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Stochastic bosonization for a $d \geq 3$ Fermi system

by

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ABSTRACT. – We consider a system of fermions interacting via an external field and we prove, in $d \geq 3$, that a suitable collective operator, bilinear in the fermionic fields, in the stochastic limit becomes a boson quantum brownian motion. The evolution operator after the limit satisfies a quantum stochastic differential equation, in which the imaginary part of the Ito correction is the ground state shift while its real part is the lifetime of the ground state.

RÉSUMÉ. – Nous étudions un système de fermions interagissant à travers un champ extérieur et nous démontrons, en dimension $d \geq 3$, qu'un opérateur collectif convenable, bilinéaire dans les chaps de fermions, devient un mouvement brownien quantique dans la limite stochastique. La limite de l'opérateur d'évolution satisfait une équation différentielle stochastique quantique, dans laquelle la partie imaginaire de la correction de Ito est le décalage de l'énergie de l'état fondamental tandis que la partie réelle est la durée de vie de l'état fondamental.

1. INTRODUCTION

The basic idea of *bosonization* consists in expressing some measurable quantities of a fermionic field (transition probabilities, correlation functions...) in terms of the corresponding quantities of a bosonic field. Some recently developed techniques concerning the *stochastic limit of*

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quantum systems suggest a new approach to the problem whose basic idea is that bosonization does not take place exactly, *but only after some limit* (*cf.* (1.5) for an example). The term *stochastic* is justified by the fact that, after the scaling limit, some nonlinear (in the present model, quadratic) expressions in the field operators become a boson field (in our case a *quantum Brownian motion*), and the Schrödinger equation becomes a quantum stochastic differential equation: this is a general feature of the stochastic limit of quantum systems. The basic idea of the stochastic bosonization procedure has been described in [AcLuVo94a]. In order to apply this procedure to more realistic physical systems more elaborated models have to be taken into account. In the present paper we start our programme by considering a quadratic, non relativistic system of fermions interacting with an external potential in $d \geq 3$. The reason why we choose to start our program with such a simple system has its roots in previous experience with the stochastic limit of quantum systems which shows that:

(i) The main analytical difficulties in the stochastic limit of quantum systems are already present in the case of interactions quadratic in the fermionic field. These are met in the deduction of the stochastic equation as a limit of the Schrödinger equation (*cf.* Theorem (5.1) below, which is the main result of the present paper).

(ii) The passage from quadratic to more general polynomial interactions introduces additional technical difficulties but does not change the overall picture (*cf.* [AcMa] in which a self interacting fermionic system (*i.e.* with quartic interaction) is considered). As mentioned at the beginning of this section, *stochastic bosonization is not the same as usual bosonization.* The main differences are:

(i) The main point of usual bosonization is the ability of writing, by an *exact* transformation, a Bose field as a functional of a Fermi field or conversely of writing a fermion field in terms of boson fields. In stochastic bosonization only in a limiting sense some functionals of the Fermi field become Bose fields (*cf.* Theorem (4.1) below). Moreover the knowledge of the limit Bose field does not allow to reconstruct uniquely the original Fermi field [in the central limit theorem the gaussian distribution can be obtained in many ways as limit of sums of random variables] (*cf.* our comments after formula (1.6) below).

(ii) In usual bosonization, the time variable does not play any role in the Boson-Fermion correspondence. In stochastic bosonization the Bose field is obtained as a limit of special time averages of products of Fermi fields (*cf.* formula (1.6) below).

(iii) In usual bosonization the Bose field is expressed as a quadratic expression in the Fermi fields. So, for example, a quartic Fermi Hamiltonian becomes quadratic in the Bose fields. In stochastic bosonization the expressions in the Fermi field shall give rise, in the limit, to a *linear* interaction in the Bose fields. This phenomenon has been called *linearization* [AcLuVo94b].

(iv) In usual bosonization the Bose field evolves according to a usual Heisenberg equation. In stochastic bosonization the Bose field satisfies a stochastic equation.

(v) In usual bosonization *rigorous* results have been obtained up to now only in 1 dimension (for a non relativistic fermionic model called Luttinger model, *see* [MaLi], or for a relativistic fermionic model called Thirring model: such two models are equivalent, *see* [Ma]. For a rewiev about one dimensional bosonization, rigourous or not, *see* [So]). A number of paper appear in recent times about bosonization in $d > 2$ for many fermion systems, motivated by the problem of high temperature superconductivity (*see* for instance [KoMeSc], [KwHoMa], [HoMa], [HoKwMa], [HoKwMaSc], [ScMeKo]). A general mathematical scheme to associate to any quasi-local system (in arbitrary dimensions) different Bose fields has been developed in [GoVeVe]. They try to obtain an *exact* bosonization and their results are not rigorous, while in our approach the bosonization is rigorous but obtained only in the limit. Moreover the method of stochastic bosonization works in dimensions greater or equal than 3, if no conditions are to be put on the cut-off introduced with exception of smoothness e.g. to be a function vanishing at infinity faster than any power. Under special assumptions on the supports of these functions we can lower the dimension to $d \geq 2$. Thus the present approach seems to be complementary rather than alternative to the exact bosonization in $d = 1$. Such a dipendence upon the dimension is not a technical, or model dependent, matter and is illustrated by several examples in solid state physics [Ha].

Besides these differences there are also similarities. In particular the expression of the evolution operator, in these papers, is given by an *exponential of a boson field,* thus showing a striking similarity with our formula (1.7).

The *stochastic limit of quantum systems* should not be confused with the *master equation approach to open systems* (weak coupling limit). The basic physical idea in this latter approach is that a *small* (tipically discrete) system interacts with a *large* (tipically continuous spectrum), representing the *energy bath* or *reservoir* and energy flows from the small to the large system, leading to an irreversible evolution of the observables of the former, when averaged over the degrees of freedom of the latter (reservoir). This idea goes back to van Hove [vH] and has found a mathematical formulation in the theory of quantum Markov semigrup and of associated master field ([Da], [Pu]). In these works it is shown that, in the limit (1.5), the evolution operation, *if averaged over the reservoir,* obeys a master equation and can give informations on the system. In the *stochastic limit of quantum systems*, on the other hand, the van Hove scaling is by no means the only one considered (*see* for instance [AcMa]) and one wants to deduce limiting equations not only for the reduced evolution of the system, but *for the whole coupled system,* including the reservoir; moreover these evolutions are not irreversible but unitary. The first result in this sense was obtained in [AcFrLu] showing that the evolution operator (*not simply its reservoir average*) in a model consisting of a quantum particle (system) interacting with an electromagnetic field satisfies, in the stochastic limit, a quantum stochastic differential equation. The development of the last years, *see* [AcLuVo94b] for a review, have shown that the existence of such limiting stochastic equation can be considered as a *universal phenomenon* in quantum theory *i.e.* not bound to a narrow class of models.

The stochastic limit approach is also different from the perturbative one (*cf.* [JaPi] for some recent results obtained by a sophisticated application of the usual perturbation theory) in which one truncates the perturbative expansion obtained, by replacing the (rescaled) right hand side of (1.2) in (1.3), and maintains only some lower order terms in the powers of λ. In the stochastic limit one distinguishes, inside *each* term of the perturbation series, those contribution which are small if the coupling is small from those which remain finite. Then one *proves* that the former tend to zero in the limit and that the remaining ones can be resummed giving rise to a *unitary evolution* satisfying a stochastic differential equation whose explicit form is determined (this is the main, and somehow surprising result of the whole theory).

The goal to derive from a *microscopic* model a stochastic equation relates the stochastic limit with the results of [FoKaMa] and [DeDuLeLi] (incidentally: both papers deal with models quadratic in the fields) in which one shows that in a system of oscillators, the reduced dynamic of a single oscillator obeys a *Langevin equation.* If the oscillators are classical, the equation is the classical Langevin equation giving the positions and momenta of a Brownian particle but if they are quantum the noise term found by the above mentioned authors is neither white (*i.e.* δ-correlated) nor Markovian, but it is a coloured gaussian noise. In the quantum stochastic limit, on the other hand, one shows that the stochastic equation which governs the system after the limit is just a quantum stochastic differential equation, driven by a Bose field which is the natural candidate to replace the classical Brownian motion in the quantum case and which, for these reasons is called *quantum Brownian motion* (*cf.* the remark after Theorem (4.1) for a precise definition; [Ac90] for a survey of the physical origins of this notion and [HuPa] for the associated notion of stochastic differential equation).

Up to now we have discussed the general idea of stochastic bosonization and its relationship to the usual bosonization. In order to formulate the results of the present paper it is necessary to say a few words on the specific model we shall study here (a more precise definition of the objects involved is in Section (2) below).

The Hamiltonian of the model we are going to discuss is given by:

$$H = H_0 + H_I = \int_\Lambda d\vec{x} \psi_x^+ \left(\frac{\partial_x^2}{2m} - \mu \right) \psi_x^- + \lambda \int_\Lambda dx \phi(x, t) \psi_x^+ \psi_x^- \quad (1.1)$$

where $\Lambda \subset R^d$ is a square box of side L, $\mu = p_F^2/2m$ is the *chemical potential,* p_F is the *Fermi momentum,* m is the fermion mass and ψ_x^ε, $\varepsilon = \pm 1$ is the *fermionic field* with periodic boundary condition:

$$\psi_x^\varepsilon = \frac{1}{L^{d/2}} \sum_k e^{i\varepsilon k x} a_k^\varepsilon$$

where $k = \frac{2n\pi}{L}$, $n = (n_1, \ldots, n_d) \in Z^d$ and $\{a_k^\varepsilon, a_{k'}^{-\varepsilon'}\} = a_k^\varepsilon a_{k'}^{-\varepsilon'} + a_{k'}^{-\varepsilon'} a_k^\varepsilon = \delta_{\varepsilon,\varepsilon'} \delta_{k,k'}$ and for any operator X we use the notation:

$$X^\varepsilon = \begin{cases} X & \text{if } \varepsilon = -1 \\ X^+ & \text{if } \varepsilon = +1. \end{cases}$$

The function $\phi(x, t)$ is an *external field* such that:

$$\phi(x, t) = \frac{1}{L^d} \sum_p [u_p e^{-ipx - i\omega_p t} + \bar{u}_p e^{ipx + i\omega_p t}]$$

with $\omega_p = c|p|$ and u_p is a cutoff function whose physical meaning is discussed after formula (2.15).

