ON THE STOCHASTIC LIMIT FOR QUANTUM THEORY

L. ACCARDI

Centro Matematico Vito Volterra, Dipartimento di Matematica,
Università di Roma Tor Vergata, Roma, Italia

J. GOUGH

School of Mathematics, Trinity College, Dublin, Ireland

Y. G. LU

Dipartimento di Matematica, Università di Bari, I-70125 Bari, Italia

(*Received June 8, 1995*)

The basic ideas of the stochastic limit for a quantum system with discrete energy spectrum, coupled to a Bose reservoir are illustrated through a detailed analysis of a general linear interaction: under this limit we have quantum noise processes substituting for the field. We prove that the usual Schrödinger evolution in interaction representation converges to a limiting evolution unitary on the system and noise space which, when reduced to system's degrees of freedom, provides the master and Langevin equations that are postulated on heuristic grounds by physicists. In addition, we give a concrete application of our results by deriving the evolution of an atomic system interacting with the electrodynamic field without recourse to either rotating wave or dipole approximations.

1. Quantum theory of damping

1.0. Introduction

Irreversible quantum evolutions now play a fundamental role in many areas of physics, especially quantum optics. A large body of physical literature has been built up around the problem of describing in a stochastic model the effect of a source of quantum noise on a given quantum mechanical system, emphasizing the quantum stochastic properties of the source of the quantum noise. The approach to stochasticity discussed in this paper takes into account the essential quantum nature of the problem by following the weak coupling approach [1] to quantum damping. As will be explained later, the noise fields used to model physical noise sources are still quantum in nature, and, because they are arrived at from a well-defined physical scaling limit, do not require us to put in the desired features of the noise by hand, but to deduce these properties from those of an underlying Hamiltonian model.

The derivations of quantum master equations and quantum Langevin equations for open quantum systems which are present in the current physics literature are well motivated from the physical point of view, cf. [2], however mathematically imprecise. The usual heuristic procedures to render the reservoir, to which the system is coupled, into a source of quantum noise via some *markovian approximation* and to approximate by fitting Quantum Brownian Motions (QBMs) (cf. [5, 6]) are generally reliant on arbitrary and often mutually contradictory assumptions (cf. the appendix of the present paper).

On the other hand, the weak and the singular coupling limit for an open quantum system gives a device for obtaining irreversible evolutions. Mathematically rigorous derivations of the master equations along these lines have been given for certain specific models by several authors: Pulé [3], Gorini et al., Davies [4], Kossakowski, Malishev,...

According to the theory of *stochastic limits of quantum fields*, developed in a series of papers [1, 8–11] by Accardi, Frigerio and Lu, a quantum reservoir can be reduced to a quantum stochastic noise source via a scaling limit procedure. The theory is mathematically rigorous — while at the same time — applicable to the wide range of phenomena considered by physicists and gives a precise description of the reservoir as a quantum noise source. The convergence of the reservoir fields can be intuitively interpreted in terms of *quantum central limit theorems*; that is, central limit theorems for quantum mechanical observables. *Quantum Probability* affords the necessary mathematical framework to interpret the limit processes in terms of the usual Fock space descriptions of Bose or Fermi reservoirs.

Our objective in this paper is to review some results of the stochastic limit of quantum field theory and to extend them so as to deal with general interactions between an atomic system and a noise source encountered in physical theories. This we do and show that the energy shifts, linewidths, master equations and Langevin equations, arising for the system as a result of its coupling to the noise field, concur with those obtained by earlier researchers [2]. However, the present theory also gives, in addition, a quantum stochastic description of the reservoir noise fields themselves. In particular we have a full microscopic description of the parameters determining the dynamics of the so-called *output fields*. As a concrete application of our theory we consider the particular case, where a quantum electrodynamical field acts as reservoir, however we stress that this is only one of the many applications of the theory.

We shall discuss only minimal coupling interacions, that is interactions linear in the creation/annihilation operators for the reservoir. In a forthcoming paper we discuss how to treat the situations where the interaction is of polynomial type.

1.1. Open quantum systems

We consider a system (S) coupled to a reservoir (R). The system (S) is to be quantum mechanical: its state space will be a separable Hilbert space $\mathcal{H}_S$. The reservoir, on the other hand, is comprised of one or several quantum fields, and so has infinitely many degrees of freedom. We shall consider a bosonic reservoir; the state space for (R) is the bosonic Fock space $\mathcal{H}_R$ over a separable Hilbert space $\mathcal{H}_R^1$; in standard notation we write $\mathcal{H}_R = \Gamma_B(\mathcal{H}_R^1)$. $\mathcal{H}_R^1$ is again to be a separable Hilbert space and may quite generally describe not only one but several individual species of particle in the reservoir. For instance, consider several species of particles $P_1, P_2, P_3, \ldots$ in the reservoir and suppose that $\mathcal{H}_R^1 = \oplus_j \mathcal{H}_{P_j}^1$, where $\mathcal{H}_{P_j}^1$ is the state space for particle type P_j; then

$$\mathcal{H}_R = \Gamma_B(\mathcal{H}_R^1) = \Gamma_B(\oplus_j \mathcal{H}_{P_j}^1) = \otimes_j \Gamma_B(\mathcal{H}_{P_j}^1). \tag{1.1.1}$$

The space $\mathcal{H}_R^1$ is referred to as the (combined) one particle state space for the reservoir. The overall state space for the combined system and reservoir is $\mathcal{H}_S \otimes \mathcal{H}_R$. The vacuum vector of the reservoir space will be denoted throughout as Ψ_R. In the following we consider only bosonic species in the reservoir, however it is also possible to work with fermions [9]. The dynamics of the combined system and reservoir is governed by the formal Hamiltonian $H^{(\lambda)}$ which we may write as

$$H^{(\lambda)} = H^{(0)} + \lambda H_I\,, \tag{1.1.2}$$

that is, as the sum of a free Hamiltonian $H^{(0)}$ and an interaction H_I, with λ a real coupling parameter. $H^{(0)}$ is to be expressible as

$$H^{(0)} = H_S \otimes 1_R + 1_S \otimes H_R\,, \tag{1.1.3}$$

where H_S and H_R are self-adjoint operators on the spaces $\mathcal{H}_S$ and $\mathcal{H}_R$, respectively. For each λ, we consider the unitary operator $V_t^{(\lambda)}$ on $\mathcal{H}_S \otimes \mathcal{H}_R$ defined by

$$V_t^{(\lambda)} = \exp\left\{\frac{t}{i\hbar} H^{(\lambda)}\right\}, \tag{1.1.4}$$

This gives the time evolution under $H^{(\lambda)}$. A standard device in pertubation theory is to transform to the interaction picture; this involves introducing the operator

$$U_t^{(\lambda)} = V_t^{(0)\dagger} V_t^{(\lambda)}. \tag{1.1.5}$$

$U_t^{(\lambda)}$ is a unitary operator on $\mathcal{H}_S \otimes \mathcal{H}_R$ called *the wave operator at time t* or, more frequently, *the Schrödinger evolution in interaction representation*. We note that $\{U_t^{(\lambda)} : t \in \mathbb{R}\}$ is a left $v_t^{(0)}$-cocycle, that is, it satisfies the relation

$$U_{t+s}^{(\lambda)} = v_t^{(0)}(U_s^{(\lambda)}) U_t^{(\lambda)}, \tag{1.1.6}$$

where for each $\lambda \geq 0$ the time-evolute of any operator $X \in \mathcal{B}(\mathcal{H}_S) \otimes \mathcal{B}(\mathcal{H}_R)$ is defined by

$$v_t^{(\lambda)}(X) = V_t^{(\lambda)\dagger} X V_t^{(\lambda)}.$$

We also introduce the evolution in interaction picture:

$$u_t^{(\lambda)}(X) = U_t^{(\lambda)\dagger} X U_t^{(\lambda)}. \tag{1.1.7}$$

The Schrödinger equation for the time evolutions is

$$\frac{\partial}{\partial t} V_t^{(\lambda)} = \frac{1}{i\hbar} H^{(\lambda)} V_t^{(\lambda)}$$

and, in interaction representation,

$$\frac{\partial}{\partial t} U_t^{(\lambda)} = \frac{\lambda}{i\hbar} v_t^{(0)}(H_I) U_t^{(\lambda)} . \tag{1.1.8}$$

With these notations we deduce the associated Heisenberg equations

$$\frac{\partial}{\partial t} v_t^{(\lambda)}(X) = \frac{1}{i\hbar}[v_t^{(\lambda)}(X), v_t^{(\lambda)}(H^{(\lambda)})] = \frac{1}{i\hbar} v_t^{(\lambda)}([X, H^{(\lambda)}])$$

and, in interaction representation,

$$\frac{\partial}{\partial t} u_t^{(\lambda)}(X) = \frac{\lambda}{i\hbar} u_t^{(\lambda)}([X, v_t^{(0)}(H_I)]) . \tag{1.1.9}$$

From (1.1.8) we obtain the integral equation

$$U_t^{(\lambda)} = 1 + \frac{\lambda}{i\hbar} \int_0^t ds\, v_s^{(0)}(H_I)\, U_s^{(\lambda)} \tag{1.1.10}$$

and, consequently, the iterated series

$$U_t^{(\lambda)} = 1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{i\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, v_{t_1}^{(0)}(H_I) \cdots v_{t_n}^{(0)}(H_I) . \tag{1.1.11}$$

This may be symbolically described as

$$U_t^{(\lambda)} = \mathcal{T} \exp\left\{ \frac{\lambda}{i\hbar} \int_0^t ds\, v_s^{(0)}(H_I) \right\} , \tag{1.1.12}$$

where $\mathcal{T}$ denotes time ordering.

Before we continue, we must say more about how to interpret the formal sum of $H^{(0)}$ and H_I. Firstly, we assume that $H^{(0)}$ and H_I are self-adjoint operators on $\mathcal{H}_S \otimes \mathcal{H}_R$. We shall assume that, for sufficiently small λ and bounded t, the iterated series (1.1.11) is weakly convergent on the domain $\mathcal{H}_S^0 \underline{\otimes} \mathcal{E}(\mathcal{H}_R^1)$, the algebraic tensor product a total subset $\mathcal{H}_S^0$ of $\mathcal{H}_S$ and of a total subset of $\Gamma(\mathcal{H}_R^1)$, for example the set of exponential or number vectors. From the cocycle relation (1.1.6) we have that if we define the unitary operator $V_t^{(\lambda)}$ by

$$V_t^{(\lambda)} = V_t^{(0)} U_t^{(\lambda)} , \tag{1.1.13}$$

then $\{V_t^{(\lambda)} : t \in \mathbb{R}\}$ gives a strongly continuous unitary group whose generator $\frac{1}{\hbar} H^{(\lambda)}$ is formally given as $H^{(\lambda)} = H^{(0)} + \lambda H_I$. The time evolution in the Heisenberg picture is then given by

$$v_t^{(\lambda)}(X) = u_t^{(\lambda)}(v_t^{(0)}(X)) . \tag{1.1.14}$$

1.2. The free evolution of the reservoir

For $\mathcal{H}_R^1 = L^2(\mathbb{R}^3)$ (momentum space), we introduce the creation and annihilation densities $a^\sharp(\mathbf{k})$ satisfying

$$[a(\mathbf{k}), a(\mathbf{k}')^\dagger] = \delta(\mathbf{k}-\mathbf{k}')\,, \qquad [a(\mathbf{k})^\dagger, a^\dagger(\mathbf{k}')] = 0\,, \qquad [a(\mathbf{k}), a(\mathbf{k}')] = 0\,. \tag{1.2.1}$$

The formal creation and annihilation densities are related to the corresponding fields on $\mathcal{H}_R$ by the relation

$$A^\dagger(g) = \int d^3k\, g(\mathbf{k}) a^\dagger(\mathbf{k})\,, \qquad A(g) = \int d^3k\, \overline{g}(\mathbf{k}) a(\mathbf{k}) \tag{1.2.2}$$

valid for any regular (e.g. in the Schwartz space of rapidly decreasing functions) test function $g \in \mathcal{H}_R^1$.

From (1.2.2) we obtain the canonical commutation relations (CCR): $[A(h), A^\dagger(f)] = \langle h, f\rangle$, while $[A(h), A(f)] = 0$. We take H_R to be the second quantization of an operator H_R^1 on $\mathcal{H}_R^1$ given by

$$(H_R^1 f)(k) = \hbar\omega(k) f(k)\,, \tag{1.2.3}$$

where $\omega(k)$ is a positive (usually strictly positive) function whose form depends on the specific model.

In terms of the creation and annihilation densities, the operator H_R may then be expressed as

$$H_R = \int dk\, \hbar\omega(k)\, a^\dagger(k) a(k)\,. \tag{1.2.4}$$

Note that $v_t^{(0)}(1_S \otimes A^\sharp(g)) = 1_S \otimes A^\sharp(S_t g)$, where we have introduced the unitary operator S_t on $\mathcal{H}_R^1$ given by

$$S_t = \exp\left\{-\frac{t}{i\hbar} H_R^1\right\}\,, \tag{1.2.5}$$

that is, $(S_t f)(k) = e^{i\omega(k)t} f(k)$.

1.3. The standard approach to the quantum Langevin equation

Consider an interaction of the type

$$H_I = i\hbar\{D \otimes A^\dagger(g) - D^\dagger \otimes A(g)\}\,, \tag{1.3.1}$$

where $D \in \mathcal{B}(\mathcal{H}_S)$ has a harmonic operator for the free-evolution; that is

$$\frac{1}{i}[D, H_S] = -i\omega D \leftrightarrow e^{itH_S} D e^{-itH_S} = e^{-it\omega} D \tag{1.3.1a}$$

and we further assume that

$$\omega > 0\,. \tag{1.3.1b}$$

Then we have

$$v_t^{(0)}(H_I) = i\hbar\{D \otimes A^\dagger(S_t^\omega g) - D^\dagger \otimes A(S_t^\omega g)\}\,, \tag{1.3.2}$$

where

$$S_t^\omega = e^{-i\omega t} S_t \,, \tag{1.3.3}$$

that is, $(S_t^\omega f)(\mathbf{k}) = e^{i(\omega(\mathbf{k})-\omega)t} f(\mathbf{k})$. Now write $X_t = u_t^{(\lambda)}(X \otimes 1_R)$, for $X \in \mathcal{B}(\mathcal{H}_S)$, then from (1.1.9) we have

$$\frac{\partial X_t}{\partial t} = \lambda\{A_t^\dagger(S_t^\omega g)[X,D]_t - [X,D^\dagger]_t A_t(S_t^\omega g)\}\,, \tag{1.3.4}$$

where $A_t^\sharp(f) = u_t^{(\lambda)}(1_S \otimes A^\sharp(f))$. Similarly, $\frac{\partial}{\partial t} A_t(f) = \lambda u_t^{(\lambda)}(D)\langle g, S_{tg}^\omega\rangle$, so that $A_t(f) = \lambda \int_0^t ds\, D_s \langle f, S_s^\omega g\rangle + 1_S \otimes A(f)$, which, substituting back into (1.3.4) gives

$$\frac{\partial X_t}{\partial t} = \lambda^2 \int_0^\infty \{D_s^\dagger \phi^\omega(s-t)[X,D]_t - [X,D^\dagger]_t \phi^\omega(t-s) D_s\} + \lambda\{\xi_t^\omega [X,D]_t - [X,D^\dagger]_t \xi_t^{\omega\dagger}\}\,, \tag{1.3.5}$$

where $\phi^\omega(t) = \langle g, S_{-t}^\omega g\rangle = \int d^3k |g(\mathbf{k}|^2) e^{-i(\omega(\mathbf{k})-\omega)t}$ and $\xi^\omega(t) = 1_S \otimes A(S_t^\omega g)$. In standard terminology $\phi(t)$ is called *the memory function* and ξ_t^ω the *fluctuating quantum force* [2,17] or *input field* [18], albeit in the interaction picture. One notes that, in the vacuum state, ξ_t^ω is gaussian-distributed and all first and second moments vanish except the two-point function

$$\langle \Psi_R, \xi_t^\omega \xi_t^{\omega\dagger} \Psi_R\rangle = \langle S_t^\omega g, S_s^\omega g\rangle = \phi^\omega(t-s)\,. \tag{1.3.6}$$

The standard approach made at this juncture is to introduce the so-called *first Markov approximation.* Here, for example, one takes $\mathcal{H}_R^1 = L^2(\mathbb{R})$, $g = \sqrt{\frac{\kappa}{2\pi}}$ (constant) and $\omega(k) = k$. Then

$$\phi^\omega(t) = \int_{-\infty}^\infty dk \frac{\kappa}{2\pi} e^{i(k+\omega)t} = \kappa e^{i\omega t}\delta(t)\,. \tag{1.3.7}$$

There are, however, several important objections to be made to this approach. Firstly any physical details specific to the reservoir must be put in by hand. Secondly, the condition $\omega(k) = k$ implies that the spectrum of H_R^1 is unbounded below, this is necessary to produce the delta function correlation of white noise. From a physical point of view this is unacceptable as H_R^1 must be bounded below for stability. Finally, the fact that the frequency spectrum $\omega(k) = k$ is unbounded below precludes any possibility of dropping the rotating wave approximation.

