

Quantum probability and the non-locality issue in quantum theory

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Abstract. Three computers, with local independent choices, generate the EPR correlations hence violating Bell's inequality.

1 Bell's inequality and its consequences according to Bell

Bell's inequality: Let A, B, C be random variables defined on the same probability space (Ω, \mathcal{F}, P) and with values in the set $\{-1, 1\}$. Denote

$$\langle AB \rangle := \int_{\Omega} A(\omega)B(\omega)P(d\omega)$$

the correlations (mean zero can be assumed without loss of generality). Then the following inequality holds:

$$|\langle AB \rangle - \langle CB \rangle| \leq 1 - \langle AC \rangle \quad (1)$$

Proof. Since the expectation value is linear

$$|\langle AB \rangle - \langle CB \rangle| = |\langle AB - CB \rangle| \quad (2)$$

Since $|\langle X \rangle| \leq \langle |X| \rangle$ and A, B, C are ± 1 -valued (2) is

$$\leq \langle |AB - CB| \rangle = \langle |AB| \cdot |1 - AC| \rangle = \langle |1 - AC| \rangle = \langle 1 - AC \rangle = 1 - \langle AC \rangle$$

Corollary (Clauser-Horne-Shimony-Holt (CHSH) inequality): Let A, B, A', B' be random variables defined on the same probability space (Ω, \mathcal{F}, P) and with values in the set $\{-1, 1\}$. Then the following inequality holds:

$$|\langle AB \rangle - \langle A'B \rangle + \langle AB' \rangle + \langle A'B' \rangle| \leq 2 \quad (3)$$

Proof. Replace C with A' in (1). Then add to this what we obtain by replacing again A' with $-A'$ and B with B' .

Theorem 1: There cannot exist a stochastic process $S_a^{(1)}, S_b^{(2)} (a, b \in [0, 2\pi])$ defined on a probability space (Ω, \mathcal{F}, P) and with values in the set $\{\pm 1\}$, whose correlations are given by:

$$\langle S_a^{(1)} S_b^{(2)} \rangle = -\cos(a - b) \quad ; \quad a, b \in [0, 2\pi] \quad (4)$$

Proof. (4) implies that, for any $a \in [0, 2\pi]$, $S_a^{(1)} = -S_a^{(2)}$. Because of (2), if such a process exists, it should satisfy

$$|\langle S_a^{(1)} S_b^{(2)} \rangle - \langle S_c^{(1)} S_b^{(2)} \rangle| - \langle S_a^{(1)} S_c^{(2)} \rangle \leq 1 \quad (5)$$

But, because of (4), with $a = 0$, $b = \pi/2$, $c = \pi/4$, the left hand side of (5) is equal to $\sqrt{2}$.

Physical meaning (Bell [Be64]): “... the statistical predictions of quantum mechanics are incompatible with local predetermination ...”

2 Criticism to Bell’s analysis

Observations:

- (i) The contradiction, pointed out by Bell, arises only from his implicit postulate that 3 statistical correlations, coming from 3 mutually incompatible experiments, can be described within a single classical probabilistic model.
- (ii) This implicit postulate is by no means a consequence of locality and reality.

Weak point of Bell’s proof. The first step (2) of Bell’s proof:

$\langle AB \rangle - \langle CB \rangle = \langle AB - CB \rangle$ reads as

$$\int_{\Omega} A(\omega)B(\omega)P_{a,b}(d\omega) - \int_{\Omega} C(\omega)B(\omega)P_{c,b}(d\omega) = \int_{\Omega} (A(\omega)B(\omega) - C(\omega)B(\omega)) P_{a,c,b}(d\omega) .$$

Thus, Bell’s proof relies on the existence of the triple joint probability $P_{a,c,b}$. While the pair joint probabilities $P_{a,b}, P_{c,b}, \dots$ are experimentally observable, there is no reason to postulate that the, experimentally unobservable, triple joint probabilities $P_{a,c,b}$ always exist. Indeed, it is well known from classical probability that there are constraints, i.e. compatibility conditions, which relate the pair with the triple joint probabilities and which are necessary conditions for the existence of the latter ones. This fact was first pointed out in [Ac81a].