The free evolution is characterized by the following property:

$$\psi_{x,t}^\varepsilon = e^{iH_0 t} \psi_x^\varepsilon e^{-iH_0 t} = \frac{1}{\sqrt{L^d}} \sum_k e^{i\varepsilon(kx + \left(\frac{k^2}{2m} - \mu\right) t)} a_k^\varepsilon.$$

The Hamiltonian in interaction representation is $H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t}$ and we define the *evolution (wave) operator* at time T in the usual way:

$$U_T = 1 + \sum_{n=1}^\infty (-i)^n \lambda^n \int_0^T dt_1 \ldots \int_0^{t_{n-1}} dt_n H_I(t_1) \ldots H_I(t_n) \quad (1.2)$$

where the series converges in norm for each finite L.

In the standard perturbation theory one estimates the expression

$$Z = \langle \phi_F, U_T \phi_F \rangle \tag{1.3}$$

where $\phi_F = \prod_{|\vec{k}| \leq p_F} a^{\varepsilon}_{\vec{k}} |0\rangle$ is the *ground state* of H_0, by replacing the series, which defines the operator U_T, by a finite sum (tipically its first two or three terms). However this procedure has a meaning only if the higher order terms are negligible or, in other words, if the behaviour of the system is "close" to the *free* ($\lambda = 0$) one. But in many examples this expectation is wrong *i.e.* the behaviour of the free system is completely different from the interacting one even if the interaction is very weak. This is not too strange as the particles interact with a weak potential, but the time in which they interact can be very long so that the final effects may be relevant, *i.e.*: in a very weakly interacting system the effects of a vanishing coupling constant $\lambda \to 0$ can be compensated by very long times of evolution T so that the cumulative effect may be non trivial.

This suggests to study the system in the Friedrichs-Van Hove limit:

$$\lambda \to 0, \quad T \to \infty, \quad \lambda^2 T \to t = \text{constant}, \quad L \to \infty \tag{1.4}$$

Notice that the limit Eq. (1.4) is equivalent to the *scaling limit:*

$$\lambda \to 0; \quad T \to \frac{T}{\lambda^2}; \quad L \to \infty \tag{1.5}$$

and in the following we shall always use the scaling (1.5). This has been the prototype model for the above mentioned stochastic limit.

According to the general scheme of the stochastic limit, the first order term in the perturbative expansion of $U_{\frac{t}{\lambda^2}}$ (Eq. (12)) suggests to study suitable time averages of the fields operators, called *collective operators,* of the form:

$$B_{t,\lambda} = \lambda \int_0^{\frac{1}{\lambda^2}} dt_1 \frac{1}{L^d} \sum_{k,p} u_p \, e^{it_1(\varepsilon_{k+p} - \varepsilon_k - \omega_p)} a^+_{k+p} a_k \tag{1.6}$$

after introducing a cut-off in the variable k, given by a test-function g, where $\varepsilon_k = \frac{k^2}{2m} - \mu$. In Theorem (4.1) it is proved that the operators $B_{t,\lambda}$, $B^+_{t,\lambda}$ in the limit $\lambda \to 0$, $L \to \infty$ are boson gaussian fields in the sense that their correlations over the ground state converges in the limit to the corresponding correlations of *boson gaussian fields,* which we call B_t, B^+_t; we denote by ψ their vacuum. This result should be considered a manifestation of some kind of *functional central limit* effect, in which $\frac{1}{\lambda^2}$ is the analogue of the number of stochastic variables. The theorem is not true, strictly speaking, with the above definition of the collective operators; we need a *regularization* of the above definition, given by Eq. (2.14) below, in order to prove it. The limiting field $W_t = B_t + B_t^+$, $t \geq 0$ is a classical $\mathcal{K}$-valued Brownian motion and the pair B_t^+, B_t is a *quantum Brownian motion i.e.* a boson gaussian field over the 1-particle space $L^2(R_+, \mathcal{K})$, where $\mathcal{K}$ is the Hilbert space described at the end of section 2, with two point vacuum correlation functions given by:

$$\langle \psi, B_t^+ B_s \psi \rangle = C_1 \min(t, s) \quad \langle \psi, B_t B_s^+ \psi \rangle = C_2 \min(t, s)$$
$$\langle \psi, B_t^+ B_s \psi \rangle = \langle \psi, B_t B_s \psi \rangle = 0$$

where C_1 and C_2 depend only on the cut-off functions necessary to make meaningful the limit (*see* Eq. (2.15)). We prove, in Theorem (5.1), that in the same sense which the collective operators converge to a Bosonic Brownian motion in the weak coupling limit, the evolution operator $U_{\frac{t}{\lambda^2}}$, $t \geq 0$ converges to the solution U_t of the quantum stochastic differential equation in the sense of [HuPa]:

$$dU_t = [idB_t^+ + idB_t - K dt] \, U_t$$

where K is a complex number with Re $K > 0$, whose solution is

$$U_t = \exp(i \, B_t + B_t^+ - \operatorname{Im} K)) \tag{1.7}$$

This implies that $\langle \psi, U_t, \psi \rangle = e^{-Kt}$ so that the imaginary part of the Ito correction of the quantum stochastic differential equation for U_t represents the shift of the ground state energy of the system due to the interaction with the external field while its real part is related to dissipation effects and it gives the lifetime of the groud state. Of course from U_t we can compute the transition probability between any state obtained applying a number of collective operators to the ground state. The evolution operator, after the limit, remains unitary but at the same time takes into account dissipation effects: this is essentially due to the stochastic description of the model. Thus we have found that, in the stochastic limit, the system is described by a bosonic operator. In this sense we claim that, we have obtained a *bosonization* of the theory in the given limit.

2. DEFINITIONS AND NOTATIONS

In this section we define more precisely our model and introduce some definitions and notations which will be useful in the following. Let A, A^+ be a representation of the CAR on $L^2(\mathbf{R}^d)$ and for each

$n = (n_1, \ldots, n_d) \in \mathbf{Z}^d$ define

$$\Lambda_n = \{x = (x_1, \ldots, x_d) \in \mathbf{R}^d : n_j \leq x_j < n_j + 1; \quad j = 1, 2, \ldots, d\}. \tag{2.1}$$

Given $L > 0$, define:

$$\Lambda_L = \left\{k = \frac{2\pi n}{L} : n \in \mathbf{Z}^d\right\}; \quad \Lambda_L^* = \Lambda_L \backslash \{0\} \tag{2.2}$$

and, for each $k \in \Lambda_L$ define

$$a_k^\varepsilon = A^\varepsilon (\chi_{\Lambda \frac{Lk}{2\pi}}); \quad \varepsilon = 0, 1 \tag{2.3}$$

where, here and in the following, for any set I, we denote χ_I its characteristic function:

$$\chi_I(x) = \begin{cases} 0 & \text{if } x \notin I \\ 1 & \text{if } x \in I. \end{cases} \tag{2.4}$$

In the interaction representation the Hamiltonian is:

$$H_I(u, L) = \frac{1}{L^d} \sum_{\substack{p \in \Lambda_L^* \\ k \in \Lambda_L}} \bar{u}_p \, e^{it(\varepsilon_k - \varepsilon_{k+p} + \omega_p)} \, a_k^+ \, a_{k+p} + c.c. \tag{2.5}$$

where, putting $2m = 1$ for simplicity ε_k is defined by:

$$\varepsilon_k = k^2 - p_F^2 \tag{2.6}$$

and the diagonal terms of the Hamiltonian, corresponding to $p = 0$, are included in the renormalization of the chemical potential. We *regularize* the interacting Hamiltonian in the following way:

$$H_I(g, u, L) = \frac{1}{L^d} \sum_{\substack{p \in \Lambda_L^* \\ k \in \Lambda_L}} F(k, p) \, e^{it(\varepsilon_k - \varepsilon_{k+p} + \omega_p)} \, a_k^+ \, a_{k+P} + c.c. \tag{2.7}$$

where $F(k, p)$ is a suitable cut-off function which will be chosen in the following.

The 1-particle dynamics is by definition:

$$S_t^L = \sum_{k \in \Lambda_L} e^{it\varepsilon_k} \, M(\chi_{\Lambda_{\frac{Lk}{2\pi}}}) \tag{2.8}$$

$$M(f) = \text{multiplication operator by } f \text{ in } L^2(\mathbf{R}^d); \; M(f)\,\phi \equiv f\,\phi. \tag{2.9}$$

The second quantized free dynamics is

$$u_t^L(A^\varepsilon(f)) = A^\varepsilon(S_t^L f). \tag{2.10}$$

If Φ is the Fock vacuum the free ground state is defined by

$$\phi_F = \prod_{\{k \in \Lambda_L : |k| \leq p_F\}} A^+ (\chi_{\Lambda_{\frac{Lk}{2\pi}}}) \Phi \tag{2.11}$$

where the order in the product is arbitrary but fixed. Introducing the set

$$B_F = \{k \in \Lambda_L : |k| \leq p_F\} \tag{2.12}$$

if $C \subseteq B_F$ is a subset of B_F, we shall write

$$\phi_{B_F \setminus C} \tag{2.13}$$

to denote the state obtained from (2.11) by suppressing in the product, all the creators with index $k \in C$. Notice that the number of points in B_F is of order $(L/2\,p_F)^d$. For any set I, we denote I^c its set theoretical complement.