1.4. The stochastic limit

We now describe the ideas behind the stochastic limit in the simplest situation, where we have taken a dipole and rotating wave approximation. From now on, unless explicitly stated, we shall take $\mathcal{H}_R^1 = L^2(\mathbb{R}^3)$. All the results remain valid in $L^2(\mathbb{R}^d)$ with $d \geq 3$. We define the following *collective annihilation* operator:

$$B_t^{(\omega,\lambda)}(g) = A(\lambda \int_0^t dt_1 S_{t_1}^\omega g) = \lambda \int_{\mathbb{R}^3} dk \int_0^t dt_1 e^{-i(\omega(k)-\omega)t_1} \overline{g}(k) a(k)\,. \tag{1.4.1}$$

Calculating the two-point vacuum expectations gives

$$\begin{aligned}\langle \Psi_R(0), B_t^{(\omega,\lambda)}(g) B_s^{(\omega,\lambda)\dagger}(f)\Psi_R(0)\rangle &= \lambda^2 \int_0^t dt_1 \int_0^s ds_1 \langle S_{t_1}^\omega g, S_{s_1}^\omega f\rangle \\ &\equiv \int_0^{\lambda^2 t} du \int_{u/\lambda^2 - s}^{u/\lambda^2} d\tau \langle S_\tau^\omega g, f\rangle\,, \qquad (1.4.2)\end{aligned}$$

where we have substituted $u = \lambda^2 t_1$ and $\tau = t_1 - s_1$. This shows that the only way, in order to obtain a nontrivial two-point function in the limit $\lambda \to 0$ is to rescale time as

$$t \hookrightarrow t/\lambda^2\,. \qquad (1.4.3)$$

This is known as the *Friedrichs-van Hove* or weak coupling limit in physics. One finds, under this scaling:

$$\lim_{\lambda\to 0} \langle \Psi_R,\, B_{t/\lambda^2}^{(\omega,\lambda)}(g)\, B_{s/\lambda^2}^{(\omega,\lambda)\dagger}(f)\,\Psi_R\rangle = \min\{t,s\} \int_{-\infty}^{\infty} d\tau\, \langle S_\tau^\omega g, f\rangle\,. \qquad (1.4.4)$$

Physically, the limit $\lambda \to 0$ with $t \hookrightarrow t/\lambda^2$ allows us to consider progressively weaker interactions which are allowed to run over increasingly larger periods of time and so we obtain the long term cumulative effect of the interaction on the system. Now the creation and annihilation operators are gaussian in the vacuum state and, as a result, so are the operators $B_{t/\lambda^2}^{(\omega,\lambda)\sharp}$.

Furthermore, the limiting two-point function (1.4.4) is suggestive of the correlation function of a Brownian motion. However, an interpretation of the above in terms of classical Brownian motion is erroneous as it ignores the essentially quantum probabilistic nature of these processes.

1.5. The interaction

For technical reasons we work with a system Hamiltonian H_S which has discrete spectrum. H_R^1 is taken to be bounded below as required from physics. The type of interaction H_I which we wish to study is of the form

$$H_I = i\hbar \sum_{\omega\in F} \sum_{j=1}^{N(\omega)} \{\, D_j^\omega \otimes A^\dagger(g_j^\omega) - D_j^{\omega\dagger} \otimes A(g_j^\omega)\,\}\,, \qquad (1.5.1)$$

where F is a discrete subset of $\mathbb{R}$. For each $\omega \in F$, we take $D_j^\omega \in \mathcal{B}(\mathcal{H}_S)$ to have harmonic free evolution with frequency ω:

$$\frac{1}{i}[D_j^\omega, H_S] = -i\omega D_j^\omega\,, \qquad j = 1,\dots,N(\omega)\,. \qquad (1.5.2)$$

Thus the superscript ω labels harmonic frequency and $j = 1, ..., N(\omega)$ the degeneracy of that frequency. An interaction similar to (1.5.1) has been treated in [10], however there the test functions g_j^ω were taken to be equal for each value of ω.

Our reasons for studying (1.5.1) above are because it allows us to treat the most general interactions encountered in physics. Typically, in quantum field theory one considers an interaction of the type

$$H_I = ih \int d^3k \, \{\theta(\mathbf{k}) \otimes a^\dagger(\mathbf{k}) - \theta^\dagger(\mathbf{k}) \otimes a(\mathbf{k})\}, \tag{1.5.3}$$

where $\{\theta(\mathbf{k}) : \mathbf{k} \in \mathbb{R}^3\}$ is a family of operators on $\mathcal{H}_S$. The operators $\theta(\mathbf{k})$ are called the *response terms*: they contain local information about the interaction. In the dipole approximation of quantum field theory one makes the replacement

$$\theta(\mathbf{k}) \hookrightarrow \theta^{\text{dipole}}(\mathbf{k}) = g(\mathbf{k})\theta(0), \tag{1.5.4}$$

where $g(\mathbf{k})$ is some suitable test function. The physical argument is, cf. [2], that the response does not vary appreciably for values of the wavelength of the reservoir particles, which are large relative to the physical dimensions of the system, though this can hardly be true for large momenta. As a result, one obtains the approximate Hamiltonian

$$H_I \hookrightarrow H_I^{\text{dipole}} = i\hbar\{\theta(0) \otimes A^\dagger(g) - \theta^\dagger(0) \otimes A(g)\}. \tag{1.5.5}$$

A further approximation often made by physicists is to replace $\theta(0)$ by an operator D having a harmonic free evolution with some frequency $\omega \in \mathbb{R}$. This approximation is just the *rotating wave approximation*.

In order to avoid these appproximations we argue as follows:

Let B be a complete basis of eigenstates of H_S, then

$$H_I = \sum_{\phi,\phi' \in B} \langle\phi|H_I|\phi'\rangle|\phi\rangle\langle\phi'|. \tag{1.5.6}$$

However, we may write

$$\langle\phi|H_I|\phi'\rangle = i\hbar \int dk\{\langle\phi|\theta(\mathbf{k})|\phi'\rangle a^\dagger(\mathbf{k}) - \langle\phi|\theta^\dagger(\mathbf{k})|\phi'\rangle a(\mathbf{k})\} \equiv i\hbar[A^\dagger(g_{\phi\phi'}) - A(g_{\phi'\phi})], \tag{1.5.7}$$

where we have introduced the test functions

$$g_{\phi\phi'}(\mathbf{k}) = \langle\phi|\theta(\mathbf{k})|\phi'\rangle. \tag{1.5.8}$$

Note that the order of ϕ and ϕ' is reversed in the second term in (1.5.7) due to the conjugate linear nature of the creation field.

This now means that the interaction can be expressed as

$$H_I = i\hbar \sum_{\phi,\phi' \in B} \{T_{\phi\phi'} \otimes A^\dagger(g_{\phi\phi'}) - T^\dagger_{\phi\phi'} \otimes A(g_{\phi\phi'})\}, \tag{1.5.9}$$

where we have introduced the transition operators $T_{\phi\phi'} = |\phi\rangle\langle\phi'|$.

We note that the transition operators $T_{\phi\phi'}$ are harmonic under the free evolution. In fact, we have

$$\frac{1}{i\hbar}[T_{\phi\phi'}, H_S] = -i\omega_{\phi\phi'}T_{\phi\phi'}\,, \tag{1.5.10}$$

where

$$\omega_{\phi\phi'} = \frac{E_{\phi'} - E_{\phi}}{\hbar}\,. \tag{1.5.11}$$

So $F = \{\omega_{\phi\phi'} : \phi, \phi' \in B\}$ is now the set of *Bohr frequencies*.

The expression (1.5.9) is now equivalent to the interaction (1.5.1) which we propose to study. Here we need only relabel the $T_{\phi\phi'}$ as D_j^{ω}, where $\omega = \omega_{\phi\phi'}$ and the j again labels degeneracy. The functions $g_{\phi\phi'}$ are relabelled accordingly.

2. The quantum stochastic limit

2.1. Quantum Brownian motions

In this section, we first of all discuss the concept of quantum Brownian motion. As this is not yet widely known amongst physicists we give an exposition below:

DEFINITION. A *quantum Brownian motion* (QBM) is a triple $(\mathcal{H}, \Phi, (B_t)_t)$, where $\mathcal{H}$ is a separable Hilbert space, $\Phi \in \mathcal{H}$ with $\|\Phi\| = 1$, $(B_t)_t$ is a family of operators on $\mathcal{H}$ such that:

(i) $q_t = \mathrm{Re}B_t$ and $p_t = \mathrm{Im}B_t$ are classical Brownian motions for the state Φ,

(ii) $[p_s, q_t] = \frac{\kappa}{2i}\min\{s,t\}$, where $\kappa \in \mathbb{R}$.

The basic example is the following: Let $\mathcal{H}_{\mathbf{C}} = \Gamma_B(L^2(\mathbb{R}))$ and $\Phi_{\mathbf{C}}$ be the vacuum state. Then define B_t to be

$$B_t = A_{\mathbf{C}}(\chi_{[0,t]})\,, \tag{2.1.1}$$

where $A_{\mathbf{C}}$ is the annihilation operator on $\mathcal{H}_{\mathbf{C}}$. From the (CCR) we have

$$[B_t, B_s^{\dagger}] = \langle \chi_{[0,t]}, \chi_{[0,s]}\rangle = \min\{t,s\}; \qquad [B_t, B_s] = 0 = [B_t^{\dagger}, B_s^{\dagger}]\,. \tag{2.1.2}$$

So setting $q_t = \frac{1}{2}(B_t + B_t^{\dagger})$ and $p_t = \frac{1}{2i}(B_t - B_t^{\dagger})$ we have from the (CCR) that

$$-[q_t, p_s] = [p_t, q_s] = \frac{1}{2i}\min\{t,s\}\,. \tag{2.1.3}$$

Now if we set for some t and $dt > 0$

$$dB_t = B_{t+dt} - B_t = A_{\mathbf{C}}(\chi_{[t,t+dt]})\,, \tag{2.1.4}$$

we have that

$$\begin{gathered}\langle \Phi_{\mathbf{C}}, dB_t^{\sharp}\Phi_{\mathbf{C}}\rangle = 0\,, \qquad \langle \Phi_{\mathbf{C}}, (dB_t^{\sharp})^2\Phi_{\mathbf{C}}\rangle = 0\,, \qquad \langle \Phi_{\mathbf{C}}, dB_t^{\dagger}dB_t\Phi_{\mathbf{C}}\rangle = 0\,, \\ \langle \Phi_{\mathbf{C}}, dB_t dB_t^{\dagger}\Phi_{\mathbf{C}}\rangle = dt\,.\end{gathered} \tag{2.1.5}$$

Now dB_t and $dB_t^\dagger$ are gaussian in the vacuum state, because the creation and annihilation fields are, therefore so are dq_t and dp_t. Furthermore,

$$\langle \Phi_{\mathbf{C}}, (dq_t)^2 \Phi_{\mathbf{C}} \rangle = \frac{1}{4} \langle \Phi_{\mathbf{C}}, (dB_t + dB_t^\dagger)^2 \Phi_C \rangle = \frac{1}{4} dt , \tag{2.1.6}$$

and similarly, $\langle \Phi_{\mathbf{C}}, (dp_t)^2 \Phi_C \rangle = \frac{1}{4} dt$. Finally, noting that at unequal times s and t

$$\langle \Phi_{\mathbf{C}}, dq_t dq_s \Phi_{\mathbf{C}} \rangle = 0 = \langle \Phi_{\mathbf{C}}, dp_t dp_s \Phi_{\mathbf{C}} \rangle \tag{2.1.7}$$

whenever $t < t + dt \leq s < s + ds$ or $s < s + ds \leq t < t + dt$, we conclude that $(q_t)_t$ and $(p_t)_t$ are each separate Brownian motions for expectations taken in the state $\Phi_{\mathbf{C}}$. So $\{\mathcal{H}_{\mathbf{C}}, \Phi_{\mathbf{C}}, (B_t)_t\}$ is a quantum Brownian motion. We can introduce formal creation and annihilation densities $b^\sharp(t)$ satisfying

$$[b(t), b(s)] = 0 = [b^+(t), b^+(s)] , \qquad [b(t), b^+(s)] = \delta(t - s) \tag{2.1.8}$$

such that

$$A_{\mathbf{C}}(g) = \int_{\mathbb{R}} ds \, \overline{g(s)} b(s) , \qquad A_{\mathbf{C}}^\dagger(g) = \int_{\mathbb{R}} ds \, g(s) b^+(s) . \tag{2.1.9}$$

From this we see

$$B_t^\sharp = \int_0^t ds \, b^\sharp(s) . \tag{2.1.10}$$

We may write $b_t^\sharp = \frac{dB_t}{dt}$ and consider these densities as "quantum white noises".

More generally, let K be a separable Hilbert space and let $L^2(\mathbb{R}, K)$ denote the set of square-integrable K-valued functions over $\mathbb{R}$. Now $h \in L^2(\mathbb{R}; K)$ is a function $h(t) \in K$ with $\int_{\mathbb{R}} dt \, \|h(t)\|_K^2 < \infty$. The inner product on $L^2(\mathbb{R}, K)$ is given by

$$\langle h, h' \rangle = \int_{\mathbb{R}} \langle h(t), h'(t) \rangle dt . \tag{2.1.11}$$

If $\{e_n\}_n$ is a complete orthonormal basis for K then we can write $h(t) = \sum_n h_n(t) e_n$, where $h_n(t) = \langle e_n, h(t) \rangle_K$; this gives a natural isomorphism

$$L^2(\mathbb{R}, K) \cong K \otimes L^2(\mathbb{R}) . \tag{2.1.12}$$

Now take $\mathcal{H}_K = \Gamma_B(L^2(\mathbb{R}, K))$ and let Φ_K denote vacuum vector of $\mathcal{H}_K$. Then a quantum Brownian motion is given by $(\mathcal{H}_K, \Phi_K, (B_t(g))_t)$, for non-zero $g \in K$, where

$$B_t(g) = A_K(g \otimes \chi_{[0,t]}) , \tag{2.1.13}$$

where A_K is the annihilation operator on $\mathcal{H}_K$. The commutation relations are

$$[B_t(g), B_s^\dagger(f)] = \langle g \otimes \chi_{[0,t]}, f \otimes \chi_{[0,s]} \rangle = \langle g, f \rangle_K \min\{t, s\} \tag{2.1.14}$$

with remaining commutators vanishing. So $\{\mathcal{H}_K = \Gamma_B(L^2(\mathbb{R}, K)), \ \Phi_K, \ (B_t(g))_t\}$ is a quantum Brownian motion. Taking $K = \mathbb{C}$ and $|g|^2 = \kappa$ leads back to the original example.