It has been claimed that the existence of the triple joint probabilities is consequence of the “realism” assumption. This is true for “ballot box” (or Einstein) realism, not for “chameleon realism”.

3 “Chameleon (or adaptive) Reality”

Starting point: von Neumann’s measurement theory

The joint evolution of system and apparatus should be taken into account to analyze measurement processes.

von Neumann’s theory was extended in [Ac93] (cf. also [AcRe00b] [AcRe01a]) to include in it the two basic conditions of locality and causality. To this goal the notion of “Chameleon reality” was introduced to denote those systems whose dynamics depends on the observables we want to measure (or the environment in general). For such systems what you measure is a *response to an interaction*. This is opposed to ballot box (or Einstein) reality in which you measure what was there (independently of the environment).

The law of classical statistics and probability were abstracted having in mind the “ballot box” reality model. However one can prove that some simple induction rules (e.g. the “counterfactual argument” cf. [AcRe00b]) which are constantly used in classical statistics and probability, lead to wrong conclusions when applied to “chameleon reality”. Therefore the statistics of adaptive systems must be based on different induction rules, hence on a different probability theory. Quantum probability is one of these non-Kolmogorovian models. Others are mathematically possible and we expect that they will find applications to other types of adaptive systems arising, for example, in medicine, in economics...

Remark 1: Such a dependence, of the dynamical evolution of a system on the observables we want to measure, is called “chameleon effect”.

Remark 2: The chameleon effect can also be considered within the framework of classical mechanics.

Remark 3: A classical system realizing the chameleon effect can violate Bell’s inequality while satisfying locality condition.

4 A local classical system violating Bell’s inequality

- (i) The classical system σ is composed of two sub-systems: σ_1 and σ_2
- (ii) The states of σ_1 and σ_2 are numbers λ_1, λ_2 in $[0, 2\pi]$; the observables are

± 1 -valued functions $\tilde{S}_a^{(1)}(\lambda_1)$, $\tilde{S}_b^{(2)}(\lambda_2)$ of the states (realistic description). Also a, b take values in $[0, 2\pi]$.

- (iii) At time 0 the states (λ_1, λ_2) , of (σ_1, σ_2) , are randomly distributed with law

$$\psi_0(d\lambda_1, d\lambda_2) = (2\pi)^{-1} \delta(\lambda_1 - \lambda_2) d\lambda_1 d\lambda_2 \quad (6)$$

- (iv) The subsystems σ_1 and σ_2 evolve independently (locality).
- (v) Time is discretized and 1 is the instant of simultaneous measurement of both systems.
- (vi) At time 1 two observables, $\tilde{S}_a^{(1)}$ of σ_1 and $\tilde{S}_b^{(2)}$ of σ_2 are measured and the result of the measurement is uniquely determined by the state (λ_1, λ_2) of the system at time 0 (pre-determination).
- (vii) The (Heisenberg) time evolution of the system realizes the chameleon effect (i.e. depends on the observables we want to measure) and it is assumed to have the form

$$P(f \otimes g)(\lambda_1, \lambda_2) = T'_{1,a}(\lambda_1) f(T_{1,a}(\lambda_1)) T'_{2,b}(\lambda_2) g(T_{2,b}(\lambda_2))$$

where

$$T'_{1,a}(\lambda_1) = \sqrt{2\pi} \quad , \quad T'_{2,b}(\lambda_2) = \frac{\sqrt{2\pi}}{4} \left| \cos(\lambda_2 - b) \right|$$

Notice that the dynamics is deterministic and entirely local: the evolution of particle 1 (resp. 2) depends only on a (resp. b). It is not an automorphism because it describes the reduced evolution after averaging over the reservoir degrees of freedom of the apparatus (cf. Section 3). Notice in addition that the dual evolution P^* ($P^*\psi(F) := \psi(P(F))$) applied to the initial probability measure (6) still gives a probability measure. Thus P is a generalized Markovian operator (in the sense that $P^*\psi$ is a probability measure not for all probability measures ψ , but only for a convex subset \mathcal{S}_o of them, called the set of admissible initial states). Within this class the interpretation of P as a reduced evolution is justified. Recently [AcImRe01] it has been proved that this evolution is indeed obtained by taking partial expectation of an appropriate automorphic evolution.