If $|k| \leq p_F$, then, writting χ_k instead of $\chi_{\Lambda_{\frac{Lk}{2\pi}}}$, we have:

$$A(S_t^L \chi_k)\, \phi_{p_F} = A(S_t^L \chi_k) \prod_{|h| \leq p_F} A^+(\chi_h)\, \Phi = (-1)^{\nu(k)}\, e^{-it\varepsilon_k}\, \phi_{p_F \setminus \{k\}}$$

where $A^+(\chi_k)$ occupies the $\nu(k)$-th place and, if $|k| > p_F$, $A(S_t^L \chi_k)\, \phi_{p_F} = 0$. In these notations, the interaction hamiltonian becomes:

$$H_I(u, g, L) = \frac{1}{L^d} \sum_{p \in \Lambda_L^*} e^{i\omega_p t} \sum_{k \in \Lambda_L} F(k, p)\, A^+(S_t^L \chi_{\Lambda_k})\, A(S_t^L \chi_{\Lambda_{k+p}}) + \text{h.c.}$$

A crutial role in the following shall be played by the so-called *collective operators* defined for $\sigma = 0,\ 1$ by:

$$B_\lambda^\sigma(S_1,\ T_1,\ g,\ u,\ L) = \frac{\lambda}{L^d} \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt$$

$$\sum_{p \in \Lambda_L^*} \sum_{k \in \Lambda_L} F^\sigma(k,\ p)\, e^{i(1-2\sigma)\,\omega_p t}\, A^+(S_t^L \chi_{\Lambda_{k+\sigma p}})\, A(S_t^L \chi_{\Lambda_{k+(1-\sigma)p}}) \tag{2.14}$$

with $F^0(k,\ p) = \bar{F}(k,\ p)$ and $F^1(k,\ p) = F(k,\ p)$. For finite λ and L these *are not* Bose fields: only in the limit $\lambda \to 0$, $L \to \infty$ they converge, in the sense specified by Theorem (4.1), to a Bose field $B(\chi_{[S_1, T_1]} \otimes F)$ which is δ-correlated in time. Any quantum field with this property is called a *quantum Brownian motion.*

Such operators are defined in the non zero subspace K_0 of $L^2(\mathbf{R}^d) \otimes L^2(\mathbf{R}^d)$ with the property that, for any pair of vectors F, G in this subspace

$$\int_{\mathbf{R}} dt \left| \int_{\mathbf{R}^d} dk \int_{\mathbf{R}^d} dp \, (\chi_{B_F}(k+p) \chi_{B_F^c}(k) \times e^{-i(\varepsilon_{k+p}-\varepsilon_k-\omega_p)t} \bar{F}(k, p) G(k, p) + \chi_{B_F^c}(k+p) \chi_{B_F}(k) \times e^{i(\varepsilon_{k+p}-\varepsilon_k-\omega_p)t} \bar{F}(k, p) G(k, p)) \right| < +\infty \tag{2.15}$$

A possible choice for the cut-off function $F(k, p)$ is $u_p g_k$ where u_p, g_k are such that $g_k = g_{|k|}$ and $u_p = u_{|p|}$ and vanishing at infinity faster than any power. It is possible in fact to check that, if $d \geq 3$, Eq. (2.15) holds. Such a choice has a clear physical meaning: the function u_p is a cut-off of the momentum that the external field exchanges with the fermions while g_k is a *bandwidth cut-off* taking into account that the band structure in a metal forbids the electrons to have large momenta (*see* [So]).

The proof of (2.15) is not completely trivial. By introducing more particular cut-off functions, the arguments drastically simplifies and one obtains a stronger result. Namely we consider the cut-off function $F(k, p)$ to be of the form $g_k g_{k+p} u_p$ where g_k is the sum of two C^∞ functions, the first, called $g_{1,k}$, with support in B_F and the second, called $g_{2,k}$, with support in B_F^c and decreasing faster then any power at infinity. Moreover we choose g_k real. The first summand in Eq. (2.15) can be written then:

$$\int_{\mathbf{R}} dt \left| \int_{\mathbf{R}^d} dk \int_{\mathbf{R}^d} dp e^{-i(\varepsilon_{k+p}-\varepsilon_k-\omega_p)t} |u_p|^2 g_{2,k+p}^2 g_{1,k}^2 \right| \tag{2.16}$$

and performing the change of variables $p' = pt$ we find:

$$\int_{\mathbf{R}} dt \frac{1}{t^d} \left| \int_{\mathbf{R}^d} dp e^{-i(p^2/t - c|p|)} |u_{p/t}|^2 f(p, t) \right| \tag{2.17}$$

where:

$$f(p, t) = \int_{\mathbf{R}^d} dk e^{-i2kp} g_{2,k+p/t}^2 g_{1,k}^2.$$

Integrating by parts and noting that the integrand vanishes at the extrema of integration we obtain that for any integer N $|f(p, t)| \leq \frac{C_N}{1+p^N}$ where C_N is a suitable constant. The condition that g_k, u_k are vanishing at p_F has the only effect to smooth χ_{B_F} and $\chi_{B_F^c}$. Note that this choice of the cut-off is quite natural if the temperature is not 0. In this case in fact the χ functions are replaced by smooth C^∞ functions, which are the densities of the Fermi distributions. The functions g_k, g_{k+p} are band-width cut-off for the two fermions operators and u_p is the cut-off on the exhanged momentum. With these cut-offs our theory holds for $d \geq 2$.

We introduce finally, for further use, the following definition:

$$(u \otimes f \mid v \otimes g) := \int_{\mathbf{R}} dt \int dk \int dp \tag{2.18}$$

$$(e^{-i(\varepsilon_{k+p} - \varepsilon_k - \omega_p)t} \chi_{B_F}(p+k) \chi_{B_F^c}(k) u_p f_k f_{k+p} \bar{v}_p \bar{g}_k \bar{g}_{k+p} + \chi_{B_F^c}(k+p) \chi_{B_F}(k) \bar{u}_p \bar{f}_k \bar{f}_{k+p} v_p g_k g_{k+p} e^{i(\varepsilon_{k+p} - \varepsilon_k - \omega_p)t})$$

In the above assumptions the expression (2.18) defines a pre-scalar product on the test functions F, and the completion of the (quotient by the zero norm elements of the) space K_0 by this scalar product, denoted $\mathcal{K}$ is interpreted as the Hilbert space were the Brownian motion takes its values. This is a general feature of the stochastic limit.

3. THE 2-POINT FUNCTION FOR THE COLLECTIVE OPERATORS

In this section we study the following limit:

$$\lim_{\lambda \to 0} \lim_{L \to \infty} \langle \phi_F, B_\lambda^{\sigma(1)}(S_1, T_1, g, u, L) B_\lambda^{\sigma(2)}(S_2, T_2, g, u, L) \phi_F \rangle \tag{3.1}$$

there are four cases:

Case I. – $\sigma(1) = 1$, $\sigma(2) = 0$.

With the definition (2.14) of the collective vectors one has:

$$\langle \phi_F, B_\lambda^+(T_1, S_1, g, u, L) B_\lambda(T_2, S_2, g, u, L) \phi_F \rangle$$
$$= \frac{\lambda^2}{L^{2d}} \sum_{k_1 \in \Lambda_L} \sum_{k_2 \in \Lambda_L} \sum_{p_1 \in \Lambda_L^*} \sum_{p_2 \in \Lambda_L^*} \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1$$
$$\int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_2 \, u_{p_1} g_{k_1} g_{k_1+p_1} \bar{u}_{p_2} g_{k_2} g_{k_2+p_2} e^{-i(\omega_{p_1} t_1)} e^{i(\omega_{p_2} t_2)}$$

$$\langle \phi_F,\, A^+(S^L_{t_1}\chi_{k_1+p_1})\, A(S^L_{t_1}\chi_{k_1})\, A^+(S^L_{t_2}\chi_{k_2})\, A(S^L_{t_2}\chi_{k_2+p_2})\,\phi_F\rangle$$
$$= \frac{\lambda^2}{L^{2d}} \sum_{k_1,\,k_2\in\Lambda_L}\ \sum_{p_1,\,p_2:|p_1+k_1|,\,|p_2+k_2|\le p_F} \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1$$
$$\int_{T_1/\lambda^2}^{T_2/\lambda^2} dt_2\, u_{p_1}\, g_{k_1}\, g_{k_1+p_1}\, \bar u_{p_2}\, g_{k_2}\, g_{k_2+p_2}$$
$$e^{-i\omega_{p_1}t_1+i\omega_{p_2}t_2}\,\{\langle A(S^L_{t_1}\chi_{k_1+p_1})\,\phi_F,\ A(S^L_{t_2}\chi_{p_2+k_2})\,\phi_F\rangle\langle\chi_{k_1},\ S^L_{t_2-t_1}\chi_{k_2}\rangle$$
$$-\langle A(S^L_{t_2}\chi_{k_2})\,A(S^L_{t_1}\chi_{k_1+p_1})\,\phi_F,\ A(S^L_{t_1}\chi_{k_1})\,A(S^L_{t_2}\chi_{k_2+p_2})\,\phi_F\rangle\} \tag{3.2}$$

Notice that we use the same symbol $\langle\cdot,\,\cdot\rangle$ to denote the scalar product both in the 1-particle space and in the Fock space.

The first term of the sum is equal to

$$\frac{\lambda^2}{L^{2d}} \sum_{k\in\Lambda_L}\ \sum_{p_1,\,p_2:|p_1+k|,\,|p_2+k_2|\le p_F} \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1$$
$$\int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2\, \bar u_{p_1}\, u_{p_2}\, g_k^2\, g_{k+p_1}\, g_{k+p_2}$$
$$\langle A(S^L_{t_1}\chi_{k_1+p_1})\,\phi_F,\ A(S^L_{t_2}\chi_{p_2+k})\,\phi_F\rangle\, e^{-i\omega_{p_1}t_1+i\omega_{p_2}t_2}. \tag{3.3}$$

The scalar product in (3.3) is equal to

$$e^{it_1\varepsilon_{k+p_1}}\, e^{-it_2\varepsilon_{k+p_2}}\,(-1)^{\nu(k+p_1)}\,\langle\phi_{F\backslash\{k+p_1\}},\ \phi_{F\backslash\{k+p_2\}}\rangle(-1)^{\nu(k+p_2)}$$
$$=\delta_{p_1,\,p_2}\, e^{i(t_1-t_2)\varepsilon_{k+p_1}}$$

so (3.3) is equal to

$$\frac{\lambda^2}{L^{2d}} \sum_{k\in\Lambda_L}\ \sum_{\substack{p\\ |p+k|\le p_F}} \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1$$
$$\int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2\,|\,u_p\,|^2\, g_k^2\, g_{k+p}^2\, e^{i(t_2-t_1)(-\varepsilon_{k+p}+\varepsilon_k+\omega_p)}. \tag{3.4}$$

Similarly, the second term in (3.2) gives

$$\frac{\lambda^2}{L^{2d}} \sum_{|k|\le p_F}\ \sum_{|k+p|\le p_F} \int_{S_1/\lambda^2}^{T_1/\lambda^2}$$
$$\int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2|\,u_p\,|^2\, g_k^2\, g_{k+p}^2\, e^{i(t_2-t_1)(-\varepsilon_{k+p}+\varepsilon_k+\omega_p)}.$$

Notice that the above formula is exactly like (3.4) with the only differences that the sum in k does not run over all Λ_L but only over the sphere $|k| \leq p_F$ and that there is a minus sign. Summing up, we find:

$$\langle \phi_F,\, B_\lambda^+ (T_1,\, S_1,\, g,\, u,\, L) B_\lambda (T_2,\, S_2,\, g,\, u,\, L)\, \phi_F \rangle$$
$$= \frac{\lambda^2}{L^{2d}} \sum_{|k|>p_F} \sum_{p\in\Lambda_L^*} \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1$$
$$\int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2\, \chi_{B_F} (p+k)\, |\, u_p\,|^2\, g_k^2\, g_{k+p}^2\, e^{i(t_2-t_1)(-\varepsilon_{k+p}+\varepsilon_k+\omega_p)}$$