However, there is a more general possibility than that above. Let $Q \geq 1_K$ and set

$$C = Q \otimes 1_{L^2(\mathbb{R})} . \tag{2.1.15}$$

Then let φ_C be the state on the Weyl algebra $W(\mathcal{H}_K)$ with covariance C. We can construct $\{G_B(\mathcal{H}_K, C), \pi^C_{\mathcal{H}_K}, \Phi^C_{\mathcal{H}_K}\}$, the GNS triple over $\{W(\mathcal{H}_K), C\}$, and define on it the operator

$$B_Q(g, t) = \pi^C_{\mathcal{H}_K} B_t(g) . \tag{2.1.16}$$

Then $\{G_B(\mathcal{H}_K, C), \Phi^C_{\mathcal{H}_K}, (B_Q(g,t))_t\}$ is a quantum Brownian motion referred to as quantum Brownian motion over $L^2(\mathbb{R}, K)$ with covariance C, or more loosely with covariance Q. We have that

$$\langle \Phi^C_{\mathcal{H}_K}, B_Q(g,t)\, B^\dagger_Q(f,s)\, \Phi^C_{\mathcal{H}_K} \rangle = \varphi_C(B_t(g) B^\dagger_s(f)) = \min\{t, s\} \langle g, \frac{Q+1}{2} f \rangle_K \tag{2.1.17}$$

and similarly

$$\langle \Phi^C_{\mathcal{H}_K}, B^\dagger_Q(f,s)\, B_Q(g,t)\, \Phi^C_{\mathcal{H}_K} \rangle = \min\{t, s\} \langle g, \frac{Q-1}{2} f \rangle_K . \tag{2.1.18}$$

2.2. Quantum stochastic calculus

As is well known, a stochastic calculus can be built up around classical Brownian motion and that the resulting theory has widespread applications in the study of noisy systems in physics and engineering. It is also possible to build up quantum stochastic calculus based on the QBMs we have just considered. This was originally done by Hudson and Parthasarathy [13,14]. The basic integrators are dt and, depending on the context, $dB^\sharp_t$ or $dB^\sharp_t(g)$ or $dB^\sharp_Q(t, g)$.

In the simplest case, for instance, we have for a partition $-\infty = t_1 < t_2 < \ldots < t_n < t_{n+1} = \infty$,

$$L^2(\mathbb{R}) = \bigoplus_{m=1}^{n} L^2([t_m, t_{m+1}]) ,$$

and consequently,

$$\Gamma_B(L^2(\mathbb{R})) = \bigotimes_{m=1}^{n} \Gamma_B(L^2([t_m, t_{m+1}]) . \tag{2.2.1}$$

This gives the required time filtration in the quantum situation. We say that a family of operators $(X_t)_t$ on $\Gamma_B((L^2(\mathbb{R})))$ is adapted if, for all t,

$$X_t \equiv \tilde{X}_t \otimes 1 \tag{2.2.2}$$

on $\Gamma_B(L^2((-\infty, t)) \otimes \Gamma_B(L^2([t, \infty)))$.

The quantum Ito table reads as

$$dB_Q(g,t) \cdot dB^\dagger_Q(f,t) \equiv \langle g, \frac{Q+1}{2} f \rangle_K dt , \qquad dB^\dagger_Q(f,t) \cdot dB_Q(g,t) \equiv \langle g, \frac{Q-1}{2} f \rangle_K dt ,$$

$$(dt)^2 = dt \cdot dB_Q^\sharp(g,t) = (dB_Q^\sharp(g,t))^2 \equiv 0\,. \tag{2.2.3}$$

Let $(X_t)_t$ be an adapted process of the form

$$X_t = \int_0^t (x_s ds + x_s^+ dB_Q^\dagger(g,s) + x_s^- dB_Q(g,s)) \tag{2.2.4}$$

and $(Y_t)_t$ a similar process. Then we have the quantum Ito formula

$$d(X_t \cdot Y_t) \equiv dX_t \cdot Y_t + X_t \cdot dY_t + dX_t \cdot dY_t \tag{2.2.5}$$

with

$$dX_t = x_t dt + x_t^+ dB_Q^\dagger(g,t) + x_t^- dB_Q(g,t))\,. \tag{2.2.6}$$

2.3. The weak coupling limit of quantum field theory

The first results of Accardi, Frigerio and Lu, which were concerned with the weak coupling limit for an interaction (1.5.8) which has undergone both a dipole and a rotating wave approximation, can be summarised as follows.

Recall that

$$\lim_{\lambda\to 0}\langle \Psi_R, B_{t/\lambda^2}^{(\omega,\lambda)}(g) B_{s/\lambda^2}^{(\omega,\lambda)\dagger}(f)\Psi_R\rangle = \min\{t,s\}\int_{-\infty}^{\infty} d\tau \langle S_\tau^\omega g, f\rangle\,.$$

Now define a sesquilinear form $(.|.)^\omega$ on $\mathcal{H}_R^1$, the one-particle reservoir space, by

$$(g|f)^\omega = \int_{-\infty}^{\infty} d\tau \langle S_\tau^\omega g, f\rangle\,. \tag{2.3.1}$$

We consider a space of suitable test-functions $T^\omega \subset \mathcal{H}_R^1$, determined by the condition

$$\int_{-\infty}^{\infty} dt\, |\langle g, S_t^\omega f\rangle| < \infty \tag{2.3.2}$$

whenever $f, g \in T^\omega$. Note that technically T^ω does not depend on ω, however we keep it in as a label. Then we construct K_ω the completion of T^ω with respect to $(.|.)^\omega$. That is K^ω is the completion of T^ω factored out by its $(.|.)^\omega$-norm null space. K^ω is a separable Hilbert space with inner product $(.|.)^\omega$

THEOREM 1. *In the limit $\lambda \to 0$ the stochastic process on the resevoir space*

$$\{\mathcal{H}_R, \Psi_R, (B_{t/\lambda^2}^{(\omega,\lambda)}(f))_t\}$$

for $f \in K^\omega$, converges weakly in the sense of matrix elements to a quantum Brownian motion on $L^2(\mathbb{R}, K_\omega)$. We denote this quantum Brownian motion by $\{\mathcal{H}^\omega = \Gamma_B(L^2(\mathbf{R}, K^\omega)), \Phi^\omega = \Phi_{K^\omega}, (B_t^\omega(f))_t\}$.

In the next theorem we show that $U^{(\lambda)}_{t/\lambda^2}$ converges to a stochastic process U_t on $\mathcal{H}_S \otimes \mathcal{H}^\omega$ in a sense to be made explicit now.

THEOREM 2. *Let $f^{(j)}, h^{(j')} \in K^\omega$; $T^{(j)}, S^{(j')} > 0$, for $j = 1, ..., n : j' = 1, ..., m$ and let $\phi, \phi' \in \mathcal{H}_S$ then the limit as $\lambda \to 0$ of the matrix element*

$$\langle \phi \otimes B^{(\omega,\lambda)\dagger}_{T^{(1)}/\lambda^2}(f^{(1)}) \cdots B^{(\omega,\lambda)\dagger}_{T^{(m)}/\lambda^2}(f^{(n)})\Psi_R | U^{(\lambda)}_{t/\lambda^2} | \phi' \otimes B^{(\omega,\lambda)\dagger}_{S^{(1)}/\lambda^2}(h^{(1)}) \cdots B^{(\omega,\lambda)\dagger}_{S^{(m)}/\lambda^2}(h^{(m)})\Psi_R\rangle \tag{2.3.3}$$

exists and equals

$$\langle \phi \otimes B^{\omega\dagger}_{T^{(1)}}(f^{(1)}) \cdots B^{\omega\dagger}_{T^{(n)}}(f^{(n)})\Phi^\omega | U_t | \phi' \otimes B^{\omega\dagger}_{S^{(1)}}(h^{(1)}) \cdots B^{\omega\dagger}_{S^{(m)}}(h^{(m)})\Phi^\omega\rangle\,, \tag{2.3.4}$$

where U_t is a process on $\mathcal{H}_S \otimes \mathcal{H}^\omega$ which is the solution to the quantum stochastic differential equation

$$dU_t = \{D \otimes dB^{\omega\dagger}_t(g) - D^\dagger \otimes dB^\omega_t(g) - (g|g)^{\omega-} D^\dagger D \otimes dt\}U_t \tag{2.3.5}$$

with

$$(g|f)^{\omega-} = \int_{-\infty}^0 d\tau \langle g, S^\omega_\tau f\rangle\,. \tag{2.3.6}$$

Note that $d(U_t U_t^\dagger) \equiv 0 \equiv d(U_t^\dagger U_t)$ by the quantum Ito formula and the Ito table. So U_t is *unitary* on $\mathcal{H}_S \otimes \mathcal{H}^\omega$, however it describes an *irreversible* evolution when restricted to $\mathcal{H}_S$. The unitarity condition corresponds to a *fluctuation-dissipation law* (cf. [22]).

THEOREM 3. *Let $X \in \mathcal{B}(\mathcal{H}_S)$, then in the notation of Theorem 2 the limit*

$$\langle \phi \otimes B^{(\omega,\lambda)\dagger}_{T^{(1)}/\lambda^2}(f^{(1)})..B^{(\omega,\lambda)\dagger}_{T^{(n)}/\lambda^2}(f^{(n)})\Psi_R | U^{(\lambda)\,\dagger}_{t/\lambda^2}\ (X \otimes 1_R) U^{(\lambda)}_{t/\lambda^2} | \phi' \otimes B^{(\omega,\lambda)\dagger}_{S^{(1)}/\lambda^2}(h^{(1)}) \cdots$$

$$\cdots B^{(\omega,\lambda)\dagger}_{S^{(m)}/\lambda^2}(h^{(m)})\Psi_R\rangle \tag{2.3.7}$$

exists and equals

$$\langle \phi \otimes B^{\omega\dagger}_{T^{(1)}}(f^{(1)}) \cdots B^{\omega\dagger}_{T^{(n)}}(f^{(n)})\Phi^\omega | U_t^\dagger (X \otimes 1) U_t | \phi' \otimes B^{\omega\dagger}_{(1)}(h^{(1)}) \cdots B^{\omega\dagger}_{(m)}(h^{(m)})\Phi^\omega\rangle\,. \tag{2.3.8}$$

Note that in these theorems we encounter vectors of the type $B^{(\omega,\lambda)\dagger}_{T/\lambda^2}(f)\Psi_R$ which are *1-particle vectors* $\lambda \int_{S/\lambda^2}^{T/\lambda^2} ds S^\omega_s f$ with test functions $\lambda \int_0^{T/\lambda^2} d\tau S^\omega_\tau f$: similarly any n–particle or exponential vector with test functions are called *collective vectors* in the terminology of [10] and they are designed to extract the long time cumulative behaviour of the reservoir fields.

2.4. Non-zero temperature reservoir

Next, for the non-vacuum case, we consider a density matrix ρ_Q on $\mathcal{H}_{\mathcal{R}}$ which is invariant under the free evolution and gaussian with covariance $Q \geq 1_{\mathcal{H}_R^1}$. That is

$$\mathrm{Tr}\{\rho_Q W(g)\} = e^{-\frac{1}{2}\langle g, Qg\rangle}, \qquad \forall g \in \mathcal{H}_R^1. \tag{2.4.1}$$

The invariance condition is equivalent to

$$[S_t, Q] = 0, \qquad \text{on} \quad \mathrm{Dom}(Q). \tag{2.4.2}$$

In particular, the choice of a heat bath at inverse temperature β and fugacity z is given by

$$Q = \frac{1 + ze^{-\beta H_R^1}}{1 - ze^{-\beta H_R^1}}, \tag{2.4.3}$$

that is

$$(Qf)(k) = \coth\frac{\beta}{2}(\hbar\omega(k) - \mu)\, f(k), \tag{2.4.4}$$

where $\mu = \frac{1}{\beta}\ln z$ is the chemical potential. Now

$$\begin{aligned}\lim_{\lambda\to 0} \mathrm{Tr}\left\{\rho_Q\, B^{(\omega,\lambda)}_{t/\lambda^2}(g) B^{(\omega,\lambda)\dagger}_{s/\lambda^2}(f)\right\} &= \min\{t,s\}\int_{-\infty}^{\infty} d\tau\, \mathrm{Tr}\{\rho_Q\, A(S_\tau^\omega g) A^\dagger(f)\} \\ &= \min\{t,s\}\int_{-\infty}^{\infty} d\tau \langle S_\tau^\omega g, (\frac{Q+1}{2}) f\rangle, \end{aligned}\tag{2.4.5}$$

and similarly

$$\lim_{\lambda\to 0} \mathrm{Tr}\left\{\rho_Q\, B^{(\omega\lambda)\dagger}_{s/\lambda^2}(f) B^{(\omega,\lambda)}_{t/\lambda^2}(g)\right\} = \min\{t,s\}\int_{-\infty}^{\infty} d\tau \langle S_\tau^\omega g, (\frac{Q-1}{2}) f\rangle. \tag{2.4.6}$$

Let T_Q^ω be the subset of $\mathrm{Dom}(Q)$ such that

$$\int_{-\infty}^{\infty} |\langle f, S_t^\omega h\rangle|\, dt < \infty \qquad \text{and} \qquad \int_{-\infty}^{\infty} |\langle f, S_t^\omega Q h\rangle|\, dt < \infty \tag{2.4.7}$$

whenever $f, h \in T_Q^\omega$. Let K_Q^ω be the Hilbert space completion of T_Q^ω with respect to the sesquilinear form $(.|.)_Q^\omega$ given by

$$(f|h)_Q^\omega = \int_{-\infty}^{\infty} \langle f, S_t^\omega h\rangle\, dt. \tag{2.4.8}$$

Note that in most cases T_Q^ω is dense in $\mathcal{H}_R^1$ and that K_Q^ω is a Hilbert space equipped with inner product $(.|.)_Q^\omega$.

