\iota(x_\iota))_{\iota\in I}$ *be a C_0-family in* $\bigotimes_{\iota\in I}\mathcal{H}_r(\Omega_\iota, \mathfrak{B}_\iota)$.

Then $(\alpha_\iota(x_\iota))_{\iota\in I}$ *is equivalent to an essentially positive C_0-family if and only if* $(\alpha_\iota(x_\iota^+))_{\iota\in I}$ *is a C_0-family. In this case* $\Sigma_{\iota\in I}x_\iota^-(\Omega) < \infty$.

In the case when $(m_\iota)_{\iota\in I}$ is a family of positive measures, the identity $\bigotimes_{\iota\in I}\sqrt{m_\iota} = \sqrt{\Pi_{\iota\in I}m_\iota}$ follows from Theorem 4.

In this way, in the case of positive product measures, the equivalence relation introduced by von Neumann in [1] and asymptotic equivalence induce one and the same classification. From Lemma 3 it follows that if the set I is infinite, then there exists a continuum of equivalence classes of C_0-families, none of which contains an essentially positive C_0-family (they do not even contain a C_0-family that is weakly equivalent (in the sense of Definition (6.1.1) of [1]) to such a C_0-family).

We denote by $\mathfrak{V}$ the subspace of the space $\bigotimes_{\iota \in I} \mathcal{H}_r(\Omega_\iota, \mathfrak{B}_\iota)$ generated by all incomplete tensor products corresponding to equivalence classes $\mathfrak{G}$ which do not contain an essentially positive C_0-family. We denote by $\mathcal{K}^\perp$ the subspace of the space $\mathcal{H}_r(\prod_{\iota \in I}\Omega_\iota, \prod_{\iota \in I}\mathfrak{B}_\iota)$ which is generated by the images of measures orthogonal to every product measure.

Theorem 5. *There exists an exact sequence*

$$0 \to \mathscr{V} \xrightarrow{J} \bigotimes_{\iota \in I} \mathscr{H}_r(\Omega_\iota, \mathfrak{B}_\iota) \xrightarrow{U} \mathscr{H}_r\Big(\prod_{\iota \in I} \Omega_\iota;\ \prod_{\iota \in I} \mathfrak{B}_\iota\Big) \xrightarrow{P} \mathscr{K}^\perp \to 0,$$

where J and P denote the natural injection and projection, respectively, and U is characterized by the fact that its restrictions to incomplete tensor products corresponding to essentially positive equivalence classes $\mathfrak{G}^+$ coincide with the mappings $U_{\mathfrak{G}^+}$ defined in Theorem 1.

In conclusion we consider the following example.

For each $\iota \in I$ (I countable) let Ω_ι be a countable set having the discrete measurable structure. Then $\mathfrak{M}_R(\Omega_\iota) = l^1_R(\Omega_\iota)$ (the space of sequences of real numbers $(x_\nu)_{\nu \in I}$ such that $\sum_{\nu \in I}|x_\nu| < \infty$), and $\mathcal{H}_r(\Omega_\iota)$ is easily seen to be $l^2_R(\Omega_\iota)$ (the space of real sequences $(y_\nu)_{\nu \in I}$ such that $\sum_{\nu \in I}|y_\nu|^2 < \infty$). Every incomplete component of the tensor product $\bigotimes_{\iota \in I}\mathcal{H}_r(\Omega_\iota)$ is isomorphic to (real) Fock space (see [3], [4] and Lemma (4.1.2) of [1]); in addition, the space $\mathcal{H}_r(\prod_{\iota \in I}\Omega_\iota)$ represents a factorizable tensor product in the sense of Araki (see [5]).

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