In the limit $L \to \infty$ this converges to

$$\lambda^2 \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1 \int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2 \int dk \int dp\, |\, u_p\,|^2\, g_k^2\, g_{k+p}^2$$
$$e^{i\,(t_2-t_1)(-\varepsilon_{k+p}+\varepsilon_k+\omega_p)}\, \chi_{B_F^c}(k)\, \chi_{B_F}(p+k).$$

With the change of variable $\lambda^2\, t_1 = \tau_1$, we obtain

$$\int_{S_1}^{T_1} d\tau_1 \int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2 \int dk \int dp\, |\, u_p\,|^2\, g_k^2\, g_{k+p}^2$$
$$e^{i\,(t_2-\tau_1/\lambda^2)(\varepsilon_{k+p}-\varepsilon_k-\omega_p)}\, \chi_{B_F^c}(k)\, \chi_{B_F}(p+k).$$

With the further change of variable $t_2 - \frac{\tau_1}{\lambda^2} = \tau_2$ we find:

$$\int_{S_1}^{T_1} d\tau_1 \int_{(S_2-\tau_1)/\lambda^2}^{(T_2-\tau_1)/\lambda^2} d\tau_2 \int dk$$
$$\int dp\, |\, u_p\,|^2\, g_k^2\, g_{k+p}^2\, e^{-i\tau_2(\varepsilon_{k+p}-\varepsilon_k-\omega_p)}\, \chi_{B_F^c}(k)\, \chi_{B_F}(p+k) \to_{\lambda\to 0}$$
$$\langle \chi_{[S_1,\,T_1]},\, \chi_{[S_2,\,T_2]} \rangle_{L^2(\mathbf{R})} (u\otimes g\,|\,u\otimes g)_1 \tag{3.5}$$

where $(u\otimes g\,|\,u\otimes g)_1$ is the first addend of Eq. (2.18).

Case II. – $\sigma(1) = 0$, $\sigma(2) = 1$. By similar computations one finds:

$$\langle \phi_F,\, B(S_1,\, T_1,\, g,\, u,\, L)\, B^+ (S_2,\, T_2,\, g,\, u,\, L)\, \phi_F \rangle \to_{L\to\infty}$$
$$\int_{S_1}^{T_1} d\tau_1 \int_{(S_2-\tau_1)/\lambda^2}^{(T_2-\tau_1)/\lambda^2} d\tau_2 \int dk \int dp\, |\, u_p\,|^2\, g_k^2\, g_{k+p}^2$$
$$e^{i\tau_2\,(\varepsilon_{k+p}-\varepsilon_k-\omega_p)}\, \chi_{B_F}(k)\, \chi_{B_F^c}(p+k) \to_{\lambda\to 0}$$
$$\langle \chi_{[S_1,\,T_1]},\, \chi_{[S_2,\,T_2]} \rangle_{L^2\,(\mathbf{R})} (u\otimes g\,|u\otimes g)_2 \tag{3.7}$$

where $(u \otimes g \,|\, u \otimes g)_2$ is given by the second summand of Eq. (2.18).

Case III. – $\sigma(1) = \sigma(2) = 0$.

$$\lim_{\lambda\to 0} \lim_{L\to\infty} \frac{\lambda^2}{L^{2d}} \sum_{k_1, k_2} \sum_{p_1, p_2} \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1$$

$$\int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2\, g_{k_1}\, g_{k_1+p_1}\, g_{k_2}\, g_{k_2+p_2}\, \bar{u}_{p_1}\, \bar{u}_{p_2}\, e^{i\omega_{p_1} t_1 + i\omega_{p_2} t_2}$$

$$\langle \phi_F,\, A^+ (S_{t_1}^L \chi_{k_1})\, A(S_{t_1}^L \chi_{k_1+p_1})\, A^+ (S_{t_2}^L \chi_{k_2})\, A(S_{t_2}^L \chi_{k_2+p_2})\, \phi_F \rangle$$

$$= \lim_{\lambda\to 0} \lim_{L\to\infty} \frac{\lambda^2}{L^{2d}} \sum_{\substack{k_1 \in \Lambda_L \\ p_1 \in \Lambda_L^*}} \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1 \int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2\, g_{k_1}^2\, g_{k_1+p_1}^2\, \bar{u}_{p_1}^2$$

$$\chi_{B_F}(k_1)\, \chi_{B_F^c}(k_1 + p_1)\, e^{-i(\varepsilon_{k_1+p_1} - \varepsilon_{k_1} - \omega_{p_1})(t_1 - t_2)}\, e^{2i\omega_{p_1} t_1}. \tag{3.8}$$

Performing the change of variables $t_2 - t_1 = \tau_2$ and $t_1 \lambda^2 = \tau_1$ we obtain:

$$\lim_{\lambda\to 0} \int_{S_1}^{T_1} d\tau_1 \int_{(S_2-\tau_1)/\lambda^2}^{(T_2-\tau_1)/\lambda^2} d\tau_2 \int dk_1\, dp_1\, \chi_{B_F}(k_1)\, \chi_{B_F^c}(k_1 + p_1)$$

$$g_{k_1}^2\, g_{k_1+p_1}^2\, \bar{u}_{p_1}\, \bar{u}_{-p_1}\, e^{-i(\varepsilon_{k_1+p_1} - \varepsilon_{k_1} - \omega_{p_1})\tau_2}\, e^{\frac{2i\omega_{p_1}\tau_1}{\lambda^2}}.$$

The integral can be rewritten as:

$$\lim_{\lambda\to 0} \int_{S_1}^{T_1} d\tau_1 \int dp_1\, e^{\frac{2ic|p_1|\tau_1}{\lambda^2}}\, F_\lambda(p_1)$$

and using polar coordinates $p_1 = \rho\hat{p}$ we have:

$$\lim_{\lambda\to 0} \int_{S_1}^{T_1} d\tau_1 \int d\hat{p} \int_{-\infty}^{\infty} d\rho\rho^{d-1}\, e^{\frac{2ic|p_1|\tau_1}{\lambda^2}}\, F_\lambda(\rho\hat{p})$$

which vanishes for the Riemann-Lebesgue lemma.

Case IV. – $\sigma(1) = \sigma(2) = 1$.

Proceding as in case III one finds that this limit vanishes.

In conclusion we have that:

$$\lim_{\lambda\to 0} \lim_{L\to\infty} \langle \phi_F,\, W_\lambda(S_1, T_1, g, u, L)\, W_\lambda(S_2, T_2, g, u, L),\, \phi_F \rangle$$

$$= \langle \chi_{[S_1, T_1]},\, \chi_{[S_2, T_2]}, \rangle_{L^2(\mathbf{R})}\, (u \otimes g \,|\, u \otimes g)$$

and this motivates the definition Eq. (2.18) in the preceding section also showing that the right hand side of (2.18) is a pre-scalar product.

4. THE COLLECTIVE OPERATORS AS BOSON GAUSSIAN FIELDS

In this section we prove that the collective operators in the weak coupling limit are Boson Gaussian fields.

THEOREM (4.1). – *In the notations* (2.11), (2.14), (2.15) *one has, for any* $N \in \mathbf{N}$, $S_1, T_1, \ldots, S_N, T_N \in \mathbf{R}$, $g_1 \otimes u_1, \ldots, g_N \otimes u_N \in K_0$, *with* $K_0 \subseteq L^2(\mathbf{R}^{2d})$ *defined by* (2.15):

$$\begin{aligned}\lim_{\lambda\to 0}\lim_{L\to\infty} &\langle \phi_F, B_\lambda^{\sigma(1)}(L_1, T_1, g_1, u_1, L) \\ &\ldots B_\lambda^{\sigma(N)}(S_N, T_N, g_N, u_N, L), \phi_F\rangle \\ &= \langle \psi, B^{\sigma(1)}(S_1, T_1, g_1, u_1)\ldots B^{\sigma(N)}(S_N, T_N, g_N, u_N)\psi\rangle\end{aligned}$$

where $\{B^\#, \psi\}$ *is the unique mean zero gauge invariant Boson Gaussian field with 2-point function given by:*

$$\begin{aligned}&\langle \psi, B^+(S_1, T_1, g_1, u_1) B(S_2, T_2, g_2, u_2)\psi\rangle \\ &\quad = \langle \chi_{[S_1, T_1]}, \chi_{[S_2, T_2]}\rangle_{L^2(\mathbf{R})} (u_1 \otimes g_1 \mid u_2 \otimes g_2)_1\end{aligned}$$

$$\begin{aligned}&\langle \psi, B(S_1, T_1, g_1, u_1) B^+(S_2, T_2, g_2, u_2)\psi\rangle \\ &\quad = \langle \chi_{[S_1, T_1]}, \chi_{[S_2, T_2]}\rangle_{L^2(\mathbf{R})} (u_1 \otimes g_1 \mid u_2 \otimes g_2)_2\end{aligned}$$

(gauge invariance implies that the other 2-point functions are zero).

Remark. – The field $\{B^\#, \psi\}$ defined in Theorem 4.1 is an example of a quantum Brownian motion.

By definition a *quantum Brownian motion* is a representation $\{B^\#, \psi\}$ of the CCR, in unbounded form, with ψ in the domain of the polynomial algebra of the fields and with the following properties:

(i) The test functions of the field operators belong to a Hilbert space of the form $L^2(\quad) \otimes \mathcal{K}$ (where $\mathcal{K}$ is an arbitrary Hilbert space).

(ii) The family $\{B^\#(\psi \otimes f) : \psi \in L^2(\quad), f \in \mathcal{K}\}$ is a mean zero Gaussian family with respect to the state ψ (this means that the odd joint correlations are zero and the even ones are obtained as products over all pair partitions of the corresponding pair correlations).

(iii) The pair correlations have the form

$$\langle \psi, B^{\varepsilon_1} (\varphi_1 \otimes f_1) B^{\varepsilon_2} (\varphi_2 \otimes f_2) \psi \rangle$$
$$= \langle \varphi_1, \varphi_2 \rangle_{q_{\varepsilon_1, \varepsilon_2}} (f_1, f_2)$$

for any pair of real valued test funtions $\varphi_1, \varphi_2 \in L^2(\quad)$ for any ε_1, $\varepsilon_2 \in \{0, 1\}$ where $B^0 = B^+$, $B^1 = B$ and $q_{\varepsilon_1, \varepsilon_2}$ are quadratic forms on a dense subspace of $\mathcal{K}$.