THEOREM 1A. *The process $(B^{(\lambda)}_{t/\lambda^2}(f))_t$ in the mixed state ρ_Q converges weakly in the sense of matrix elements to a quantum Brownian motion over $L^2(\mathbb{R}, K_Q^\omega)$ with covariance Q. This is denoted as $\{\mathcal{H}_Q^\omega = G_B((L^2(\mathbb{R}, K_Q^\omega), Q\otimes 1), \Phi_Q^\omega, (B_Q^\omega(f,t))_t\}$.*

THEOREM 2A. *For $\phi, \phi' \in \mathcal{H}_S$, $f^{(j)}, h^{(j')} \in K_Q^\omega$ and $T^{(j)}, S^{(j')} > 0$, for $j = 1, \ldots, n$; $j' = 1, \ldots, m$, the limit as $\lambda \to 0$ of*

$$\operatorname{Tr}\left\{|\phi'\rangle\langle\phi| \otimes B^{(\omega,\lambda)\dagger}_{S^{(m)}/\lambda^2}(h^{(m)}) \cdots B^{(\omega,\lambda)\dagger}_{S^{(1)}/\lambda^2}(h^{(1)})\rho_Q B^{(\omega,\lambda)}_{T^{(1)}/\lambda^2}(f^{(1)}) \cdots B^{(\omega,\lambda)}_{T^{(n)}/\lambda^2}(f^{(n)})\, U^{(\lambda)}_{t/\lambda^2}\right\} \tag{2.4.9}$$

exists and equals

$$\langle \phi \otimes B_Q^{\omega\dagger}(f^{(1)}, T^{(1)}) \cdots B_Q^{\omega\dagger}(f^{(n)}, T^{(n)})\Phi_Q^\omega | U_t | \phi' \otimes B_Q^{\omega\dagger}(h^{(1)}, S^{(1)}) \cdots B_Q^{\omega\dagger}(h^{(m)}, S^{(m)})\Phi_Q^\omega\rangle\,, \tag{2.4.10}$$

where U_t is a unitary operator on $\mathcal{H}_S \otimes G_B(L^2(\mathbb{R}, K_Q^\omega), C)$, with $C = Q \otimes 1$, satisfying the quantum stochastic differential equation

$$dU_t = \left\{D \otimes dB_Q^{\omega\dagger}(g,t) - D^\dagger \otimes dB_Q^\omega(g,t) - (g|g)^{\omega-}_{Q+} D^\dagger D \otimes dt - \overline{(g|g)^{\omega-}_{Q-}} DD^\dagger \otimes dt\right\} U_t\,, \tag{2.4.11}$$

where

$$(g|f)^{\omega-}_{Q\pm} = \int_{-\infty}^0 d\tau \langle g, S_\tau^\omega \frac{Q \pm 1}{2} f\rangle\,. \tag{2.4.12}$$

THEOREM 3A. *Let $X \in \mathcal{B}(\mathcal{H}_S)$, then in the notation of Theorem 2*

$$\lim_{\lambda\to 0} \operatorname{Tr}\Big\{|\phi'\rangle\langle\phi| \otimes B^{(\omega,\lambda)\dagger}_{S^{(m)}/\lambda^2}(h^{(m)}) \cdots B^{(\omega,\lambda)\dagger}_{S^{(1)}/\lambda^2}(h^{(1)})\rho_Q B^{(\omega,\lambda)}_{T^{(1)}/\lambda^2}(f^{(1)}) \cdots$$
$$\cdots B^{(\omega,\lambda)}_{T^{(n)}/\lambda^2}(f^{(n)})\, u^{(\lambda)}_{t/\lambda^2}(X \otimes 1)\Big\} \tag{2.4.13}$$

exists and equals

$$\langle \phi \otimes B_Q^{\omega\dagger}(f^{(1)}, T^{(1)}) \cdots B_Q^{\omega\dagger}(f^{(n)}, T^{(n)})\Phi_Q^\omega | U_t^\dagger (X \otimes 1) U_t | \phi' \otimes B_Q^{\omega\dagger}(h^{(1)}, S^{(1)}) \cdots$$
$$\cdots B_Q^{\omega\dagger}(h^{(m)}, S^{(m)})\Phi_Q^\omega\rangle\,. \tag{2.4.14}$$

2.5. The quantum stochastic limit for the full interaction

Now suppose that the interaction is of the form (1.5.1). The problem of dropping the rotating wave approximation was first tackled by Accardi and Lu in [10] for an interaction similar to (1.5.1), except that all the test functions were taken to be the same. The result is that for each Bohr frequency ω we obtain a separate independent quantum Brownian motion.

First of all, note that $\Gamma_B(\bigoplus_{\omega\in F} L^2(\mathbb{R}, K_Q^\omega)) = \bigotimes_{\omega\in F} \Gamma_G(L^2(\mathbb{R}, K_Q^\omega))$. Then consider the Weyl algebra $W(\bigoplus_{\omega\in F} L^2(\mathbb{R}, K_Q^\omega)) = \bigotimes_{\omega\in F} W(L^2(\mathbb{R}, K_Q^\omega))$ with quasi-free state $\varphi_{\tilde{C}}$ and covariance $\tilde{C} = \bigotimes_{\omega\in F} C$, where $C = Q \otimes 1$ on $\mathcal{H}_{K_Q^\omega} \cong K_Q^\omega \otimes L^2(\mathbb{R})$.

The GNS triple over $\{W(\bigoplus_{\omega\in F} L^2(\mathbb{R}, K_Q^\omega)), \varphi_{\tilde{C}}\}$ is $\{\mathcal{H}_Q^\omega = G_B(\bigoplus_{\omega\in F} L^2(\mathbb{R}, K_Q^\omega)), \tilde{C})), \pi_Q^F, \Phi_Q^\omega\}$. Now observe that

$$\mathcal{H}_Q^F = \bigotimes_{\omega\in F} \mathcal{H}_Q^\omega\,, \qquad \pi_Q^F = \bigotimes_{\omega\in F} \pi_Q^\omega\,, \qquad \Phi_Q^F = \bigotimes_{\omega\in F} \Phi_Q^\omega\,. \tag{2.5.1}$$

For each $f \in K_Q^\omega$ we have

$$B_Q^\omega(f,t) = \pi_Q^\omega A_Q^\omega(f \otimes \chi_{[0,t]})\,, \tag{2.5.2}$$

where A_Q^ω is an annihilation operator on $L^2(\mathbb{R}, K_Q^\omega)$, so for $t_\omega > 0, f_\omega \in K_Q^\omega$, for each $\omega \in F$, we have

$$B_Q^F(\bigotimes_{\omega\in F} f_\omega, (t_\omega)_{\omega\in F}) = \bigotimes_{\omega\in F} B_Q^\omega(f_\omega, t_\omega)\,. \tag{2.5.3}$$

THEOREM 1B. *For each $\omega \in F$ and $f \in K_Q^\omega$ the limit $\lambda \to 0$, $B_{t/\lambda^2}^{(\omega,\lambda)}(f)$ taken in the state ρ_Q converges in the sense of matrix elements to a quantum Brownian motion over $L^2(\mathbb{R}, K_Q^\omega)$ with covariance Q and each of these limiting processes are independent for different values of ω.*

THEOREM 2B. *Let $f_\omega^{(j)}, h_\omega^{(j')} \in K_Q^\omega; T_\omega^{(j)}, S_\omega^{(j')} > 0$ for each $\omega \in F$ $\quad j = 1, \ldots, n_\omega; j' = 1, \ldots, m_\omega;$ and $t \geq 0$; $\phi, \phi' \in \mathcal{H}_S$ then the limit as $\lambda \to 0$ of the matrix element*

$$\mathrm{Tr}\Big\{|\phi'\rangle\langle\phi| \otimes [\bigotimes_{\omega\in F} B^{(\omega,\lambda)}_{S_\omega^{(m_\omega)}/\lambda^2}(h_\omega^{(m_\omega)}) \cdots B^{(\omega,\lambda)}_{S_\omega^{(1)}/\lambda^2}(h_\omega^{(1)})]^\dagger \rho_Q [\bigotimes_{\omega\in F} B^{(\omega,\lambda)}_{T_\omega^{(1)}/\lambda^2}(f_\omega^{(1)}) \cdots$$
$$\cdots B^{(\omega,\lambda)}_{T_\omega^{(n_\omega)}/\lambda^2}(f_\omega^{(n_\omega)})]\, U_{t/\lambda^2}^{(\lambda)}\Big\} \tag{2.5.4}$$

exists and equals

$$\langle \phi \otimes [\bigotimes_{\omega\in F} B_Q^\omega(f_\omega^{(1)}, T_\omega^{(1)}) \cdots B_Q^\omega(f_\omega^{(n_\omega)}, T_\omega^{(n_\omega)})]^\dagger \Phi_Q^F | U_t | \phi' \otimes [\bigotimes_{\omega\in F} B_Q^\omega(h_\omega^{(1)}, S_\omega^{(1)}) \cdots$$
$$\cdots B_Q^\omega(h_\omega^{(m_\omega)}, S_\omega^{(m_\omega)})]^\dagger \Phi_Q^F\rangle\,, \tag{2.5.5}$$

where U_t is unitary on $\mathcal{H}_S \otimes \mathcal{H}_Q^F$ and satisfies quantum stochastic differential equation

$$dU_t = \Big\{\sum_{\omega\in F}\sum_{j=1}^{N(\omega)} [D_j^\omega \otimes dB_\omega^\dagger(g_j^\omega, t) - D_j^{\omega\dagger} \otimes dB_\omega(g_j^\omega, t)]$$
$$- \sum_{\omega\in F}\sum_{j,k=1}^{N(\omega)} D_j^{\omega\dagger} D_k^\omega\ (g_j^\omega|g_k^\omega)_{Q+}^{\omega-}\, dt - \sum_{\omega\in F}\sum_{j,k=1}^{N(\omega)} D_j^\omega D_k^{\omega\dagger}\ \overline{(g_j^\omega|g_k^\omega)_{Q-}^{\omega-}}\, dt\Big\}\, U_t \tag{2.5.6}$$

with $U_0 = 1$. The scalar coefficients are given by

$$(f|h)_{Q\pm}^{\omega-} = \int_{-\infty}^0 dt\, \langle f, S_t^\omega(\frac{Q\pm 1}{2})h\rangle \tag{2.5.7}$$

for $f, h \in K_\omega$. The Ito table is given by

$$\begin{aligned} dB_Q^\omega(f,t) dB_Q^{\omega'\dagger}(g,t) &\equiv \delta_{\omega,\omega'}(f|g)_{Q+}^\omega dt\,,\\ dB_Q^{\omega\dagger}(g,t) dB_Q^{\omega'}(f,t) &\equiv \delta_{\omega,\omega'}(f|g)_{Q-}^\omega dt\,. \end{aligned} \tag{2.5.8}$$

A more explicit form of the coefficients in (2.5.7) *is*

$$\begin{aligned}(f|g)^{\omega}_{Q+} &= \textstyle\int_{-\infty}^{\infty} dt\langle f, S^{\omega}_t(\frac{Q+1}{2})g\rangle = (f|g)^{\omega-}_{Q+} + \overline{(g|f)^{\omega-}_{Q+}}\,,\\ (f|g)^{\omega}_{Q-} &= \textstyle\int_{-\infty}^{\infty} dt\langle f, S^{\omega}_t(\frac{Q-1}{2})g\rangle = \overline{(g|f)^{\omega-}_{Q-}} + (f|g)^{\omega-}_{Q-}\,.\end{aligned} \tag{2.5.9}$$

The idea of the proof of Theorem 2A is as follows; first of all we know that different ω give rise to independent Q-quantum Brownian motions. This is done in [10]. The next step is to consider the effect of the degeneracy which may arise for each $\omega \in F$. In this case we must, therefore, generalize the results of [8,10] accordingly. This is done in Appendices B and C.

THEOREM 3A. *In the notations of Theorem* 2A, *for any* $X \in \mathcal{B}(\mathcal{H}_S)$, *the limit as* $\lambda \to 0$ *of*

$$\begin{aligned}\mathrm{Tr}\Big\{&|\phi'\rangle\langle\phi| \otimes [\bigotimes_{\omega\in F} B^{(\omega,\lambda)}_{S^{(1)}_{\omega}/\lambda^2}(h^{(1)}_{\omega})\cdots B^{(\omega,\lambda)}_{S^{(m_\omega)}_{\omega}/\lambda^2}(h^{(m_\omega)}_{\omega})]^{\dagger}\rho_Q[\bigotimes_{\omega\in F} B^{(\omega,\lambda)}_{T^{(1)}_{\omega}/\lambda^2}(f^{(1)}_{\omega})\cdots\\ &\cdots B^{(\omega,\lambda)}_{T^{(n_\omega)}_{\omega}/\lambda^2}(f^{(n_\omega)}_{\omega})]u^{(\lambda)}_{t/\lambda^2}(X\otimes 1)\Big\}\end{aligned} \tag{2.5.10}$$

exists and equals

$$\begin{aligned}&\langle\phi \otimes [\bigotimes_{\omega\in F} B^{\omega}_{Q}(f^{(1)}_{\omega}, T^{(1)}_{\omega})\cdots B^{\omega}_{Q}(f^{(n_\omega)}_{\omega}, T^{(n_\omega)}_{\omega})]^{\dagger}\Phi^{F}_{Q}|\\ &U^{\dagger}_t(X\otimes 1)U_t|\phi' \otimes [\bigotimes_{\omega\in F} B^{\omega}_{Q}(h^{(1)}_{\omega}, S^{(1)}_{\omega})\cdots B^{\omega}_{Q}(h^{(m_\omega)}_{\omega}, S^{(m_\omega)}_{\omega})]^{\dagger}\Phi^{F}_{Q}\rangle\,,\end{aligned} \tag{2.5.11}$$

where U_t *is the solution to the quantum stochastic differential equation* (2.5.6).

2.6. The Langevin and master equations

In each of the cases (2.3.5), (2.4.11) and (2.5.6) the right hand side of the expression for dU_t contains a term of the form $-(Y\otimes 1)dtU_t$. For instance, in (2.5.6) above we have

$$Y = \sum_{\omega\in F}\sum_{j,k=1}^{N(\omega)}\left\{ D^{\omega\dagger}_j D^{\omega}_k\ (g^{\omega}_j|g^{\omega}_k)^{\omega-}_{Q+} + D^{\omega}_j D^{\omega\dagger}_k\ \overline{(g^{\omega}_j|g^{\omega}_k)^{\omega-}_{Q-}}\right\}. \tag{2.6.1}$$

The Langevin equation then reads as follows

$$d[U^{\dagger}_t(X\otimes 1)U_t] \equiv [dU_t]^{\dagger}(X\otimes 1)U_t + U^{\dagger}_t(X\otimes 1)dU_t + [dU_t]^{\dagger}(X\otimes 1)dU_t \tag{2.6.2}$$

$$\equiv U^{\dagger}_t[L_0(X)\otimes dt + \sum_{\omega\in F}\sum_{j,k=1}^{N(\omega)} L^{\omega}_{j+}(X)\otimes dB^{\omega\dagger}_{Q}(g^{\omega}_j,t) + \sum_{\omega\in F}\sum_{j,k=1}^{N(\omega)} L^{\omega}_{j-}(X)\otimes dB^{\omega}_{Q}(g^{\omega}_j,t)\}U_t\,,$$

where

$$L_0(X) = -XY - Y^\dagger X + \Theta(X) \tag{2.6.3}$$

with

$$\Theta(X) = \sum_{\omega\in F} \sum_{j,k=1}^{N(\omega)} \Big\{ D_j^{\omega\dagger} X D_k^\omega \; [(g_j^\omega|g_k^\omega)_{Q+}^{\omega-} + \overline{(g_k^\omega|g_j^\omega)}_{Q+}^{\omega-}] + D_j^\omega X D_k^{\omega\dagger} \; [\overline{(g_j^\omega|g_k^\omega)}_{Q-}^{\omega-}$$

$$+(g_k^\omega|g_j^\omega)_{Q-}^{\omega-}] \Big\} = \sum_{\omega\in F} \sum_{j,k=1}^{N(\omega)} \Big\{ D_j^{\omega\dagger} X D_k^\omega \; (g_j^\omega|g_k^\omega)_{Q+}^\omega + D_k^\omega X D_j^{\omega\dagger} \; (g_j^\omega|g_k^\omega)_{Q-}^\omega] \Big\} \tag{2.6.4}$$

and

$$L_{j+}^\omega(X) = XD_j^\omega - D_j^\omega X\,, \qquad L_{j-}^\omega(X) = D_j^{\omega\dagger} X - XD_j^{\omega\dagger}\,. \tag{2.6.5}$$