Under these conditions, if the observables are defined by the conditions

$$\tilde{S}_a^{(1)}(T_{1,a}(\lambda_1)) = \text{sgn}(\cos(\lambda_1 - a)) \quad , \quad \tilde{S}_b^{(2)}(T_{2,b}(\lambda_2)) = -\text{sgn}(\cos(\lambda_2 - b))$$

then their correlations

$$\begin{aligned} \langle S_a^{(1)} S_b^{(2)} \rangle &:= \int \int P \left(\tilde{S}_a^{(1)} \otimes \tilde{S}_b^{(2)} \right) (\lambda_1, \lambda_2) \psi_0(d\lambda_1, d\lambda_2) = \\ &= \int_0^{2\pi} \frac{d\lambda}{2\pi} \tilde{S}_a^{(1)}(T_{1,a}(\lambda_1)) T'_{1,a}(\lambda) \tilde{S}_b^{(2)}(T_{2,b}(\lambda_2)) T'_{2,b}(\lambda) \end{aligned} \quad (7)$$

satisfy condition (4) hence they violate Bell's inequality. The Schrödinger representation of the correlations (7) is

$$\begin{aligned} \langle S_a^{(1)} S_b^{(2)} \rangle &:= \int \int \tilde{S}_a^{(1)} \otimes \tilde{S}_b^{(2)} (\lambda_1, \lambda_2) P^*(\psi_0)(d\lambda_1, d\lambda_2) = \\ &= \int \int \tilde{S}_a^{(1)}(\mu_1) \tilde{S}_b^{(2)}(\mu_2) \delta(T_{1,a}^{-1}\mu_1 - T_{2,b}^{-1}\mu_2) \frac{d\mu_1 d\mu_2}{2\pi} \end{aligned} \quad (8)$$

which, since $P^*(\psi_0)$ is easily seen to be a probability measure, shows that our model indeed produces standard correlations of ± 1 -valued observables.

Notice that, in terms of $S_x^{(j)}(\lambda) := \tilde{S}_x^{(j)}(T_{j,x}(\lambda))$ ($j = 1, 2$, $x = a, b$) and $P_{a,b}(d\lambda) := T'_{1,a}(\lambda) T'_{2,b}(\lambda) d\lambda / 2\pi$, (7) reads as an average over an observable-dependent probability measure:

$$\langle S_a^{(1)} S_b^{(2)} \rangle = \int_0^{2\pi} S_a^{(1)}(\lambda) S_b^{(2)}(\lambda) P_{a,b}(d\lambda)$$

The measure $P_{a,b}(d\lambda)$, which arises from integration of the δ -function, is a global object, however the factorization of its density $T'_{1,a}(\lambda) T'_{2,b}(\lambda)$ reflects the local action of the environment and allows local simulation.

In summary: we have proved that the correlations (6), of our dynamical system, are equal to (5) and we use (5) for the local simulation.

Further details on this model can be found in [AcRe01a], [AcRe00b].

Description of the experiment:

The experiment uses three computers, A , B and C , which can be situated in different towns and exchange data via internet. Due to the identity between the right hand side of (7) and the right hand side of (6), we use the form (6) of the integral, which is simpler to implement in the computer. It consists of the following steps:

- (1) Computer C generates a random number $\lambda \in [0, 2\pi]$ and it is sent to computers A and B .
- (2) Computer A choses a value $a \in [0, 2\pi]$ and evaluates $T'_{1,a}(\lambda)\tilde{S}_a^{(1)}(T_{1,a}(\lambda))$ from the λ -value generated by Computer C . The value is sent back to Computer C .
- (3) Computer B choses a value $b \in [0, 2\pi]$ and evaluates $T'_{2,b}(\lambda)\tilde{S}_b^{(2)}(T_{2,b}(\lambda))$ from the λ -value generated by Computer C . The value is sent back to Computer C .
- (4) Based on the data sent back from Computers A and B , the correlation $\langle S_a^{(1)}S_b^{(2)} \rangle$ is calculated following (5) and the violation of Bell's inequality is checked.

References

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