Usually, instead of considering a generic (time) test function φ, one restricts oneself to functions of the form $\chi_{[0, t)}$ and one writes $B_t(g)$ instead of $B(\chi_{[0, t]} \otimes g)$. Fixing g and considering the 1-parameter family $\{B_t(g) + B_t(g) : t \geq 0\}$ (resp. $\{(1/i)[B_t(g) - B_t(g)] : t \geq 0\}$) one obtains a classical process (*i.e.* a family of commuting operators) whose point correlations in the state ψ are precisely those of a classical Brownian motion. This justified the above definition.

Proof. – We shall need to compute expectation values of the form

$$\langle \phi_F, a^+_{x_1} a_{y_1} a^+_{x_2} a_{y_2} \cdot \ldots \cdot a^+_{x_n} a_{y_n} \phi_F \rangle$$
$$= \langle \Pi_{B_F} a_h \Phi, a^+_{x_1} a_{y_1} a^+_{x_2} a_{y_2} \ldots a^+_{x_n} a_{y_n} \Pi_{B_F} a^+_h \Phi \rangle \tag{4.1}$$

with $x_j, y_j \in \Lambda_L$ and (since $p \in \Lambda_L \backslash \{0\}$)

$$x_j \neq y_j.$$

To this goal notice that:

$$a^+_x a_y \phi_F = \chi_{B_F}(y)(-1)^{\nu(y)} a^+_x \Pi_{B_F \backslash \{y\}} a^+_h \Phi$$
$$= \chi_{B^c_F}(x) \chi_{B_F}(y) \Pi_{B_F \{y, x\}} a^+_h \Phi$$

where the notation

$$\Pi_{B_F \{y, x\}} a^+_h \tag{4.1a}$$

means that the creator a^+_y, in the product indexed by B_F, has been replaced by a^+_x: this replacement makes the factor $(-1)^{\nu(y)}$ disappear. More generally we denote

$$B_F \{y_n, x_n, y_{n-1}, x_{n-1}, \ldots, y_k, x_k\}$$

the set defined inductively as follows: $B_F \{y_n, x_n, \ldots, y_{k+1}, x_{k+1}, y_k, x_k\}$ is obtained from $B_F \{y_n, x_n, \ldots, y_{k+1}, x_{k+1}\}$ by replacing y_k by x_k, if $y_k \in B_F \{y_n, x_n, \ldots, y_{k+1}, x_{k+1}\}$, and $B_F \{y_n, x_n, \ldots, y_k, x_k\}$ is empty if $y_k \notin B_F \{y_n, x_n, \ldots, y_{k+1}, x_{k+1}\}$. We introduce the convenction that a product of operators indexed by the empty set is equal to the identity.

With these notations we have the identity

$$
\begin{aligned}
& a_{x_1}^+ a_{y_1} a_{x_2}^+ a_{y_2} \cdot \ldots \cdot a_{x_n}^+ a_{y_n} \Pi_{B_F} a_h^+ \Phi \\
& = \chi_{B_F}(y_n) \chi_{B_F^c}(x_n) a_{x_1}^+ a_{y_1} \cdot \ldots \cdot a_{x_{n-1}}^+ a_{y_{n-1}} \Pi_{B_F \{y_n, x_n\}} a_h^+ \Phi \\
& = \chi_{B_F}(y_n) \chi_{B_F \{y_n, x_n\}}(y_{n-1}) \chi_{B_F^c}(x_n) \chi_{B_F \{y_n, x_n, y_{n-1}\}^c} \\
& \qquad a_{x_1}^+ a_{y_1} \ldots a_{x_{n-2}}^+ a_{y_{n-2}} \Pi_{B_F \{y_n, x_n, y_{n-1}, x_{n-1}\}} a_h^+ \Phi \\
& = \cdots \\
& = \chi_{B_F}(y_n) \chi_{B_F \{y_n, x_n\}}(y_{n-1}) \\
& \quad \times \ldots \chi_{B_F \{y_n, x_n, y_{n-1}, x_{n-1}, \ldots, y_2, x_2\}} \\
& \chi_{B_F^c}(x_n) \chi_{B_F \{y_n, x_n, y_{n-1}\}^c}(x_{n-1}) \\
& \quad \times \ldots \chi_{B_F \{y_n, x_n, y_{n-1}, x_{n-1}, \ldots, y_2, x_2, y_1\}^c}(x_1) \\
& \qquad \qquad \Pi_{B_F \{y_n, x_n, y_{n-1}, x_{n-1}, \ldots, y_1, x_1\}} a_h^+ \Phi
\end{aligned}
$$

Giben the above identity, we see that the scalar product (4.1) is zero unless

$$B_F = B_F (y_n, x_n, \ldots, y_1, x_1\}$$

and this can happen if and only if

$$(y_n, \ldots, y_1\} = \{x_n, \ldots, x_1\} \tag{4.2}$$

the identity being meant in the sense of sets.

In order to have a more explicit form for the scalar product (4.1) in the nonzero case, we consider the expression

$$\langle \phi_F, a_{k_1}^+ a_{k_1'} \cdot a_{k_2}^+ a_{k_2'} \cdot \ldots \cdot a_{k_n}^+ a_{k_n'} \phi_F \rangle \tag{4.3}$$

From the discussion above we know that for each k_j there exists an index, denoted $\pi(j)$ such that

$$k_j = k'_{\pi(j)} \tag{4.4}$$

Moreover we shall assume that

$$k_i \neq k_j'; \quad \text{if } j \neq \pi(i). \tag{4.5}$$

Notice that the assumption (4.5) means that the index $\pi(j)$, for which (4.4) is satisfied, is unique, *i.e.* that π is a permutation of the set $\{1, \ldots, n\}$.

Moreover it also implies, if (4.3) is non zero, that

$$k_i \neq k_j; \quad k_i' \neq k_j', \quad \text{for } i \neq j. \tag{4.6}$$

Under these assumptions we can compute the scalar product (4.3) as follows.

Since $a_{k_n'} \phi_F = 0$ for $k_n' \notin B_F$ we introduce a factor $\chi_{B_F}(k_n')$. Then we anticommute back $a_{k_n'}$ until it meets $a^+_{k_{\pi(n)}}$: this produces a factor $(-1)^{\nu(k_n')}$. Applying the CAR to this pair produces the factor $a^+_{k_n'} a_{k_n'} = 1 - a_{k_n'} a^+_{k_n'}$ but, since $k_n' \in B_F$, because of the assumption (4.6), the operator term $a_{k_n'} a^+_{k_n'}$ gives zero contribution. Similarly, $a^+_{k_n}$ produces the factor $\chi_{B_F^c}(k_n)$ times a factor $(-1)^{\nu(k_n)}$. In the next step we have two possibilities: either $\pi(n) = n-1$, and in the product there is no more the factor $a_{k_{n-1}}$, otherwise we proceed as for $a^+_{k_n}$.

Iterating the procedure by (4.4) and (4.5) we obtain:

$$\begin{aligned} &\langle \phi_F, a^+_{k_1} a_{k_2} \cdot a^+_{k_3} a_{k_4} \ldots a^+_{k_{2n-1}} a_{k_{2n}}, \phi_F \rangle \\ &\quad = \sum_{\pi} (-1)^{\nu(\pi)} \prod_{i,j} \delta_{i,\pi(j)} \langle \phi_F, a^{\varepsilon_i}_{k_i} a^{\varepsilon_j}_{k_j}, \phi_F \rangle \end{aligned} \tag{4.7}$$

where π is a permutation of the index set $\{1, \ldots, 2n\}$, $\nu(\pi)$ is the parity of the permutation and

$$\langle \phi_F, a^+_{k_1} a_{k_2} \phi_F \rangle = \delta_{k_1,k_2} \chi_{B_F}(k_1) \quad \langle \phi_F, a_{k_1} a^+_{k_2} \phi_F \rangle = \delta_{k_1,k_2} \chi_{B_F^c}(k_1)$$
$$\langle \phi_F, a^+_{k_1} a^+_{k_2} \phi_F \rangle = \langle \phi_F, a_{k_1} a_{k_2} \phi_F \rangle = 0$$

Notice that in expressions of the form

$$\begin{aligned} &\frac{1}{L^{nd}} \sum_{\substack{k_1, \ldots, k_n \\ k_1', \ldots, k_n'}} \langle \phi_F, a^+_{k_1} a_{k_1'} \cdot \ldots \cdot a^+_{k_n} a_{k_n'}, \phi_F \rangle g_{k_1} \bar{g}_{k_1'} \ldots g_{k_n} \bar{g}_{k_n'} \\ &\quad = \frac{1}{L^{nd}} \sum_{k_1, \ldots, k_n} | g_{k_1} |^2 \chi_{B_F^\pi}(k_1) \ldots | g_{k_n} |^2 \chi_{B_F^\pi}(k_n) \end{aligned}$$

where each of the $\chi_{B_F^\pi}(k_1)$ can be $\chi_{B_F}(k_1)$ or $\chi_{B_F^c}(k_1)$, the terms in which condition (4.6) is not satisfied, behave, as $L \to \infty$ as

$$\frac{1}{L} \left(\int_{\mathbf{R}} | g_k |^2 \chi_{B_F^\pi}(k)\, dk \right)^{n-1} \left(\int_{\mathbf{R}} | g_k |^2 \chi_{B_F^\pi}(k)\, dk \right) = O\left(\frac{1}{L}\right).$$

Recalling the definition (2.14) of the collective operators, with $\sigma = 0,\ 1$:

$$B_\lambda^\sigma(S,\,T,\,g,\,v,\,L) = \frac{\lambda}{L^d}\int_{S/\lambda^2}^{T/\lambda^2} dt$$

$$\sum_{k\in\Lambda_L}\sum_{p\subset\Lambda_L^*} g_k\,u_p^\sigma\,A^+(S_t^L\,\chi_{k+\sigma_p})\,e^{i(1-2\sigma)\omega_p t}\,A(S_t^L\,\chi_{k+(1-\sigma)p})$$

$$= \frac{\lambda}{L^d}\int_{S/\lambda^2}^{T/\lambda^2} dt\sum_{k\in\Lambda_L}\sum_{p\subset\Lambda_L^*}$$

$$g_k\,u_p^\sigma\,A^+\,e^{it(\varepsilon_{k+\sigma_p}-\varepsilon_{k+1(1-\sigma)p}+(1-2\sigma)\omega_p t}\,a_{k+\sigma_p}^+\,a_{k+(1-\sigma)p}$$