Note that unitarity follows from

$$L_0(1_S) = -(Y+Y^\dagger) + \Theta(1_S) = 0\,, \qquad L_{j\pm}^\omega(1_S) = 0\,. \tag{2.6.6}$$

It is instructive to set $Y = \frac{1}{2}\Gamma + \frac{i}{\hbar}H'_S$ where both Γ and H'_S are self-adjoint; we then have that

$$L_0(X) = -\frac{1}{2}(X\Gamma + \Gamma X) + \Theta(X) + \frac{1}{i\hbar}[X, H'_S]\,. \tag{2.6.7}$$

The unitarity condition is then 2Re $Y = \Gamma = \Theta(1_S)$; this is the fluctuation-dissipation relation of [22]. The presence of the imaginary term H'_S does not effect the unitarity. For ρ_S a density matrix on $\mathcal{H}_S$ we define the expectation $\langle . \rangle_t$ by

$$\langle X\rangle_t = \mathrm{Tr}\left\{\rho_S \otimes |\Phi_Q^F\rangle\langle\Phi_Q^F|\, U_t^\dagger (X\otimes 1) U_t\right\} = \mathrm{Tr}\{s_t X\}\,, \tag{2.6.8}$$

where s_t denotes the effective density matrix on (S) and the second trace is a partial trace over the system space (the trace over the reservoir space assumed to be taken already). Now

$$\frac{d}{dt}\langle X\rangle_t = \mathrm{Tr}\Big\{\rho_S \otimes |\Phi_Q^F\rangle\rangle\Phi_Q^F|\, \frac{d}{dt} U_t^\dagger (X\otimes 1) U_t\Big\}\,, \tag{2.6.9}$$

so in terms of the effective density matrix s_t

$$\frac{d}{dt}\mathrm{Tr}\{s_t X\} = \mathrm{Tr}\{s_t L_0(X)\} = \mathrm{Tr}\{L_0^*(s_t) X\}\,, \tag{2.6.10}$$

where L_0^* denotes the adjoint operation to L_0 on the dual of $\mathcal{B}(\mathcal{H}_S)$. This gives the master equation

$$\begin{aligned} \frac{ds_t}{dt} \;&=\; L_0^*(s_t) = -\sum_{\omega\in F}\sum_{j,k=1}^{N(\omega)} \Big\{ [D_j^\omega D_k^{\omega\dagger} s_t - D_k^{\omega\dagger} s_t D_j^\omega]\overline{(g_j^\omega|g_k^\omega)}_{Q-}^{\omega-} \\ &+ [D_j^{\omega\dagger} D_k^\omega s_t - D_k^\omega s_t D_j^{\omega\dagger}](g_j^\omega|g_k^\omega)_{Q+}^{\omega-} - [D_j^{\omega\dagger} s_t D_k^\omega - s_t D_k^\omega D_j^{\omega\dagger}](g_j^\omega|g_k^\omega)_{Q-}^{\omega-} \\ &- [D_j^\omega s_t D_k^{\omega\dagger} - s_t D_k^{\omega\dagger} D_j^\omega]\overline{(g_j^\omega|g_k^\omega)}_{Q+}^{\omega-} \Big\}\,. \end{aligned} \tag{2.6.11}$$

From the relation (1.5.2) we see that $[Y, H_S] = 0$. If we define the *effective evolution operator* by

$$V_t = (e^{\frac{t}{i\hbar}H_S} \otimes 1)U_t \tag{2.6.12}$$

which satisfies the quantum stochastic differential equation

$$dV_t \equiv \left(\frac{1}{i\hbar}H_S e^{\frac{t}{i\hbar}H_S} \otimes dt\right) U_t + (e^{\frac{t}{i\hbar}H_S} \otimes 1)dU_t\,. \tag{2.6.13}$$

Explicitly, this gives

$$\begin{aligned} dV_t &\equiv (e^{\frac{t}{i\hbar}H_S} \otimes 1)\Bigg[\sum_{\omega\in F}\sum_{\phi,\phi'\in B}^{(\omega_{\phi\phi'}=\omega)} \left\{T_{\phi\phi'} \otimes dB_Q^{\omega\dagger}(g_{\phi\phi'},t) - T^\dagger_{\phi\phi'} \otimes dB_Q^{\omega}(g_{\phi\phi'},t)\right\} \\ &\quad - \left\{Y + \frac{i}{\hbar}H_S\right\}dt\Bigg]U_t 1 \equiv \Bigg[\sum_{\omega\in F}\sum_{\phi,\phi'\in B}^{(\omega_{\phi\phi'}=\omega)} \Big\{T_{\phi\phi'} \otimes dB_Q^{\omega\dagger}(g_{\phi\phi'},t)e^{i\omega t} \\ &\quad - T^\dagger_{\phi\phi'} \otimes dB_Q^{\omega}(g_{\phi\phi'},t)e^{-i\omega t}\Big\} - \left\{\frac{1}{2}\Gamma + \frac{i}{\hbar}(H_S + H'_S)\right\}dt\Bigg] V_t\,, \end{aligned} \tag{2.6.14}$$

$$\langle\phi, Y\phi\rangle = \sum_{\omega\in F}\sum_{j,k=1}^{N(\omega)}\left\{\langle\phi, D_j^{\omega\dagger}D_k^\omega\phi\rangle\,(g_j^\omega|g_k^\omega)_{Q+}^{\omega-} + \langle\phi, D_j^\omega D_k^{\omega\dagger}\phi\rangle\,\overline{(g_j^\omega|g_k^\omega)_{Q-}^{\omega-}}\right\}. \tag{2.6.15}$$

This is, however, equivalent to the (complex) shift one calculates using second order perturbation theory. For example, taking the zero temperature for simplicity, one calculates in second order shift [19]

$$Y_\phi^{(2)} = \frac{1}{i\hbar}\left\langle \phi\otimes\Psi_R, H_I\frac{1}{H^{(0)} - E_\phi - i0^+}H_I \quad \phi\otimes\Psi_R\right\rangle \tag{2.6.16}$$

$$= \sum_{\omega,\omega'\in F}\sum_{j}^{N(\omega)}\sum_{j'}^{N(\omega')}\int_{-\infty}^0 d\tau\,\langle\phi\otimes\Psi_R, D_j^\omega\otimes A(g_j^\omega)\,e^{i(H^{(0)}-E_\phi)\tau/\hbar}\,D_{j'}^{\omega'}\otimes A^\dagger(g_{j'}^{\omega'})\,\phi\otimes\Psi_R\rangle\,.$$

Here we have used the well known identity

$$\int_{-\infty}^0 dt\; e^{ixt} = \frac{1}{i(x - i0^+)} = \pi\delta(x) - i\wp\left(\frac{1}{x}\right), \qquad (x\in\mathbb{R})\,, \tag{2.6.17}$$

where $\wp$ means that we take the principal part of the integral. Now $D_j^\omega\phi$ is an eigenstate of H_S with eigenvalue $E_\phi - \hbar\omega$, so the summation need only be considered over $\omega = \omega'$ in (2.6) above. Therefore, we have

$$Y_\phi^{(2)} = \sum_{\omega\in F}\sum_{j,k}^{N(\omega)}\int_{-\infty}^0 d\tau\,\langle\phi\otimes\Psi_R, D_j^{\omega\dagger}\otimes A(g_j^\omega)\,e^{i(H_R-\hbar\omega)\tau/\hbar}\,D_k^\omega\otimes A^\dagger(g_k^\omega)\,\phi\otimes\Psi_R\rangle$$

$$= \sum_{\omega \in F} \sum_{j,k}^{N(\omega)} \langle \phi, D_j^{\omega\dagger} D_k^{\omega} \phi \rangle \int_{-\infty}^{0} d\tau \, \langle \Psi_R, A(g_j^{\omega}) A^{\dagger}(S_\tau^{\omega} g_k^{\omega}) \Psi_R \rangle \,. \tag{2.6.18}$$

Hence $Y_\phi^{(2)} = \langle \phi, Y\phi \rangle$. The real and imaginary parts of Y are therefore the linewidth and energy shift as would normally be calculated using second order perturbation theory; this is true in the non-vacuum cases also.

2.7. Transition probabilities

Let $P_t(\psi|\phi)$ denote the probability that the sytem will be measured in state ψ at time t if it initially was in state ϕ. Then

$$p_t(\psi|\phi) = \langle \phi \otimes \Phi_Q^F, U_t^{\dagger}(|\psi\rangle\langle\psi| \otimes 1) U_t \, \phi \otimes \Phi_Q^F \rangle \,. \tag{2.7.1}$$

From Theorems 3 and 3A we have

$$\frac{d}{dt} p_t(\psi|\phi) = \langle \phi \otimes \Phi_Q^F, U_t^{\dagger} L_0(|\psi\rangle\langle\psi|) \otimes 1 U_t \, \phi \otimes \Phi_Q^F \rangle \,. \tag{2.7.2}$$

Therefore, setting $t = 0$,

$$\frac{d}{dt} p_t(\psi|\phi)|_{t=0} = \langle \phi, L_0(|\psi\rangle\langle\psi|) \, \phi \rangle \,. \tag{2.7.3}$$

If $\psi = \phi$ we obtain the relation

$$\frac{d}{dt} p_t(\phi|\phi)|_{t=0} = -\langle \phi, \Gamma \phi \rangle \,, \tag{2.7.4}$$

while if $\langle \phi, \psi \rangle = 0$ then (2.7.3) gives

$$\frac{d}{dt} p_t(\psi|\phi)|_{t=0} = \langle \phi, \Theta(|\psi\rangle\langle\psi|) \, \phi \rangle \tag{2.7.5}$$

$$= \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \left\{ \langle \phi, D_j^{\omega\dagger} \psi \rangle \langle \psi, D_k^{\omega} \phi \rangle (g_j^{\omega} | g_k^{\omega})_{Q+}^{\omega} + \langle \phi, D_k^{\omega} \psi \rangle \langle \psi, D_j^{\omega\dagger} \phi \rangle (g_j^{\omega} | g_k^{\omega})_{Q-}^{\omega} \right\} .$$

Using the relation $\int_{-\infty}^{\infty} dt \, e^{ixt} = 2\pi\delta(x)$ we can rewrite this as

$$\frac{d}{dt} p_t(\phi|\phi)|_{t=0} = 2\pi \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \int dk \, \overline{g_j^{\omega}}(\mathbf{k}) g_k^{\omega}(\mathbf{k}) \, \delta(\omega(k) - \omega)$$

$$\left\{ \langle \phi, D_j^{\omega\dagger} \psi \rangle \langle \psi, D_k^{\omega} \phi \rangle \frac{q(\mathbf{k}) + 1}{2} + \langle \phi, D_k^{\omega} \psi \rangle \langle \psi, D_j^{\omega\dagger} \phi \rangle \frac{q(\mathbf{k}) - 1}{2} \right\} , \tag{2.7.6}$$

where $Q(\mathbf{k})$ is the spectral function associated with Q, cf. (3.11). This is our formulation of the *Fermi golden rule* for transitions of the systems state and it corresponds to the usual expressions, cf. formulae (1.21.27a,b) of [2].

3. The weak coupling limit in QED

As an illustration of our theory we consider the case of quantum electrodynamics. We stress however, that the theory encompasses a wide range of physical phenomena. For instance, some qualitative new results concerning exciton models in solid state physics, such as phonon models or the Frölich [15,16] polaron model, were recently obtained on the basis of a natural generalization of the following treatment. The electromagnetic field acts as a reservoir for our system (S) which we take to consist of a single electron. The electromagnetic field can be derived from the potential $\mathbf{A}$ given by

$$\mathbf{A}(\mathbf{r}) = \sum_{\sigma=1,2} \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\epsilon_o c|\mathbf{k}|}} \left\{a_\sigma^\dagger(\mathbf{k})e^{-i\mathbf{k}.\mathbf{r}} + a_\sigma(\mathbf{k})e^{i\mathbf{k}.\mathbf{r}}\right\} \epsilon^\sigma(\hat{\mathbf{k}})\,. \tag{3.1}$$

Here we consider two transverse polarizations ($\sigma = 1, 2$) for each mode $\mathbf{k}$. In our notation $\{\epsilon^1(\hat{\mathbf{k}}), \epsilon^2(\hat{\mathbf{k}}), \hat{\mathbf{k}} = |\mathbf{k}|^{-1}\mathbf{k}\}$ form a right-handed triad for each $\mathbf{k}$. This ensures that we are working with the radiation gauge $\nabla \cdot \mathbf{A} = 0$. The operators $a_\sigma^\sharp(\mathbf{k})$ on the reservoir state space $\mathcal{H}_R$ satisfy Bose commutation relations,

$$[a_\sigma(\mathbf{k}), a_{\sigma'}^\dagger(\mathbf{k}')] = \delta_{\sigma\sigma'}\delta(\mathbf{k} - \mathbf{k}')\,. \tag{3.2}$$

The total Hamiltonian for the system and reservoir is

$$H = \frac{1}{2m}|\mathbf{p} - e\mathbf{A}|^2 + \Phi(\mathbf{r}) + H_R = H_S + H_R + H_I + H'_I\,, \tag{3.3}$$

where the unperturbed system Hamiltonian (with potential $\Phi(\mathbf{r})$) is

$$H_S = \frac{1}{2m}|\mathbf{p}|^2 + \Phi(\mathbf{r})\,, \tag{3.4}$$

$$H_R = \sum_{\sigma=1,2} \int d^3k\, \hbar c|k|\; a_\sigma^\dagger(\mathbf{k})a_\sigma(\mathbf{k})\,, \tag{3.5}$$

$$H_I = -\frac{e}{m} \sum_{\sigma=1,2} \int d^3k \left\{a_\sigma^\dagger(\mathbf{k})e^{-i\mathbf{k}.\mathbf{r}} + a_\sigma(\mathbf{k})e^{i\mathbf{k}.\mathbf{r}}\right\} \mathbf{G}^\sigma(\mathbf{k}).\mathbf{p}\,, \tag{3.6}$$

with

$$\mathbf{G}^\sigma(\hat{\mathbf{k}}) = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\epsilon_o c|\mathbf{k}|}}\epsilon^\sigma(\hat{\mathbf{k}})$$

and

$$H'_I = \frac{e^2}{2m}|\mathbf{A}|^2\,.$$

If we rescale the electronic charge as $e \hookrightarrow \lambda e$, we find that

$$H \hookrightarrow H_S + H_R + \lambda H_I + \lambda^2 H'_I\,.$$

In the subsequent analysis we drop the term $\lambda^2 H'_I$ and consider only

$$H^{(\lambda)} = H_S + H_R + \lambda H_I \,. \tag{3.7}$$

It has been established rigorously that this does not affect the final result in the weak coupling limit. Now the interaction H_I given by (3.6) has response terms described by the vectors $\theta_j^\sigma(\mathbf{k}) = \frac{ie}{\hbar m} e^{-i\mathbf{k}.\mathbf{r}} G_j^\sigma(\hat{\mathbf{k}}) \cdot \mathbf{p}$. We assume, as usual, that the unperturbed system Hamiltonian H_S has a complete orthonormal set of eigenstates B. In the case of the hydrogen atom, this means that we consider only the bound states and ignore the effect of the ionized states. In general, $\mathcal{H}_S$ can be decomposed into complementary subspaces generated by the discrete, the absolutely continuous and the singular parts of the spectrum of H_S. It is enough to prepare the system in the discrete spectrum subspace to apply our results. It is the standard approach in atomic physics to study only the behaviour of bound states anyway, so we are justified in this restriction. We introduce the test-functions

$$g^\sigma_{\phi\phi'}(\mathbf{k}) = \frac{ie}{\hbar m} \langle \phi | e^{-i\mathbf{k}.\mathbf{r}} \mathbf{p} | \phi' \rangle \mathbf{G}^\sigma(\hat{\mathbf{k}}) \,. \tag{3.8}$$

The interaction H_I can be expressed as

$$\begin{aligned} H_I &= \sum_{\phi,\phi' \in B} \sum_{\sigma=1,2} \int d^3k \, \{ a_\sigma^\dagger(\mathbf{k}) g^\sigma_{\phi\phi'}(\mathbf{k}) - a_\sigma(\mathbf{k}) \overline{g}^\sigma_{\phi\phi'}(\mathbf{k}) \} \otimes T_{\phi\phi'} \\ &= \sum_{\phi,\phi' \in B} \sum_{\sigma=1,2} \int d^3k \, \{ T_{\phi\phi'} \otimes a_\sigma^\dagger(\mathbf{k}) g^\sigma_{\phi\phi'}(\mathbf{k}) - T_{\phi'\phi} \otimes a_\sigma(\mathbf{k}) \overline{g}^\sigma_{\phi\phi'}(\mathbf{k}) \} \\ &= \sum_{\phi,\phi' \in B} \{ T_{\phi\phi'} \otimes A^\dagger(g_{\phi\phi'}) - T^\dagger_{\phi\phi'} \otimes A(g_{\phi\phi'}) \} \,, \end{aligned} \tag{3.9}$$

where $g_{\phi\phi'} = g^1_{\phi\phi'} \oplus g^2_{\phi\phi'} \in L^2(R^3) \oplus L^2(R^3) = \mathcal{H}^1_R$ and $A^\sharp$ are the creation/annihilation operators on $\Gamma_B(\mathcal{H}^1_R) = \otimes \Gamma_B(L^2(\mathbb{R}))$:

$$\begin{aligned} A^\dagger(f^1 \oplus f^2) &= \textstyle\sum_{\sigma=1,2} \int d^3k f^\sigma(\mathbf{k}) a_\sigma^\dagger(\mathbf{k}) \,, \\ A(f^1 \oplus f^2) &= \textstyle\sum_{\sigma=1,2} \int d^3k \overline{f^\sigma}(\mathbf{k}) a_\sigma(\mathbf{k}) \,. \end{aligned} \tag{3.10}$$

Our choice of $\mathcal{H}^1_R$ above for one particle of the reservoir space is quite natural; namely, it consists of wave-functions in the momentum representation with two transverse polarizations. The state of the reservoir is in our case determined by the covariance operator Q which we shall now specify as that of a thermal field at inverse temperature $\beta > 0$, that is

$$Q : \mathcal{H}^1_R \mapsto \mathcal{H}^1_R : h_1 \oplus h_2 \mapsto \tilde{h}_1 \oplus \tilde{h}_2 \,, \qquad \text{with} \qquad \tilde{h}_\sigma(\mathbf{k}) = q(c|\mathbf{k}|) h_\sigma(\mathbf{k}) \,, \tag{3.11}$$

where $q(\omega) = \coth \frac{\beta\hbar\omega}{2}$. With $\omega_{\phi\phi'} = (E_{\phi'} - E_\phi)/\hbar \in F$, we define

$$S_t^{\omega_{\phi\phi'}} : \mathcal{H}^1_R \mapsto \mathcal{H}^1_R : h_1 \oplus h_2 \mapsto \tilde{h}_1 \oplus \tilde{h}_2$$

with

$$\tilde{h}_\sigma(\mathbf{k}) = e^{i(c|\mathbf{k}|-\omega_{\phi\phi'})t} h_\sigma(\mathbf{k}) . \tag{3.12}$$

In this setup we have allowed for the most general coupling, that is, where all the fundamental frequencies $F = \{\omega_{\phi\phi'} : \phi, \phi' \in B\}$ are to be considered. This set is always degenerate in general; however it is important to consider two classes of degeneracy arising. The first is the *secular class*; these are the situations in which degeneracies always arise regardless of the spectrum $\{E_\phi : \phi \in B\}$ of H_S; they are the pairs (ϕ, ϕ') and (ψ, ψ') which have $\omega_{\phi\phi'} = \omega_{\psi\psi'}$ due to one of the following reasons

$$\begin{array}{ll} 1. & \phi = \phi' = \psi = \psi' , \\ 2. & \phi = \phi' , \quad \psi = \psi'; \quad (\phi \neq \psi) , \\ 3. & \phi = \psi , \quad \phi' = \psi'; \quad (\phi \neq \phi') . \end{array} \tag{3.13}$$

Any solution to the equation $\omega_{\phi\phi'} = \omega_{\psi\psi'}$, or equivalently $E_\phi - E_{\phi'} = E_\psi - E_{\psi'}$, not of the secular type will be called an *extraneous solution.* The extraneous solutions are, of course, dependent on the spectrum of H_S. It is a standard procedure in physical literature to assume that such possibilities do not arise, however this is a requirement on H_S, which cannot be satisfied in many important examples. For a particle in a rectangular box, apart from the natural degeneracies arising if the ratios of the sides are rational, we also have to consider the fact that the contribution to the energy for the mode of vibration n_i along the i^{th}-axis is proportional to n_i^2. This means solving the Diophantine equations for the harmonics

$$n_i^2 - m_i^2 = n'^2_i - m'^2_i .$$

For the hydrogen atom we have to consider, apart from the spherical harmonical degeneracies, the integer solutions to the Diophantine equations

$$\frac{1}{n^2} - \frac{1}{m^2} = \frac{1}{n'^2} - \frac{1}{m'^2}$$

for the principal atomic numbers. After simple manipulations this leads to the study of the intersection of the algebraic projective curve in $\mathbb{R}^4$;

$$x_1^2 x_3^2 x_4^2 - x_2^2 x_3^2 x_4^2 - x_1^2 x_2^2 x_4^2 + x_1^2 x_2^2 x_3^2 = 0$$

with the lattice of positive integers.

In the weak coupling limit we obtain the quantum stochastic differential equation

$$dU_t = \left[\sum_{\omega \in F} \sum_{\phi,\phi' \in B}^{(\omega_{\phi\phi'}=\omega)} \{T_{\phi\phi'} \otimes dB_Q^{\omega\dagger}(g_{\phi\phi'}, t) + T^\dagger_{\phi\phi'} \otimes dB_Q^{\omega}(g_{\phi\phi'}, t)\} + Y dt \right] U_t ,$$

where

$$Y = \sum_{\omega \in F} \sum_{\phi,\phi',\psi,\psi' \in B}^{(\omega_{\phi\phi'}=\omega=\omega_{\psi\psi'})} \left[T^\dagger_{\phi\phi'} T_{\psi\psi'} (g_{\phi\phi'} | g_{\psi\psi'})^{\omega -}_{Q+} + T_{\phi\phi'} T^\dagger_{\psi\psi'} \overline{(g_{\phi\phi'} | g_{\psi\psi'})^{\omega -}_{Q-}} \right] \tag{3.14}$$

with $U_0 = 1$. However, using the fact that $T^{\dagger}_{\phi\phi'} T_{\psi\psi'} = \langle\phi, \psi\rangle T_{\phi'\psi'}$ etc., we may write Y as

$$Y = \sum_{\omega\in F} \sum_{\phi,\psi,\phi'\in B}^{(\omega=\omega_{\phi\phi'}=\omega_{\psi\phi'})} \left\{ (g_{\phi'\phi}|g_{\phi'\psi})^{(-\omega)-}_{Q+} + \overline{(g_{\phi\phi'}|g_{\psi\phi'})^{\omega-}_{Q-}} \right\} T_{\phi\psi} . \tag{3.15}$$

In the summation we consider only ϕ and ψ for which there exists a ϕ' so that $\omega_{\phi\phi'} = \omega_{\psi\phi'}$, however this is equivalent to demanding that $\omega_{\phi\psi} = 0$ as we always have the identity $\omega_{\phi\psi} = \omega_{\phi\phi'} - \omega_{\psi\phi'}$. Therefore, Y is a linear combination of terms $T_{\phi\psi}$ with $\omega_{\phi\psi} = 0$ and this, in particular, implies that Y commutes with H_S. It is natural in light of this to write Y as

$$Y = \sum_{\phi,\psi\in B}^{(\omega_{\phi\psi}=0)} y_{\phi\psi}\; T_{\phi\psi} = \sum_{\phi,\psi\in B}^{(E_\phi=E_\psi)} y_{\phi\psi}\; T_{\phi\psi} , \tag{3.16}$$

where

$$y_{\phi\psi} = \sum_{\phi'\in B} \left\{ (g_{\phi'\phi}|g_{\phi'\psi})^{\omega_{\phi'\phi}-}_{Q+} + \overline{(g_{\phi\phi'}|g_{\psi\phi'})^{\omega_{\phi\phi'}-}_{Q-}} \right\} . \tag{3.17}$$

According to the general rule, the generator of the master equation associated with equation (3.14) is determined by the drift term of this equation according to the rule

$$\frac{ds_t}{dt} = L_0^*(s_t) = -(Y s_t + s_t Y^{\dagger})$$

$$+ \sum_{\omega\in F} \sum_{\phi,\phi',\psi,\psi'\in B}^{(\omega=\omega_{\phi\phi'}=\omega_{\psi\psi'})} [T^{\dagger}_{\phi\phi'} s_t T_{\psi\psi'} (g_{\phi\phi'}|g_{\psi\psi'})^{\omega}_{Q+} + T_{\phi\phi'} s_t T^{\dagger}_{\psi\psi'} (g_{\psi\psi'}|g_{\phi\phi'})^{\omega}_{Q-}] . \tag{3.18}$$

In order to find the general expression for H'_S, we return to equation (3.19). Now

$$y_{\phi\psi} = \sum_{\phi'\in B} \int_{-\infty}^{0} d\tau \left\{ \langle g_{\phi'\phi}, S_\tau^{\omega_{\phi'\phi}} \frac{Q+1}{2} g_{\phi'\psi}\rangle + \overline{\langle g_{\phi\phi'}, S_\tau^{\omega_{\phi\phi'}} \frac{Q-1}{2} g_{\psi\phi'}\rangle} \right\}$$

$$= \sum_{\phi'\in B} \int_{-\infty}^{0} d\tau \sum_{\sigma=1,2} \int d^3k \left\{ \overline{g^{\sigma}_{\phi'\phi}}(\mathbf{k}) g^{\sigma}_{\phi'\psi}(\mathbf{k}) e^{-ic|(\mathbf{k}|t} \frac{q(c|\mathbf{k}|)+1}{2} \right.$$

$$\left. + g^{\sigma}_{\phi\phi'}(\mathbf{k}) \overline{g^{\sigma}_{\psi\phi'}}(\mathbf{k}) e^{ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|)-1}{2} \right\} e^{i\omega_{\phi'\phi}t} .$$

But using $\overline{g^{\sigma}_{\phi\phi'}}(\mathbf{k}) = -g^{\sigma}_{\phi'\phi}(-\mathbf{k})$ we have

$$y_{\phi\psi} = \sum_{\phi'\in B} \int_{-\infty}^{0} d\tau \sum_{\sigma=1,2} \int d^3k\, g^{\sigma}_{\phi\phi'}(\mathbf{k}) \overline{g^{\sigma}_{\psi\phi'}}(\mathbf{k}) \times$$

$$\times \left\{ e^{-ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|)+1}{2} + e^{ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|)-1}{2} \right\} e^{i\omega_{\phi'\phi}t}$$

$$= \frac{e^2}{\hbar^2 m^2} \sum_{\phi' \in B} \int_{-\infty}^{0} d\tau \sum_{\sigma=1,2} \int d^3k \sum_{j,j'=1,2,3} \langle \phi | e^{-i\mathbf{k}\cdot\mathbf{r}} p_j | \phi' \rangle \langle \phi' | e^{i\mathbf{k}\cdot\mathbf{r}} p_{j'} | \psi \rangle G_j^{\sigma}(\mathbf{k}) G_{j'}^{\sigma}(\mathbf{k}) \times$$

$$\times \left\{ e^{-ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|)+1}{2} + e^{ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|)-1}{2} \right\} e^{i\omega_{\phi'\phi} t}$$

$$= \frac{e^2}{\hbar^2 m^2} \sum_{\phi' \in B} \int_{-\infty}^{0} d\tau \sum_{\sigma=1,2} \int d^3k \sum_{j,j'=1,2,3} \langle \phi | e^{\frac{t}{i\hbar} E_\phi} e^{-i\mathbf{k}\cdot\mathbf{r}} p_j e^{\frac{-t}{i\hbar} H_S} e^{i\mathbf{k}\cdot\mathbf{r}} p_{j'} | \psi \rangle G_j^{\sigma}(\mathbf{k}) G_{j'}^{\sigma}(\mathbf{k})$$

$$\times \left\{ e^{-ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|)+1}{2} + e^{ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|)-1}{2} \right\} . \tag{3.19}$$

We remark that the effect of the response term is as follows; from the commutation relations of $\mathbf{r}$ and $\mathbf{p}$ we have that

$$e^{-i\mathbf{k}\cdot\mathbf{r}} e^{\frac{t}{i\hbar} H_S} e^{i\mathbf{k}\cdot\mathbf{r}} = e^{-i\mathbf{k}\cdot\mathbf{r}} \exp \frac{t}{i\hbar} \left(\frac{|\mathbf{p}|^2}{2m} + V(r) \right) e^{i\mathbf{k}\cdot\mathbf{r}} = \exp \frac{t}{i\hbar} \left(\frac{|\mathbf{p}+\hbar\mathbf{k}|^2}{2m} + V(r) \right) , \tag{3.20}$$

that is, H_S is replacd by $H_S + \frac{\hbar \mathbf{k}\cdot\mathbf{p}}{m} + \frac{\hbar^2 |\mathbf{k}|^2}{2m}$. Now the $\mathbf{k}$ dependence in the above expression prevents us from using the well-known isotropic identity

$$\sum_{\sigma=1,2} \int_{|\mathbf{k}|=\omega/c} d^2\hat{k}\, G_j^{\sigma}(\mathbf{k}) G_{j'}^{\sigma}(\mathbf{k}) = \frac{1}{(2\pi)^3} \frac{\hbar}{2\epsilon_0 \omega} \frac{8\pi}{3} \delta_{j,j'} \tag{3.21}$$

to calculate $y_{\phi\psi}$ as in the dipole approximation. Note that the Lamb shift and the damping coefficients are affected by inclusion of the response terms.

The complex shift $Y_{\phi\phi} = \frac{1}{2}\Gamma_\phi + \frac{i}{\hbar} E'_\phi$, giving the linewidth Γ_ϕ and energy shift E'_ϕ for a state $\phi \in B$ can the be written as

$$Y_{\phi\phi} = \frac{e^2}{2i\hbar m^2} \sum_{\sigma=1,2} \int d^3k \sum_{j,j'=1,2,3} \langle \phi | p_j \left[\frac{q(c|\mathbf{k}|)+1}{\mathcal{D}^+(\mathbf{k}) - i0^+} + \frac{q(c|\mathbf{k}|)-1}{\mathcal{D}^-(\mathbf{k}) - i0^+} \right] p_{j'} | \phi \rangle G_j^{\sigma}(\mathbf{k}) G_{j'}^{\sigma}(\mathbf{k}) \tag{3.22}$$

where the denominators in the above expression are

$$\mathcal{D}^{\pm}(\mathbf{k}) = H_S + \frac{\hbar \mathbf{k}\cdot\mathbf{p}}{m} + \frac{\hbar^2 |\mathbf{k}|^2}{2m} \pm \hbar c|\mathbf{k}| - E_\phi . \tag{3.23}$$

This expression has been derived several times in the zero temperature case, cf. [20], but for the nonzero temperature case there seems not to be universal agreement, see e.g. [21]: the result coming from the present theory seems to be free from any ambiguity.