We want to calculate

$$\lim_{\lambda\to 0}\lim_{L\to\infty}\langle\phi_F,\,B_\lambda^{\sigma_1}(S_1,\,T_1,\,g_1,\,v_1,\,L)\ldots B_\lambda^{\sigma_n}(S_n,\,T_n,\,g_n,\,v_n,\,L)\,\phi_F\rangle$$

$$= \frac{\lambda^n}{L^{nd}}\int_{S_1/\lambda^2}^{T/\lambda^2} dt_1\ldots\int_{S_n/\lambda^2}^{T/\lambda^2} dt_n\prod_{i=1}^n$$

$$\Big(\sum_{k_i'\in\Lambda_L}\sum_{p_i\in\Lambda_L^*} g_{k_i'}\,u_{p_i}^{\sigma_i}\,e^{it_i[\varepsilon_{k_i'+\sigma_i p_i}-\varepsilon_{k_i'+(1-\sigma_i)p_i}+(1-2\sigma_i)\omega_{p_i}]}\Big)$$

$$\langle\phi_F,\,a_{k_1'+\sigma_1 p_1}^+\,a_{k_1'+(1-\sigma_1)p_1}\ldots a_{k_n'+\sigma_n p_n}^+\,a_{k'+(1-\sigma_n)p_n}\,\phi_F\rangle. \tag{4.9}$$

Performing the change of variables

$$k_i' + \sigma_i\,p_i \to k_{2i-1},\quad k_i' + (1-\sigma_i)\,p_i \to k_{2i};\quad i = 1,\,\ldots,\,n \tag{4.10}$$

and neglecting terms of order $O\left(\frac{1}{L}\right)$, we can assume that (4.6) holds. We shall see that the index i, which appears in (4.10) is precisely the index that labels the limiting bosonic operators. With this notation, the creators are labeled by the *odd* indices; the annihilators by even indices. But, by the preceeding argument, each annihilator variable k_{2j} is equal to one (and only one) creator variable: this shall be denoted $k_{2\pi(j)-1}$. Thus, by definition of π : $k_{2j} = k_{2\pi(j)-1}$. With these notations, from (4.9), we obtain that, in the limit $L \to \infty$:

$$\lambda^n\int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1\ldots\int_{S_n/\lambda^2}^{T/\lambda^2} dt_n\sum_\pi(-1)^{\nu(\pi)}\prod_{j=1}^n\int dk_{2j-1} \tag{4.11}$$

$$\chi_{B_F^\pi}(k_{2j-1})\,e^{it_j(\varepsilon_{k_{2j-1}}-\varepsilon_{k_{2\pi(j)-1}}+(1-2\sigma_j)\,\omega_j)}\,g_j\,u_j^{\sigma_j}$$

where $\chi_{B_F^\pi}(k_{2j-1})$ is equal to $\chi_{B_F}(k_{2j-1})$ or $\chi_{B_F^\pi}(k_{2j-1})$ depending on the permutation π, $\omega_j = \omega_{k_{2\pi(j)-1}-k_{2j-1}}$ and we have introduced the notation:

$$u_j\, g_j = g_{k_{2j-1}}\, g_{k_{2\pi(j)-1}}\, u_{k_{2\pi(j)-1}-k_{2j-1}}.$$

To find the limit of (4.11), as $\lambda \to 0$, let us start considering the case in which n is even. Recall that we want to prove that, in the limit (4.9), the B_λ^σ tend to some boson operator B^σ. But each B_λ^σ is a sum (integral) of fermionic operators $a_{k_{2j-1}}^+\, a_{k_{2j}}$, therefore *each such pair should behave like a single object.* In order to prove this we have to show that, if in the gaussian expansion of the left hand side of (4.9) the creator $a_{k_{2j-1}}^+$ produces a scalar product with the annihilator $a_{k_{2\pi(j)}}$ then the annihilator $a_{k_{2j}}$ must be paired with the creator $a_{k_{2\pi(j)-1}}$. We shall prove that the terms for which this condition is not satisfied become negligible in the limit $\lambda \to 0$.

In order to evidentiate the negligible terms, it is convenient to rewrite the product (4.11) as a product over $n/2$ pairs (since, in the limit, each such pair shall define a scalar product in the bosonic one-particle space). To this goal, recall that, in (4.11) the permutation π of $\{1, \ldots, n\}$ is fixed and define inductively the subset $A_\pi = \{\nu_1, \ldots, \nu_{n/2}\} \subseteq \{1, \ldots, n\}$ as follows:

$$\nu_1 = 1$$

$$\nu_{j+1} = \operatorname{Min}\{\{1, \ldots, n\}\backslash\{\nu_1, \pi(\nu_1), \pi^{-1}(\nu_1), \ldots, \nu_j, \pi(\nu_j), \pi^{-1}(\nu_j)\}$$

With these notations the expression (4.11) can be written:

$$\lambda^n \int_{S/\lambda^2}^{T/\lambda^2} dt_1 \ldots \int_{S_n/\lambda^2}^{T/\lambda^2} dt_n \sum_\pi (-1)^{\nu(\pi)} \prod_{j\in A_\pi} \tag{4.11}$$

$$\int dk_{2j-1}\, dk_{2\pi(j)-1}\, g_j\, u_j^{\sigma_j}\, g_{\pi(j)}\, u_{\pi(j)}^{\sigma_{\pi(j)}}$$

$$\chi_{B_F^\pi}(k_{2j-1})\, \chi_{B_F^\pi}(k_{2\pi(j)-1})\, e^{it_j[\varepsilon_{k_{2j-1}} - \varepsilon_{k_{2\pi(j)-1}} + (1-2\sigma_j)\,\omega_j]}$$

$$e^{it_{\pi(j)}[\varepsilon_{k_{2\pi(j)-1}} - \varepsilon_{k_{2\pi(\pi(j))-1}} + (1-2\sigma_{\pi(j)})\,\omega_{\pi(j)}]}$$

$$= \lambda^n \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1 \ldots \int_{S_n/\lambda^2}^{T_n/\lambda^2} dt_n \sum_\pi (-1)^{\nu(\pi)} \prod_{j\in A_\pi}$$

$$\int dk_{2j-1}\, dk_{2\pi(j)-1}\, g_j\, u_j\, g_{\pi(j)}\, u_{\pi(j)}$$

$$\chi_{B_F^\pi}(k_{2j-1})\, \chi_{B_F^\pi}(k_{2\pi(j)-1})\, e^{i(t_j - t_{\pi(j)})[\varepsilon_{k_{2j-1}} - \varepsilon_{k_{2\pi(j)-1}} + (1-2\sigma_j)\,\omega_j]}$$

$$e^{it_{\pi(j)}[\varepsilon_{k_{2\pi(j)-1}} - \varepsilon_{k_{2\pi(\pi(j))-1}} + (1-2\sigma_j)\,\omega_j + (1-2\,\sigma_{\pi(j)})\omega_{\pi(j)}]}.$$

Performing the change of variables $t_j - t_{\pi(j)} = \tau_j$ and $\lambda^2 t_{\pi(j)} = \tau_{\pi(j)}$ we obtain:

$$\sum_{\pi} (-1)^{\nu(\pi)} \prod_{j \in A_\pi} \int_{S_{\pi(j)}}^{T_{\pi(j)}} d\tau_{\pi(j)} \int_{(S_j - \tau_{\pi(j)})/\lambda^2}^{(T_j - \tau_{\pi(j)})/\lambda^2} d\tau_j$$

$$\int dk_{2j-1}\, dk_{2\pi(j)-1}\, g_j\, u_j^{\sigma_j}\, g_{\pi(j)}\, u_{\pi(j)}^{\sigma_{\pi(j)}}\, \chi_{B_F^\pi}(k_{2j-1})\, \chi_{B_F^\pi}(k_{2\pi(j)-1})$$

$$e^{i\tau_j [\varepsilon_{k_{2j-1}} - \varepsilon_{k_{2\pi(j)-1}} + \omega_j (1 - 2\sigma_j)]}$$
$$\times\, e^{i \frac{\tau_{\pi(j)}}{\lambda^2} [\varepsilon_{k_{2j-1}} - \varepsilon_{k_{2\pi(\pi(j))-1}} + (1-2\sigma_j)\omega_j + (1 - 2\sigma_{\pi(j)})\omega_{\pi(j)}]}. \tag{4.12}$$

In the limit $\lambda \to 0$ we distinguish two kinds of terms in the sum over π in Eq. (4.12):

1) Let us consider a term in the sum over π in Eq. (4.12) such that, for some $\bar{j}$, $k_{2\bar{j}-1} \neq k_{2\pi(\pi(\bar{j}))-1}$: such term is vanishing for the Riemann Lebesge lemma.

2) If $\forall j$, $k_{2j-1} = k_{2\pi(\pi(j))-1}$ than $\omega_j = \omega_{\pi(j)}$ and from (4.12) we obtain:

$$\lim_{\lambda \to 0,\, L \to \infty} \langle \phi_F,\, B_\lambda^{\sigma_1}(S_1, T_1, g_1, v_1)$$
$$\ldots B_\lambda^{\sigma_1}(S_n, T_n, g_n, v_n, L),\, \phi_F \rangle$$
$$= \sum_p \prod_{i, j \in p} \langle \psi,\, B^{\sigma_i}(S_i, T_i, g_i, v_i)$$
$$\times\, B^{\sigma_j}(S_j, T_j, g_j, v_j)\, \psi \rangle \tag{4.13}$$

where p is a pair partition of the indices $1, \ldots, n$.

A similar argument shows that if n is odd the left hand side of (4.9) vanishes in the $\lambda \to 0$, $L \to \infty$ limit. Finally, it is not difficult to check (*cf.* [AcLuVo94a]) that the communators of the $B^\#$ fields are scalars (given the explicit form of the limit this proof is not necessary). Thus the $B^\#$ effectively realize a representation of the CCR in unbounded from.