Appendices

A. The traditional derivation of the master equation

For sake of comparison we give the standard arguments used in the derivation of the master equation. This section follows closely the development of Louisell [2]. The interaction is taken to be of the form

$$H_I = \sum_j D_j \otimes F_j\,, \tag{A.1}$$

where D_j and F_j act nontrivially on the system and reservoir spaces, respectively. We assume that D_j evolves harmonically in time under the free evolution with frequency ω_j. We assume that at time $t = 0$ the system and reservoir are uncoupled, that is the density operator $\rho(t)$ at time zero factors as

$$\rho(0) = \rho_0^{(S)} \otimes \rho^{(R)}\,. \tag{A.2}$$

No subscript is required for $\rho^{(R)}$ as we assume that it is invariant under the free-evolution. In particular this is true for the choice of a thermal state $\rho^{(R)} = e^{-\beta(H_R^1-\mu)}/\mathrm{Tr}e^{-\beta(H_R^1-\mu)}$. We define the reduced system state at time t in the interaction dynamics to be the density operator

$$s_t = \mathrm{Tr}_{\mathcal{H}_R}\left\{U_t^{(\lambda)}(\rho_0^{(S)} \otimes \rho^{(R)})U_t^{(\lambda)\dagger}\right\}\,. \tag{A.3}$$

The iterated series expansion of s_t, truncated to second order, is

$$\begin{aligned} s_t &= s_0 + \frac{1}{i\hbar}\int_0^t dt_1 \mathrm{Tr}_{\mathcal{H}_R}[v_{t_1}^{(0)}(H_I), s_0 \otimes \rho^{(R)}] \\ &\quad + \frac{1}{(i\hbar)^2}\int_0^t dt_1 \int_0^{t_1} dt_2 \mathrm{Tr}_{\mathcal{H}_R}\left[v_{t_1}^{(0)}(H_I), [v_{t_2}^{(0)}(H_I), s_0 \otimes \rho^{(R)}]\right]\,, \end{aligned} \tag{A.4}$$

where we have set $\lambda = 1$. Substituting in for the potential H_I we find

$$\begin{aligned} s_t &= s_0 + \sum_j \int_0^t dt_1 \langle v_{t_1}^{(0)}(F_j)\rangle_R e^{-i\omega_j t_1}[D_j, s_0] \\ &\quad + \sum_{j,k}\int_0^t dt_1 \int_0^{t_1} dt_2\, e^{-i(\omega_j t_1 + \omega_k t_2)}\Big\{[D_j D_k s_0 - D_k s_0 D_j]\langle v_{t_1}^{(0)}(F_j) v_{t_2}^{(0)}(F_k)\rangle_R \\ &\quad - [D_j s_0 D_k - s_0 D_k D_j]\langle v_{t_2}^{(0)}(F_k) v_{t_1}^{(0)}(F_j)\rangle_R\Big\}\,, \end{aligned} \tag{A.5}$$

where $\langle.\rangle_R = \mathrm{Tr}_{\mathcal{H}_R}[\rho^{(R)}.]$. Due to the invariance of the reservoir fields under the free evolution we have that

$$\langle v_t^{(0)}(F_j)\rangle_R = \langle F_j\rangle_R\,,$$
$$\langle v_t^{(0)}(F_j) v_s^{(0)}(F_k)\rangle_R = \langle v_{t-s}^{(0)}(F_j) F_k\rangle_R\,.$$

Therefore, if we let $\tau = t_1 - t_2, y = t_2$ then (A.5) can be written as

$$\begin{aligned} s_t &= s_0 + \sum_j \langle F_j\rangle_R [D_j, s_0]\int_0^t e^{-i\omega_j y} dy \\ &\quad + \sum_{j,k}\int_0^t dy\, e^{-i(\omega_j+\omega_k)y}\int_0^{t-y} d\tau\, e^{-i\omega_j\tau}\Big\{[D_j D_k s_0 - D_k s_0 D_j]\langle v_\tau^{(0)}(F_j) F_k\rangle_R \\ &\quad - [D_j s_0 D_k - s_0 D_k D_j]\langle F_k v_\tau^{(0)}(F_j)\rangle_R\Big\}\,. \end{aligned}$$

The approximation procedure is based on the following four steps:

Step I. One postulates that the contributions coming from the sum of all terms higher that second order in the iterated series are negligible.

Step II. One postulates a finite autocorrelation time τ_c such that

$$\langle v_\tau^{(0)}(F_j)F_k\rangle_R = 0 = \langle F_j v_\tau^{(0)}(F_k)\rangle_R$$

whenever $|\tau| > \tau_c$. Thus for $t \gg \tau_c$ one may replace the upper limit of the τ-integral by $+\infty$. This gives

$$\begin{aligned} s_t &= s_0 + \sum_j \langle F_j\rangle_R [D_j, s_0] I^t(\omega_j) \\ &\quad + \sum_{j,k} \left\{ [D_j D_k s_0 - D_k s_0 D_j] w_{j,k}^+ - [D_j s_0 D_k - s_0 D_k D_j] w_{k,j}^- \right\} I^t(\omega_j + \omega_k)\,, \end{aligned}$$

where

$$\begin{aligned} w_{j,k}^+ &= \textstyle\int_0^\infty e^{-i\omega_j\tau} \langle v_\tau^{(0)}(F_j)F_k\rangle_R d\tau\,, \\ w_{k,j}^- &= \textstyle\int_0^\infty e^{-i\omega_j\tau} \langle F_k v_\tau^{(0)}(F_j)\rangle_R d\tau \end{aligned} \tag{A.6}$$

and

$$I^t(\omega) = \int_0^t e^{-i\omega y} dy\,.$$

Step III. For t large with respect to τ_c one makes the replacement

$$I^t(\omega) \hookrightarrow t\delta(\omega)\,.$$

Step IV. One postulates that the formulae deduced under the previous assumptions, when t is large with respect to τ_c, hold also in the limit $t \to 0$. This gives

$$\begin{aligned} \frac{ds}{dt}\Big|_0 = \lim_{t\to 0} \frac{s_t - s_0}{t} &= \sum_j^{\omega_j = 0} [D_j, s_0] \langle F_j\rangle_R \\ &+ \sum_{j,k}^{\omega_j+\omega_k=0} \left\{ [D_j D_k s_0 - D_k s_0 D_j] w_{j,k}^+ - [D_j s_0 D_k - s_0 D_k D_j] w_{k,j}^- \right\}\,. \end{aligned} \tag{A.7}$$

The assumptions leading to the derivation of (A.7) have a decidedly *ad hoc* nature, especially those introduced in steps III and IV. The replacement for I^t, put in by hand, in step III is precisely what is needed to allow the limit to be taken easily.

It is instructive to calculate explicitly the master equation (A.7) in a particular case. We consider as reservoir a free Bose gas at inverse temperature β and fugacity $z = e^{\beta\mu}$. This can be described by the quasi-free state φ_Q, on $L^2(\mathbb{R}^n)$ for example, characterized by

$$\langle A^\dagger(f)A(g)\rangle_R \equiv \varphi_Q(A^\dagger(f)A(g)) = \langle f, \frac{Q-1}{2} g\rangle\,, \tag{A.8}$$

where

$$Q = \frac{1 + ze^{-\beta H_R^1}}{1 - ze^{-\beta H_R^1}} = \coth \frac{\beta}{2}(H_R^1 - \mu)\,.$$

We may take H_R^1 to be for instance $-\Delta$.

We may write the interaction H_I of (1.5.8) in the form (A.1) with the notations

$$H_I = i\hbar \sum_j \{D_j \otimes A^\dagger(g_j) - h.c.\} \equiv i\hbar \sum_{(j,\alpha)} D_{(j,\alpha)} \otimes F_{(j,\alpha)}\,, \tag{A.9}$$

where we have a summation also over an index $\alpha \in \{0,1\}$ with the notations

$$D_{(j,0)} = D_j\,, \qquad D_{(j,1)} = -D_j^\dagger\,, \qquad F_{(j,0)} = A^\dagger(g_j)\,, \qquad F_{(j,1)} = A(g_j) \tag{A.10}$$

and consequently

$$\omega_{(j,0)} = \omega_j\,, \qquad \omega_{(j,1)} = -\omega_j\,. \tag{A.11}$$

We then have

$$\begin{aligned} w^+_{(j0),(k1)} &= \int_0^\infty d\tau\, e^{-i\omega_j\tau} \langle v_\tau^{(0)}(A^\dagger(g_j))A(g_k)\rangle_R = \int_0^\infty d\tau\, \varphi_Q(A^\dagger(S_\tau^{\omega_j} g_j)A(g_k)) \\ &= \int_0^\infty d\tau\, \langle S_\tau^{\omega_j} g_j, \frac{Q-1}{2} g_k\rangle = \overline{(g_j|g_k)_{Q-}^{\omega_j-}}\,. \end{aligned} \tag{A.12a}$$

Similarly, using the CCR, we find

$$w^+_{(j1),(k0)} = \int_0^\infty d\tau\, e^{i\omega_j\tau} \langle v_\tau^{(0)}(A(g_j))A^\dagger(g_k)\rangle_R = (g_j|g_k)_{Q+}^{\omega_j-}\,, \tag{A.12b}$$

$$w^-_{(k0),(j1)} = (g_j|g_k)_{Q-}^{\omega_j-}\,, \tag{A.12c}$$

$$w^-_{(j1),(k0)} = \overline{(g_j|g_k)_{Q+}^{\omega_j-}}\,. \tag{A.12d}$$

while $w^\pm_{(j\epsilon),(j'\epsilon')} = 0$ if $\epsilon = \epsilon'$ as we have $\langle A(f)A(g)\rangle_R = 0 = \langle A^\dagger(f)A^\dagger(g)\rangle_R$. We note that $\langle F_{(j,\alpha)}\rangle_R = 0$ in all cases.

The master equation then reads

$$\begin{aligned} \left.\frac{ds}{dt}\right|_0 &= \sum_{j,k;\alpha,\alpha'}^{(-1)^\alpha\omega_j + (-1)^{\alpha'}\omega_k = 0} \Big\{ [D_{j,\alpha} D_{k,\alpha'} s_0 - D_{k,\alpha'} s_0 D_{j\alpha}] w^+_{(\alpha),(k\alpha')} \\ &\quad - [D_{j,\alpha} s_0 D_{k,\alpha'} - s_0 D_{k,\alpha'} D_{j,\alpha}] w^-_{(k\alpha'),(j\alpha)} \Big\} \\ &= -\sum_{j,k}^{\omega_j - \omega_k = 0} \Big\{ [D_j D_k^\dagger s_0 - D_k^\dagger s_0 D_j] w^+_{(j0),(k1)} + [D_j^\dagger D_k s_0 - D_k s_0 D_j^\dagger] w^+_{(j1),(k0)} \\ &\quad - [D_j^\dagger s_0 D_k - s_0 D_k D_j^\dagger] w^-_{(k0),(j1)} - [D_j s_0 D_k^\dagger - s_0 D_k^\dagger D_j] w^-_{(k1),(j0)} \Big\} \end{aligned} \tag{A.13}$$

or, writing in our notation (and employing the relabeling in terms of the frequency degeneracies as in (1.5.1)),

$$\frac{ds}{dt}\Big|_0 = -\sum_{\omega\in F}\sum_{j,k=1}^{N(\omega)}\Big\{[D_j^\omega D_k^{\omega\dagger}s_0 - D_k^{\omega\dagger}s_0 D_j^\omega]\overline{(g_j^\omega|g_k^\omega)_{Q-}^{\omega-}} + [D_j^{\omega\dagger}D_k^\omega s_0 - D_k^\omega s_0 D_j^{\omega\dagger}](g_j^\omega|g_k^\omega)_{Q+}^{\omega-}$$

$$- [D_j^{\omega\dagger}s_0 D_k^\omega - s_0 D_k^\omega D_j^{\omega\dagger}](g_j^\omega|g_k^\omega)_{Q-}^{\omega-} - [D_j^\omega s_0 D_k^{\omega\dagger} - s_0 D_k^{\omega\dagger}D_j^\omega]\overline{(g_j^\omega|g_k^\omega)_{Q+}^{\omega-}}\Big\}. \qquad \text{(A.14)}$$

But this is exactly $\frac{ds_t}{dt}|_0 = L_0^*(s_0)$, where L_0^* is given by (2.6.11).

B. The convergence of the collective processes to the noise processes

The mathematical theory behind the weak coupling limit developed in [1] and subsequent papers is the following. We estimate the behaviour as $\lambda \to 0$ of matrix elements of $U_{t/\lambda^2}^{(\lambda)}$ with respect to collective number or coherent vectors, that is vectors of the form $B_{t^{(1)}/\lambda^2}^{(\omega,\lambda)}(f^{(1)})\cdots B_{t^{(n)}/\lambda^2}^{(\omega,\lambda)}(f^{(n)})\Psi_R$. This involves substituting $v_t(H_I)$, as expressed in (1.3.2) for example, into the series expansion (1.1.11) for $U_{t/\lambda^2}^{(\lambda)}$ and examining each of the terms arising. The detailed analysis of [1] shows that each term, upon normal ordering, leads to two classes of terms: relevant ones (type I) and negligible ones (type II). The type I terms are exactly those put into normal order by commuting *time consecutive* pairs of reservoir variables, the type II terms account for all others. Following this resummation it is shown that the type II terms give vanishing contribution in the limit $\lambda \to 0$ while the explicit limit for the type I terms is calculated; uniform convergence is established, the main technical device used here is one of various generalizations of the Pulé inequality [3].

The independence of the noise processes for different frequencies follows from the next two lemmas.

LEMMA 1. *For each ω', $\omega \in F$ let $f'_\omega \in K_{\omega'}$ and $S_\omega, T_\omega, S'_{\omega'}, T'_{\omega'} \in \mathbb{R}$ then*

$$\lim_{\lambda\to 0}\left\langle \lambda\int_{S_\omega/\lambda^2}^{T_\omega/\lambda^2} S_u^\omega f_\omega du, \lambda\int_{S'_{\omega'}/\lambda^2}^{T'_{\omega'}/\lambda^2} S_v^\omega f'_\omega dv\right\rangle$$

$$= \delta_{\omega,\omega'}\sum_{\omega\in F}\langle\chi_{[S_\omega,T_\omega]},\chi_{[S'_\omega,T'_\omega]}\rangle_{L^2(\mathbb{R})}(f_\omega|f'_\omega)_\omega$$

$$= (\oplus_{\omega\in F}(\chi_{[S_\omega,T_\omega]}\otimes f_\omega)|\oplus_{\omega'\in F}(\chi_{[S'_{\omega'},T'_{\omega'}]}\otimes f'_{\omega'}) . \qquad \text{(B.1)}$$

Proof: The left hand side of (B.1) can be written as a sum over $\omega, \omega' \in F$ of terms

$$\lim_{\lambda\to 0}\int_{S_\omega}^{T_\omega} du \int_{(S'_{\omega'}-u)/\lambda^2}^{(T'_{\omega'}-u)/\lambda^2} du' \langle f_\omega, S_{u'}^\omega f'_{\omega'}\rangle e^{i(\omega-\omega')u/\lambda^2} .$$

By the Riemann–Lesbegue Lemma the terms $\omega \neq \omega'$ vanish while the $\omega = \omega'$ terms converge by inspection to

$$\langle \chi_{[S_\omega, T_\omega]}, \chi_{[S'_\omega, T'_\omega]} \rangle_{L^2(\mathbb{R})} (f_\omega | f'_\omega)_\omega \, . \qquad \square$$

LEMMA 2. *For $n \in \mathbb{N}$ let $f_\omega^{(k)} \in K_\omega, x_\omega^{(k)} \in \mathbb{R}, S_\omega^{(k)} < T_\omega^{(k)}$, for $1 \leq k \leq n$ and each $\omega \in F$,*

$$\lim_{\lambda \to 0} \left\langle \Phi_Q^F, B_Q^{F\dagger} \left(\sum_{\omega \in F} x_\omega^{(1)} \lambda \int_{S_\omega^{(1)}}^{T_\omega^{(1)}} S_{u_1}^\omega f_\omega^{(1)} du_1 \right) \cdots \right.$$

$$\left. \cdots B_Q^{F\dagger} \left(\sum_{\omega \in F} x_\omega^{(n)} \lambda \int_{S_\omega^{(n)}}^{T_\omega^{(n)}} S_{u_n}^\omega f_\omega^{(n)} du_n \right) \Phi_Q^F \right\rangle \tag{B.2}$$

exists uniformly for the x's and $[S,T]$'s in a bounded set of $\mathbb{R}$ and is equal to

$$\langle \Phi_R^Q, W(\oplus_{\omega \in F}(x_\omega^{(1)} \chi_{[S_\omega^{(1)}, T_\omega^{(1)}]} \otimes f_\omega^{(1)})) \cdots W(\oplus_{\omega \in F}(x_\omega^{(n)} \chi_{[S_\omega^{(n)}, T_\omega^{(n)}]} \otimes f_\omega^{(n)})) \, \Phi_R^Q \rangle \, . \tag{B.3}$$

For a proof, see [10].

C. The quantum stochastic differential equation for U_t

For convenience we consider only one coupling frequency ω so that

$$H_I \equiv i\hbar \sum_j (D_j \otimes A^\dagger(g_j)) + \text{ h.c.}, \tag{C.1}$$

where we have dropped the superscript ω from the operators. Also we shall consider only the Fock (vacuum) case $Q = 1$. We define, for ψ, $\phi \in \mathcal{H}_S$, $G_\lambda(t) \in \mathcal{H}_S$ by

$$\langle \psi, G_\lambda(t) \rangle = \operatorname{Tr} \left\{ |\phi\rangle\langle\psi| \otimes B_{T'/\lambda^2}^{(\omega,\lambda)\dagger}(f') |\Psi_R\rangle\langle\Psi_R| B_{T/\lambda^2}^{(\omega,\lambda)}(f) U_{t/\lambda^2}^{(\lambda)} \right\}, \tag{C.2}$$

where the right hand side inner product is meant on $\mathcal{H}_S \otimes \Gamma_B(L^2(\mathbb{R}, K_\omega))$: we know that the limit $\lim_{\lambda \to 0} \langle \psi, G_\lambda(t) \rangle$ exists and equals

$$\langle \psi \otimes B_Q^{\omega\dagger}(f,T) \Phi_Q^\omega, U_t \phi \otimes B_Q^{\omega\dagger}(f',T') \Phi_Q^\omega \rangle \, . \tag{C.3}$$

It is easy to show that this limit has the form $\langle \psi, G(t) \rangle$ where $t \mapsto G(t) \in \mathcal{H}_S$ is weakly differentiable. In order to obtain a differential equation for $G(t)$ we note, that for fixed λ one has

$$\frac{d}{dt} \langle \psi, G_\lambda(t) \rangle = \operatorname{Tr} \left\{ |\phi\rangle\langle\psi| \otimes B_{T'/\lambda^2}^{(\omega,\lambda)\dagger}(f') |\Psi_R\rangle\langle\Psi_R| B_{T/\lambda^2}^{(\omega,\lambda)}(f) \times \right.$$

$$\left. \times \; \frac{1}{\lambda} \sum_j (D_j \otimes A^\dagger(S_{t/\lambda^2}^\omega g_j) - D_j^\dagger \otimes A(S_{t/\lambda^2}^\omega g_j)) U_{t/\lambda^2}^{(\lambda)} \right\} = \Gamma_\lambda + \Xi_\lambda \, , \tag{C.4}$$

where

$$\varGamma_\lambda = \frac{1}{\lambda}\mathrm{Tr}\Big\{ |\phi\rangle\langle\psi| \otimes B^{(\omega,\lambda)\dagger}_{T'/\lambda^2}(f')|\varPsi_R\rangle\langle\varPsi_R|B^{(\omega,\lambda)}_{T/\lambda^2}(f) \sum_j D_j \otimes A^\dagger(S^\omega_{t/\lambda^2}g_j)U^{(\lambda)}_{t/\lambda^2}\Big\}, \quad \text{(C.4a)}$$

$$\varXi_\lambda = -\frac{1}{\lambda}\mathrm{Tr}\Big\{ |\phi\rangle\langle\psi| \otimes B^{(\omega,\lambda)\dagger}_{T'/\lambda^2}(f')|\varPsi_R\rangle\langle\varPsi_R|B^{(\omega,\lambda)}_{T/\lambda^2}(f) \sum_j D_j^\dagger \otimes A(S^\omega_{t/\lambda^2}g_j)U^{(\lambda)}_{t/\lambda^2}\Big\}. \quad \text{(C.4b)}$$

Now

$$\begin{aligned}\varGamma_\lambda &= \sum_j \frac{1}{\lambda}\lambda\int_0^{T/\lambda^2} \langle S^\omega_u f, S^\omega_{t/\lambda^2}g_j\rangle du \langle D_j^\dagger\psi, G_\lambda(t)\rangle \\ &= \sum_j \frac{1}{\lambda}\lambda\int_{(-t)/\lambda^2}^{(T-t)/\lambda^2} \langle S^\omega_v f, g_j\rangle dv \langle D_j^\dagger\psi, G_\lambda(t)\rangle\,, \end{aligned} \quad \text{(C.5)}$$

where we have made the substitution $u - t/\lambda^2 = v$. We see that for bounded D this converges as $\lambda \to 0$ a.e. to

$$\sum_j \chi_{[S,T]}(f|g_j)_\omega \langle D_j^\dagger\psi, G_\lambda(t)\rangle\,. \quad \text{(C.5a)}$$

Next of all, the term $\varXi_\lambda$ must be reordered as follows

$$\begin{aligned}\varXi_\lambda = &-\frac{1}{\lambda}\sum_j \mathrm{Tr}\Big\{|\phi\rangle\langle\psi| \otimes B^{(\omega,\lambda)\dagger}_{T'/\lambda^2}(f')|\varPsi_R\rangle\langle\varPsi_R|B^{(\omega,\lambda)}_{T/\lambda^2}(f)\ (D_j^\dagger\otimes 1)U^{(\lambda)}_{t/\lambda}(1\otimes A(S^\omega_{t/\lambda^2}))\Big\} \\ &-\frac{1}{\lambda}\sum_j \mathrm{Tr}\Big\{|\phi\rangle\langle\psi| \otimes B^{(\omega,\lambda)\dagger}_{T'/\lambda^2}(f')|\varPsi_R\rangle\langle\varPsi_R|B^{(\omega,\lambda)}_{T/\lambda^2}(f)\ (D_j^\dagger\otimes 1)[(1\otimes A(S^\omega_{t/\lambda^2})), U^{(\lambda)}_{t/\lambda}]\Big\} \\ &= \varXi^a_\lambda + \varXi^b_\lambda\,. \end{aligned} \quad \text{(C.6)}$$

In a fashion similar to the calculation of $\varGamma_\lambda$, one easily arrives at

$$\lim_{\lambda\to 0} \varXi^a_\lambda = -\sum_j \chi_{[0,T']}(g_j|f')_\omega \langle D_j^\dagger\psi, G_\lambda(t)\rangle\,, \qquad \text{a.e.} \quad \text{(C.7)}$$

To evaluate the limit of $\varXi^b_\lambda$, we note that from (2.4.13)

$$\begin{aligned} &[(1\otimes A(S^\omega_{t/\lambda^2}g_j)), U^{(\lambda)}_{t/\lambda^2}] \\ &= \sum_{n=1}^\infty \left(\frac{\lambda}{i\hbar}\right)^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n [(1\otimes A(S^\omega_{t/\lambda^2}g_j)), H_I(t_1)\cdots H_I(t_n)] \\ &= \sum_{n=1}^\infty \left(\frac{\lambda}{i\hbar}\right)^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \Big\{[(1\otimes A(S^\omega_{t/\lambda^2}g_j)), H_I(t_1)]H_I(t_2)\cdots H_I(t_n) \end{aligned}$$

$$+H_I(t_1)[(1\otimes A(S_{t/\lambda^2}g_j),H_I(t_2)\cdots H_I(t_n)]\}\ . \tag{C.8}$$

It can be shown that in the limit $\lambda\to 0$ only the commutator involving $H_I(t_1)$ contributes. Hence

$$\lim_{\lambda\to 0}\Xi_\lambda^b=-\lim_{\lambda\to 0}\sum_{n=1}^{\infty}\left(\frac{\lambda}{i\hbar}\right)^{n-1}\int_0^{t/\lambda^2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\ \times$$

$$\sum_{j,k}\langle S^\omega_{t/\lambda^2}g_j,S^\omega_{t_1}g_k\rangle\mathrm{Tr}\left\{|\phi\rangle\langle\psi|\otimes B^{(\omega,\lambda)\dagger}_{T'/\lambda^2}(f')|\Psi_R\rangle\langle\Psi_R|B^{(\omega,\lambda)}_{T/\lambda^2}(f)\ D_j^\dagger D_kH_I(t_2)\cdots H_I(t_n)\right\},$$

but this is the same as

$$-\lim_{\lambda\to 0}\frac{1}{\lambda^2}\sum_{j,k}\int_0^t ds\langle S^\omega_{t/\lambda^2}g_j,S^\omega_{s/\lambda^2}g_k\rangle D_j^\dagger D_kU^{(\lambda)}_{s/\lambda^2}= \tag{C.9}$$

$$-\lim_{\lambda\to 0}\frac{1}{\lambda^2}\sum_{j,k}\int_0^t ds\langle S^\omega_{t/\lambda^2}g_j,S^\omega_{s/\lambda^2}g_k\rangle\langle D_k^\dagger D_j\psi,G_\lambda(t)\rangle=-\sum_{j,k}(g_j|g_k)^{\omega-}\langle\psi,D_j^\dagger D_kG(t)\rangle,$$

where we have used a technical lemma (6.3) of [1].

We now have

$$\langle\psi,G(t)\rangle=\lim_{\lambda\to 0}\langle\psi,G_\lambda(t)\rangle=\langle\psi,G(0)\rangle+\lim_{\lambda\to 0}\int_0^t(\Gamma_\lambda(s)+\Xi_\lambda(s))ds$$

$$=\langle\psi,G(0)\rangle+\int_0^t ds\Bigg\{\sum_j\chi_{[S,T]}(s)(f|g_j)_\omega\langle D_j^\dagger\psi,G(s)\rangle-\sum_j\chi_{[S',T']}(s)(g_j|f)^\omega\langle\psi,D_jG(s)\rangle$$

$$-\sum_{j,k}(g_j|g_k)^{\omega-}\langle\psi,D_j^\dagger D_kG(s)\rangle\Bigg\}. \tag{C.10}$$

Here we have written $(g|f)^{\omega-}$ for $(g|f)^{\omega-}_{Q+}$ when $Q=1$.

The quantum stochastic differential equation corresponding to this integral equation is

$$dU_t=\left\{\sum_j(D_j\otimes dB_Q^{\omega\dagger}(g_j,t)-D_j^\dagger\otimes dB_Q^\omega(g_j,t))-\sum_{j,k}(g_j|g_k)^{\omega-}D_j^\dagger D_kdt\right\}U_t\ . \tag{C.11}$$

The generalization to $Q>1$ and several coupling frequencies ω is now obvious.

Aknowledgements

The authors acknowledge partial support from the Human Capital and Mobility programme, contract number: erbchrxct930094. They also express their gratitude to Igor Volovich for interesting comments.

REFERENCES

[1] Accardi, L., Frigerio, A. and Lu, Y.G.: *Commun. Math. Phys.* **131** (1990), 537.

[2] Louisell, W.: *Quantum Statistical Properties of Radiation*, John Wiley and Sons, 1973.

[3] Pulé, J.V.: *Commun. Math. Phys.* **38** (1974), 241.

[4] Davies, E.B.: *Commun. Math. Phys.* **39** (1974), 91.

[5] Haken, H.: *Laser Theory*, Springer, Berlin 1984.

[6] Lax, M.: *Phys. Rev.* **145** (1965), 111.

[7] von Waldenfels, W.: *Ito solution of the linear quantum stochastic differential equation describing light emission and absorption*, SLNM 1055 (eds. L. Accardi, A. Frigerio and V. Gorini).

[8] Accardi, L., Frigerio, A. and Lu, Y.G.: *The weak coupling limit (II): The Langevin equation and finite temperature case*, Preprint Volterra Centro, No. 13 (1989).

[9] Accardi, L., Frigerio, A. and Lu, Y.G.: *The weak coupling limit for Fermions*, Preprint Volterra Centro No. 12 (1989).

[10] Accardi, L., Frigerio, A. and Lu, Y.G.: *The weak coupling limit without rotating wave approximation*, Preprint Volterra Centro No. 23 (1990).

[11] Accardi, L., Frigerio, A. and Lu, Y.G.: *Unified Approach to the Quantum Master and Langevin Equations*, Preprint Centro Volterra No. 69 (1991).

[12] Accardi, L. and Lu, Y.G.: *On the weak coupling limit for quantum electrodynamics*, in Prob. Meth. in Math. Phys. (eds. F. Guerra, M.I. Loffredo, C. Marchioro) World Scientific, Singapore 1992, pp. 16–22.

[13] Hudson, R.L. and Parthasarathy, K.R.: *Commun. Math. Phys.* **131** (1990), 537.

[14] Parthasarathy, K.R.: *An Introduction to Quantum Statistical Calculus*, Monographs in Mathematics, Birkhaüser, Basel 1992.

[15] Kittel, G.: *Quantum Theory of Solids*, John Wiley and Sons, 1963.

[16] N. N. Bogolubov and N. N. Bogolubov Jr.: *Some Applications of the Polaron Theory*, World Scientific Lecture Notes in Mathematics, vol. **4**, Singapore 1992.

[17] Sewell, G.L.: *Quantum Theory of Collective Phenomena*, Monographs on the Physics and Chemistry of Materials, Oxford Science Publications, Oxford 1986.

[18] Collett, M.J. and Gardiner, C.W.: *Phys. Rev. A* **0** (1984), 1386.

[19] Messiah, A.: *Quantum Mechanics*, Vol. II, North Holland, 1961.

[20] Au, C–K. and Feinberg, G.: *Phys. Rev. A* **9** (1974), 1974, and **12** (1975), 1772.

[21] Ford, G.W. and von Waldenfels, W.: *Radiative Energy Shifts for a Nonrelativistic Atom.*

[22] Accardi L.: *Rev. Math. Phys.* **2** (1990), 127.

[23] Frigerio A. and V. Gorini: *J. Math. Phys.* **17** (1976), 2123.

[24] Frigerio A., C. Novellone and M. Verri: *Rep. Math. Phys.* **12** (1977), 133.