5. LIMIT PROCESS

Because of the remark after Theorem (4.1), the fact that the collective creators and annihilators appear, in the interaction Hamiltonian, in the form

$B^+_{t,\lambda}(u \otimes g) + B_{t,\lambda}(u \otimes g)$, it $B_{t,\lambda}(u \otimes g) = B(0, t, g, u)$ suggests that the limit wave operator U_t satisfies a quantum stochastic differential equation driven by the *classical Brownian motion* $W_t(u \otimes g)$. The following theorem shows that this is indeed the case. Notice that, if we write the limit equation (5.3), in informal notations, in terms not of the Brownian motion $B^+_t + B_t$ but of its formal time derivative, *i.e.* the white noise $\dot{B}^+_t + \dot{B}_t$, then this equation takes the form

$$\frac{d}{dt} U(t) = \Big[i(\dot{B}^+_t(u \otimes g) + \dot{B}_t(u \otimes g)) - i\,\mathrm{Im}\,(u \otimes g \,|\, u \otimes g) - \frac{1}{2}(u \otimes g \,|\, u \otimes g) \Big] U_t$$

which looks like a non-self-adjoint Hamiltonian with a dissipation term $-\frac{1}{2}(u \otimes g \,|\, u \otimes g)(\leq 0)$. This non-self-adjointness is however only apparent as seen from the explicit solution (5.4) which is unitary. The reason of this apparent paradox is that the formal identification of (5.3) with an ordinary differential equation is wrong: the dissipative part in (5.3) is the real part of the Ito correction term which restores the unitarity due to the quantum fluctuation-dissipation theorem of [Ac90]. In the proof of Theorem (5.1) below we use the uniform estimate of [Lu92] but we introduce a new idea which avoids the combinatorial complications due to the non Fock character of the ground state ϕ_F.

THEOREM 5.1. – *In the notations* (1.2), (2.11), (2.14), (2.18) *the limit*

$$\begin{aligned}\lim_{\lambda\to 0}\lim_{L\to\infty} \langle\, & B^{\sigma(1)}_\lambda(S_1, T_1, f_1, u_1, L) \\ & \dots B^{\sigma(1)}_\lambda(S_N, T_N, f_N, u_N, L)\,\phi_F, \\ & U^{(\lambda, L)}_{t/\lambda^2}\, B^{\tau(1)}_\lambda(S'_1, T'_1, f'_1, u'_1, L) \\ & \dots B^{\tau(N')}_\lambda \times (S'_N, T'_N, f'_N, u'_N, L)\,\phi_F\,\rangle \end{aligned} \tag{5.1}$$

exists and is equal to

$$\left\langle \prod_{h=1}^{N} B^{\sigma(h)}(S_h, T_h, f_h, u_h)\,\psi,\ U_t \prod_{h=1}^{N'} B^{\tau(h)}(S'_h, T'_h, f'_h, u'_h)\,\psi \right\rangle \tag{5.2}$$

where $\sigma \in \{0, 1\}^N$, $\tau \in \{0, 1\}^{N'}$, *and* $B^{\#}(S, T, u, f)$ *is the quantum Brownian motion defined in Theorem* (4.1).

U_t is the unique (unitary) solution of the Stochastic Differential Equation:

$$U_t = 1 + \int_0^t \{id\, B_s^+ (u \otimes g) + id\, B_s (u \otimes g) - (u \otimes g \,|\, u \otimes g)_- ds\} \, U_s \quad (5.3)$$

where we use the notation:

$$dB_s^{\#} (u \otimes g) := dB^{\#} (0,\, s,\, u,\, g)$$

with $(u \otimes g \,|\, u \otimes g)_-$ *defined as* $(u \otimes g \,|\, u \otimes g)$ (cf 2.18) *but with* $\int_{-\infty}^{+\infty}$ *replaced by* $\int_{-\infty}^{0}$. *Moreover the solution of* (5.3) *is*

$$U_t = e^{i\,(B_t^+ (u \otimes g) + B_t (u \otimes g) - \mathrm{Im}\,(u \otimes g \,|\, u \otimes g)_-)}. \quad (5.4)$$

Remark. – The Ito correction term can be written as $(u \otimes g \,|\, u \otimes g)_- = i\, E_0 + \Gamma$ with E_0 and Γ real and $\Gamma > 0$. Using the well known distribution formula:

$$\int_{-\infty}^{0} e^{i\omega t}\, dt = i\, P \left(\frac{1}{\omega} \right) + \pi \delta(\omega)$$

where P denotes the principal part, we have that:

$$E_0 = P \int dk\, dp \, \frac{|\, u_p\,|^2\, g_k^2\, g_{k+p}^2}{\varepsilon\,(k) - \varepsilon\,(k+p) + \omega_p}$$

$$\times \, [\chi_{B_F}\,(k)\, \chi_{B_F^c}\,(k+p) - \chi_{B_F^c}\,(k)\, \chi_{B_F}\,(k+p)]$$

$$\Gamma = \int dk\, dp \,|\, u_p\,|^2\, g_k^2\, g_{k+p}^2\, \delta\,(\varepsilon\,(k) - \varepsilon\,(k+p) + \omega_p)$$

$$[\chi_{B_F}\,(k\, \chi_{B_F^c}\,(k+p) + \chi_{B_F^c}\,(k)\, \chi_{B_F}\,(k+p)]$$

If

$$\lim_{\lambda \to 0} \lim_{L \to \infty} \langle\, \phi_F,\, U_{t/\lambda^2}\, \phi_F\, \rangle = \langle\, \psi,\, U_t\, \psi\, \rangle$$

from (5.3) one obtains:

$$\frac{d}{dt} \langle\, \psi,\, U_t\, \psi\, \rangle = -(i\, E_0 - \Gamma) \langle\, \psi,\, U_t\, \psi\, \rangle$$

so that E_0 and Γ are respectively the ground state energy shift and the lifetime of the ground state; they coincide with the correspond quantities computed by standarding second order (in λ) perturbation theory (*see* for instance [Ha]).

Proof. – The proof of the Theorem 5.1 will be done in several steps. Let us define:

$$W_\lambda\,(S,\, T,\, g_1,\, u_1,\, L) = \lambda \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt\, B_\lambda\,(t,\, g_1,\, u_1,\, L)$$

and the operator Ψ_λ as:

$$\Psi_\lambda(g_1, u_1) = \sum_{n=1}^{\infty} \frac{(-i)^n \lambda^n}{n!} \left[\int_{S_1/\lambda^2}^{T_1/\lambda^2} dt\, B_\lambda(t, g_1, u_1) \right]^n.$$

We consider:

$$\langle \phi_F, \Psi_\lambda(g_1, u_1) U_{\frac{t}{\lambda^2}} \Psi_\lambda(g_2, u_2) \phi_F \rangle \tag{5.6}$$

Expanding eq. (5.6) with the iterated series one obtains:

$$\langle \phi_F, \Psi_\lambda(g_1, u_1) \Psi_\lambda(g_2, u_2), \phi_F \rangle + \sum_{n=1}^{\infty} (-i)^n I_n$$

$$I_n = \lambda^n \int_0^{\frac{t}{\lambda^2}} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$

$$\langle \phi_F, \Psi_\lambda(g_1, u_1) B_\lambda(t_1, g, u, L) \ldots B_\lambda(t_n, g, u, L) \Psi_\lambda(g_2, u_2) \phi_F \rangle.$$

It is possible to prove, by an adaptation of the technique used in [Lu92], that $\sum_n (-i)^n I_n$ is a series absolutely convergent, uniformly in the pair (λ, t). It is convenient to write $I_n = I_n^1 + I_n^2 + I_n^3$ where I_n^3 is given by the terms in which the fermionic operators belonging to the same B_λ^σ are *paired* in the sense of eq. (4.7) with operators belonging to different B_λ^σ, and by the analysis of section 4 $\lim_{\lambda \to 0} \lim_{L \to \infty} I_n^3 = 0$ and:

$$I_n^1 = \sum_{\{m\}} \int_0^{\frac{t}{\lambda^2}} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{i, j \in \{m\};\, i-j=1}$$

$$\langle \phi_F, \lambda B_\lambda(t_i, g, u, L) \lambda B_\lambda(t_j, g, u, L), \phi_F \rangle$$

$$\prod_{i \in \{m_1\}} \int_{S_1/\lambda^2}^{T_1/\lambda^2} d\tau \langle \phi_F, \lambda B_\lambda(t_i, g, u, L) \lambda B_\lambda(\tau, g_1, u_1, L), \phi_F \rangle$$

$$\prod_{i \in \{m_2\}} \int_{S_1/\lambda^2}^{T_1/\lambda^2} d\tau \langle \phi_F, \lambda B_\lambda(t_i, g, u, L) \lambda B_\lambda(\tau, g_2, u_2, L), \phi_F \rangle$$

$$\times \langle \phi_F, \Psi_\lambda(g_1, u_1) \Psi_\lambda(g_2, u_2), \phi_F \rangle$$

where $\{m\} \cup \{m_1\} \cup \{m_2\} = 1 \ldots n$ and:

$$\langle \phi_F, \lambda B_\lambda(t_i, g_1, u_1, L) \lambda B_\lambda(t_2, g_2, u_2, L), \phi_F \rangle$$

$$= \frac{\lambda^2}{L^2} \sum_{k, p} e^{i(p^2 + 2kp - \omega_p)(t_1 - t_2)}$$

$$[g_1 \bar{u}_1 g_2 u_2 \chi_{B_F}(k) \chi_{B_F^c}(k+p) + u_1 g_1 \bar{u}_2 g_2 \chi_{B_F^c}(k) \chi_{B_F}(k+p)]$$

It is straightforward to check that I_n^1 converges to a value different from zero in the weak coupling limit. On the other hand by definition I_n^2 is given by:

$$I_n^2 = \int_0^{\frac{t}{\lambda^2}} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{i,\, j \in \{\bar{m}\}}^{*}$$
$$\times \langle \phi_F,\, \lambda B_\lambda (t_i,\, g,\, u,\, L)\, \lambda B_\lambda (t_j,\, g,\, u,\, L)\, \phi_F \rangle$$

$$\prod_{\alpha = \pm 1} \prod_{k \in \{\bar{m}_\alpha\}} \int_{S_1/\lambda^2}^{T_1/\lambda^2} d\tau \, \langle \phi_F,\, \lambda B_\lambda (t_k,\, g,\, u,\, L)\, \lambda B_\lambda (\tau,\, g_\alpha,\, u_\alpha,\, L)\, \phi_F \rangle$$

$$\langle \phi_F,\, \Psi_\lambda (g_1,\, u_1)\, \Psi_\lambda (g_2,\, u_2),\, \phi_F \rangle$$

where $\prod^*$ means that at least a pair $(i,\, j)$ is such that $j - i > 1$ and $\{\bar{m}\} \cup \{\bar{m}_1\} \cup \{m_2\} = 1 ... n$.

Performing the change of variables $\tau_j = \frac{t_j - t_i}{\lambda^2}$ we have that I_2 contains at least an integral of the form:

$$\int_0^{t_{i-1}} dt_i \int_{\frac{-t_i}{\lambda^2}}^{\frac{(t_{j-1} - t_i)}{\lambda^2}} d\tau_j \, \langle \sigma_F,\, \lambda B_\lambda (t_i,\, g,\, u,\, L)\, \lambda B_\lambda (\lambda^2 \tau_j + t_i,\, g,\, u,\, L)$$

with $t_{j-1} - t_i < 0$ and in the limit $\lambda \to 0$ the above expression is vanishing. Deriving eq. (5.6) with respect to t we obtain:

$$\langle \phi_F,\, \Psi_\lambda (g_1,\, u_1)\, \frac{1}{\lambda}\, B_\lambda \left(\frac{t}{\lambda^2},\, g,\, u,\, L \right) U_{\frac{t}{\lambda^2}}\, \Psi_\lambda (g_2,\, u_2),\, \phi_F \rangle. \tag{5.7}$$

The above expression is an average over the ground state of a product of fermionic operator, and by eq. (4.7) it is given by a sum of terms in which each fermionic operator is *paired* in the sense of the preceding section with some other. We call I_1 the sum of the terms in which the fermionic operators in $B_\lambda \left(\frac{t}{\lambda^2},\, g,\, u\right)$ are paired with the operators in $\Psi_\lambda (g_1,\, u_1)$, I_2 the analogue of I_3 with $\Psi_\lambda (g_2,\, u_2)$, I_3 the sum of terms in which the operators in $B_\lambda \left(\frac{t}{\lambda^2},\, g,\, u\right)$ are paired with fermionic operators in $U_{\frac{t}{\lambda^2}}$; the other terms *i.e.* the terms in which the fermionic operators in $B_\lambda \left(\frac{t}{\lambda^2}\right)$ are paired with operators belonging to different B operators vanish in the $\lambda \to 0$, $L \to \infty$ limit, as follows from the computation of the preceding section.

Let us start considering the I_1 term. We need the following lemma:

LEMMA. – *It holds that:*

$$\lim_{\lambda\to 0,\, L\to\infty} \langle\, \phi_F,\ \lambda B_\lambda\,(S_1,\ T_1,\ g_1,\ u_1,\ L)$$

$$\cdots \lambda B_\lambda\,(S_m,\ T_m,\ g_m,\ u_m,\ L)\, \frac{1}{\lambda}\, B_\lambda\left(\frac{t}{\lambda^2},\ g,\ u,\ L\right),\ \phi_F\,\rangle$$

$$= \sum_{\tilde{i}} \chi_{S_{\tilde{i}},\,T_{\tilde{i}}}\,(t)\,(g_{\tilde{i}} \otimes u_{\tilde{i}} | g \otimes u)$$

$$\times \sum_\pi \prod_{i,\, j\in\pi/\tilde{i}} \langle\, \psi,\ B\,(T_i,\ S_i,\ g_i,\ u_i)\, B\,(T_j,\ S_j,\ g_j,\ u_j)\, \psi\,\rangle$$

where $\pi/\tilde{i}$ is a partition in pairs of the indices $1,\ \ldots,\ m$ without $\tilde{i}$ and $\sum_\pi$ is the sum over such partitions.

Proof. – The left hand side of the above equation is given, by eq. (4.7), by a sum of terms; noting that the terms in which the fermionic operators of the $B_\lambda\left(\frac{t}{\lambda^2},\ g,\ u\right)$ operator are paired with fermionic operators with different times are vanishing in the limit $\lambda\to 0$, $L\to\infty$, the left hand side of the above equation in given by:

$$\lim_{\lambda\to 0} \sum_{\tilde{i}=1}^{m} \int_{S_{\tilde{i}}/\lambda^2}^{T_{\tilde{i}}/\lambda^2} dt_{\tilde{i}} \int dk_{\tilde{i}}\, dp_{\tilde{i}}\, [e^{i(p_{\tilde{i}}^2+2p_{\tilde{i}}k_{\tilde{i}}-\omega_{p_{\tilde{i}}})(\frac{t}{\lambda^2}-t_{\tilde{i}})}$$

$$\times\ g_{\tilde{i}}\, g\bar{u}_{\tilde{i}}\, u\chi_{B_F}(k_{\tilde{i}})\, \chi_{B_F^c}\,(k_{\tilde{i}}+p_{\tilde{i}})$$

$$+\, e^{-i\,(p_{\tilde{i}}^2+2p_{\tilde{i}}k_{\tilde{i}}-\omega_{p_{\tilde{i}}})(\frac{t}{\lambda^2}-t_{\tilde{i}})}\, g_{\tilde{i}}\, gu_{\tilde{i}}\, \bar{u}g\, \chi_{B_F^c}\,(k_{\tilde{i}})\, \chi_{B_F}\,(k_{\tilde{i}}+p_{\tilde{i}})$$

$$\lambda^{m-1} \sum_\pi \prod_{i,\, j\in\pi/\tilde{i}} \int_{S_j/\lambda^2}^{T_j/\lambda^2} dt_j \int_{S_i/\lambda^2}^{T_i/\lambda^2} dt_i$$

$$\langle\, \phi_F,\ B_\lambda\,(t_j,\ g_j,\ u_j,\ L)\, B_\lambda\,(t_i,\ g_i,\ u_i)\, \phi_F\,\rangle$$

$$= \lim_{\lambda\to 0} \sum_{\tilde{i}=1}^{m} \int_{(S_{\tilde{i}}-t)/\lambda^2}^{(T_{\tilde{i}}-t)/\lambda^2} d\tau_{\tilde{i}} \int dk_{\tilde{i}}\, dp_{\tilde{i}}$$

$$[e^{i\,(p_{\tilde{i}}^2+2p_{\tilde{i}}k_{\tilde{i}}-\omega_{p_{\tilde{i}}})\tau}\, g_{\tilde{i}}\, \bar{u}_i\, ug\, \chi_{B_F}\,(k_{\tilde{i}})\, \chi_{B_F^c}\,(k_{\tilde{i}}+p_{\tilde{i}})$$

$$+\, e^{-i(p_{\tilde{i}}^2+2p_{\tilde{i}}k_{\tilde{i}}-\omega_{p^{\tilde{i}}})^\tau}\, u_{\tilde{i}}\, g_{\tilde{i}}\, \bar{u}g\, \chi_{B_F^c}\,(k_{\tilde{i}})\, \chi_{B^F}\,(k_{\tilde{i}}\, p_{\tilde{i}})]$$

$$\lambda^{m-1} \sum_\pi \prod_{i,\, j\in\pi/\tilde{i}} \int_{S_j/\lambda^2}^{T_j/\lambda^2} dt_j \int_{S_i/\lambda^2}^{T_i/\lambda^2} dt_i$$

$$\langle\, \phi_F,\ B_\lambda(t_j,\ g_j,\ u_j,\ L)\, B_\lambda(t_i,\ g_i,\ u_i)\, \phi_F\,\rangle$$

where n_π/i is the subset of $1, \ldots, i-1, i+i, \ldots, m$ such that the lemma holds. By straightforward application of the above lemma we obtain:

$$\lim_{\lambda\to 0,\, L\to\infty} I_1 = (u_1 \otimes g_1 \,|\, u \otimes g)\, \chi_{[S_1,\, T_1]}(t)\, \langle\, \psi, \Psi_\lambda(g_1, u_1)\, U_t\, \Phi_\lambda(g_2, u_2), \psi\,\rangle$$

$$\lim_{\lambda\to 0,\, L\to\infty} I_2 = \langle\, u \otimes g \,|\, u_2 \otimes g_2)\, \chi_{[S_2,\, T_2]}(t)\ \psi, \Psi_\lambda(g_1, u_1)\, U_t\, \Psi_\lambda(g_2, u_2), \psi\,\rangle.$$

In order to compute the I_3 term we need another lemma:

LEMMA. – *It holds that:*

$$\lim_{\lambda\to 0,\, L\to\infty} \int_0^{\frac{t}{\lambda^2}} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$

$$\langle \phi_F,\ \lambda B_\lambda(t_1,\, g,\, u,\, L) \cdots \lambda B_\lambda(t_n,\, u,\, g,\, L)]\, \frac{1}{\lambda}\, B\left(\frac{t}{\lambda^2},\, u,\, g,\, L\right),\ \phi_F\,\rangle$$

$$= (u \otimes g | u \otimes g)_- \langle\, \psi,\ u_t\, \psi\,\rangle$$

Proof. – The proof consits in showing that the terms in which $B_\lambda(t/\lambda^2,\, g,\, u,\, L)$ is paired with $B_\lambda(t_j,\, g_j,\, u_j,\, L)$ with $j \neq 1$ are vanishing in the limit $\lambda \to 0$, $L \to \infty$. The left hand side of the above expression can be written as:

$$\lim_{\lambda\to 0} \sum_{\tilde{i}=1}^{n} \int_0^{\frac{t}{\lambda^2}} dt_1 \cdots \int_{-t/\lambda^2}^{t_{\tilde{i}-1}-t/\lambda^2} d\tau_{\tilde{i}} \int_0^{t_{n-1}} dt_n \int dk_{\tilde{i}}\, dp_{\tilde{i}}\ [e^{i(p_{\tilde{i}}^2+2p_{\tilde{i}}k_{\tilde{i}}-\omega_{p_{\tilde{i}}})\tilde{\tau}_i}$$

$$g u\, g\bar{u}\, \chi_{B_F}(k_{\tilde{i}})\, \chi_{B_F^c}(k_{\tilde{i}} + p_{\tilde{i}})$$
$$+\, e^{-i\,(p_{\tilde{i}}^2+2p_{\tilde{i}}k_{\tilde{i}}-\omega_{p_{\tilde{i}}})\tilde{\tau}_i}\ g\,\bar{u} u g\, \chi_{B_F^c}(k_{\tilde{i}})\, \chi_{B_F}(k_{\tilde{i}} + p_{\tilde{i}})]$$

$$\lambda^{n-1} \sum_\pi \prod_{i,\, j \in \pi/\tilde{i}} \langle\, \phi_F,\ B_\lambda(t_j,\, g,\, u,\, L)\, B_\lambda(t_i,\, g,\, u,\, L)\, \phi_F\,\rangle$$

where $t_0 = t\,\lambda^2$ and, is $\tilde{i} \neq 1$, than $t_{\tilde{i}-1} - t/\lambda^2 < 0$ and the above term is vanishing in the limit.

Using this lemma we have that:

$$\lim_{\lambda\to 0,\, L\to\infty} I_3 = (u \otimes g \,|\, u \otimes g)_- \langle\, \Psi_\lambda(g_1\, u_1)\, U_t\, \Psi_\lambda(g_2,\, u_2)\,\rangle$$

and the theorem is proved.

The last part of the theorem is proved by a straigthforward adaptation of [AcFriLu].

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